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Lisa Carbone[1] · **Kyu-Hwan Lee[2]** · **Dongwen Liu[3]**

Abstract We define Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$, induced from quasi–characters. We prove convergence of the constant term and hence the almost everywhere convergence of the Eisenstein series. We define and calculate the degenerate Fourier coefficients. We also consider Eisenstein series induced from cusp forms and show that these are entire functions.

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1 Introduction

After being developed by Langlands [26,27] in great generality, the theory of Eisenstein series has played a fundamental role in the formulation of the Langlands

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functoriality conjecture and in the study of *L*-functions by means of the Langlands–Shahidi method. Eisenstein series also appear in many other places throughout number theory and representation theory. The scope of applications is being extended to geometry and mathematical physics. On the other hand, since we have seen many successful generalizations of finite dimensional constructions to infinite dimensional Kac–Moody groups [22,25], it is a natural question to ask whether one can generalize the theory of Eisenstein series to Kac–Moody groups. Such an attempt is not merely for the sake of generalization. Even though it is hypothetical for the present, a satisfactory theory of Eisenstein series on Kac–Moody groups would have significant impact on some of the central problems in number theory [3,34].

In pioneering work, H. Garland developed a theory of Eisenstein series for the affine Kac–Moody groups over $\mathbb{R}$ in a series of papers [13–20], and he established absolute convergence and meromorphic continuation. The absolute convergence result has been generalized to the case of number fields by Liu [30]. In a recent preprint [21], Garland, Miller and Patnaik showed that Eisenstein series induced from cups forms are entire functions. Garland's idea was extended to the function field case by Kapranov [24] through geometric methods and was systematically developed by Patnaik [33]. An algebraic approach to this case was made by Lee and Lombardo [28]. Braverman and Kazhdan's recent preprint [1] announces more results in the function field case.

The purpose of this paper is to construct Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$, generalizing Garland's work in the affine case. The rank 2 hyperbolic Kac–Moody groups form the first family beyond the affine case. However, contrary to the affine case, our understanding of hyperbolic Kac–Moody groups (and algebras) is far from being complete. In particular, information regarding imaginary root multiplicities of hyperbolic Kac–Moody algebras is limited. A recent survey on this topic can be found in [5].

Nevertheless, we have the necessary information to construct Eisenstein series induced from quasi–characters on rank 2 hyperbolic Kac–Moody groups and to prove their almost everywhere convergence, thanks to the works of Lepowsky and Moody [29], Feingold [10] and Kang and Melville [23]. We can also prove entirety of the Eisenstein series induced from cusp forms. Indeed, one of the benefits of working in the Kac–Moody group rather than its Kac–Moody algebra, is that the group is generated by root groups corresponding to only 'real' roots. The 'real' part of Kac–Moody is sufficiently well understood and carries many properties similar to finite dimensional simple Lie algebras [6].

We assume that G is a rank 2 hyperbolic Kac–Moody group attached to a symmetric 2×2 generalized Cartan matrix, and we define Eisenstein series on the 'arithmetic' quotient $K(G_\mathbb{R})\backslash G_\mathbb{R}/G_\mathbb{Z}$, where $K = K(G_\mathbb{R})$ is the unitary form of G, an infinite dimensional analogue of a maximal compact subgroup. Our method is to choose a quasi–character ν on a Borel subgroup and then extend it to the whole of $G_\mathbb{R}$ via Iwasawa decomposition $G_\mathbb{R} = KA^+N$, which is given uniquely. Here $A^+ \cong (\mathbb{R}^+)^{rank(G)}$ is an abelian subgroup and N is the completion of the subgroup generated by all positive real root groups.

We then average over an appropriate quotient of $G_\mathbb{Z}$ to obtain a $G_\mathbb{Z}$–invariant function $E_\nu(g)$ on $K\backslash G_\mathbb{R}/G_\mathbb{Z}$. Our first main result is:

Theorem 1.1 *Assume that ν satisfies Godement's criterion, and consider the cone*

$$A' = \{a \in A^+ : a^{\alpha_i} < 1, \ i = 1, 2\}.$$

Then for any compact subset A'_c of A', there is a measure zero subset S_0 of $A'_c N$ such that $E_\nu(g)$ converges absolutely for $g \in K A'_c N$ off the set $K S_0$.

Although the idea of the proof is similar to that of [14], our proof heavily depends on a concrete description of root systems of rank 2 hyperbolic Kac–Moody algebras. We compute the constant term of the series $E_\nu(g)$ and show that the constant term is absolutely convergent, which implies almost everywhere convergence of the series. We conjecture that the Eisenstein series actually converges everywhere under a weaker condition than Godement's criterion (See Conjecture 5.3). As the argument in [15] does not generalize to the hyperbolic case, the conjecture seems out of reach at the current time.

We also calculate Fourier coefficients of the Eisenstein series in Sect. 6. Let ψ be a non-trivial character of $N/(G_\mathbb{Z} \cap N)$. Then we can write $\psi = \psi_1 \psi_2$, where ψ_i corresponds to the simple root α_i for $i = 1, 2$. We call ψ *generic* if each ψ_i is non-trivial for $i = 1, 2$. We first show that the Fourier coefficients attached to generic characters vanish (Lemma 6.1). Then we consider characters of the form $\psi = \psi_i$ (i.e. either ψ_1 or ψ_2 is trivial) and compute the corresponding Fourier coefficients. The resulting formula is an infinite sum of products of the n-th Whittaker coefficient of the analytic Eisenstein series on SL_2 and quotients of the completed Riemann zeta function (Theorem 6.2).

The next main result is the entirety of the Eisenstein series $E_{s,f}(g)$ induced from a cusp form f. Our approach is similar to that of Garland, Miller and Patnaik in [21]; however, our method requires us to use information about the structure of the root system of G. We obtain:

Theorem 1.2 *Let f be an unramified cusp form on SL_2. For any compact subset A'_c of A', there is a measure zero subset S_0 of $A'_c N$ such that $E_{s,f}(g)$ is an entire function of $s \in \mathbb{C}$ for $g \in K A'_c N$ off the set $K S_0$.*

As mentioned earlier, rank 2 hyperbolic Kac–Moody algebras and groups form the first family beyond the affine case. It would be interesting to generalize the results of this paper to other hyperbolic Kac–Moody groups, for example, to the Kac–Moody group corresponding to the Feingold and Frenkel's rank 3 hyperbolic Kac–Moody algebra [11]. Actually, in a subsequent paper [8] with Miller and Garland, we will prove that for general Kac–Moody groups, the Eisenstein series $E_\nu(g)$ converges almost everywhere in the full region satisfying the Godement's criterion Re $\nu(h_{\alpha_i}) < -2$. It will be very exciting to see further developments toward a satisfactory theory of Eisenstein series on Kac–Moody groups.

2 Rank 2 hyperbolic Kac–Moody algebras and $\mathbb{Z}$-forms

Let $\mathfrak{g} = \mathfrak{g}_{\mathbb{C}}$ be the rank 2 hyperbolic Kac–Moody algebra associated with the symmetric generalized Cartan matrix

$$\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}, \quad m \geq 3.$$

Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{C}}$ be a Cartan subalgebra. Let Φ be the corresponding root system and let $\Phi_{\pm}$ denote the positive and negative roots respectively. Let

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$$

be the triangular decomposition of $\mathfrak{g}$, where

$$\mathfrak{g}^- = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}_\alpha, \quad \mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha.$$

Let $W = W(A)$ be the Weyl group of $\mathfrak{g}$. We have

$$W(A) = \langle r_1, \, r_2 \mid r_1^2 = 1, \, r_2^2 = 1 \rangle$$

which is the infinite dihedral group

$$W = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \rtimes \{\pm 1\},$$

where $\langle (r_1 r_2) \rangle \cong \mathbb{Z}$.

A root $\alpha \in \Phi$ is called a *real root* if there exists $w \in W$ such that $w\alpha$ is a simple root. A root α which is not real is called *imaginary*. We denote by Φ^{re} the real roots and Φ^{im} the imaginary roots.

