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A combinatorial description of the affine Gindikin-Karpelevich formula of type $A_n^{(1)}$

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Abstract. The classical Gindikin-Karpelevich formula appears in Langlands' calculation of the constant terms of Eisenstein series on reductive groups and in Macdonald's work on p-adic groups and affine Hecke algebras. The formula has been generalized in the work of Garland to the affine Kac-Moody case, and the affine case has been geometrically constructed in a recent paper of Braverman, Finkelberg, and Kazhdan. On the other hand, there have been efforts to write the formula as a sum over Kashiwara's crystal basis or Lusztig's canonical basis, initiated by Brubaker, Bump, and Friedberg. In this paper, we write the affine Gindikin-Karpelevich formula as a sum over the crystal of generalized Young walls when the underlying Kac-Moody algebra is of affine type $A_n^{(1)}$. The coefficients of the terms in the sum are determined explicitly by the combinatorial data from Young walls.

0. Introduction

The classical Gindikin-Karpelevich formula originated from a certain integration on real reductive groups [**GK62**]. When Langlands calculated the constant terms of Eisenstein series on reductive groups [**Lan71**], he considered a p-adic analogue of the integration and called the resulting formula the *Gindikin-Karpelevich formula*. In the case of GL_{n+1}, the formula can be described as follows: let F be a p-adic field with residue field of q elements and let N_- be the maximal unipotent subgroup of $\mathrm{GL}_{n+1}(F)$ with maximal torus T. Let f° denote the standard spherical vector corresponding to an unramified character χ of T, let $T(\mathbf{C})$ be the maximal torus in the L-group $\mathrm{GL}_{n+1}(\mathbf{C})$ of $\mathrm{GL}_{n+1}(F)$, and let $\boldsymbol{z} \in T(\mathbf{C})$ be the element corresponding to χ via the Satake isomorphism. Then the Gindikin-Karpelevich

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formula is given by

$$\int_{N_-(F)} f^\circ(\boldsymbol{n})\,\mathrm{d}\boldsymbol{n} = \prod_{\alpha\in\Delta^+} \frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha}, \tag{0.1}$$

where Δ^+ is the set of positive roots of $\mathrm{GL}_{n+1}(\mathbf{C})$. The formula appears in Macdonald's study on p-adic groups and affine Hecke algebras as well [**Mac71**], and the product side of (0.1) is also known as Macdonald's c-function.

In the paper [**Gar04**], Garland generalized Langlands' calculation to affine Kac-Moody groups and obtained an affine Gindikin-Karpelevich formula as a product over $\Delta^+ \cap w^{-1}(\Delta^-)$ for each $w \in W$, where Δ^+ (resp. Δ^-) is the set of positive (resp. negative) roots of the corresponding affine Kac-Moody algebra and W is the Weyl group. In a recent paper of Braverman, Finkelberg, and Kazhdan [**BFK12**], the authors interpreted the classical Gindikin-Karpelevich formula in a geometric way, and generalized the formula to affine Kac-Moody groups and obtained another version of affine Gindikin-Karpelevich formula, which has an additional "correction factor" in the product side.

On the other hand, in the works of Brubaker, Bump and Friedberg [**BBF11**], Bump and Nakasuji [**BN10**], and McNamara [**McN11**], the product side of the classical Gindikin-Karpelevich formula in type A_n was written as a sum over the crystal $\mathcal{B}(\infty)$. (For the definition of a crystal, see [**HK02**, **Kas02**].) More precisely, they proved

$$\prod_{\alpha\in\Delta^+} \frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha} = \sum_{b\in\mathcal{B}(\infty)} G_{\mathbf{i}}^{(e)}(b) q^{\langle \mathrm{wt}(b),\rho\rangle} \boldsymbol{z}^{-\mathrm{wt}(b)},$$

where ρ is the half-sum of the positive roots, $\mathrm{wt}(b)$ is the weight of b, and the coefficients $G_{\mathbf{i}}^{(e)}(b)$ are defined using so-called BZL paths or Kashiwara's parametrization. As shown in [**KL11**] by H. Kim and K.-H. Lee, one can also choose a reduced word for the longest element of the Weyl group and use Lusztig's parametrization of canonical bases ([**Lus90**, **Lus91**]), and the product can be written as

$$\prod_{\alpha\in\Delta^+} \frac{1-q^{-1}\boldsymbol{z}^\alpha}{1-\boldsymbol{z}^\alpha} = \sum_{b\in\mathcal{B}(\infty)} (1-q^{-1})^{\mathscr{N}(\phi_{\mathbf{i}}(b))} \boldsymbol{z}^{-\mathrm{wt}(b)}, \tag{0.2}$$

where $\mathscr{N}(\phi_{\mathbf{i}}(b))$ is the number of nonzero entries in Lusztig's parametrization $\phi_{\mathbf{i}}(b)$. The equation (0.2) was proved for all finite roots systems Δ, and was generalized in a subsequent paper [**KL12**] to the affine Kac-Moody case using the results of Beck, Chari, and Pressley [**BCP99**] and Beck and Nakajima [**BN04**] on PBW-type bases. The parametrizations of basis elements in simply-laced affine cases can be found in [**BCP99**, Theorem 3]. We will call them *canonical parametrizations.*

