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REPRESENTATIONS OF ARIKI–KOIKE ALGEBRAS AND GRÖBNER–SHIRSHOV BASES

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Introduction

In the development of representation theory, the Hecke algebras have played a significant role, providing combinatorial and geometric aspects of the theory. A natural generalization of the finite Hecke algebra of type A or type B is the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$, since it is a deformation of the group algebra of the complex reflection group $G(r,1,n)$. We recover the finite Hecke algebra of type A or B when $r=1$ or 2, respectively. (See § 2 for the definitions.) The significance of the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$ is growing even more rapidly after it was discovered that there is a close connection between the representation ring of $\mathcal{H}_{r,1,n}$ and the crystal bases of the quantum affine algebra $U_q(A_{r-1}^{(1)})$ [**1**, **3**, **10**].

In [**5**], Graham and Lehrer showed that for any ring R and any multipartition λ of n, there exists a right $\mathcal{H}_{r,1,n}$-module S_R^λ, called the Specht module, and that when R is a field, every irreducible $\mathcal{H}_{r,1,n}$-module appears as the simple quotient of S_R^λ for some λ. Actually, the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$ is semisimple if and only if the Specht module S_R^λ is irreducible for each multipartition λ of n. Thus, the study of Specht modules is the first step towards the understanding of the representations of Ariki–Koike algebras.

The purpose of this paper is to investigate the structure of Specht modules over Ariki–Koike algebras $\mathcal{H}_{r,1,n}$. We construct the Specht module S^λ as a quotient of $\mathcal{H}_{r,1,n}$, obtaining a presentation given by generators and relations. One of the main ingredients of our approach is the Gröbner–Shirshov basis theory for the representations of associative algebras developed in [**7**, **8**], where one can find the motivation and applications as well as the exposition of the theory. This approach naturally enables us to construct a linear basis of S^λ consisting of standard monomials. The other main ingredient of our approach is the combinatorics of Young tableaux: all the relations that hold in S^λ are expressed in terms of Young tableaux of shape λ. Moreover, we show that the standard monomials are in 1-1 correspondence with the cozy tableaux of shape λ. The results of this paper can be considered not only as a generalization but also as a refinement of the results on the finite Hecke algebras of type A obtained in [**9**]. In some of the proofs in this paper, we quote the results and notation of [**9**] without presenting them thoroughly. The readers may refer to [**9**] for more details.

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In [**4**], Dipper, James and Mathas defined the cellular structure on the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$ and constructed a linear basis of S^λ whose elements are parametrized by standard tableaux of shape λ. Compared with their results, our approach provides a different combinatorial method to yield an explicit monomial basis that can be identified with the set of cozy tableaux of shape λ.

The contents of this paper is organized as follows. In the first section, we briefly explain the Gröbner–Shirshov basis theory for the representations of associative algebras developed in [**7**, **8**]. In § 2, we recall the definition of the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$ and determine its Gröbner–Shirshov basis, which yields a monomial basis of $\mathcal{H}_{r,1,n}$. In § 3, we recollect some of the standard facts about Specht modules over Ariki–Koike algebras. In the final section, we determine the Gröbner–Shirshov pair for the Specht module S^λ and obtain a presentation given by generators and relations. Furthermore, we construct a linear basis of S^λ consisting of standard monomials with respect to the Gröbner–Shirshov pair.

1. *Gröbner–Shirshov pair*

In this section, we briefly recall the Gröbner–Shirshov basis theory for the representations of associative algebras which was developed in [**7**, **8**].

Let X be a set and let X^* be the free monoid generated by X; that is, the set of all associative monomials on X including the empty monomial 1. We denote by $l(u)$ the *length* (or *degree*) of a monomial u with the convention $l(1) = 0$. A well-ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$.

Example 1.1. Let $X = \{x_1, x_2, \ldots\}$. For

$$u = x_{i_1} x_{i_2} \ldots x_{i_k} \quad \text{and} \quad v = x_{j_1} x_{j_2} \ldots x_{j_l} \in X^*,$$

we define $u \prec_{\text{deg-lex}} v$ if $k < l$ or $k = l$ and $i_r < j_r$ for the first r such that $i_r \neq j_r$. Then it is a monomial order on X^* called the *degree lexicographic order*.

Fix a monomial order $\prec$ on X^* and let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a non-zero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial appearing in p under the ordering $\prec$. If the coefficient of $\overline{p}$ is 1, p is said to be *monic*.

Let (S, T) be a pair of subsets of monic elements of $\mathcal{A}_X$, let J be the two-sided ideal of $\mathcal{A}_X$ generated by S, and let I be the right ideal of the algebra $A = \mathcal{A}_X/J$ generated by (the image of) T. Then we say that the algebra $A = A_S = \mathcal{A}_X/J$ is *defined by* S and that the right A-module $M = M_{S,T} = A/I$ is *defined by the pair* (S, T). The images of $p \in \mathcal{A}_X$ in A and in M under the canonical quotient maps will also be denoted by p.

In the rest of this section, we assume that (S, T) is a pair of subsets of monic elements of $\mathcal{A}_X$.

Definition 1.2. A monomial $u \in X^*$ is said to be (S, T)-*standard* if $u \neq a\overline{s}b$ and $u \neq \overline{t}c$ for any $s \in S$, $t \in T$ and $a, b, c \in X^*$. Otherwise, the monomial u is said to be (S, T)-*reducible*. If $T = \emptyset$, we will simply say that u is S-*standard* or S-*reducible*.

PROPOSITION 1.3 [**7**, **8**]. *The set of (S,T)-standard monomials spans the right A-module $M = A/I$ defined by the pair (S,T).*

DEFINITION 1.4. Let $M = A/I$ be the right A-module defined by the pair (S,T). We say that (S,T) is a *Gröbner–Shirshov pair* for the module M if the set of (S,T)-standard monomials forms a linear basis of the right A-module M. If a pair $(S,\emptyset)$ is a Gröbner–Shirshov pair, then we also say that S is a *Gröbner–Shirshov basis* for the algebra $A = \mathcal{A}_X/J$ defined by S.

Let p and q be monic elements of $\mathcal{A}_X$ with leading terms $\overline{p}$ and $\overline{q}$. We define the composition of p and q as follows.

DEFINITION 1.5. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p,q)_w = pa - bq$. Furthermore, if $b = 1$, the composition $(p,q)_w$ is called *left-justified*.

(b) If there exist a and b in X^* such that $b \neq 1$ and $\overline{p} = a\overline{q}b = w$, then the *composition of inclusion* is defined to be $(p,q)_w = p - aqb$.

REMARK. To see some examples of compositions, one may refer to [**8**, Example 2.2].

For $p, q \in \mathcal{A}_X$ and $w \in X^*$, we define a *congruence relation* on $\mathcal{A}_X$ as follows:

$$p \equiv q \bmod (S,T;w) \iff p - q = \sum \alpha_i a_i s_i b_i + \sum \beta_j t_j c_j,$$

where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, $t_j \in T$, $a_i \overline{s_i} b_i \prec w$, and $\overline{t_j} c_j \prec w$. When $T = \emptyset$, we simply write $p \equiv q \bmod (S;w)$.

DEFINITION 1.6. A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if

(i) $(p,q)_w \equiv 0 \bmod (S;w)$ for all $p, q \in S$ and $w \in X^*$ whenever the composition $(p,q)_w$ is defined,

(ii) $(p,q)_w \equiv 0 \bmod (S,T;w)$ for all $p, q \in T$ and $w \in X^*$ whenever the left-justified composition $(p,q)_w$ is defined,

(iii) $(p,q)_w \equiv 0 \bmod (S,T;w)$ for all $p \in T$, $q \in S$ and $w \in X^*$ whenever the composition $(p,q)_w$ is defined.

If $T = \emptyset$, we will simply say that S is closed under composition.

The main ingredient of Gröbner–Shirshov basis theory is the following theorem.