Set $I = \{1, 2\}$. We let $\Lambda \subseteq \mathfrak{h}^*$ be the $\mathbb{R}$–linear span of the simple roots α_i, for $i \in I$, and $\Lambda^\vee \subseteq \mathfrak{h}$ be the $\mathbb{R}$–linear span of the simple coroots h_{α_i}, for $i \in I$. Let $e_i = e_{\alpha_i}$ and $f_i = f_{\alpha_i}$ be root vectors in $\mathfrak{g}$ corresponding to simple roots α_i, $i \in I$. Let $\mathcal{U}_{\mathbb{C}}, \mathcal{U}_{\mathbb{C}}^+$ and $\mathcal{U}_{\mathbb{C}}^-$ be the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{g}^+$ and $\mathfrak{g}^-$ respectively. We define the following $\mathbb{Z}$-subalgebras: Let

(1) $\mathcal{U}_{\mathbb{Z}}^+ \subseteq \mathcal{U}_{\mathbb{C}}^+$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{e_i^n}{n!}$ for $i \in I$ and $n \geq 0$,

(2) $\mathcal{U}_{\mathbb{Z}}^- \subseteq \mathcal{U}_{\mathbb{C}}^-$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{f_i^n}{n!}$ for $i \in I$ and $n \geq 0$,

(3) $\mathcal{U}_{\mathbb{Z}}^0 \subseteq \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ be the $\mathbb{Z}$-subalgebra generated by $\dbinom{h}{n}$, for $h \in \Lambda^\vee$ and $n \geq 0$,

where

$$\binom{h}{n} = \frac{h(h-1)\cdots(h-n+1)}{n!},$$

(4) $\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{C}}$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{e_i^n}{n!}$, $\dfrac{f_i^n}{n!}$ for $i \in I$ and $\dbinom{h}{n}$, for $h \in \Lambda^{\vee}$ and $n \geq 0$.

It follows ([35]) that $\mathcal{U}_{\mathbb{Z}}$ is a $\mathbb{Z}$-*form* of $\mathcal{U}_{\mathbb{C}}$, i.e. the canonical map

$$\mathcal{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathcal{U}_{\mathbb{C}}$$

is bijective. We have a triangular decomposition $\mathcal{U}_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}}^+ \mathcal{U}_{\mathbb{Z}}^0 \mathcal{U}_{\mathbb{Z}}^-$.

Recall that $\mathfrak{g}^+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$. Let V be a representation of $\mathfrak{g}$. Then V is called a *highest weight representation* with highest weight $\lambda \in \mathfrak{h}^*$ if there exists $0 \neq v_\lambda \in V$ such that

$$\mathfrak{g}^+(v_\lambda) = 0,$$

$$h(v_\lambda) = \lambda(h) v_\lambda$$

for $h \in \mathfrak{h}$ and

$$V = \mathcal{U}_{\mathbb{C}} \cdot v_\lambda.$$

Since $\mathfrak{g}^+$ annihilates v_λ and $\mathfrak{h}$ acts as scalar multiplication on v_λ, we have

$$V = \mathcal{U}_{\mathbb{C}}^- \cdot v_\lambda.$$

We write $V = V^\lambda$ for the unique irreducible highest weight module with highest weight λ. We will assume that $V = V^\lambda$ is integrable.

We shall construct a lattice $V_{\mathbb{Z}}$ in V by taking the orbit of a highest weight vector v_λ under $\mathcal{U}_{\mathbb{Z}}$. We have $\mathcal{U}_{\mathbb{Z}}^+ \cdot v_\lambda = \mathbb{Z} v_\lambda$ since all elements of $\mathcal{U}_{\mathbb{Z}}^+$ except for 1 annihilate v_λ. Also $\mathcal{U}_{\mathbb{Z}}^0$ acts as scalar multiplication on v_λ by a $\mathbb{Z}$-valued scalar, since $\dbinom{h}{n}$ for $h \in \Lambda^{\vee}$ and $n \geq 0$ acts on v_λ as

$$\binom{\lambda(h)}{n} = \frac{\lambda(h)(\lambda(h) - 1) \cdots (\lambda(h) - n + 1)}{n!} \in \mathbb{Z}.$$

Thus

$$\mathcal{U}_{\mathbb{Z}}^0 \cdot v_\lambda = \mathbb{Z} v_\lambda, \quad \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^- \cdot (\mathbb{Z} v_\lambda) = \mathcal{U}_{\mathbb{Z}}^- \cdot v_\lambda.$$

Let α be any real root and let e_α and f_α be root vectors corresponding to α. Then

$$\frac{f_\alpha^n}{n!} v_\lambda \in V_{\lambda - n\alpha}.$$

For a weight $\mu < \lambda$, i.e., $\lambda - \mu$ is a linear combination of α_i, $i \in I$ with nonnegative integral coefficients, we have

$$\frac{e_\alpha^n}{n!} v_\mu \in V_{\mu+n\alpha}.$$

We set

$$V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^- \cdot v_\lambda$$

Then $V_{\mathbb{Z}}$ is a lattice in $V_{\mathbb{C}}$ and a $\mathcal{U}_{\mathbb{Z}}$-module.

For each weight μ of V, let V_μ be the corresponding weight space, and we set

$$V_{\mu,\mathbb{Z}} = V_\mu \cap V_{\mathbb{Z}}.$$

We have

$$V_{\mathbb{Z}} = \oplus_\mu V_{\mu,\mathbb{Z}},$$

where the sum is taken over the weights of V. Thus $V_{\mathbb{Z}}$ is a direct sum of its weight spaces. We set

$$V_{\mu,\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} V_{\mu,\mathbb{Z}}$$

so that

$$V_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} V_{\mathbb{Z}} = \oplus_\mu V_{\mu,\mathbb{R}}.$$

For each weight μ of V, we have $\mu = \lambda - (k_1\alpha_1 + k_2\alpha_2)$, where λ is the highest weight and $k_i \in \mathbb{Z}_{\geq 0}$. Define the *depth* of μ to be

$$depth(\mu) = k_1 + k_2.$$

A basis $\Psi = \{v_1, v_2, \dots\}$ of V is called *coherently ordered relative to depth* if

(1) Ψ consists of weight vectors;
(2) If $v_i \in V_\mu$, $v_j \in V_{\mu'}$ and $depth(\mu') > depth(\mu)$, then $j > i$;
(3) $\Psi \cap V_\mu$ consists of an interval $v_k, v_{k+1}, \dots, v_{k+m}$.

The following was established in [12] in the affine case and in [6] in the general symmetrizable Kac–Moody case.

Theorem 2.1 *The lattice $V_{\mathbb{Z}}$ has a coherently ordered $\mathbb{Z}$-basis $\{v_1, v_2, \dots\}$ where $v_i \in V_{\mathbb{Z}}$, $v_i = \xi_i v_\lambda$, for some $\xi_i \in \mathcal{U}_{\mathbb{Z}}$. Let $w_i = k_i \otimes v_i$, where we choose an arbitrary $k_i \in \mathbb{R} \backslash \{0\}$. Then the set $\{w_1, w_2, \dots\}$ is a coherently ordered basis for $V_{\mathbb{R}}$. There exists a positive definite, Hermitian inner product $\langle , \rangle$ on V such that any vector in $V_{\mathbb{Z}}$ has an integer valued norm.*

3 The Kac–Moody group *G* and Iwasawa decomposition

Our next step is to construct our Kac–Moody group G over $\mathbb{R}$. The construction below can be used to construct G over any field F [6]. As before, let V be an integrable highest weight module for $\mathfrak{g}_{\mathbb{C}}$. Then the simple root vectors e_i and f_i are locally nilpotent on V.

We let $V_{\mathbb{Z}}$ be a $\mathbb{Z}$-form of V as in Sect. 2. Since $V_{\mathbb{Z}}$ is a $\mathcal{U}_{\mathbb{Z}}$-module, we have

$$\frac{e_i^n}{n!}(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}} \quad \text{and} \quad \frac{f_i^n}{n!}(V_{\mathbb{Z}}) \subseteq V_{\mathbb{Z}} \quad \text{for } n \in \mathbb{N}, \ i \in I.$$

Let $G_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$. For $s, t \in \mathbb{R}$ and $i \in I$, set

$$\chi_{\alpha_i}(s) = \sum_{n=0}^{\infty} s^n \frac{e_i^n}{n!} = exp(se_i), \quad \chi_{-\alpha_i}(t) = \sum_{n=0}^{\infty} t^n \frac{f_i^n}{n!} = exp(tf_i).$$

Then $\chi_{\alpha_i}(s)$, $\chi_{-\alpha_i}(t)$ define elements in $\mathrm{Aut}(V_{\mathbb{R}})$, thanks to the local nilpotency of e_i, f_i. More generally, for a real root α, we write $\alpha = w\alpha_i$ for $i = 1$ or 2 and define

$$\chi_{\alpha}(s) = \chi_{w\alpha_i}(s) = w\chi_{\alpha_i}(s)w^{-1} \in \mathrm{Aut}(V_{\mathbb{R}}), \quad s \in \mathbb{R}.$$

For $t \in \mathbb{R}^{\times}$, we set

$$w_{\alpha_i}(t) = \chi_{\alpha_i}(t)\chi_{-\alpha_i}(-t^{-1})\chi_{\alpha_i}(t) \quad \text{for } i \in I,$$

and define

$$h_{\alpha_i}(t) = w_{\alpha_i}(t)w_{\alpha_i}(1)^{-1}.$$

We let $G_{\mathbb{R}}^0$ be the subgroup of $\mathrm{Aut}(V_{\mathbb{R}})$ generated by the linear automorphisms $\chi_{\alpha_i}(s)$ and $\chi_{-\alpha_i}(t)$ of $V_{\mathbb{R}}$, for $s, t \in \mathbb{R}, i \in I$. That is, we define

$$G_{\mathbb{R}}^0 = \langle exp(se_i), \ exp(tf_i) : s, t \in \mathbb{R}, \ i \in I \rangle.$$

One can see that $\chi_{\alpha}(s) \in G_{\mathbb{R}}^0$ for real roots α.