The use of crystals connects the Gindikin-Karpelevich formula to combinatorial representation theory, since much work has been done on realizations of crystals through various combinatorial objects (e.g., [**Kam10**, **Kan03**, **KN94**, **KS97**, **Lit95**]). Indeed, for type A_n, K.-H. Lee and Salisbury [**LS12**] expressed the right side of (0.2) as a sum over marginally large Young tableaux using J. Hong and H. Lee's [**HL08**] description of $\mathcal{B}(\infty)$ and the coefficients were determined by a simple statistic $\mathrm{seg}(b)$ of the tableau b. Furthermore, the meaning of $\mathrm{seg}(b)$ was studied in the frameworks of Kamnitzer's MV polytope model [**Kam10**] and Kashiwara-Saito's geometric realization [**KS97**] of the crystal $\mathcal{B}(\infty)$. The segment statistic was then generalized to types B_n, C_n, D_n, and G_2 in [**LS14**].

The goal of this paper is to extend this approach to affine type $A_n^{(1)}$ through generalized Young walls. The notion of a Young wall was first introduced by Kang [**Kan03**] in his extensive study of affine crystals. In the case of $\mathcal{B}(\infty)$ in type $A_n^{(1)}$, J.-A. Kim and D.-U. Shin [**KS10**] considered a set of *generalized* Young walls to obtain a realization of $\mathcal{B}(\infty)$, while H. Lee [**Lee07**] established a different realization. These constructions in type $A_n^{(1)}$ are closely related to Zelevinsky's *multisegments* [**Zel80**] and Lusztig's *aperiodic* multisegments [**Lus91**], whose crystal structure was studied by Leclerc, Thibon and Vasserot [**LTV99**]. In this paper, we will adopt Kim and Shin's realization and prove (Theorem 3.23)

$$\prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}\boldsymbol{z}^{\alpha}}{1-\boldsymbol{z}^{\alpha}}\right)^{\mathrm{mult}(\alpha)} = \sum_{Y\in\mathcal{Y}(\infty)}(1-q^{-1})^{\mathscr{N}(Y)}\boldsymbol{z}^{-\mathrm{wt}(Y)},$$

where $\mathcal{Y}(\infty)$ is the set of reduced proper generalized Young walls and $\mathscr{N}(Y)$ is a certain statistic on $Y\in\mathcal{Y}(\infty)$.

There are two main constructions in the proof. The first one is to establish natural bijections starting from $\mathcal{Y}(\infty)$ so that we may assign a Kostant partition to an element Y of $\mathcal{Y}(\infty)$. The second is to develop an algorithm to calculate the number $\mathscr{N}(Y)$ of distinct parts in the Kostant partition corresponding to Y. Note that if one can read off a canonical parametrization established by Beck, Chari, Nakajima and Pressley, directly from Y, then the corresponding Kostant partition is readily obtained. However, to the authors' knowledge, an efficient way to read off a canonical parametrization from Y in the affine setting is not known. Instead, our construction uses the more combinatorial nature of $\mathcal{Y}(\infty)$ and produces an explicit correspondence between $\mathcal{Y}(\infty)$ and the set of Kostant partitions. Our method then assigns a canonical-type parametrization to Y through the corresponding Kostant partition. We do not know at the moment whether our parametrization coincides with a canonical parametrization of Beck, Chari, Nakajima and Pressley.

In type $A_n^{(1)}$, the correction factor in the formula of Braverman, Finkelberg, and Kazhdan, mentioned above is given by

$$\prod_{i=1}^{n}\prod_{j=1}^{\infty}\frac{1-q^{-i}\boldsymbol{z}^{j\delta}}{1-q^{-(i+1)}\boldsymbol{z}^{j\delta}}, \tag{0.3}$$

where δ is the minimal positive imaginary root. In the last section we will write this correction factor as a sum over a subset of reduced proper generalized Young walls (Proposition 4.4), obtain an expansion of the whole product as a sum over pairs of reduced proper generalized Young walls (Corollary 4.5), and derive a combinatorial formula for the number of points in the intersection $T^{-\gamma}\cap S^0$ of certain orbits $T^{-\gamma}$ and S^0 in the (double) affine Grassmannian (Corollary 4.6).