THEOREM 1.7 [**7**, **8**]. *A pair (S,T) of subsets of monic elements of $\mathcal{A}_X$ is a Gröbner–Shirshov pair if and only if the pair (S,T) is closed under composition.*

2. *Ariki–Koike algebras*

From now on, we take $\prec = \prec_{\text{deg-lex}}$ as our monomial order. Let $\mathbb{F}$ be a field and fix a non-zero scalar $q \in \mathbb{F}^\times$. For each $r \geqslant 1$ and $n \geqslant 1$, let $G(r,1,n)$ be the wreath product of the cyclic group $C_r = \{1, s_0, \ldots, s_0^{r-1}\}$ and the symmetric group $S_n = \langle s_1, \ldots, s_{n-1} \rangle$. Then the group $G(r,1,n)$ is a complex reflection group generated by the elements $s_0, s_1, \dots, s_{n-1}$ with defining relations

$$\begin{aligned} &s_0^r = s_i^2 = 1 && \text{for } 1 \leqslant i \leqslant n-1, \\ &s_i s_j = s_j s_i && \text{for } 1 \leqslant j+1 < i \leqslant n-1, \\ &s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i && \text{for } 1 \leqslant i \leqslant n-2, \\ &s_1 s_0 s_1 s_0 = s_0 s_1 s_0 s_1. \end{aligned} \tag{2.1}$$

DEFINITION 2.1. Let $q \in \mathbb{F}^\times$ and $Q_1, \dots, Q_r \in \mathbb{F}$. The *Ariki–Koike algebra* $\mathcal{H}_{r,1,n}$ is an associative algebra over $\mathbb{F}$ generated by the elements $T_0, T_1, \dots, T_{n-1}$ subject to the defining relations

$$R_{\mathrm{AK}}\colon\quad \begin{aligned} &(T_0 - Q_1)(T_0 - Q_2) \dots (T_0 - Q_r) = 0, \\ &T_i^2 = (q-1)T_i + q && \text{for } 1 \leqslant i \leqslant n-1, \\ &T_i T_j = T_j T_i && \text{for } 1 \leqslant j+1 < i \leqslant n-1, \\ &T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i && \text{for } 1 \leqslant i \leqslant n-2, \\ &T_1 T_0 T_1 T_0 = T_0 T_1 T_0 T_1. \end{aligned} \tag{2.2}$$

We write $T_{i,j} = T_i T_{i-1} \dots T_j$ and $T^{j,i} = T_j T_{j+1} \dots T_i$ for $i \geqslant j$ (hence $T_{i,i} = T_i$ and $T^{i,i} = T_i$) with the convention $T_{i,i+1} = T^{i+1,i} = 1$ (for $i \geqslant 0$).

LEMMA 2.2. *The following relations hold in* $\mathcal{H}_{r,1,n}$.
(a) *For* $1 \leqslant j \leqslant i$, *we have*

$$T_{i+1,j} T_{i+1} = T_i T_{i+1,j}.$$

(b) *For* $i > j+1 \geqslant 1$ *and* $1 \leqslant k \leqslant r-1$, *we have*

$$T_{i,1} T_0^k T^{1,j} T_i = T_{i-1} T_{i,1} T_0^k T^{1,j}.$$

(c) *For* $1 \leqslant k \leqslant r-1$, *we have*

$$T_1 T_0^k T_1 T_0 = T_0 T_1 T_0^k T_1 - (q-1) T_0 T_1 T_0^k + (q-1) T_0^k T_1 T_0.$$

(d) *For* $i \geqslant j \geqslant 2$ *and* $1 \leqslant k \leqslant r-1$, *we have*

$$T_{i,1} T_0^k T^{1,j} T_{j-1} = T_{j-1} T_{i,1} T_0^k T^{1,j}.$$

Proof. (a) Since the subalgebra of $\mathcal{H}_{r,1,n}$ generated by $T_1, T_2, \dots, T_{n-1}$ is nothing but the Hecke algebra $\mathcal{H}_n(q)$ of type A, our relations follow from [**9**, Lemma 3.1].

(b) For $2 \leqslant i \leqslant n-1$, the relation $T_i T_0 = T_0 T_i$ yields $T_i T_0^k = T_0^k T_i$ (for $k \geqslant 1$). It follows that

$$T_{i,1} T_0^k T_i = T_{i,1} T_i T_0^k = T_{i-1} T_{i,1} T_0^k.$$

Since $T^{1,j}$ commutes with T_i for $1 \leqslant j \leqslant i$, we obtain

$$T_{i,1} T_0^k T^{1,j} T_i = T_{i,1} T_0^k T_i T^{1,j} = T_{i-1} T_{i,1} T_0^k T^{1,j} \quad (1 \leqslant k \leqslant r-1),$$

as desired.

(c) The relations $T_0^k T_1 T_0 T_1 = T_1 T_0 T_1 T_0^k$ and $T_1^{-1} = q^{-1}(T_1 - (q-1))$ imply

$$\begin{aligned} T_1 T_0^k T_1 T_0 &= T_1^2 T_0 T_1 T_0^k T_1^{-1} \\ &= (q + (q-1)T_1) T_0 T_1 T_0^k T_1^{-1} \\ &= q T_0 T_1 T_0^k T_1^{-1} + (q-1) T_1 T_0 T_1 T_0^k T_1^{-1} \\ &= T_0 T_1 T_0^k (T_1 - (q-1)) + (q-1) T_0^k T_1 T_0 \\ &= T_0 T_1 T_0^k T_1 - (q-1) T_0 T_1 T_0^k + (q-1) T_0^k T_1 T_0. \end{aligned}$$

(d) From (b), we have $T_{j,1} T_0^k T_j = T_{j-1} T_{j,1} T_0^k$, which implies

$$T_{i,1} T_0^k T_j = T_{i,j+1} T_{j-1} T_{j,1} T_0^k = T_{j-1} T_{i,1} T_0^k.$$

Hence we obtain

$$\begin{aligned} T_{i,1} T_0^k T^{1,j} T_{j-1} &= T_{i,1} T_0^k T^{1,j-2} T_{j-1} T_j T_{j-1} \\ &= T_{i,1} T_0^k T^{1,j-2} T_j T_{j-1} T_j \\ &= T_{i,1} T_0^k T_j T^{1,j} \\ &= T_{j-1} T_{i,1} T_0^k T^{1,j}. \end{aligned}$$

□

In the following proposition, we determine a Gröbner–Shirshov basis for $\mathcal{H}_{r,1,n}$ with respect to the monomial order $\prec$.

PROPOSITION 2.3. *The following relations form a Gröbner–Shirshov basis for the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$ with respect to the monomial order $\prec$ $(= \prec_{\text{deg-lex}})$:*

$$\mathcal{R}_{\text{AK}}: \begin{array}{ll} T_i T_j - T_j T_i & (i > j+1 \geqslant 1), \\ T_i^2 - (q-1)T_i - q & (1 \leqslant i \leqslant n-1), \\ T_{i+1,j} T_i - T_i T_{i+1,j} & (i \geqslant j \geqslant 1), \\ (T_0 - Q_1)(T_0 - Q_2) \dots (T_0 - Q_r), & \\ T_{i,1} T_0^k T^{1,j} T_i - T_{i-1} T_{i,1} T_0^k T^{1,j} & (i > j+1 \geqslant 1, 1 \leqslant k \leqslant r-1), \\ T_1 T_0^k T_1 T_0 - T_0 T_1 T_0^k T_1 & \\ \quad +(q-1) T_0 T_1 T_0^k - (q-1) T_0^k T_1 T_0 & (1 \leqslant k \leqslant r-1), \\ T_{i,1} T_0^k T^{1,j} T_{j-1} - T_{j-1} T_{i,1} T_0^k T^{1,j} & (i \geqslant j \geqslant 2, 1 \leqslant k \leqslant r-1). \end{array} \qquad (2.3)$$

The set of $\mathcal{R}_{\text{AK}}$-standard monomials is given by

$$\mathcal{B}_{\text{AK}} = \{ T_{0,a_0}^{(k_0)} T_{1,a_1}^{(k_1)} T_{2,a_2}^{(k_2)} \dots T_{n-1,a_{n-1}}^{(k_{n-1})} \mid 1 \leqslant a_i \leqslant i+1,\ 0 \leqslant k_i \leqslant r-1 \}, \qquad (2.4)$$

where

$$T_{i,j}^{(k)} = \begin{cases} T_{i,j} & \text{for } k = 0, \\ T_{i,1} T_0^k T^{1,j-1} & \text{for } 1 \leqslant k \leqslant r-1. \end{cases}$$