We choose a coherently ordered basis $\Psi = \{v_1, v_2, \dots\}$ of $V_{\mathbb{R}}$ and denote by B^0 the subgroup of $G_{\mathbb{R}}^0$ consisting of the elements represented by upper triangular matrices with respect to Ψ. For $t \in \mathbb{Z}_{>0}$, we let V_t be the span of the $v_s \in \Psi$ for $s \leq t$. Then $B^0 V_t \subseteq V_t$ for each t. Let B_t be the image of B^0 in $\mathrm{Aut}(V_t)$. We then have surjective homomorphisms

$$\pi_{tt'} : B_{t'} \longrightarrow B_t, \quad t' \geq t.$$

We define B to be the projective limit of the projective family $\{B_t, \pi_{tt'}\}$.

Next we consider the completion of $G_{\mathbb{R}}^0$. We can define a topology on $G_{\mathbb{R}}^0$ as follows: for a base of neighborhoods of the identity, we take sets V_t defined by

$$V_t = \{g \in G_{\mathbb{R}}^0 : g v_i = v_i, \ i = 1, 2, \ldots, t\}.$$

Let $U_p = \mathrm{Span}\{v_1, \ldots, v_p\}$. Let $g = (g_i)$ and $h = (h_i)$ be Cauchy sequences in $G_{\mathbb{R}}^0$. For any p, we can find sufficiently large $q \geq p$ such that $\{g_i^{-1}(U_p)\}$ is contained in U_q. Take N sufficiently large so that g_i and h_i represent the restrictions of g and h to U_q, respectively, whenever $i > N$. Assume that $i, j > N$. Then $h_i h_j^{-1} \in V_q$, and we have

$$g_i h_i h_j^{-1} g_j^{-1}(u) = g_i g_j^{-1}(u) = u \quad \text{for any } u \in U_p,$$

since $g_j^{-1}(u) \in U_q$. This proves that $g_i h_i h_j^{-1} g_j^{-1} \in V_p$ whenever $i, j > N$. Therefore $gh = (g_i h_i)$ is Cauchy as well.

Let $G_{\mathbb{R}}$ be the completion of $G_{\mathbb{R}}^0$, i.e. the equivalence classes of all Cauchy sequences of $G_{\mathbb{R}}^0$. By replacing $\mathbb{R}$ with F in the above construction, we obtain the group G_F for any field F.

Note that B is naturally a subgroup of $G_{\mathbb{R}}$. We define the following subgroups of $G_{\mathbb{R}}$:

(1) $N =$ the completion of the subgroup generated by all positive real root groups and is a subgroup of B,
(2) $K = \{k \in G_{\mathbb{R}} : k \text{ preserves the inner product } \langle, \rangle \text{ on } V_{\mathbb{R}}^\lambda\}$,
(3) $A = \langle h_{\alpha_i}(s) : s \in \mathbb{R}^\times, i \in I \rangle$ and $A^+ = \langle h_{\alpha_i}(s) : s \in \mathbb{R}_+, i \in I \rangle$.

Theorem 3.1 ([9]) *We have the Iwasawa decomposition:*

$$G_{\mathbb{R}} = K A^+ N \tag{3.1}$$

with uniqueness of expression.

As in [4], we now define the '$\mathbb{Z}$-form' $G_{\mathbb{Z}}$ of $G_{\mathbb{R}}$ in the following way. We set

$$G_{\mathbb{Z}} = G_{\mathbb{R}} \cap \mathrm{Aut}(V_{\mathbb{Z}}).$$

Then

$$G_{\mathbb{Z}} = \{\gamma \in G_{\mathbb{R}} : \gamma \cdot V_{\mathbb{Z}} \subseteq V_{\mathbb{Z}}\}.$$

Remark 3.2 For a discussion on dependence on the choice of V and $V_{\mathbb{Z}}$, we refer the reader to [4]. In this paper, we work with fixed V and $V_{\mathbb{Z}}$.

4 Eisenstein series on rank 2 hyperbolic Kac–Moody groups

Let $g = k_g a_g n_g \in G_{\mathbb{R}}$ be the Iwasawa decomposition according to (3.1). Let $\nu : A^+ \to \mathbb{C}^{\times}$ be a quasi—character and define

$$\Phi_{\nu} : G_{\mathbb{R}} \to \mathbb{C}^{\times}$$

to be the function

$$\Phi_{\nu}(g) = \nu(a_g)$$

Then ν is well defined since the Iwasawa decomposition is unique and Φ_{ν} is left K-invariant and right N-invariant. For convenience, we write $\Gamma = G_{\mathbb{Z}}$.

Let B denote the minimal parabolic subgroup of $G_{\mathbb{R}}$. Relative to a coherently ordered basis Ψ for $V_{\mathbb{Z}}^{\lambda}$, Γ has a representation in terms of infinite matrices with integral entries. Define the Eisenstein series on $G_{\mathbb{R}}$ to be the infinite formal sum

$$E_{\nu}(g) := \sum_{\gamma \in \Gamma/\Gamma \cap B} \Phi_{\nu}(g\gamma).$$

Recall that $\mathfrak{h}$ is the Lie algebra of A and that $h_{\alpha_i}, i \in I$, are the simple coroots. We say that the coroot ν satisfies *Godement's criterion* if

$$\mathrm{Re}\ \nu(h_{\alpha_i}) < -2, \quad i \in I.$$

The usual Godement criterion would give convergence in the complex half–space for the whole group $G_{\mathbb{R}} = KAN$. Here we consider convergence in the half–space $\mathrm{Re}\,(\nu(h_{\alpha_i})) < -2$ for the subspace $G'_{\mathbb{R}} = KA'N$, where the A–component is replaced by the 'group' A' corresponding to the Tits cone. We do not expect that the Eisenstein series will be convergent for the whole group $G_{\mathbb{R}} = KAN$. Rather the general Kac–Moody theory [22] suggests that the subspace $G'_{\mathbb{R}}$ is the natural space to work on. In the next section, we prove almost everywhere convergence of the Eisenstein series from convergence of the constant term.

We will not comment on the difficult question of meromorphic continuation to the whole complex plane here. However, we will prove in Sect. 7 that cuspidal Eisenstein series are entire.

5 Convergence of the constant term

In this section we prove convergence of the constant term and thus almost everywhere convergence of the Eisenstein series itself. Assume first that $\nu : A^+ \to \mathbb{C}^{\times}$ is real valued and positive. Then we may interpret the infinite sum $E_{\nu}(g)$ as a function taking values in $\mathbb{R}_+ \cup \{\infty\}$. The function E_{ν} may be regarded as a function on

$$K \backslash G_{\mathbb{R}}/(\Gamma \cap N) \cong A^+ \times N/(\Gamma \cap N).$$

Under the identification $\mathbb{R}_+^2 \cong A^+$:

$$(x_1, x_2) \mapsto h_{\alpha_1}(x_1)h_{\alpha_2}(x_2),$$

we have the measure da on A^+, corresponding to the measure

$$\frac{dx_1}{x_1}\frac{dx_2}{x_2}$$

on $\mathbb{R}_+^2$.

As in [14] we know that $N/(\Gamma \cap N)$ is the projective limit of a projective family of finite-dimensional compact nil-manifolds and thus admits a projective limit measure dn, which is a left N-invariant probability measure. More precisely, recall from Sect. 3 that we have a coherently basis Ψ and the spaces V_t for $t \in \mathbb{Z}_{>0}$. Denote by N^0 the subgroup of $G_{\mathbb{R}}^0$ consisting of the elements represented by upper triangular matrices with respect to Ψ with 1's on the diagonal. Let N_t be the image of N^0 in $\text{Aut}(V_t)$. Then the group N is the the projective limit of the projective family $\{N_t, \pi_{tt'}\}$. We also have the canonical projection $\pi_t : N \to N_t$ for each t. Consider $\Gamma_t := \pi_t(\Gamma \cap N)$. Then the space N_t/Γ_t is a finite-dimensional compact nil-manifold, on which we define a probability measure compatible with the projections $\pi_{tt'}$. Now we obtain a projective limit measure dn on $N/(\Gamma \cap N)$.