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1. General definitions

Let $I=\{0,1,\dots,n\}$ be an index set and let $(A,\Pi,\Pi^\vee,P,P^\vee)$ be a Cartan datum of type $A_n^{(1)}$; i.e.,

- $A=(a_{ij})_{i,j\in I}$ is a generalized Cartan matrix of type $A_n^{(1)}$,
- $\Pi=\{\alpha_i : i\in I\}$ is the set of simple roots,

- $\Pi^\vee = \{h_i : i \in I\}$ is the set of simple coroots,
- $P^\vee = \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_n \oplus \mathbf{Z}d$ is the dual weight lattice,
- $\mathfrak{h} = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$ is the Cartan subalgebra,
- and $P = \{\lambda \in \mathfrak{h}^* : \lambda(P^\vee) \subset \mathbf{Z}\}$ is the weight lattice.

In addition to the above data, we have a bilinear pairing $\langle\ ,\ \rangle\colon P^\vee \times P \longrightarrow \mathbf{Z}$ defined by $\langle h_i, \alpha_j\rangle = a_{ij}$ and $\langle d, \alpha_j\rangle = \delta_{0,j}$.

Let $\mathfrak{g}$ be the affine Kac-Moody algebra associated with this Cartan datum, and denote by $U_v(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$. We denote the generators of $U_v(\mathfrak{g})$ by e_i, f_i $(i \in I)$, and v^h $(h \in P^\vee)$. The subalgebra of $U_v(\mathfrak{g})$ generated by f_i $(i \in I)$ will be denoted by $U_v^-(\mathfrak{g})$.

A $U_v(\mathfrak{g})$-crystal is a set $\mathcal{B}$ together with maps

$$\widetilde{e}_i, \widetilde{f}_i\colon \mathcal{B} \longrightarrow \mathcal{B} \sqcup \{0\}, \qquad \varepsilon_i, \varphi_i\colon \mathcal{B} \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \qquad \mathrm{wt}\colon \mathcal{B} \longrightarrow P$$

satisfying certain conditions (see [**HK02**, **Kas95**]). The negative part $U_v^-(\mathfrak{g})$ has a crystal base (see [**Kas91**]) which is a $U_v(\mathfrak{g})$-crystal. We denote this crystal by $\mathcal{B}(\infty)$, and denote its highest weight element by u_∞.

Finally, we will describe the set of roots Δ for $\mathfrak{g}$. Since we are fixing $\mathfrak{g}$ to be of type $A_n^{(1)}$, we may make this explicit. Define

$$\Delta_{\mathrm{cl}} = \{\pm(\alpha_i + \cdots + \alpha_j) : 1 \le i \le j \le n\},$$
$$\Delta_{\mathrm{cl}}^+ = \{\alpha_i + \cdots + \alpha_j : 1 \le i \le j \le n\}$$

to be set of classical roots and positive classical roots; *i.e.*, roots in the root system of $\mathfrak{g}_{\mathrm{cl}} = \mathfrak{sl}_{n+1}$. The minimal imaginary root is $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$. Then

$$\Delta_{\mathrm{Im}} = \{m\delta : m \in \mathbf{Z} \setminus \{0\}\}, \qquad \Delta_{\mathrm{Im}}^+ = \{m\delta : m \in \mathbf{Z}_{>0}\}.$$

We have $\Delta = \Delta_{\mathrm{Re}} \sqcup \Delta_{\mathrm{Im}}$ and $\Delta^+ = \Delta_{\mathrm{Re}}^+ \sqcup \Delta_{\mathrm{Im}}^+$, where

$$\Delta_{\mathrm{Re}} = \{\alpha + m\delta : \alpha \in \Delta_{\mathrm{cl}},\ m \in \mathbf{Z}\}$$
$$\Delta_{\mathrm{Re}}^+ = \{\alpha + m\delta : \alpha \in \Delta_{\mathrm{cl}},\ m \in \mathbf{Z}_{>0}\} \cup \Delta_{\mathrm{cl}}^+.$$

Recall $\mathrm{mult}(\alpha) = 1$ for any $\alpha \in \Delta_{\mathrm{Re}}$ and $\mathrm{mult}(\alpha) = n$ for any $\alpha \in \Delta_{\mathrm{Im}}$. For notational convenience, since $\mathrm{mult}(m\delta) = n$, we write

$$\Delta_{\mathrm{Im}}^+ = \{m_1\delta_1, \dots, m_n\delta_n : m_1, \dots, m_n \in \mathbf{Z}_{>0}\},$$

where each δ_j is a copy of the imaginary root δ.

2. Generalized Young walls

In this section we describe generalized Young walls. We refer the reader to [**Kan03**, **Zel80**, **Lus91**, **LTV99**] for related constructions and background. We start by defining the board on which all generalized Young walls will be built.