Proof. Observe that if $i = 0$, then $a_0 = 1$ and $T_{0,a_0}^{(k_0)} = T_0^{k_0}$ for $0 \leqslant k_0 \leqslant r-1$. By the definition of $\mathcal{H}_{r,1,n}$ and Lemma 2.2, it is easy to see that all the relations in $\mathcal{R}_{\text{AK}}$ hold in $\mathcal{H}_{r,1,n}$ and that $\mathcal{B}_{\text{AK}}$ is the set of $\mathcal{R}_{\text{AK}}$-standard monomials. By

Proposition 1.3, the set $\mathcal{B}_{\mathrm{AK}}$ spans the algebra $\mathcal{H}_{r,1,n}$. Note that the number of elements in $\mathcal{B}_{\mathrm{AK}}$ is $r^n n!$. Since $\dim \mathcal{H}_{r,1,n} = r^n n!$, $\mathcal{B}_{\mathrm{AK}}$ must be a linear basis of $\mathcal{H}_{r,1,n}$. Hence, by Definition 1.4, $\mathcal{R}_{\mathrm{AK}}$ is a Gröbner–Shirshov basis for the algebra $\mathcal{H}_{r,1,n}$. □

3. *Specht modules*

A *composition* λ of n, denoted by $\lambda \vDash n$, is a sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of non-negative integers such that $|\lambda| := \sum_{j=1}^{k} \lambda_j = n$. A *partition* λ of n, denoted by $\lambda \vdash n$, is a composition such that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k$.

The *diagram* of a composition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is defined to be the set

$$[\lambda] = \{(i, j) \mid 1 \leqslant j \leqslant \lambda_i \text{ and } i \geqslant 1\}.$$

A *tableau of shape* λ (or a λ-*tableau*) is a map $t : [\lambda] \to \{1, 2, \ldots, n\}$. We also denote by $[t]$ the diagram $[\lambda]$ corresponding to the tableau t. A λ-tableau t is *row standard* if the entries in t are increasing from left to right in each row. A row standard λ-tableau is said to be *standard* if λ is a partition and the entries in t are increasing from top to bottom in each column.

A *multicomposition* of n is an ordered r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of compositions such that $|\lambda| := |\lambda^{(1)}| + \ldots + |\lambda^{(r)}| = n$. We call $\lambda^{(k)}$ the *kth component* of λ. A multicomposition λ is called a *multipartition* if each $\lambda^{(k)}$ is a partition.

For a multicomposition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of n, we define

$$(k, i)_\lambda = \sum_{l=1}^{k-1} |\lambda^{(l)}| + \sum_{l=1}^{i} \lambda_l^{(k)} \quad (i \geqslant 0,\ 1 \leqslant k \leqslant r). \tag{3.1}$$

We introduce a partial ordering on the set of multicompositions as follows:

$$\lambda \trianglerighteq \mu \quad \text{if and only if} \quad (k, i)_\lambda \geqslant (k, i)_\mu \text{ for all } i \geqslant 0,\ 1 \leqslant k \leqslant r.$$

Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be a multicomposition of n. The *diagram* of λ is the set

$$[\lambda] = \{(i, j, k) \mid 1 \leqslant j \leqslant \lambda_i^{(k)}, i \geqslant 1, 1 \leqslant k \leqslant r\}.$$

A *tableau of shape* λ (or a λ-*tableau*) is a map $\mathfrak{t} : [\lambda] \to \{1, 2, \ldots, n\}$. There is a natural action of S_n on the set of all bijective λ-tableaux. We identify a diagram $[\lambda]$ with an r-tuple of diagrams $([\lambda^{(1)}], \ldots, [\lambda^{(r)}])$ in an obvious way. Similarly, we identify a λ-tableau $\mathfrak{t}$ with an r-tuple of tableaux $(\mathfrak{t}^{(1)}, \ldots, \mathfrak{t}^{(r)})$.

A tableau $\mathfrak{t}$ is called *row standard* (respectively *standard*) if each $\mathfrak{t}^{(k)}$ is row standard (respectively standard). We denote by $\mathrm{RS}(\lambda)$ (respectively $\mathrm{ST}(\lambda)$) the set of all row standard (respectively standard) tableaux of shape λ.

For a row standard tableau $\mathfrak{t}$, we denote by $\mathfrak{t} \downarrow m$ the tableau obtained from $\mathfrak{t}$ by deleting all the entries greater than m. Given two row standard tableaux, we define

$$\mathfrak{s} \trianglerighteq \mathfrak{t} \quad \text{if and only if} \quad [\mathfrak{s} \downarrow m] \trianglerighteq [\mathfrak{t} \downarrow m] \text{ for all } 1 \leqslant m \leqslant n.$$

Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ be a multipartition of n. We define a λ-tableau $\mathfrak{t}^\lambda$ by

$$\mathfrak{t}^\lambda(i, j, k) = (k, i-1)_\lambda + j. \tag{3.2}$$

For each standard λ-tableau $\mathfrak{t}$, we denote by $d(\mathfrak{t})$ the element of S_n such that $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$. We also denote by $W_\lambda = W_{\lambda^{(1)}} \times \ldots \times W_{\lambda^{(r)}}$ the row stabilizer of $\mathfrak{t}^\lambda$, where $W_{\lambda^{(k)}}$ is regarded as a subgroup of the symmetric group on the set $\{(k,0)_\lambda + 1, \dots, (k+1,0)_\lambda\}$.

EXAMPLE 3.1. If

$$\lambda = \left(\begin{array}{|c|c|c|} \hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline \ & \ \\ \hline \ & \ \\ \hline \end{array} \right),$$

then

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|} \hline 7 & 8 \\ \hline 9 & 10 \\ \hline \end{array} \right),$$

and we have

$$\begin{aligned} W_\lambda &= W_{(3,2,1)} \times W_{(2,2)} \\ &= \big(S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6\}}\big) \times \big(S_{\{7,8\}} \times S_{\{9,10\}}\big) \\ &\cong S_3 \times S_2 \times S_2 \times S_2. \end{aligned}$$

We would like to introduce a *cellular structure* on the Ariki–Koike algebra $\mathcal{H}_{r,1,n}$. For $i = 1, \dots, n$, we define

$$L_i = q^{-i+1} T_{i-1,1} T_0 T^{1,i-1}. \tag{3.3}$$

Then the following lemma is well known.

LEMMA 3.2 [**2**, **4**]. *For* $1 \leqslant i \leqslant n-1$ *and* $1 \leqslant j \leqslant n$, *we have*

$$\begin{aligned} &L_i L_j = L_j L_i, \\ &T_i L_j = L_j T_i \quad \text{for } i \neq j-1, j, \\ &T_i L_i L_{i+1} = L_{i+1} L_i T_i, \\ &T_i(L_i + L_{i+1}) = (L_i + L_{i+1}) T_i, \\ &T_i(L_1 - a) \dots (L_j - a) = (L_1 - a) \dots (L_j - a) T_i \quad \text{for } i \neq j, a \in \mathbb{F}. \end{aligned} \tag{3.4}$$

For a reduced expression $w = \tau_{i_1} \dots \tau_{i_k} \in S_n$ (with $1 \leqslant i_j \leqslant n-1$), set

$$T_w = T_{i_1} \dots T_{i_k} \in \mathcal{H}_{r,1,n}$$

and define an anti-automorphism $* : \mathcal{H}_{r,1,n} \to \mathcal{H}_{r,1,n}$ by

$$T_i^* = T_i \quad \text{for } i = 0, 1, \dots, n-1.$$

We also define

$$\begin{aligned} u_{\lambda,k} &= \prod_{i=1}^{(k,0)_\lambda} (L_i - Q_k) \quad (1 \leqslant k \leqslant r), \\ u_\lambda &= u_{\lambda,1} u_{\lambda,2} \dots u_{\lambda,r}, \\ x_\lambda &= \sum_{w \in W_\lambda} T_w, \quad m_\lambda = u_\lambda x_\lambda, \\ m_{\mathfrak{s},\mathfrak{t}} &= T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}, \end{aligned} \tag{3.5}$$

where $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux of shape λ.

With this notation, we have the following result.