We define for all $g \in G_{\mathbb{R}}$ the constant term

$$E_\nu^\sharp(g) = \int_{N/(\Gamma \cap N)} E_\nu(gn)dn$$

which is left K-invariant and right N-invariant. In particular $E_\nu^\sharp(g)$ is determined by the A^+-component of g in the Iwasawa decomposition. Let $\rho \in \mathfrak{h}^*$ satisfying $\rho(h_{\alpha_i}) = 1, i \in I$. Then

$$\rho = \frac{\alpha_1 + \alpha_2}{2 - m}.$$

Applying the Gindikin–Karpelevich formula, a formal calculation as in [14] yields that for $a \in A^+$

$$E_\nu^\sharp(a) = \sum_{w \in W} a^{w(\nu+\rho)-\rho}c(\nu, w),$$

where

$$c(\nu, w) = \prod_{\alpha \in \Phi_+ \cap w^{-1}\Phi_-} \frac{\xi(-(\nu + \rho)(h_\alpha))}{\xi(1 - (\nu + \rho)(h_\alpha))},$$

and $\xi(s)$ is the completed Riemann zeta function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\prod_p \frac{1}{1-p^{-s}}.$$

Before proving the convergence of the constant term, let us first give some preliminaries for the structure of the root system of $\mathfrak{g}$, following [23]. Let

$$\gamma = \frac{m + \sqrt{m^2 - 4}}{2}, \tag{5.1}$$

which is a root of the polynomial $x^2 - mx + 1$. Let r_1, r_2 be the simple reflections corresponding to the simple roots α_1, α_2. Then the Weyl group W is generated by r_1, r_2 subject to the relations $r_1^2 = r_2^2 = 1$, and has an explicit description

$$W = \{1, r_1(r_2r_1)^n, r_2(r_1r_2)^n, (r_1r_2)^{n+1}, (r_2r_1)^{n+1} : n \geq 0\}.$$

We introduce a sequence $\{A_n\}$ defined by

$$A_0 = 0, \quad A_1 = 1, \quad A_{n+2} = mA_{n+1} - A_n + 1, \quad n \geq 0.$$

Then we have the explicit formula

$$A_n = \frac{\gamma^{2n+1} - \gamma^n(1+\gamma) + 1}{\gamma^{n-1}(\gamma+1)(\gamma-1)^2} = \frac{\gamma^{n+2}}{(\gamma+1)(\gamma-1)^2} + O(1), \quad n \geq 0. \tag{5.2}$$

We also need another sequence

$$B_n = A_n - A_{n-1} = \frac{\gamma^{2n} - 1}{\gamma^{n-1}(\gamma^2 - 1)} = \frac{\gamma^{n+1}}{\gamma^2 - 1} + o(1), \quad n \geq 0. \tag{5.3}$$

In the above, $O(1)$ (resp. $o(1)$) stands for a quantity that is bounded (resp. tends to zero) as $n \to \infty$. Then we have the following formulas for the actions of $r_1(r_2r_1)^n$ and $(r_1r_2)^{n+1}$ on simple roots:

$$\begin{cases} r_1(r_2r_1)^n\alpha_1 = -B_{2n+1}\alpha_1 - B_{2n}\alpha_2, \\ r_1(r_2r_1)^n\alpha_2 = B_{2n+2}\alpha_1 + B_{2n+1}\alpha_2. \end{cases} \tag{5.4}$$

$$\begin{cases} (r_1r_2)^{n+1}\alpha_1 = B_{2n+3}\alpha_1 + B_{2n+2}\alpha_2, \\ (r_1r_2)^{n+1}\alpha_2 = -B_{2n+2}\alpha_1 - B_{2n+1}\alpha_2. \end{cases} \tag{5.5}$$

Switching α_1 and α_2, we may also obtain the similar actions of $r_2(r_1r_2)^n$ and $(r_2r_1)^{n+1}$. Regarding $w\rho - \rho$ we have

$$\begin{cases} r_1(r_2r_1)^n\rho - \rho = -A_{2n+1}\alpha_1 - A_{2n}\alpha_2, \\ r_2(r_1r_2)^n\rho - \rho = -A_{2n}\alpha_1 - A_{2n+1}\alpha_2, \\ (r_1r_2)^{n+1}\rho - \rho = -A_{2n+2}\alpha_1 - A_{2n+1}\alpha_2, \\ (r_2r_1)^{n+1}\rho - \rho = -A_{2n+1}\alpha_1 - A_{2n+2}\alpha_2. \end{cases} \tag{5.6}$$

Theorem 5.1 *Assume that ν satisfies Godement's criterion. Then the constant term $E_\nu^\sharp(g)$ converges absolutely for $g \in KA'N$, where A' is the cone*

$$A' = \{a \in A^+ : a^{\alpha_i} < 1, i \in I\}.$$

Proof We may without loss of generality assume that $\nu(h_{\alpha_i})$ has real values, $i \in I$. Then we may write

$$\nu = s_1\alpha_1 + s_2\alpha_2$$

where $s_1, s_2 \in \mathbb{R}$. Godement's criterion then reads

$$\nu(h_{\alpha_1}) = 2s_1 - ms_2 < -2, \quad \nu(h_{\alpha_2}) = 2s_2 - ms_1 < -2. \tag{5.7}$$

In particular we have $s_1, s_2 > 0$. Let us consider a typical term

$$a^{w(\nu+\rho)-\rho}c(\nu, w)$$

in $E_\nu^\sharp(a)$, where $a \in A'$. By symmetry we only need to consider $w = r_1(r_2r_1)^n$ and $w = (r_1r_2)^{n+1}, n \geq 0$.

For $w = r_1(r_2r_1)^n$, by (5.4) and (5.6) we have

$$w(\nu + \rho) - \rho = (-s_1B_{2n+1} + s_2B_{2n+2} - A_{2n+1})\alpha_1$$
$$+(-s_1B_{2n} + s_2B_{2n+1} - A_{2n})\alpha_2.$$

By (5.2) and (5.3) we have

$$-s_1B_{2n+1} + s_2B_{2n+2} - A_{2n+1}$$
$$= (\gamma s_2 - s_1)\frac{\gamma^{2n+2}}{\gamma^2 - 1} - \frac{\gamma^{2n+3}}{(\gamma + 1)(\gamma - 1)^2} + O(1)$$
$$= (\gamma s_2 - s_1 - \frac{\gamma}{\gamma - 1})\frac{\gamma^{2n+2}}{\gamma^2 - 1} + O(1).$$

From (5.7) it follows that

$$\gamma s_2 - s_1 = \frac{m\gamma - 2}{4 - m^2}(2s_1 - ms_2) + \frac{2\gamma - m}{4 - m^2}(2s_2 - ms_1)$$
$$> \frac{2(m\gamma - 2 + 2\gamma - m)}{m^2 - 4}$$
$$= \frac{2(\gamma - 1)}{m - 2} = \frac{2\gamma}{\gamma - 1},$$

where the last equation follows from $\gamma^2 - m\gamma + 1 = 0$. If we introduce a constant

$$C_\nu = (\gamma s_2 - s_1 - \frac{\gamma}{\gamma - 1})\frac{1}{\gamma^2 - 1} > 0,$$

then

$$r_1(r_2r_1)^n(\nu+\rho)-\rho = (C_\nu\gamma^{2n+2}+O(1))\alpha_1+(C_\nu\gamma^{2n+1}+O(1))\alpha_2.$$

Similarly, we have

$$(r_1r_2)^{n+1}(\nu+\rho)-\rho = (D_\nu\gamma^{2n+3}+O(1))\alpha_1+(D_\nu\gamma^{2n+2}+O(1))\alpha_2,$$

where

$$D_\nu = (\gamma s_1 - s_2 - \frac{\gamma}{\gamma-1})\frac{1}{\gamma^2-1} > 0.$$

By Godement's criterion we have

$$-\mathrm{Re}\,(\nu+\rho)(h_{\alpha_i}) > 1$$

for $i \in I$. Then there exists $\varepsilon > 0$ such that

$$-\mathrm{Re}\,(\nu+\rho)(h_\alpha) > 1+\varepsilon$$

for any positive real root α. Using the properties of Riemann zeta functions we can find a constant $C_\varepsilon > 0$ depending on ε such that

$$|c(\nu,w)| < C_\varepsilon^{\ell(w)}$$

where $\ell(w)$ is the length of w.