Define

(2.1)

	⋮	⋮	⋮		⋮	⋮	⋮
⋯	0	1	2	⋯	0	1	$(n+3)$rd row
⋯	n	0	1	⋯	n	0	$(n+2)$nd row
⋯	$n-1$	n	0	⋯	$n-1$	n	$(n+1)$st row
	⋮	⋮	⋮		⋮	⋮	⋮
⋯	0	1	2	⋯	0	1	2nd row
⋯	n	0	1	⋯	n	0	1st row

In particular, the color of the jth site from the bottom of the ith column from the right in (2.1) is $j - i \bmod n+1$.

DEFINITION 2.1. A *generalized Young wall* is a finite collection of i-colored boxes $\boxed{i}$ $(i \in I)$ on the board (2.1) satisfying the following building conditions.

(1) The colored boxes should be located according to the colors of the sites on the board (2.1).
(2) The colored boxes are put in rows; that is, one stacks boxes from right to left in each row.

For a generalized Young wall Y, we define the *weight* $\mathrm{wt}(Y)$ of Y to be

$$\mathrm{wt}(Y) = -\sum_{i\in I} m_i(Y)\alpha_i,$$

where $m_i(Y)$ is the number of i-colored boxes in Y.

DEFINITION 2.2. A generalized Young wall is called *proper* if for any $k > \ell$ and $k - \ell \equiv 0 \bmod n+1$, the number of boxes in the kth row from the bottom is less than or equal to that of the ℓth row from the bottom.

DEFINITION 2.3. Let Y be a generalized Young wall and let Y_k be the kth column of Y from the right. Set $a_i(k)$, with $i \in I$ and $k \geq 1$, to be the number of i-colored boxes in the kth column Y_k.

(1) We say Y_k contains a *removable* δ if we may remove one i-colored box for all $i \in I$ from Y_k and still obtain a generalized Young wall. In other words, Y_k contains a removable δ if $a_{i-1}(k+1) < a_i(k)$ for all $i \in I$.
(2) Y is said to be *reduced* if no column Y_k of Y contains a removable δ.

Let $\mathcal{Y}(\infty)$ denote the set of all reduced proper generalized Young walls. In [**KS10**], Kim and Shin defined a crystal structure on $\mathcal{Y}(\infty)$ and proved the following theorem. We refer the reader to [**KS10**] for the details.

THEOREM 2.4 ([**KS10**]). *We have* $\mathcal{B}(\infty) \cong \mathcal{Y}(\infty)$ *as crystals.*

3. Kostant partitions

Let

$$\alpha_i^{(\ell)} = \alpha_i + \alpha_{i-1} + \cdots + \alpha_{i-\ell+1}, \qquad i \in I,\ 1 \le \ell \le n,$$

where the indices are understood mod $n+1$.

EXAMPLE 3.1. Let $n = 2$. Then

$$\alpha_0^{(1)} = \alpha_0 \qquad \alpha_1^{(1)} = \alpha_1 \qquad \alpha_2^{(1)} = \alpha_2$$
$$\alpha_0^{(2)} = \alpha_0 + \alpha_2 \qquad \alpha_1^{(2)} = \alpha_1 + \alpha_0 \qquad \alpha_2^{(2)} = \alpha_2 + \alpha_1.$$

Let

$$S_1 = \left\{ (m_k\delta_k),\ (c_{i,\ell}\,\delta + \alpha_i^{(\ell)}) : {}^{m_k > 0,\ 1 \le k \le n,}_{c_{i,\ell} \ge 0,\ i \in I,\ 1 \le \ell \le n} \right\}.$$

We introduce the generator $\boldsymbol{\delta}^{(m)}$ for $m \in \mathbf{Z}_{>0}$ and set

$$S_2 = \{ \boldsymbol{\delta}^{(m)} : m \in \mathbf{Z}_{>0} \}.$$

Let $\widetilde{\mathcal{G}}$ be the free abelian group generated by $S_1 \cup S_2$. Consider the subgroup L of $\widetilde{\mathcal{G}}$ generated by the elements: for $m > 0$,

$$(3.1) \qquad \begin{cases} \boldsymbol{\delta}^{(m)} - \sum_{i \in I} (k\delta + \alpha_i^{(\ell)}), & m = (n+1)k + \ell,\ 1 \le \ell \le n; \\ \boldsymbol{\delta}^{(m)} - \boldsymbol{\delta}^{(k)} - \sum_{i=1}^{n} (k\delta_i), & m = (n+1)k. \end{cases}$$

We set $\mathcal{G} = \widetilde{\mathcal{G}}/L$ and let $\mathcal{G}^+$ be the $\mathbf{Z}_{\ge 0}$-span of $S_1 \cup S_2$ in $\mathcal{G}$. The following observation will play an important role.