PROPOSITION 3.3 [**4**]. *The Ariki–Koike algebra $\mathcal{H}_{r,1,n}$ has a cellular basis*

$$\{m_{\mathfrak{s},\mathfrak{t}} \mid \mathfrak{s} \text{ and } \mathfrak{t} \text{ are standard } \lambda\text{-tableaux for some multipartition } \lambda \text{ of } n\}.$$

For a multipartition λ of n, let N^λ (respectively $\overline{N}^\lambda$) be the $\mathbb{F}$-subspace of $\mathcal{H}_{r,1,n}$ spanned by all $m_{\mathfrak{s},\mathfrak{t}}$, where $\mathfrak{s}$ and $\mathfrak{t}$ are taken over all standard μ-tableaux for some multipartition μ of n with $\mu \trianglerighteq \lambda$ (respectively $\mu \triangleright \lambda$). Let $M^\lambda = m_\lambda \mathcal{H}_{r,1,n}$ and set $\overline{M}^\lambda = M^\lambda \cap \overline{N}^\lambda$.

DEFINITION 3.4. The $\mathcal{H}_{r,1,n}$-module $S^\lambda = M^\lambda / \overline{M}^\lambda$ is called the *Specht module* corresponding to the multipartition λ.

PROPOSITION 3.5 [**4**]. *The Specht module S^λ has a basis consisting of the vectors $m_{\mathfrak{t}^\lambda \mathfrak{s}} + \overline{M}^\lambda$, where $\mathfrak{s}$ runs over all standard tableaux of shape λ.*

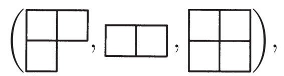
Let $x = (a, b, c) \in [\lambda]$ be a node in $[\lambda]$. The *residue* of x is defined to be

$$\operatorname{res}(x) = q^{b-a} Q_c. \tag{3.6}$$

For $i = 1, 2, \ldots, n$, we write

$$\operatorname{res}(i) = \operatorname{res}(x),$$

where x is the unique node in $[\lambda]$ such that $\mathfrak{t}^\lambda(x) = i$. For example, if

$\lambda =$ 

then

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 8 & 9 \\ \hline \end{array} \right),$$

and we have

$$\operatorname{res}(1) = Q_1, \quad \operatorname{res}(5) = qQ_2, \quad \operatorname{res}(8) = q^{-1}Q_3.$$

LEMMA 3.6 [**6**, Proposition 3.7]. *In S^λ, we have*

$$m_\lambda L_i = \operatorname{res}(i) m_\lambda \quad \textit{for all } i = 1, 2, \ldots, n.$$

In the next section, (some of) the relations in the above lemma will be included in the set of defining relations of S^λ. (See (4.1) and the argument following it.) As a result, we will obtain a monomial basis of S^λ consisting of *T_0-free monomials*; that is, the monomials on $T_1, T_2, \ldots, T_{n-1}$ only. Keeping this in our mind, we introduce the notion of semi-cozy tableaux and the corresponding set of monomials $\mathcal{B}_H$ as follows.

DEFINITION 3.7. A λ-tableau $\mathfrak{t} : [\lambda] \to \{1, 2, \ldots, n\}$ is said to be *semi-cozy* if

$$1 \leqslant \mathfrak{t}(i, j, k) \leqslant \mathfrak{t}^\lambda(i, j, k) \quad \text{for all } (i, j, k) \in [\lambda]. \tag{3.7}$$

The set of all semi-cozy λ-tableaux will be denoted by $\mathrm{SC}(\lambda)$. We define a set $\mathcal{B}_H$

of monomials to be

$$\mathcal{B}_H = \{T_{1,a_1}T_{2,a_2}\dots T_{n-1,a_{n-1}} \,|\, 1 \leqslant a_k \leqslant k+1, k=1,2,\dots,n-1\}.$$

For a monomial $T_{1,a_1}T_{2,a_2}\dots T_{n-1,a_{n-1}} \in \mathcal{B}_H$, we associate the semi-cozy λ-tableau $\mathfrak{t} : [\lambda] \to \{1,2,\dots,n\}$ defined by

$$(i,j,k) \mapsto a_{(k,i-1)_\lambda + j - 1} \quad (a_0 = 1). \tag{3.8}$$

Conversely, if $\mathfrak{t}$ is a semi-cozy tableau, we can read off the corresponding monomial by defining

$$a_l = \mathfrak{t}(i,j,k) \quad (l \geqslant 1), \tag{3.9}$$

where (i,j,k) is the node of the $(l+1)$th entry of $\mathfrak{t}$ reading a (single) tableau from left to right and from top to bottom and proceeding from left to right in the r-tuple of tableaux. In this way, the set $\mathrm{SC}(\lambda)$ of semi-cozy λ-tableau is identified with the set $\mathcal{B}_H$. (Note that our identification of $\mathrm{SC}(\lambda)$ is not with $\mathcal{B}_{\mathrm{AK}}$ but with $\mathcal{B}_H$.)

EXAMPLE 3.8. Assume that

$$\lambda = \left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline \\ \hline\end{array}, \begin{array}{|c|c|}\hline & \\ \hline\end{array} \right).$$

Then

$$T_{1,2}T_{2,3}T_{3,4}T_{4,5}T_{5,6} = 1 \longleftrightarrow \mathfrak{t}^\lambda = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline\end{array} \right),$$

$$T_{1,2}T_{2,1}T_{3,4}T_{4,3}T_{5,1} = T_{2,1}T_{4,3}T_{5,1} \longleftrightarrow \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 1 \\ \hline\end{array} \right),$$

$$T_{1,1}T_{2,1}T_{3,1}T_{4,1}T_{5,1} \longleftrightarrow \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \right).$$

For each $i \geqslant 2$, $j \geqslant 1$ and $1 \leqslant k \leqslant r$, we define the (i,j,k)-*Garnir tableau* $\mathfrak{t}^\lambda_{i,j,k}$ by

$$\mathfrak{t}^\lambda_{i,j,k}(a,b,c) = \begin{cases} (k,i-2)_\lambda + j + b - 1 & \text{if } c = k,\ a = i,\ 1 \leqslant b \leqslant j, \\ \mathfrak{t}^\lambda(a,b,c) & \text{otherwise.} \end{cases} \tag{3.10}$$

We also define $\Sigma^\lambda_{i,j,k}$ to be the sum of all λ-tableaux $\mathfrak{t}$ (or the corresponding monomials in $\mathcal{B}_H$) in $\mathrm{RS}(\lambda) \cap \mathrm{SC}(\lambda)$ such that $\mathfrak{t}^\lambda \trianglelefteq \mathfrak{t} \trianglelefteq \mathfrak{t}^\lambda_{i,j,k}$; that is,

$$\Sigma^\lambda_{i,j,k} := \sum_{\substack{\mathfrak{t} \in \mathrm{RS}(\lambda) \cap \mathrm{SC}(\lambda), \\ \mathfrak{t}^\lambda \trianglelefteq \mathfrak{t} \trianglelefteq \mathfrak{t}^\lambda_{i,j,k}}} \mathfrak{t}. \tag{3.11}$$

For example, if

$$\lambda = \left(\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline & \\ \hline\end{array}, \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array} \right),$$

then we have

$$\mathfrak{t}^{\lambda}_{2,1,3} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 6 & 9 \\ \hline \end{array} \right),$$

$$\Sigma^{\lambda}_{2,1,3} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 8 & 9 \\ \hline \end{array} \right) + \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 7 & 9 \\ \hline \end{array} \right)$$
$$+ \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 6 & 9 \\ \hline \end{array} \right).$$

LEMMA 3.9. *The following relations hold in* S^{λ}.
(a) *For* $i = 1, \dots, n-1$ *with* $i \neq (k,l)_{\lambda}$ (*for* $1 \leqslant k \leqslant r$ *and* $l \geqslant 1$), *we have*

$$m_{\lambda}(T_i - q) = 0.$$

(b) *For each* $i \geqslant 2$, $j \geqslant 1$, $1 \leqslant k \leqslant r$, *we have*

$$m_{\lambda}\Sigma^{\lambda}_{i,j,k} = 0.$$

Proof. (a) Since m_{λ} has a right factor $T_i + 1$ for $i \neq (k,l)_{\lambda}$ and $(T_i+1)(T_i-q) = 0$, we obtain the desired relation.