Since $a \in A'$, we have $a^{\alpha_i} < 1$, $i \in I$. Combining above estimates, we see that there exist positive constants C_1, C_2 and M depending on ν such that

$$E_\nu^\sharp(a) = \sum_{w\in W} a^{w(\nu+\rho)-\rho}c(\nu,w)$$
$$\leq M \sum_{w\in W} a^{C_1\gamma^{\ell(w)}\alpha_1+C_2\gamma^{\ell(w)}\alpha_2}C_\varepsilon^{\ell(w)}$$

which converges absolutely. $\qquad\square$

Corollary 5.2 *Assume that ν satisfies Godement's criterion. Then for any compact subset A'_c of A', there is a measure zero subset S_0 of A'_cN such that $E_\nu(g)$ converges absolutely for $g \in KA'_cN$ off the set KS_0.*

Proof The proof is essentially the same as that of Theorem 9.1 in [14]. The constant term $E_\nu^\sharp(a)$ is continuous for $a \in A'$ and integrable on any compact subset A'_c of A'. When ν is real, it follows from Tonelli's theorem that the series $E_\nu(g)$ is locally integrable on $A'N$. In particular, for any compact subset A'_c of A', there is a measure zero subset $S_0 \subset A'_cN$ such that $E_\nu(g)$ converges absolutely for $g \in KA'_cN$ off the set KS_0. When ν is complex, the series $E_\nu(g)$ is dominated by $E_{\mathrm{Re}(\nu)}(g)$ and we are done. $\qquad\square$

We propose the following conjecture, which weakens Godement's criterion and asserts everywhere convergence instead of almost everywhere convergence.

Conjecture 5.3 $E_\nu(g)$ *converges absolutely for* $g \in KA'N$ *and* ν *satisfying* $\mathrm{Re}\, \nu(h_{\alpha_i}) < -1$, $i \in I$.

6 Fourier coefficients

In this section we shall define and calculate the Fourier coefficients of $E_\nu(g)$. To facilitate the computation, we work with adelic groups. Let $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p$ and $\mathbb{I} = \mathbb{A}^\times$ be the adele ring and idele group of the rational number field $\mathbb{Q}$, respectively. According to Sect. 3, for any prime p we have the group $G_{\mathbb{Q}_p} \subset \mathrm{Aut}(V_{\mathbb{Q}_p})$, and we let $K_p \subset G_{\mathbb{Q}_p}$ be the subgroup $K_p = \{g \in G_{\mathbb{Q}_p} : g \cdot V_{\mathbb{Z}_p} = V_{\mathbb{Z}_p}\}$. Let $G_{\mathbb{A}} = G_{\mathbb{R}} \times \prod_p' G_{\mathbb{Q}_p}$ and $G_{\mathbb{A}_f} = \prod_p' G_{\mathbb{Q}_p}$ (the adele and finite adele groups respectively) be the restricted products with respect to the family of subgroups K_p. Note that we have the diagonal embedding $\iota : G_{\mathbb{Q}} \hookrightarrow G_{\mathbb{R}} \times \prod_p G_{\mathbb{Q}_p}$. Set $\Gamma_{\mathbb{Q}} = \iota^{-1}(G_{\mathbb{A}})$ and $K_{\mathbb{A}} = K \times \prod_p K_p$.

We shall extend the definition of $E_\nu(g)$ to $g \in G_{\mathbb{A}}$. For each prime p we have an Iwasawa decomposition [9]

$$G_{\mathbb{Q}_p} = K_p A_{\mathbb{Q}_p} N_{\mathbb{Q}_p},$$

where $A_{\mathbb{Q}_p}$ is generated by $h_{\alpha_i}(s)$, $i = 1, 2$, $s \in \mathbb{Q}_p^\times$, and $N_{\mathbb{Q}_p}$ is generated by $\chi_\alpha(s)$, $\alpha \in \Phi_+^{re}$, $s \in \mathbb{Q}_p$. From the local Iwasawa decompositions we have

$$G_{\mathbb{A}} = K_{\mathbb{A}} A_{\mathbb{A}} N_{\mathbb{A}}.$$

If $\iota = (\iota_\infty, \iota_p) \in \mathbb{I}$ is an idele, define the usual norm $|\iota|$ of ι by

$$|\iota| = |\iota_\infty| \prod_p |\iota_p|_p.$$

An element $a \in A_{\mathbb{A}}$ can be decomposed as $a = h_{\alpha_1}(s_1)h_{\alpha_2}(s_2)$, $s_1, s_2 \in \mathbb{I}$. We let $|a| \in A^+$ be the element $h_{\alpha_1}(|s_1|)h_{\alpha_2}(|s_2|)$. Let $g \in G_{\mathbb{A}}$ be decomposed as

$$g = k_g a_g n_g, \quad k_g \in K_{\mathbb{A}}, \quad a_g \in A_{\mathbb{A}}, \quad n_g \in N_{\mathbb{A}}.$$

Note that $|a_g|$ is uniquely determined by g, although above decomposition is not unique. Then we may define

$$\Phi_\nu(g) = |a_g|^\nu.$$

The Eisenstein series is defined by

$$E_\nu(g) = \sum_{\gamma \in \Gamma_{\mathbb{Q}}/\Gamma_{\mathbb{Q}} \cap B_{\mathbb{Q}}} \Phi_\nu(g\gamma), \quad g \in \mathbb{A}.$$

When $g \in G_{\mathbb{R}}$ this coincides with our previous definition, since $\Gamma_{\mathbb{Q}}/\Gamma_{\mathbb{Q}} \cap B_{\mathbb{Q}} \cong \Gamma/\Gamma \cap B$.

For a positive real root α, let U_{α} be the root subgroup $\{\chi_{\alpha}(u) : u \in \mathbb{R}\}$.

Let ψ be a non-trivial character of $N/(\Gamma \cap N)$. Then we have $\psi = \psi_1 \psi_2$, where ψ_i is a character of $U_{\alpha_i}/\Gamma \cap U_{\alpha_i}$. This follows from the fact that $N/[N, N] \cong U_{\alpha_1} \times U_{\alpha_2}$. We extend ψ to a character of $N_{\mathbb{A}}/N_{\mathbb{Q}}$, where $N_{\mathbb{Q}} := \Gamma_{\mathbb{Q}} \cap N_{\mathbb{A}}$ and define the ψ-th Fourier coefficient of $E_\nu(g)$ along B by

$$E_{\nu,\psi}(g) = \int_{N/(\Gamma \cap N)} E_\nu(gn)\bar{\psi}(n)dn = \int_{N_{\mathbb{A}}/N_{\mathbb{Q}}} E_\nu(gn)\bar{\psi}(n)dn.$$

Then $E_{\nu,\psi}(g)$ is a Whittaker function on G, that is, a function W satisfying the relation

$$W(gn) = \psi(n)W(g),$$

for each $n \in N$.

We call ψ *generic* if each ψ_i is non-trivial for $i = 1, 2$. Then we have the following vanishing result for generic characters, which in fact holds generally for infinite-dimensional Kac–Moody groups (cf. [30]).

Lemma 6.1 *If ψ is generic, then $E_{\nu,\psi}(g) = 0$.*

Proof Recall that we have the Bruhat decomposition

$$\Gamma_{\mathbb{Q}} = \bigsqcup_{w \in W} N_{w,\mathbb{Q}}w(\Gamma_{\mathbb{Q}} \cap B_{\mathbb{Q}}),$$

where $N_{w,\mathbb{Q}} = \Gamma_{\mathbb{Q}} \cap \prod_{\alpha \in \Phi_+ \cap w\Phi_-} U_{\alpha}$. Then we have

$$E_\nu(g) = \sum_{\gamma \in \Gamma_{\mathbb{Q}}/\Gamma_{\mathbb{Q}} \cap B_{\mathbb{Q}}} \Phi_\nu(g\gamma) = \sum_{w \in W} \sum_{\gamma \in N_{w,\mathbb{Q}}} \Phi_\nu(g\gamma w).$$

We introduce $N'_w = \prod_{\alpha \in \Phi_+ \cap w\Phi_+} U_{\alpha}$. Then $N = N_w N'_w$ and it follows that

$$E_{\nu,\psi}(g) = \sum_{w \in W} \int_{N_{\mathbb{A}}/N_{\mathbb{Q}}} \sum_{\gamma \in N_{w,\mathbb{Q}}} \Phi_\nu(gn\gamma w)\bar{\psi}(n)dn$$

$$= \sum_{w \in W} \int_{N_{\mathbb{A}}/N'_{w,\mathbb{Q}}} \Phi_\nu(gnw)\bar{\psi}(n)dn$$

$$= \sum_{w \in W} \int_{N_{w,\mathbb{A}}} \bar{\psi}(n_w) \int_{N'_{w,\mathbb{A}}/N'_{w,\mathbb{Q}}} \Phi_\nu(gn_w n'_w w)\bar{\psi}(n'_w)dn'_w dn_w.$$

For each $w \in W$, at least one of the two roots $w^{-1}\alpha_i$, $i = 1, 2$, is positive. Since Φ_ν is right N-invariant, the inner integral of the last equation involves a factor

$$\int_{U_{\alpha_i,\mathbb{A}}/U_{\alpha_i,\mathbb{Q}}} \bar{\psi}_i(u)du = \int_{U_{\alpha_i}/\Gamma_{\mathbb{Q}} \cap U_{\alpha_i}} \bar{\psi}_i(u)du$$

for some i, which is zero by the assumption that ψ is generic.   □

Using this lemma, we may assume that $\psi = \psi_1$ or ψ_2, and is non-trivial. Any character of U_{α_i}, which is trivial on $\Gamma \cap U_{\alpha_i}$, is of the form

$$\psi_{i,n} : \chi_{\alpha_i}(u) \mapsto e^{2\pi i n u}, \quad u \in \mathbb{R}$$

for some $n \in \mathbb{Z}$.