REMARK 3.2. If we slightly abuse language, we may say that, in $\mathcal{G}$, the element $\boldsymbol{\delta}^{(m)}$ is equal to the sum of $n+1$ distinct positive "roots" of equal length m whose total weight is $m\delta$. In particular, if $m = (n+1)k + \ell$ $(1 \le \ell \le n)$, then $\boldsymbol{\delta}^{(m)}$ is equal to the sum of $n+1$ distinct positive real roots of equal length m, and if $m = (n+1)k$, then $\boldsymbol{\delta}^{(m)}$ is equal to the sum of $(k\delta_1), \dots, (k\delta_n), \boldsymbol{\delta}^{(k)}$ of equal length m.

EXAMPLE 3.3. Let $n = 2$. Then in $\mathcal{G}$,

$$\begin{aligned}
\boldsymbol{\delta}^{(1)} &= (\alpha_0) + (\alpha_1) + (\alpha_2) \\
\boldsymbol{\delta}^{(2)} &= (\alpha_0 + \alpha_2) + (\alpha_1 + \alpha_0) + (\alpha_2 + \alpha_1) \\
\boldsymbol{\delta}^{(3)} &= \boldsymbol{\delta}^{(1)} + (\delta_1) + (\delta_2) = (\alpha_0) + (\alpha_1) + (\alpha_2) + (\delta_1) + (\delta_2) \\
\boldsymbol{\delta}^{(4)} &= (\delta + \alpha_0) + (\delta + \alpha_1) + (\delta + \alpha_2) \\
\boldsymbol{\delta}^{(5)} &= (\delta + \alpha_0 + \alpha_2) + (\delta + \alpha_1 + \alpha_0) + (\delta + \alpha_2 + \alpha_1) \\
\boldsymbol{\delta}^{(6)} &= \boldsymbol{\delta}^{(2)} + (2\delta_1) + (2\delta_2) = (\alpha_0 + \alpha_2) + (\alpha_1 + \alpha_0) + (\alpha_2 + \alpha_1) + (2\delta_1) + (2\delta_2) \\
&\ \vdots \\
\boldsymbol{\delta}^{(9)} &= \boldsymbol{\delta}^{(3)} + (3\delta_1) + (3\delta_2) = (\alpha_0) + (\alpha_1) + (\alpha_2) + (\delta_1) + (\delta_2) + (3\delta_1) + (3\delta_2) \\
&\ \vdots
\end{aligned}$$

DEFINITION 3.4. Let $\mathfrak{p} \in \mathcal{G}^+$, and write $\mathfrak{p}$ as a $\mathbf{Z}_{\ge 0}$-linear combination of elements in $S_1 \cup S_2$.

(1) We say an expression of $\mathfrak{p}$ contains a *removable* δ if it contains some parts that can be replaced by $\boldsymbol{\delta}^{(k)}$ for some $k > 0$.
(2) We say an expression of $\mathfrak{p}$ is *reduced* if it does not contain a removable δ.

Let $\mathcal{K}(\infty)$ denote the set of reduced expressions of elements in $\mathcal{G}^+$. We define the set $\mathcal{K}$ of *Kostant partitions* to be the $\mathbf{Z}_{\geq 0}$-span of the set S_1 in $\mathcal{G}^+$. Notice that the set S_1 is linearly independent.

Definition 3.5. For $\mathfrak{p} \in \mathcal{K}$, we denote by $\mathscr{N}(\mathfrak{p})$ the number of distinct parts in $\mathfrak{p}$.

Example 3.6. If $\mathfrak{p} = 2(\alpha_0 + \alpha_1) + 5(\alpha_2 + \alpha_1) + 2(\delta_1) + (\delta_2) + (\alpha_0) + 4(\alpha_1)$, then $\mathscr{N}(\mathfrak{p}) = 6$.

Define a *reduction map* $\psi\colon \mathcal{K} \longrightarrow \mathcal{K}(\infty)$ as follows: Given $\mathfrak{p} \in \mathcal{K}$, write it as a $\mathbf{Z}_{\geq 0}$-linear combination of elements in S_1. Replace $k_1 \sum_{i\in I}(\alpha_i^{(1)})$ in the expression, where k_1 is the largest possible, with $k_1\boldsymbol{\delta}^{(1)}$. The resulting expression is denoted by $\mathfrak{p}^{(1)}$. Next, replace $k_2 \sum_{i\in I}(\alpha_i^{(2)})$ (or $k_2(\boldsymbol{\delta}^{(1)} + (\delta_1))$ if $n = 1$), where k_2 is the largest possible, with $k_2\boldsymbol{\delta}^{(2)}$. The result is denoted by $\mathfrak{p}^{(2)}$. Continue this process with $\boldsymbol{\delta}^{(k)}$ $(k \geq 3)$ using the relations in (3.1). The process stops with $\mathfrak{p}^{(s)}$ for some s. By construction, $\mathfrak{p}^{(s)} \in \mathcal{K}(\infty)$, and we define $\psi(\mathfrak{p}) = \mathfrak{p}^{(s)}$.