(b) Note that $m_{\lambda} = u_{\lambda}x_{\lambda^{(1)}} \dots x_{\lambda^{(r)}}$, where $x_{\lambda^{(k)}} = \sum_{w \in W_{\lambda^{(k)}}} T_w$. Clearly, the $x_{\lambda^{(k)}}$ commute with each other.

Assume that $h \in \mathcal{H}_{r,1,n}$ is contained in the subalgebra of $\mathcal{H}_{r,1,n}$ generated by

$$\{T_{(k,0)_{\lambda}+1}, \dots, T_{(k+1,0)_{\lambda}-1}\},$$

where the indices $(k,i)_{\lambda}$ are defined in (3.1). Then we can consider h as an element of the Hecke algebra $\mathcal{H}_{|\lambda^{(k)}|}(q)$ generated by $\{T_{(k,0)_{\lambda}+1}, \dots, T_{(k+1,0)_{\lambda}-1}\}$, since the relations among $T_{(k,0)_{\lambda}+1}, \dots, T_{(k+1,0)_{\lambda}-1}$ are the same as the defining relations of $\mathcal{H}_{|\lambda^{(k)}|}(q)$ given in [**9**]. Also we get the Specht module $S_q^{\lambda^{(k)}}$ over $\mathcal{H}_{|\lambda^{(k)}|}(q)$ defined in [**9**].

If $x_{\lambda^{(k)}}h = 0$ in $S_q^{\lambda^{(k)}}$, then by the definition of the Specht module $S_q^{\lambda^{(k)}}$ over $\mathcal{H}_{|\lambda^{(k)}|}(q)$, $x_{\lambda^{(k)}}h$ is a linear combination of elements of the form $x_{r,s}$, where r and s are standard ν-tableaux with entries $\{(k,0)_{\lambda}+1, \dots, (k+1,0)_{\lambda}\}$ for some partition $\nu \vdash |\lambda^{(k)}|$ with $\nu \rhd \lambda^{(k)}$. It follows from Lemma 3.2 that $m_{\lambda}h$ is a linear combination of elements of the form $m_{\mathfrak{s},\mathfrak{t}}$, where $\mathfrak{s}$ and $\mathfrak{t}$ are standard μ-tableaux with some multipartition $\mu \rhd \lambda$. Hence, by the definition of the Specht module over $\mathcal{H}_{r,1,n}$, we have $m_{\lambda}h = 0$ in S^{λ}.

Now, note that the element $\Sigma^{\lambda}_{i,j,k}$ is contained in the subalgebra of $\mathcal{H}_{r,1,n}$ generated by $\{T_{(k,0)_{\lambda}+1}, \dots, T_{(k+1,0)_{\lambda}-1}\}$. Furthermore, using the notation in [**9**], we see that $\Sigma^{\lambda}_{i,j,k}$ is exactly the same as

$$\sum_{\mathbf{a} \in C_{j,i,\lambda^{(k)}}} \langle (k,i-1)_{\lambda} + \mathbf{a} \rangle_{(k,i)_{\lambda}}.$$

Therefore, by [**9**, Lemma 3.4], we have $x_{\lambda^{(k)}}\Sigma^{\lambda}_{i,j,k} = 0$ in $S_q^{\lambda^{(k)}}$, and hence $m_{\lambda}\Sigma^{\lambda}_{i,j,k} = 0$ in S^{λ}. □

REMARK. The above lemma is well known to the experts and the relations in part (b) are called *Garnir relations*. One may deduce from the literature that the relations in the lemma are necessary to define the Specht module. But it is a non-trivial problem to present a sufficient (or minimal) set of relations. Using Gröbner–Shirshov basis theory, we will obtain such a set of relations in the next section.

4. *Gröbner–Shirshov pair for* S^λ

In this section, we will introduce an $\mathcal{H}_{r,1,n}$-module $\widehat{S}^\lambda$ defined by generators and relations and show that $\widehat{S}^\lambda$ is isomorphic to the Specht module S^λ by determining a Gröbner–Shirshov pair for $\widehat{S}^\lambda$.

Let e be the smallest positive integer such that $1+q+\ldots+q^{e-1}=0$; set $e=\infty$ if no such integer exists. And let $\lambda=(\lambda^{(1)},\ldots,\lambda^{(r)})$ be a multipartition of n. We define $\widehat{S}^\lambda$ to be the $\mathcal{H}_{r,1,n}$-module generated by $T_0,T_1,\ldots,T_{n-1}$ with defining relations given by the pair $(R_{\mathrm{AK}},R^\lambda_{\mathrm{AK}})$, where R^λ_{AK} is the set of relations:

$$\begin{aligned}
&T_i-q && (i\neq(k,l)_\lambda \text{ for } 1\leqslant k\leqslant r \text{ and } l\geqslant 1),\\
&\Sigma^\lambda_{i,1,k} && (i\geqslant 2, 1\leqslant k\leqslant r),\\
&\Sigma^\lambda_{i,le,k} && (e<\infty, i\geqslant 2, l\geqslant 1, 1\leqslant k\leqslant r),\\
&T_{(k,0)_\lambda,1}T_0T^{1,(k,0)_\lambda}-q^{(k,0)_\lambda}Q_k && (1\leqslant k\leqslant r).
\end{aligned}\tag{4.1}$$

In Lemma 3.9, we have seen that the first three relations hold in S^λ. The last relations follow from Lemma 3.6 and the simple observations

$$T_{(k,0)_\lambda,1}T_0T^{1,(k,0)_\lambda}=q^{(k,0)_\lambda}L_{(k,0)_\lambda+1} \quad\text{and}\quad \mathrm{res}((k,0)_\lambda+1)=Q_k.$$

Hence there exists a surjective $\mathcal{H}_{r,1,n}$-module homomorphism

$$\Psi:\widehat{S}^\lambda\to S^\lambda \quad \text{given by } 1\mapsto m_\lambda. \tag{4.2}$$

We claim that the map Ψ is actually an isomorphism. In other words, the Specht module S^λ can be viewed as the $\mathcal{H}_{r,1,n}$-module defined by the pair $(R_{\mathrm{AK}},R^\lambda_{\mathrm{AK}})$. The rest of this section will be devoted to proving this claim.

LEMMA 4.1. *The following relations hold in* $\widehat{S}^\lambda$.

(a) *For each* $\mathfrak{t}\in\mathrm{SC}(\lambda)$ *with* $\mathfrak{t}(i,j,k)\geqslant\mathfrak{t}(i,j+1,k)$ *for some* $(i,j,k)\in[\lambda]$, *we have*

$$\mathfrak{t}=q\,\mathfrak{t}',$$

where

$$\mathfrak{t}'(a,b,c)=\begin{cases}\mathfrak{t}(i,j+1,k) & \text{if } (a,b,c)=(i,j,k),\\ \mathfrak{t}(i,j,k)+1 & \text{if } (a,b,c)=(i,j+1,k),\\ \mathfrak{t}(a,b,c) & \text{otherwise.}\end{cases}$$

(b) *For each* $i\geqslant 2,\ j\geqslant 1,\ 1\leqslant k\leqslant r$, *we have*

$$\Sigma^\lambda_{i,j,k}=0.$$

(c) *For each row standard semi-cozy* λ*-tableau* $\mathfrak{t}\in\mathrm{SC}(\lambda)\cap\mathrm{RS}(\lambda)$ *with* $\mathfrak{t}(i,j,k)+j>\mathfrak{t}(i+1,j,k)$ *for some* $(i,j,k)\in[\lambda]$, *we have*

$$\mathfrak{t}=\sum_{\mathfrak{s}\prec\mathfrak{t}}a_{\mathfrak{t},\mathfrak{s}}\mathfrak{s} \quad \text{for some } \mathfrak{s}\in\mathrm{SC}(\lambda) \text{ and } a_{\mathfrak{t},\mathfrak{s}}\in\mathbb{F}.$$

(d) *Given* $1 \leqslant k \leqslant r$, *assume that* $\mathfrak{t}$ *is a semi-cozy* λ*-tableau such that* $\mathfrak{t}^{(l)} = (\mathfrak{t}^{\lambda})^{(l)}$ *for all* $l \geqslant k$. *Then for each* $0 \leqslant i \leqslant (k,0)_\lambda$, *we have*

$$\mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i} = q^i Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i+1} + \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}} a_{k,\mathfrak{s}} \mathfrak{s}$$

for some $\mathfrak{s} \in \mathrm{SC}(\lambda)$ *and* $a_{k,\mathfrak{s}} \in \mathbb{F}$.