Before we state and prove the main result of this section, let us first recall some Fourier coefficients for SL_2. For $F = \mathbb{R}$ or $\mathbb{Q}_p$, one has the Iwasawa decomposition $SL_2(F) = KAN$, where $K = SO(2, \mathbb{R})$ or $SL_2(\mathbb{Z}_p)$ is a maximal compact subgroup of $SL_2(F)$,

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Let $g \in SL_2(F)$ be decomposed as

$$g = k \begin{pmatrix} a_g & 0 \\ 0 & a_g^{-1} \end{pmatrix} n.$$

For $s \in \mathbb{C}$, define a function $\Phi_s(g)$ on $SL_2(F)$ by $\Phi_s(g) = |a_g|^{-s}$. Clearly Φ_s is well-defined. Given a character ψ of F, we shall consider the Fourier coefficient

$$\int_F \Phi_s \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \bar{\psi}(x) dx, \tag{6.1}$$

which is convergent for Re $s > 0$. If we write $\begin{pmatrix} a_x & 0 \\ 0 & a_x^{-1} \end{pmatrix}$ for the A-component of $\begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$ in the Iwasawa decomposition, then $|a_x| = \sqrt{1 + x^2}$ for $F = \mathbb{R}$ and $|a_x| = \max(1, |x|_p)$ for $F = \mathbb{Q}_p$.

The character $\psi(u) = e^{2\pi i u}$ of $\mathbb{R}/\mathbb{Z}$ corresponds to the character $\psi_\infty \prod_p \psi_p$ of $\prod_p \mathbb{Z}_p \backslash \mathbb{A}/\mathbb{Q}$, where

$$\psi_\infty(x) = e^{2\pi i x}, \quad x \in \mathbb{R}$$
$$\psi_p(x) = e^{-2\pi i (\text{fractional part of } x)}, \quad x \in \mathbb{Q}_p.$$

In the above the fractional part of

$$x = \sum_{i > -\infty}^{\infty} a_i p^i \in \mathbb{Q}_p, \quad a_i \in \{0, \dots, p-1\}$$

written as a base p expansion, is given by $\sum_{i < 0} a_i p^i$. Fix $y \in \mathbb{R}_+$ and associate an idele $(y_v)_v \in \mathbb{I}$ by $y_\infty = y$, $y_p = 1$. For $n \in \mathbb{Z}$, $n \neq 0$, we twist the nth power of ψ

by y, i.e. consider the characters $\psi_\infty(nyx)$ of $\mathbb{R}$ and $\psi_p(nx)$ of $\mathbb{Q}_p$. Then the Fourier coefficients (6.1) are given by the following functions. For $F = \mathbb{R}$, $\mathrm{Re}\, s > 1$ we have (cf. [2, pp. 66–67])

$$W_n^\infty(y, s) = \int_{-\infty}^{\infty} (1 + x^2)^{-\frac{s}{2}} \bar\psi_\infty(nyx)\,dx$$

$$= 2\pi^{s/2} \Gamma(s/2)^{-1} |ny|^{\frac{s-1}{2}} K_{\frac{s-1}{2}}(2\pi|n|y),$$

where $K_s(y)$ is the *K-Bessel function*, also known as the *Macdonald Bessel function*, defined by

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})/2}\, t^s\, \frac{dt}{t}, \quad \mathrm{Re}\, s > 0.$$

Write $|n| = \prod_p p^{n_p}$ into the primary decomposition. Then for $p < \infty$, $\mathrm{Re}\, s > 1$ we have

$$W_n^p(s) = \int_{\mathbb{Q}_p} \max(1, |x|_p)^{-s} \bar\psi_p(nx)\,dx$$

$$= 1 + \sum_{i=1}^\infty p^{-is} \int_{p^{-i}\mathbb{Z}_p^\times} \bar\psi_p(nx)\,dx$$

$$= \frac{(1 - p^{-s})(1 - p^{(n_p+1)(1-s)})}{1 - p^{1-s}}.$$

In the above computations we have made use of the Iwasawa decomposition for SL_2. Now we form a product

$$W_n(y, s) = W_n^\infty(y, s) \prod_p W_n^p(s)$$

$$= 2\sigma_{1-s}(|n|)|ny|^{\frac{s-1}{2}} K_{\frac{s-1}{2}}(2\pi|n|y)\frac{1}{\xi(s)}, \quad \mathrm{Re}\, s > 1, \tag{6.2}$$

where σ_s is the divisor power sum function defined by $\sigma_s(n) = \sum_{d|n} d^s$ for $n \in \mathbb{N}$.

Theorem 6.2 *Assume that v satisfies Godement's criterion. Then for $a \in A'$, $i \in I$, $n \in \mathbb{Z}$, $n \neq 0$, one has the $\psi_{i,n}$-th Fourier coefficient*

$$E_{v,\psi_{i,n}}(a) = \sum_{w \in W,\ w^{-1}\alpha_i < 0} a^{w(v+\rho)-\rho} c_{\psi_{i,n}}(v, w)(a),$$

where

$$c_{\psi_{i,n}}(v, w)(a) = W_n(a^{-\alpha_i}, 1 + w(v + \rho)(h_{\alpha_i})) \prod_{\substack{\alpha \in \Phi_+ \cap w^{-1}\Phi_- \\ \alpha \neq -w^{-1}\alpha_i}} \frac{\xi(-(v + \rho)(h_\alpha))}{\xi(1 - (v + \rho)(h_\alpha))}$$

with $W_n(y, s)$ *being defined by* (6.2).

Proof As in the proof of Lemma 6.1, we see that

$$E_{\nu,\psi}(a) = \sum_{w \in W, \; w^{-1}\alpha_i < 0} \int_{N_{w,\mathbb{A}}} \Phi_\nu(an_w w)\bar{\psi}(n_w)dn_w.$$

We follow the computation in [14] and [30, 4.4]. Let $w^{-1} = r_{k_1} \cdots r_{k_\ell}$ be the reduced expression of w^{-1}, where $\ell = \ell(w)$ and $k_j = 1$ or 2 for $j = 1, \ldots, \ell$. Let

$$\Phi_{w^{-1}} = \Phi_+ \cap w^{-1}\Phi_- = \{\beta_1, \ldots, \beta_\ell\},$$

where $\beta_j = r_{k_1} \cdots r_{k_{j-1}}\alpha_{k_j}$. Then

$$\Phi_w = \Phi_+ \cap w\Phi_- = \{\gamma_1, \ldots, \gamma_\ell\},$$

where $\gamma_j = -w\beta_j = r_{k_\ell} \cdots r_{k_{j+1}}\alpha_{k_j}$. Note that

$$\beta_1 + \cdots + \beta_\ell = \rho - w^{-1}\rho, \quad \gamma_1 + \cdots + \gamma_\ell = \rho - w\rho.$$

From these formulas it is clear that if $w^{-1}\alpha_i < 0$, then $\gamma_\ell = \alpha_i = -w\beta_\ell$. By decomposing N_w into a product of root subgroups, we get

$$\int_{N_{w,\mathbb{A}}} \Phi_\nu(an_w w)\bar{\psi}(n_w)dn_w$$

$$= \int_{\mathbb{A}^\ell} \Phi_\nu\big(a\chi_{\gamma_\ell}(u_\ell) \cdots \chi_{\gamma_1}(u_1)w\big)\bar{\psi}_{i,n}(u_\ell)du_\ell \cdots du_1$$

$$= \int_{\mathbb{A}^\ell} \Phi_\nu\big(\chi_{\gamma_\ell}(a^{\gamma_\ell}u_\ell) \cdots \chi_{\gamma_1}(a^{\gamma_1}u_1)aw\big)\bar{\psi}_{i,n}(u_\ell)du_\ell \cdots du_1$$

$$= \int_{\mathbb{A}^\ell} a^{w\nu - \gamma_1 - \cdots - \gamma_\ell}\Phi_\nu\big(\chi_{\gamma_\ell}(u_\ell) \cdots \chi_{\gamma_1}(u_1)w\big)\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell \cdots du_1$$

$$= a^{w(\nu+\rho)-\rho}\int_{\mathbb{A}^\ell} \Phi_\nu\big(\chi_{-\beta_\ell}(u_\ell) \cdots \chi_{-\beta_1}(u_1)\big)\bar{\psi}_{i,n}(a^{-\alpha_i}u_\ell)du_\ell \cdots du_1.$$