Conversely, we define the *unfolding map* $\phi\colon \mathcal{K}(\infty) \longrightarrow \mathcal{K}$ by unfolding the $\boldsymbol{\delta}^{(k)}$'s consecutively. That is, given $\mathfrak{q} \in \mathcal{K}(\infty)$, find $\boldsymbol{\delta}^{(r)}$ with the largest r and replace it with the corresponding sum from (3.1). The resulting expression is denoted by $\mathfrak{q}^{(r)}$. Next, replace $\boldsymbol{\delta}^{(r-1)}$ with the corresponding sum from (3.1). The result is denoted by $\mathfrak{q}^{(r-1)}$. Continue this process until we replace $\boldsymbol{\delta}^{(1)}$ with $\sum_{i\in I}(\alpha_i^{(1)})$ and obtain $\mathfrak{q}^{(1)}$. By construction, $\mathfrak{q}^{(1)} \in \mathcal{K}$, and we define $\phi(\mathfrak{q}) = \mathfrak{q}^{(1)}$.

It is clear from the definitions that ψ and ϕ are inverses to each other. Hence, we have proven the following lemma.

Lemma 3.7. *The reduction map $\psi\colon \mathcal{K} \longrightarrow \mathcal{K}(\infty)$ is a bijection, whose inverse is the unfolding map $\phi\colon \mathcal{K}(\infty) \longrightarrow \mathcal{K}$.*

For later use, we need to describe the unfolding map ϕ more explicitly.

Lemma 3.8. *For $p \in \mathbf{Z}_{\geq 0}$ and $q \in \mathbf{Z}_{>0}$, we have*

$$\phi\big(\boldsymbol{\delta}^{((n+1)^p q)}\big) = \sum_{j=1}^{n+1}(r\delta + \alpha_{j-1}^{(s)}) + \sum_{i=0}^{p-1}\Big(\sum_{j=1}^{n}\big((n+1)^i q\,\delta_j\big)\Big), \tag{3.2}$$

where we write $q = (n+1)r + s$, $1 \leq s \leq n$. In particular, $\boldsymbol{\delta}^{((n+1)^p q)}$ has $n+1+np$ parts.

Proof. We use induction on p. Assume that $p = 0$. Then it follows from (3.1) that

$$\phi\big(\boldsymbol{\delta}^{(q)}\big) = \sum_{j=1}^{n+1}(r\delta + \alpha_{j-1}^{(s)}).$$

Now assume that $p \geq 1$. From (3.1) and the induction hypothesis, we obtain

$$\begin{aligned}\phi\big(\boldsymbol{\delta}^{((n+1)^p q)}\big) &= \phi\big(\boldsymbol{\delta}^{((n+1)^{p-1} q)}\big) + \sum_{j=1}^{n} \big((n+1)^{p-1} q\,\delta_j\big) \\ &= \sum_{j=1}^{n+1} (r\delta + \alpha_{j-1}^{(s)}) + \sum_{i=0}^{p-2} \Big(\sum_{j=1}^{n} \big((n+1)^i q\,\delta_j\big)\Big) + \sum_{j=1}^{n} \big((n+1)^{p-1} q\,\delta_j\big) \\ &= \sum_{j=1}^{n+1} (r\delta + \alpha_{j-1}^{(s)}) + \sum_{i=0}^{p-1} \Big(\sum_{j=1}^{n} \big((n+1)^i q\,\delta_j\big)\Big). \qquad \blacksquare\end{aligned}$$

In what follows, we will establish a bijection between $\mathcal{Y}(\infty)$ and $\mathcal{K}(\infty)$. For $Y \in \mathcal{Y}(\infty)$, we define $N_k(Y)$ $(k \geq 1)$ to be the number of boxes in the kth row of Y. We first define a map $\Psi\colon \mathcal{Y}(\infty) \longrightarrow \mathcal{K}(\infty)$ by describing how the blocks in a reduced proper generalized Young wall Y contribute to the parts in a reduced Kostant partition. For $Y \in \mathcal{Y}(\infty)$, $1 \leq j \leq n+1$ and $m \geq 0$, define $\Psi(Y; j, m)$ by

$$\Psi(Y; j, m) =$$

$$(3.3) \quad \begin{cases} (k\delta_j) & \text{if } 1 \leq j \leq n \text{ and} \\ & N_{(n+1)m+j}(Y) = (n+1)k \text{ for some } k > 0, \\ (k\delta + \alpha_{j-1}^{(\ell)}) & \text{if } 1 \leq j \leq n \text{ and} \\ & N_{(n+1)m+j}(Y) = (n+1)k + \ell \text{ for some } 1 \leq \ell \leq n, k \geq 0, \\ \boldsymbol{\delta}^{(k)} & \text{if } j = n+1 \text{ and} \\ & N_{(n+1)(m+1)}(Y) = (n+1)k \text{ for some } k > 0, \\ (k\delta + \alpha_n^{(\ell)}) & \text{if } j = n+1 \text{ and} \\ & N_{(n+1)(m+1)}(Y) = (n+1)k + \ell \text{ for some } 1 \leq \ell \leq n, k \geq 0. \end{cases}$$