Proof. In this proof, we will write $\mathfrak{t} = (\mathfrak{t}^{(1)}, \mathfrak{t}^{(2)}, \ldots, \mathfrak{t}^{(r)}) = \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(r)}$, as each $\mathfrak{t}^{(l)}$ is considered as a monomial. If $\mathfrak{t}^{(l)}$ corresponds to 1 for some l, we will suppress it from the notation. Thus, $\mathfrak{t} = \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(p)}$ means $\mathfrak{t}^{(l)} = 1$ for all $l > p$.

(a) Write $\mathfrak{t} = \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(r)}$ and $\mathfrak{t}' = \mathfrak{t}'^{(1)}\mathfrak{t}'^{(2)} \ldots \mathfrak{t}'^{(r)}$. Note that $\mathfrak{t}^{(l)} = \mathfrak{t}'^{(l)}$ for $l \neq k$. Since the general case can be obtained by multiplying both sides by the monomial $\mathfrak{t}^{(k+1)}\mathfrak{t}^{(k+2)} \ldots \mathfrak{t}^{(r)}$, we may assume that $\mathfrak{t} = \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(k)}$ and $\mathfrak{t}' = \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}'^{(k)}$. As the starting point of the induction, we let $\mathfrak{t}^{(k)} = T_{(k,i)_\lambda + j}$ (for $j \geqslant 1$) and $\mathfrak{t}'^{(k)} = 1$. From the defining relation $T_{(k,i)_\lambda+j} - q$, we have $\mathfrak{t}^{(k)} = q\mathfrak{t}'^{(k)}$. Since $\mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(k-1)}$ commutes with $\mathfrak{t}^{(k)} = T_{(k,i)_\lambda+j}$ and $\mathfrak{t}'^{(k)} = 1$, we obtain

$$\begin{aligned}\mathfrak{t} &= \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(k)} = \mathfrak{t}^{(k)}\mathfrak{t}^{(1)} \ldots \mathfrak{t}^{(k-1)} \\ &= q\mathfrak{t}'^{(k)}\mathfrak{t}^{(1)} \ldots \mathfrak{t}^{(k-1)} = q\, \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}'^{(k)} = q\mathfrak{t}'.\end{aligned}$$

We may write $\mathfrak{t}^{(k)}$ as $\left(\prod_{l=0}^{i-2} \langle \mathbf{a}_l \rangle_{(k,l)_\lambda}\right) \langle \mathbf{b} \rangle_{(i-1)_\lambda}$ using the notation in [**9**]. The remaining induction argument and calculations are similar to those of Lemma 3.3 in [**9**].

(b) Consider the subalgebra $H_{|\lambda^{(k)}|}(q)$ of $\mathcal{H}_{r,1,n}$ generated by the set

$$\{T_{(k,0)_\lambda+1}, \ldots, T_{(k+1,0)_\lambda-1}\}$$

as in the proof of Lemma 3.9, where the indices $(k,i)_\lambda$ are defined in (3.1). Let $R_q^{\lambda^{(k)}}$ be the set of relations given in Definition 4.1 of [**9**] for the module $\widehat{S}_q^{\lambda^{(k)}}$ over $H_{|\lambda^{(k)}|}(q)$. (In Definition 4.1 of [**9**], $(\mathcal{R}_q, \mathcal{R}_q^{\lambda})$ should be read as (R_q, R_q^{λ}).)

Since $R_q^{\lambda^{(k)}} \subset R_{\mathrm{AK}}^{\lambda}$, the intersection of the right ideal of $\mathcal{H}_{r,1,n}$ generated by $R_{\mathrm{AK}}^{\lambda}$ and the algebra $H_{|\lambda^{(k)}|}(q)$ contains the right ideal of $H_{|\lambda^{(k)}|}(q)$ generated by $R_q^{\lambda^{(k)}}$. Thus all the relations in $\widehat{S}_q^{\lambda^{(k)}}$ also hold in $\widehat{S}^{\lambda}$. Using the notation in [**9**], one can easily see that $\Sigma_{i,j,k}^{\lambda}$ corresponds to

$$\sum_{\mathbf{a} \in C_{j,i,\lambda^{(k)}}} \langle (k, i-1)_\lambda + \mathbf{a} \rangle_{(k,i)_\lambda}.$$

Now our assertion follows from [**9**, Lemma 3.4] and [**9**, Lemma 4.2].

(c) Let $\mathfrak{t}' = \mathfrak{t}^{(1)}\mathfrak{t}^{(2)} \ldots \mathfrak{t}^{(k-1)}$. Then $\mathfrak{t}'$ commutes with all the tableaux $\mathfrak{t}''$ such that $\mathfrak{t}^{\lambda} \trianglelefteq \mathfrak{t}'' \trianglelefteq \mathfrak{t}_{i,j,k}^{\lambda}$. Thus we obtain

$$(\Sigma_{i,j,k}^{\lambda})\mathfrak{t}' = \mathfrak{t}'(\Sigma_{i,j,k}^{\lambda}) = 0.$$

The remaining part of the proof is similar to the proof of [**9**, Lemma 4.4], and we omit it.

(d) We use a downward induction on i. By the defining relations for $\widehat{S}^{\lambda}$, we obtain

$$T_{(k,0)_\lambda,1} T_0 T^{1,(k,0)_\lambda} = q^{(k,0)_\lambda} Q_k.$$

From the assumption we see that $\mathfrak{t}$ consists of those monomials on the T_j with $j \leqslant (k,0)_\lambda - 1$ only. By Lemma 3.2, $\mathfrak{t}$ commutes with $L_{(k,0)_\lambda+1}$. Hence we have

$$\mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,(k,0)_\lambda} = q^{(k,0)_\lambda} Q_k \mathfrak{t},$$

and our assertion is valid for $i = (k,0)_\lambda$.

Assume that

$$\mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i} = q^i Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i+1} + \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}} a_{k,\mathfrak{s}} \mathfrak{s}.$$

Multiplying the above equation by T_i from the right, we get

$$\begin{aligned}
0 &= \mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i} T_i - q^i Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i} - \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}} a_{k,\mathfrak{s}} \mathfrak{s}\, T_i \\
&= (q-1)\mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i} + q\, \mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i-1} \\
&\quad - q^i Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i} - \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}} a_{k,\mathfrak{s}} \mathfrak{s}\, T_i \\
&= (q-1) q^i Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i+1} + \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}} (q-1) a_{k,\mathfrak{s}} \mathfrak{s} \\
&\quad + q\, \mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i-1} - q^i Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i} - \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}} a_{k,\mathfrak{s}} \mathfrak{s}\, T_i.
\end{aligned}$$

Note that $\mathfrak{t}\, T_{(k,0)_\lambda,i+1} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i}$. Thus if $\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i+1}$, then we have

$$\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i} \quad \text{and} \quad \mathfrak{s} T_i \prec \mathfrak{t}\, T_{(k,0)_\lambda,i}.$$

It follows that

$$0 = q\, \mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i-1} - q^i Q_k\, \mathfrak{t} T_{(k,0)_\lambda,i} - \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i}} a'_{k,\mathfrak{s}} \mathfrak{s}.$$

Since q is invertible, we obtain

$$\mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 T^{1,i-1} = q^{i-1} Q_k \mathfrak{t}\, T_{(k,0)_\lambda,i} + \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,i}} q^{-1} a'_{k,\mathfrak{s}} \mathfrak{s}.$$