Let $\chi_{-\beta_\ell}(u) = k(u)a(u)n(u)$ be the Iwasawa decomposition, with $k(u) \in K, n(u) \in U_{\beta_\ell}, a(u) \in A$. Put $w'^{-1} = r_{k_1} \cdots r_{k_{\ell-1}}$, then $\{\beta_1, \ldots, \beta_{\ell-1}\} = \Phi_+ \cap w'^{-1}\Phi_-$. Consider the decomposition $N = N_{w'}N'_{w'}$ (see the proof of Lemma 6.1). Then we have

$$U_{-\beta_{\ell-1}} \cdots U_{-\beta_1} = w'^{-1}N_{w'}w'.$$

Let us define the projection

$$\pi : w'^{-1}Nw' \to w'^{-1}N_{w'}w'.$$

Since $U_{-\beta_1}, \ldots, U_{-\beta_{\ell-1}}, U_{\beta_\ell} \subset w'^{-1} N w'$, the following map

$$\pi \circ \mathrm{Ad}(n(u)) : w'^{-1} N_{w'} w' \to w'^{-1} N_{w'} w',$$

is well-defined and unimodular. From this fact, and noting that Φ_ν is right invariant under $w'^{-1} N'_{w'} w' \subset N$, it follows

$$\int_{\mathbb{A}^\ell} \Phi_\nu \big(\chi_{-\beta_\ell}(u_\ell) \cdots \chi_{-\beta_1}(u_1) \big) \bar\psi_{i,n}(a^{-\alpha_i} u_\ell) du_\ell \cdots du_1$$

$$= \int_{\mathbb{A}} a(u_\ell)^{\nu+\beta_1+\cdots\beta_{\ell-1}} \bar\psi_{i,n}(a^{-\alpha_i} u_\ell) du_\ell$$

$$\int_{\mathbb{A}^{\ell-1}} \Phi_\nu \big(\chi_{-\beta_{\ell-1}}(u_{\ell-1}) \cdots \chi_{-\beta_1}(u_1) \big) du_{\ell-1} \cdots du_1$$

$$= \int_{\mathbb{A}} a(u_\ell)^{\nu+\rho-w'^{-1}\rho} \bar\psi_{i,n}(a^{-\alpha_i} u_\ell) du_\ell$$

$$\int_{\mathbb{A}^{\ell-1}} \Phi_\nu \big(\chi_{-\beta_{\ell-1}}(u_{\ell-1}) \cdots \chi_{-\beta_1}(u_1) \big) du_{\ell-1} \cdots du_1.$$

Note that $w'^{-1}\rho(h_{\beta_\ell}) = \rho(w'h_{\beta_\ell}) = \rho(h_{\alpha_i}) = 1$. Then the first integral in the last equation equals $W_n(a^{-\alpha_i}, 1 - (\nu+\rho)(h_{\beta_\ell})) = W_n(a^{-\alpha_i}, 1 + w(\nu+\rho)(h_{\alpha_i}))$, and the second one, by Gindikin–Karpelevich formula, equals

$$\prod_{j=1}^{\ell-1} \frac{\xi(-(\nu+\rho)(h_{\beta_j}))}{\xi\big(1 - (\nu+\rho)(h_{\beta_j})\big)}.$$

Hence our assertion follows. □

7 Entirety of cuspidal Eisenstein series

In analogy with [1, Theorem 5.2] and [21], we shall prove that the Eisenstein series on G induced from cusp forms on SL_2 are entire functions. Let P be the maximal parabolic subgroup of G generated by B and the simple reflection r_1, and $P = MU$ be the Levi decomposition, where M is the Levi subgroup and U is the pro-unipotent radical of P. Let $L \cong SL_2$ be the subgroup generated by $\chi_{\pm\alpha_1}(t)$, $t \in \mathbb{R}$, and let $A_1 = \langle h_{\alpha_1}(t) : t \in \mathbb{R}^\times \rangle$, $H = \{a \in A : a^{\alpha_1} = \pm 1\}$. Then we have an almost direct product $A = A_1 \times H$, and $M = LH$. We introduce

$$H^+ = \{h_{\alpha_1}(t^m)h_{\alpha_2}(t^2) : t \in \mathbb{R}_+\} \cong K \cap H\backslash H,$$
$$A_1^+ = \{h_{\alpha_1}(t) : t \in \mathbb{R}_+\} \cong K \cap A_1\backslash A_1.$$

From the Iwasawa decomposition $G = KP = KMU$, the following maps

$$\mathrm{Iw}_L : G \to K \cap L\backslash L, \quad \mathrm{Iw}_{H^+} : G \to H^+$$

are well-defined. Similarly, using the Iwasawa decomposition for L we may define the map

$$\mathrm{Iw}_{A_1^+} : K \cap L \backslash L \to A_1^+.$$

For convenience we also denote by Iw_{A^+} the map $G \to A^+$, $g = k_g a_g n_g \mapsto a_g$ which we used previously. Then it is clear that

$$\mathrm{Iw}_{A^+} = (\mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L) \times \mathrm{Iw}_{H^+} : G \to A^+ \cong A_1^+ \times H^+. \tag{7.1}$$

Let ϖ_2 be the fundamental weight corresponding to α_2, that is

$$\varpi_2 = \frac{m\alpha_1 + 2\alpha_2}{4 - m^2}.$$

Note that ϖ_2 is trivial on $A_1 = L \cap A$, hence (7.1) implies that

$$\mathrm{Iw}_{A^+}(\cdot)^{\varpi_2} = \mathrm{Iw}_{H^+}(\cdot)^{\varpi_2}. \tag{7.2}$$

Similarly, since α_1 is trivial on H we also have

$$\mathrm{Iw}_{A^+}(\cdot)^{\alpha_1} = \mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L(\cdot)^{\alpha_1}. \tag{7.3}$$

We may regard ϖ_2 as an algebraic character of M. For $s \in \mathbb{C}$ define the Eisenstein series

$$E_s(g) = \sum_{\gamma \in \Gamma / \Gamma \cap P} \mathrm{Iw}_{H^+}(g\gamma)^{s\varpi_2}.$$

Moreover, for an unramified cusp form f on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, that is, a $SO(2)$–invariant cusp form, we define

$$E_{s,f}(g) = \sum_{\gamma \in \Gamma / \Gamma \cap P} \mathrm{Iw}_{H^+}(g\gamma)^{s\varpi_2} f\big(\mathrm{Iw}_L(g\gamma)\big).$$

Theorem 7.1 *Assume that* $\mathrm{Re}\ s < -2$. *Then for any compact subset* A'_c *of* A', *there is a measure zero subset* S_0 *of* $A'_c N$ *such that* $E_s(g)$ *converges absolutely for* $g \in K A'_c N$ *off the set* $K S_0$.

Proof Our proof is similar to the previous one for the Eisenstein series induced from characters on Borel subgroups. (See also II. I. 5. of [32].) We have the Bruhat decomposition

$$G = \bigsqcup_{w \in W_1} B w P = \bigsqcup_{w \in W_1} N_w w P,$$

where $W_1 = \{w \in W : w\alpha_1 > 0\}$ is a set of representatives of minimal length for the quotient $W/\langle r_1 \rangle$. Formal calculations show that the constant term of $E_s(a)$ for $a \in A'$ is

$$E_s^\sharp(a) = \sum_{w \in W_1} a^{w(s\varpi_2 + \rho) - \rho} c(s\varpi_2, w).$$

Then we only need to prove the absolute convergence of $E_s^\sharp(a)$ under the conditions of the theorem. We may assume that s is real. There are two cases for $w \in W_1$: $w = (r_1 r_2)^n$ or $r_2(r_1 r_2)^n$. We shall only deal with the first case, and the second case can be treated similarly.