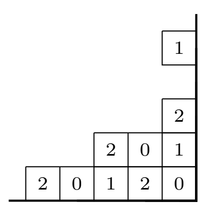
Then

$$\Psi(Y) = \sum_{m \geq 0} \sum_{j=1}^{n+1} \Psi(Y; j, m).$$

Lemma 3.9. *For any $Y \in \mathcal{Y}(\infty)$, we have $\Psi(Y) \in \mathcal{K}(\infty)$.*

Proof. Let $\mathfrak{p} = \Psi(Y)$. It is clear that $\mathfrak{p} \in \mathcal{G}^+$, so it remains to show the expression of $\mathfrak{p}$ is reduced. On the contrary, assume that $\mathfrak{p}$ contains a removable δ. By Remark 3.2, the expression of $\mathfrak{p}$ contains a sum of $n+1$ distinct positive "roots" of equal length, and the sum corresponds through (3.3) to a collection of rows of Y with equal length in non-congruent positions. Then Y contains a removable δ, which is a contradiction. Thus $\mathfrak{p}$ does not contain a removable δ, so $\mathfrak{p}$ is reduced. $\blacksquare$

Example 3.10. Let $Y = \widetilde{f}_2^3 \widetilde{f}_0^2 \widetilde{f}_1^2 \widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 Y_\infty$. That is, let

$Y =$ 

.

Then $\Psi(Y) = (\delta + \alpha_0 + \alpha_2) + (\delta_2) + (\alpha_2) + (\alpha_1)$.

Now define a function $\Phi\colon \mathcal{K}(\infty) \longrightarrow \mathcal{Y}(\infty)$ in the following way. Let $\mathfrak{p}$ be a reduced Kostant partition. To each part of the partition, we assign a row of a generalized Young wall using the following prescription. For $1 \le j \le n$ and $1 \le \ell \le n$,

$$(3.4) \qquad \Phi\colon \begin{cases} (k\delta_j) & \mapsto (n+1)k \text{ boxes in row} \equiv j \pmod{n+1}, \\ (k\delta + \alpha_{j-1}^{(\ell)}) & \mapsto (n+1)k + \ell \text{ boxes in row} \equiv j \pmod{n+1}, \\ \boldsymbol{\delta}^{(k)} & \mapsto (n+1)k \text{ boxes in row} \equiv 0 \pmod{n+1}, \\ (k\delta + \alpha_n^{(\ell)}) & \mapsto (n+1)k + \ell \text{ boxes in row} \equiv 0 \pmod{n+1}. \end{cases}$$

To construct the Young wall $\Phi(\mathfrak{p})$ from this data, we arrange the rows so that the number of boxes in each row of the form $(n+1)k + j$, for a fixed j, is weakly decreasing as k increases. Hence $\Phi(\mathfrak{p})$ is proper.

Lemma 3.11. *For any $\mathfrak{p} \in \mathcal{K}(\infty)$, we have $\Phi(\mathfrak{p}) \in \mathcal{Y}(\infty)$.*

Proof. We set $Y = \Phi(\mathfrak{p})$. Since $\mathfrak{p}$ is reduced, $\mathfrak{p}$ does not contain a removable δ. Using a similar argument as in the proof of Lemma 3.9, we see that a removable δ of Y corresponds to a removable δ of $\mathfrak{p}$. Thus Y does not contain a removable δ, so $Y \in \mathcal{Y}(\infty)$. ∎

Example 3.12. Let $\mathfrak{p} = (\alpha_0) + (2\delta + \alpha_1 + \alpha_0) + \boldsymbol{\delta}^{(3)} + (\alpha_2)$. Then

$$\Phi(\mathfrak{p}) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \multicolumn{8}{c|}{} & 2 \\ \hline 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\ \hline \multicolumn{8}{c|}{} & 0 \\ \hline \end{array}\,.$$

Proposition 3.13. *The maps Ψ and Φ are bijections which are inverses to each other. In particular, we have $\mathcal{Y}(\infty) \cong \mathcal{K}(\infty)$ as sets.*

The existence of a bijection is guaranteed by the theory of Kostant partitions and crystal bases. The importance of the proposition is that we have constructed an explicit, combinatorial description of a bijection.

Proof. Assume that $Y \in \mathcal{Y}(\infty)$. It is enough to check that a row j of Y is mapped onto the same stack of boxes in a row $\equiv j \pmod{n+1}$ by $\Phi \circ \Psi$, since the rows are arranged uniquely so that the number of boxes in each row of the form $(n+1)k + j$ for a fixed j is weakly decreasing as k increases. It follows from (3.3) and (3.4) that a row j of Y is mapped onto the same stack of boxes in a row $\equiv j$ $\pmod{n+1}$.