Hence by (downward) induction, our assertion follows. □

Let $\mathcal{R}^\lambda_{\mathrm{AK}}$ be the set of relations derived in Lemma 4.1:

$$\mathcal{R}^\lambda_{\mathrm{AK}} : \begin{array}{ll}
\mathfrak{t} - q\,\mathfrak{t}' & \text{for } \mathfrak{t} \in \mathrm{SC}(\lambda) \text{ with } \mathfrak{t}(i,j,k) \geqslant \mathfrak{t}(i,j+1,k) \\
& \quad \text{for some } (i,j,k) \in [\lambda], \\
\mathfrak{t} - \sum_{\mathfrak{s} \prec \mathfrak{t}} a_{\mathfrak{t},\mathfrak{s}} \mathfrak{s} & \text{for } \mathfrak{t} \in \mathrm{SC}(\lambda) \cap \mathrm{RS}(\lambda) \text{ with } \mathfrak{t}(i,j,k) + j > \mathfrak{t}(i+1,j,k) \\
& \quad \text{for some } (i,j,k) \in [\lambda], \\
\multicolumn{2}{l}{\mathfrak{t}\, T_{(k,0)_\lambda,1} T_0 - Q_k \mathfrak{t}\, T_{(k,0)_\lambda,1} - \sum_{\mathfrak{s} \prec \mathfrak{t}\, T_{(k,0)_\lambda,1}} a_{k,\mathfrak{s}} \mathfrak{s}} \\
& \text{for } 1 \leqslant k \leqslant r \text{ and } \mathfrak{t} \in \mathrm{SC}(\lambda) \text{ with } \mathfrak{t}^{(l)} = (\mathfrak{t}^\lambda)^{(l)} \text{ for } l \geqslant k.
\end{array} \tag{4.3}$$

Definition 4.2. A semi-cozy λ-tableau $\mathfrak{t} \in \mathrm{SC}(\lambda)$ is said to be *cozy* if it satisfies:

(i) $\mathfrak{t}$ is row standard,
(ii) $\mathfrak{t}(i,j,k)+j\leqslant\mathfrak{t}(i+1,j,k)$ for all $(i,j,k)\in[\lambda]$.

Example 4.3. Let

$$\lambda=\left(\begin{array}{|c|c|c|}\hline &&\\ \hline &&\\ \hline &\\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline &\\ \hline &\\ \hline\end{array}\right).$$

Some examples of cozy tableaux are:

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5&6\\ \hline 7&8\\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline 9&10\\ \hline 11&12\\ \hline\end{array}\right),\quad \left(\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 3&5&6\\ \hline 6&7\\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline 5&6\\ \hline 7&8\\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&5&6\\ \hline 3&8\\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline 3&4\\ \hline 6&7\\ \hline\end{array}\right),\quad \left(\begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 2&4&6\\ \hline 3&6\\ \cline{1-2}\end{array},\ \begin{array}{|c|c|}\hline 1&2\\ \hline 2&4\\ \hline\end{array}\right).$$

We denote by $\mathrm{CZ}(\lambda)$ the set of cozy λ-tableaux.

Proposition 4.4. *Under the identification of $\mathcal{B}_H$ with* $\mathrm{SC}(\lambda)$*, the set of $(\mathcal{R}_{\mathrm{AK}},\mathcal{R}^{\lambda}_{\mathrm{AK}})$-standard monomials with respect to $\prec_{\text{deg-lex}}$ is identified with the set* $\mathrm{CZ}(\lambda)$ *of cozy tableaux of shape λ.*

Proof. Let $T=T_{i_1}\dots T_{i_a}$ be an $(\mathcal{R}_{\mathrm{AK}},\mathcal{R}^{\lambda}_{\mathrm{AK}})$-standard monomial. Since T is $\mathcal{R}_{\mathrm{AK}}$-standard, we have $T\in\mathcal{B}_{\mathrm{AK}}$.

If $i_b\neq 0$ for all $b=1,\dots,a$, then T belongs to $\mathcal{B}_H$. Hence T can be identified with a semi-cozy tableau $\mathfrak{t}$ in $\mathrm{SC}(\lambda)$. By the first two relations in $\mathcal{R}^{\lambda}_{\mathrm{AK}}$, we conclude that $\mathfrak{t}$ is a cozy tableau.

Suppose $i_b=0$ for some b and let c be the smallest index such that $i_c=0$. Then by the above argument, the monomial $T_{i_1}\dots T_{i_{c-1}}$ is $(\mathcal{R}_{\mathrm{AK}},\mathcal{R}^{\lambda}_{\mathrm{AK}})$-standard and corresponds to a cozy tableau.

If $i_{c-1}>1$, then we have

$$T_{i_{c-1}}T_{i_c}=T_{i_{c-1}}T_0=T_0T_{i_{c-1}},$$

which lies in $\mathcal{R}_{\mathrm{AK}}$. Hence we must have $i_{c-1}=1$ and we may write

$$T_{i_1}\dots T_{i_{c-1}}=T_{i_1}\dots T_{i_b}T_{j,1}\quad\text{for some } b \text{ and } j.$$

Note that, in a cozy tableau, the only boxes that can have the entry 1 are the ones lying in the upper-left corner of each component. Hence $j=(k,0)_\lambda$ for some $1\leqslant k\leqslant r$ and we have

$$T_{i_1}\dots T_{i_{c-1}}T_{i_c}=T_{i_1}\dots T_{i_b}T_{j,1}T_0=T_{i_1}\dots T_{i_b}T_{(k,0)_\lambda,1}T_0.$$

However, by the third relation in $\mathcal{R}^{\lambda}_{\mathrm{AK}}$, this monomial cannot be $(\mathcal{R}_{\mathrm{AK}},\mathcal{R}^{\lambda}_{\mathrm{AK}})$-standard, which is a contradiction to the assumption that $T=T_{i_1}\dots T_{i_c}\dots T_{i_a}$ is $(\mathcal{R}_{\mathrm{AK}},\mathcal{R}^{\lambda}_{\mathrm{AK}})$-standard. Therefore, we must have $i_b\neq 0$ for all b, and T corresponds to a cozy tableau.

Conversely, given a cozy tableau $\mathfrak{t}$ of shape λ, let T be the corresponding monomial in $\mathcal{B}_H$. Since T contains no T_0 in itself, T is $\mathcal{R}_{\mathrm{AK}}$-standard and the third relation in $\mathcal{R}_{\mathrm{AK}}^{\lambda}$ does not have any effect on T. Moreover, since $\mathfrak{t}$ is cozy, T is *reduced* with respect to the first two relations in $\mathcal{R}_{\mathrm{AK}}^{\lambda}$. Hence the monomial T is $(\mathcal{R}_{\mathrm{AK}}, \mathcal{R}_{\mathrm{AK}}^{\lambda})$-standard, which completes the proof. □

Recall that the symmetric group S_n acts naturally on the set of bijective λ-tableaux. We define a map ζ from the set $\mathrm{CZ}(\lambda)$ of cozy λ-tableaux to the set $\mathrm{ST}(\lambda)$ of standard λ-tableaux by

$$\zeta(\mathfrak{t}) = \zeta(T_{i_1}T_{i_2}\dots T_{i_l}) = \mathfrak{t}^{\lambda}\tau_{i_1}\tau_{i_2}\dots\tau_{i_k}, \tag{4.4}$$

where $T_{i_1}T_{i_2}\dots T_{i_l}$ is the monomial identified with the cozy tableau $\mathfrak{t}$ and τ_j is the transposition $(j, j+1) \in S_n$.

We can explain the map ζ in a more combinatorial way. Assume that $\mathfrak{t}$ is a cozy tableau. For (i, j, k)-entry of $\mathfrak{t}$, set $p = (k, i-1)_{\lambda} + j$. If we read off the entries of $\mathfrak{t}$ in the *Western reading*; that is, if we proceed from left to right and from top to bottom, then p increases by 1 in each move. The pth box with the entry q corresponds to the monomial $T_{p-1,q}$. Observe that the action of $\tau_{p-1}\dots\tau_q$ (with $p > q$) on a tableau changes $q, q+1, \dots, p-1$ to $q+1, q+2, \dots, p$, respectively, and p to q. Actually, the image of ζ is obtained by successive actions of such elements $\tau_{p-1}\dots\tau_q$ corresponding to the 'boxes' $T_{p-1,q}$ in the tableau. This observation gives us the following combinatorics of tableaux.