Assume $w = (r_1 r_2)^n$. Then using (5.5) and (5.6), for $a \in A'$ there exists a positive constant M only depending on a such that

$$a^{w(s\varpi_2 + \rho) - \rho} \leq M a^{\gamma C_n \alpha_1 + C_n \alpha_2},$$

where

$$C_n = \frac{\gamma^{2n}}{(m^2 - 4)(\gamma + 1)(\gamma - 1)^2} \left[-s(m\gamma - 2)(\gamma - 1) - (m^2 - 4)\gamma \right].$$

It follows that $C_n \to \infty$ if and only if

$$s < -\frac{(m^2 - 4)\gamma}{(m\gamma - 2)(\gamma - 1)} = -1 - \gamma^{-1}. \tag{7.4}$$

We also need to consider the factor $c(s\varpi_2, w)$. One can show that

$$\Phi_+ \cap w^{-1} \Phi_- = \{B_i \alpha_1 + B_{i+1} \alpha_2 : i = 0, \dots, 2n - 1\}.$$

For $\alpha = B_i \alpha_1 + B_{i+1} \alpha_2$ we have

$$-(s\varpi_2 + \rho)(h_\alpha) = (-s - 1)B_{i+1} - B_i > B_{i+1} - B_i \geq 1$$

when $s < -2$. This implies that $c(s\varpi_2, w) \leq C^{\ell(w)}$ for some constant C depending on s.

Combining above analysis it is easy to see the convergence of $E_s^\sharp(a)$ for $a \in A'$. $\square$

We remark that, from $B_{i+1} > \gamma B_i$ it follows that

$$(-s - 1)B_{i+1} - B_i \to \infty \quad \text{as} \quad i \to \infty$$

when $s < -1 - \gamma^{-1}$. Together with (7.4), this suggests the following

Conjecture 7.2 $E_s(g)$ *converges absolutely for* $g \in KA'N$ *and* $\mathrm{Re}\, s < -1 - \gamma^{-1}$.

Now let us state the main result of this section.

Theorem 7.3 *Let f be an unramified cusp form on SL_2. For any compact subset A'_c of A', there is a measure zero subset S_0 of $A'_c N$ such that $E_{s,f}(g)$ is an entire function of $s \in \mathbb{C}$ for $g \in K A'_c N$ off the set $K S_0$.*

We shall follow the strategy in [21] to prove Theorem 7.3. The following lemma is in analogy with [21, Lemma 3.2], where we set $x^y := yxy^{-1}$ for $x, y \in G$.

Lemma 7.4 *If $\gamma \in \Gamma \cap BwB$, then*

$$\mathrm{Iw}_{A^+}(g\gamma) = \left(\mathrm{Iw}_{A^+} g\right)^{w^{-1}} \cdot \mathrm{Iw}_{A^+}(n_w w)$$

for some $n_w \in N_{w,\mathbb{A}}$ depending on γ and g.

Recall from [17, Lemma 6.1] that for $n_w \in N_{w,\mathbb{A}}$,

$$\ln\left(\mathrm{Iw}_{A^+}(n_w w)\right) = \sum_{\alpha \in \Phi_w} c_\alpha h_\alpha \quad \text{with} \quad c_\alpha \geq 0, \tag{7.5}$$

where $\Phi_w = \Phi_+ \cap w\Phi_-$. Now we can establish the Iwasawa inequalities:

Lemma 7.5 *There exists a constant $D > 0$ such that*

$$\mathrm{Iw}_{A^+}(g\gamma)^{\alpha_1} \geq \mathrm{Iw}_{A^+}(g\gamma)^{D\varpi_2}$$

for any $g \in K A'N$, $w \in W_1$ and $\gamma \in \Gamma \cap BwB$, where the constant D is independent of w.

Proof Put $a = \mathrm{Iw}_{A^+} g \in A'$. From Lemma 7.4 and (7.5), it suffices to find a constant D such that the following two inequalities

$$a^{w^{-1}\alpha_1} \geq a^{Dw^{-1}\varpi_2}, \tag{7.6}$$

$$\alpha_1(h_\alpha) \geq D\varpi_2(h_\alpha) \tag{7.7}$$

hold for any $w \in W_1$ and $\alpha \in \Phi_w$. We only consider the case $w = (r_1 r_2)^n$, and similar arguments apply to the other case $w = r_2(r_1 r_2)^n$. Put $w = (r_1 r_2)^n$, then $\Phi_w = \{B_{i+1}\alpha_1 + B_i\alpha_2 : i = 0, \ldots, 2n - 1\}$. For any α of the form $B_{i+1}\alpha_1 + B_i\alpha_2$ one has

$$\alpha_1(h_\alpha) = 2B_{i+1} - mB_i \geq (2\gamma - m)B_i = (2\gamma - m)\varpi_2(h_\alpha).$$

This proves (7.7) with $D = 2\gamma - m > 0$.

Now we prove (7.6). Again, there are two possibilities for $w \in W_1$: $w = (r_1 r_2)^n$ or $r_2(r_1 r_2)^n$. Here we treat the first case, while the second one is similar. For $w = (r_1 r_2)^n$, we see from the formula (5.5) that

$$\begin{cases} w^{-1}\alpha_1 = -B_{2n-1}\alpha_1 - B_{2n}\alpha_2, \\ w^{-1}\alpha_2 = B_{2n}\alpha_1 + B_{2n+1}\alpha_2. \end{cases}$$

Since we have

$$\varpi_2 = \frac{m\alpha_1 + 2\alpha_2}{4 - m^2},$$

it follows from the above formulas that

$$w^{-1}\varpi_2 = \frac{1}{4 - m^2}\left[(2B_{2n} - mB_{2n-1})\alpha_1 + (2B_{2n+1} - mB_{2n})\alpha_2\right].$$

Recall that $B_n = \gamma^n + o(1)$, which implies

$$2B_{n+1} - mB_n = (2\gamma - m)\gamma^n + o(1).$$

Note that $2\gamma - m > 0$. Then one can find a positive constant D independent of n such that

$$B_n \geq D\,\frac{2B_{n+1} - mB_n}{m^2 - 4}. \tag{7.8}$$

In fact, for any $0 < D < \frac{m^2 - 4}{2\gamma - m}$, the inequality (7.8) is true as long as n is large. Since $a \in A'$, we have $a^{\alpha_1}, a^{\alpha_2} < 1$, and the above inequality (7.8) implies that

$$a^{w^{-1}\alpha_1} \geq a^{Dw^{-1}\varpi_2}.$$

Since any such $D < \frac{m^2 - 4}{2\gamma - m} < 2\gamma - m$, (7.7) also holds as we proved in the first part. Note that we can choose the constant D to be independent of $a \in A'$ and $w \in W_1$. $\square$

We also need the rapid decay of cuspidal automorphic forms.[1] Most recent results on rapid decay, which generalize classical results in various ways, can be found in [31]. In our case, f is an $SO(2)$-finite cusp form on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, and we may assume that $g \in SL_2(\mathbb{R})$ is in a Siegel set. Then the rapid decay implies that for any natural number $n \geq 1$, there exists a constant $C > 0$ depending on n such that

$$|f(g)| \leq C\,\mathrm{Iw}_{A_1^+}(g)^{-n\alpha_1}. \tag{7.9}$$

Proof of Theorem 7.3 Since cusp forms are bounded, the assertion follows from Theorem 1.1 when Re $s < -2$. Assume that Re $s \geq -2$. Choose $s_0 \in \mathbb{R}$ with $s_0 < -2$. Choose a real number $d > $ Re $s - s_0 > 0$ such that $\frac{d}{D} \in \mathbb{N}$, where D is given in Lemma 7.5. Then Re $s - d < s_0 < -2$ and there exists a subset N' of N with measure zero complement such that $E_{\mathrm{Re}\,s-d}(g)$ converges for any $g \in KA_c'N'$.

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[1] In an earlier version of this paper, an exponential decay in [21] was used. It was pointed out by Steve D. Miller that the rapid decay was enough to obtain our result.

Put $n = \frac{d}{D} \in \mathbb{N}$ as above. From (7.2), (7.3), Lemma 7.5 and (7.9), we obtain that for any $\gamma \in \Gamma/\Gamma \cap P$,

$$
\begin{aligned}
\left| \mathrm{Iw}_{H^+}(g\gamma)^{s\varpi_2} f\left(\mathrm{Iw}_L(g\gamma)\right) \right| &\leq C \, \mathrm{Iw}_{H^+}(g\gamma)^{(\mathrm{Re}\, s)\varpi_2} \mathrm{Iw}_{A_1^+} \circ \mathrm{Iw}_L(g\gamma)^{-n\alpha_1} \\
&\leq C \, \mathrm{Iw}_{H^+}(g\gamma)^{(\mathrm{Re}\, s)\varpi_2} \mathrm{Iw}_{H^+}(g\gamma)^{-nD\varpi_2} \\
&= C \, \mathrm{Iw}_{H^+}(g\gamma)^{(\mathrm{Re}\, s - d)\varpi_2}.
\end{aligned}
$$

Note that the constants C and D are independent of w. Taking the summation over γ, it follows that $E_{s,f}(g)$ is absolutely convergent. $\qquad \square$

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