To begin with, let $\mathfrak{t}_{\lambda}$ be the cozy tableau corresponding to the longest monomial. The last tableau in Example 4.3 is $\mathfrak{t}_{\lambda}$ for $\lambda = ((3,3,2),(2,2))$. Now, for a cozy tableau $\mathfrak{t} \in \mathrm{CZ}(\lambda)$, $\zeta(\mathfrak{t})$ can be obtained in the following process. At first, let $\mathfrak{t}_1 = \mathfrak{t}$ and $\mathfrak{t}^1 = \mathfrak{t}^{\lambda}$, the standard tableau defined in (3.2). We describe the pth step of the process.

If the (i, j, k)-entry of $\mathfrak{t}_p$ is a and that of $\mathfrak{t}^p$ is b, then change b in $\mathfrak{t}^p$ into a, and add 1 to each box of $\mathfrak{t}^p$ with entry c satisfying $a \leqslant c < b$. Denote the resulting tableau by $\mathfrak{t}^{p+1}$. Also, change the (i, j, k)-entry of $\mathfrak{t}_p$ into that of $\mathfrak{t}_{\lambda}$ and denote the resulting tableau by $\mathfrak{t}_{p+1}$.

Apply the above process inductively, while $\mathfrak{t}$ is read off in the Western reading. The process ends with $\mathfrak{t}_n = \mathfrak{t}_{\lambda}$ and $\mathfrak{t}^n = \zeta(\mathfrak{t})$. It is straightforward to see that each $\mathfrak{t}_i$ is cozy, while each $\mathfrak{t}^i$ is standard.

Conversely, let $\mathfrak{t} \in \mathrm{ST}(\lambda)$ be a standard tableau of shape λ. Then, by letting $\mathfrak{t}_n = \mathfrak{t}_{\lambda}$ and $\mathfrak{t}^n = \mathfrak{t}$, we can reverse the above process to end with $\mathfrak{t}_1 = \zeta^{-1}(\mathfrak{t})$ and $\mathfrak{t}^1 = \mathfrak{t}^{\lambda}$. In this reversing process, we also obtain a family of cozy tableaux $\mathfrak{t}_n, \mathfrak{t}_{n-1}, \dots, \mathfrak{t}_1$ and a family of standard tableaux $\mathfrak{t}^n, \mathfrak{t}^{n-1}, \dots, \mathfrak{t}^1$.

Therefore, we obtain the following.

PROPOSITION 4.5. *The map* $\zeta : \mathrm{CZ}(\lambda) \to \mathrm{ST}(\lambda)$ *defined by* (4.4) *is a bijection. In particular, we have*

$$\dim S^{\lambda} = \#(\mathrm{ST}(\lambda)) = \#(\mathrm{CZ}(\lambda)).$$

EXAMPLE 4.6. If

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \right),$$

then

$$\begin{aligned}
\mathfrak{t} = \mathfrak{t}_1 = \mathfrak{t}_2 = \mathfrak{t}_3 &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 1 & 5 \\ 3 & \end{smallmatrix} \right), & \mathfrak{t}^\lambda = \mathfrak{t}^1 = \mathfrak{t}^2 = \mathfrak{t}^3 &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{smallmatrix}, \begin{smallmatrix} 6 & 7 \\ 8 & \end{smallmatrix} \right), \\
\mathfrak{t}_4 &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 1 & 5 \\ 3 & \end{smallmatrix} \right), & \mathfrak{t}^4 &= \left(\begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & \end{smallmatrix}, \begin{smallmatrix} 6 & 7 \\ 8 & \end{smallmatrix} \right), \\
\mathfrak{t}_5 &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 1 & 5 \\ 3 & \end{smallmatrix} \right), & \mathfrak{t}^5 &= \left(\begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 6 & 7 \\ 8 & \end{smallmatrix} \right), \\
\mathfrak{t}_6 &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 1 & 5 \\ 3 & \end{smallmatrix} \right), & \mathfrak{t}^6 &= \left(\begin{smallmatrix} 2 & 4 & 6 \\ 3 & 5 & \end{smallmatrix}, \begin{smallmatrix} 1 & 7 \\ 8 & \end{smallmatrix} \right), \\
\mathfrak{t}_7 &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix} \right), & \mathfrak{t}^7 &= \left(\begin{smallmatrix} 2 & 4 & 7 \\ 3 & 6 & \end{smallmatrix}, \begin{smallmatrix} 1 & 5 \\ 8 & \end{smallmatrix} \right), \\
\mathfrak{t}_8 = \mathfrak{t}_\lambda &= \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 4 & \end{smallmatrix}, \begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix} \right), & \mathfrak{t}^8 = \zeta(\mathfrak{t}) &= \left(\begin{smallmatrix} 2 & 5 & 8 \\ 4 & 7 & \end{smallmatrix}, \begin{smallmatrix} 1 & 6 \\ 3 & \end{smallmatrix} \right).
\end{aligned}$$

We now return to the $\mathcal{H}_{r,1,n}$-module homomorphism $\Psi : \widehat{S}^\lambda \to S^\lambda$ given by $\Psi(1) = m_\lambda$. Since Ψ is surjective, we have

$$\dim \widehat{S}^\lambda \geqslant \dim S^\lambda.$$

By Propositions 1.3, 3.5, 4.4 and 4.5, we have

$$\dim \widehat{S}^\lambda \leqslant \#(\mathrm{CZ}(\lambda)) = \#(\mathrm{ST}(\lambda)) = \dim S^\lambda,$$

which implies that $\dim \widehat{S}^\lambda = \dim S^\lambda$. Hence we conclude that the $\mathcal{H}_{r,1,n}$-module $\widehat{S}^\lambda$ is isomorphic to the Specht module S^λ, and the pair $(\mathcal{R}_{\mathrm{AK}}, \mathcal{R}_{\mathrm{AK}}^\lambda)$ is a Gröbner–Shirshov pair for $\widehat{S}^\lambda$. Therefore, the set $\mathrm{CZ}(\lambda)$ can be viewed as a linear basis of the Specht module S^λ consisting of the $(\mathcal{R}_{\mathrm{AK}}, \mathcal{R}_{\mathrm{AK}}^\lambda)$-standard monomials.

To summarize, we have the following.

THEOREM 4.7. *Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ be a multipartition of n.*

(a) *The Specht module S^λ is isomorphic to the $\mathcal{H}_{r,1,n}$-module $\widehat{S}^\lambda$ defined by the pair $(R_{\mathrm{AK}}, R_{\mathrm{AK}}^\lambda)$. Hence we obtain a presentation of the Specht module S^λ by generators and relations.*

(b) *The pair $(\mathcal{R}_{\mathrm{AK}}, \mathcal{R}_{\mathrm{AK}}^\lambda)$ is a Gröbner–Shirshov pair for $\widehat{S}^\lambda$ with respect to the monomial order $\prec$ $(=\prec_{\text{deg-lex}})$.*

(c) *The set $\mathrm{CZ}(\lambda)$ of cozy λ-tableaux is a linear basis of $\widehat{S}^\lambda$, and can be viewed as a linear basis of the Specht module S^λ consisting of the $(\mathcal{R}_{\mathrm{AK}}, \mathcal{R}_{\mathrm{AK}}^\lambda)$-standard monomials.*

The isomorphism $\Psi : \widehat{S}^\lambda \to S^\lambda$ gives a canonical bijection between our monomial basis $\mathrm{CZ}(\lambda)$ for $\widehat{S}^\lambda$ and the standard basis for S^λ constructed in [**4**]. Moreover, the map $\zeta : \mathrm{CZ}(\lambda) \to \mathrm{ST}(\lambda)$ gives a bijection between the labelling schemes of these bases. We would like to emphasize that the set $\mathrm{CZ}(\lambda)$ is itself a monomial basis for the Specht module S^λ, not just a labelling scheme.

We close this paper with an example.

Example 4.8. Let

$$\lambda = \left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1}\end{array}\, , \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \right).$$

Then the set $\mathrm{CZ}(\lambda)$ of cozy λ-tableaux is given below:

$$\left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&5\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&5\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&5\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&5\\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&4\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&4\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&4\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{3}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 4&5\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&5\\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&5\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&5\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&4\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&4\\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&4\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2&3\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\ \hline\end{array}\right), \left(\begin{array}{|c|c|}\hline 1&2\\ \hline \multicolumn{1}{|c|}{2}\\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\right).$$

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