Conversely, assume that $\mathfrak{p} \in \mathcal{K}(\infty)$. It is enough to check that each part of $\mathfrak{p}$ is mapped onto itself through $\Psi \circ \Phi$. Using (3.3) and (3.4), we see that it is the case. ∎

Remark 3.14. While one may define a crystal structure on $\mathcal{K}(\infty)$ directly in order to show that the bijection in Proposition 3.13 is a crystal isomorphism, the bijection given is very explicit and easily understood, so one my simply pull back the crystal structure on $\mathcal{Y}(\infty)$ to $\mathcal{K}(\infty)$ in order to obtain a crystal isomorphism.

For $1 \le j \le n+1$ and $Y \in \mathcal{Y}(\infty)$, define $S_j(Y)$ be the set of distinct $N_{(n+1)m+j}(Y)$'s for $m \ge 0$; *i.e.*, set

$$S_j(Y) = \bigcup_{m \ge 0} \{N_{(n+1)m+j}(Y)\}.$$

When $j = n+1$, for each $m \ge 0$, define $(p_m, q_m) \in \mathbf{Z}_{\ge 0} \times \mathbf{Z}_{\ge 0}$ by

$$N_{(n+1)(m+1)}(Y) = (n+1)^{p_m} q_m,$$

with q_m not divisible by $n+1$. If $N_{(n+1)(m+1)}(Y) = 0$, then we put $(p_m, q_m) = (0, 0)$. We set

$$\mathscr{Q}(Y) = \Big(\bigcup_{m \ge 0} \{(n+1)^s q_m : s = 0, 1, \dots, p_m - 1\} \Big) \cup \{0\},$$

and let

$$\mathscr{P}(Y) = \sum_{\substack{t \ge 1 \\ (n+1) \nmid t}} \max\{p_m : q_m = t,\ m \ge 0\}.$$

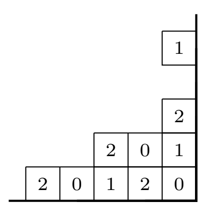
Define

$$\mathscr{N}(Y) = n\mathscr{P}(Y) + \sum_{j=1}^{n+1} \#\big(S_j(Y) \setminus \mathscr{Q}(Y)\big). \tag{3.5}$$

Proposition 3.15. *Assume that $Y \in \mathcal{Y}(\infty)$, and let $\mathfrak{p} = (\phi \circ \Psi)(Y) \in \mathcal{K}$, where ϕ is the unfolding map defined in the proof of Lemma 3.7. Then $\mathscr{N}(Y)$ is equal to the number of distinct parts in the Kostant partition $\mathfrak{p}$; i.e., we have $\mathscr{N}(Y) = \mathscr{N}(\mathfrak{p})$.*

Before we prove this proposition, we provide a pair of examples. In the first example, we do not have $\boldsymbol{\delta}^{(k)}$ in $\Psi(Y)$, and in the second example, we have $\boldsymbol{\delta}^{(k)}$ in $\Psi(Y)$. We will see how the formula for $\mathscr{N}(Y)$ works.

Example 3.16. Suppose that

$Y =$  .

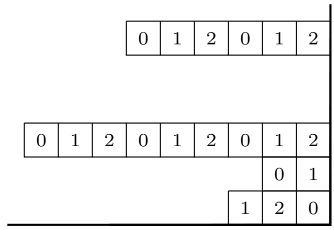
Then $\mathfrak{p} = (\phi \circ \Psi)(Y) = (\delta + \alpha_0 + \alpha_1) + (\delta_2) + (\alpha_2) + (\alpha_1)$, and the number of distinct parts is 4. On the other hand,

$$S_1(Y) = \{5, 0\}, \quad S_2(Y) = \{3, 1, 0\}, \quad S_3(Y) = \{1, 0\}.$$

Now setting $N_{3(m+1)}(Y) = 3^{p_m} q_m$ implies $(p_0, q_0) = (0, 1)$ and $(p_m, q_m) = (0, 0)$ for $m \ge 1$. Thus $\mathscr{Q}(Y) = \{0\}$ and $\mathscr{P}(Y) = 0$. So

$$\mathscr{N}(Y) = 1 + 2 + 1 + 2 \cdot 0 = 4.$$

Example 3.17. Suppose that

$Y =$ 

Then we have

$$\begin{aligned}\mathfrak{p} = (\phi\circ\Psi)(Y) &= \phi\Big((\delta_1) + (\alpha_0+\alpha_1) + \boldsymbol{\delta}^{(3)} + \boldsymbol{\delta}^{(2)}\Big)\\ &= (\delta_1) + (\alpha_0+\alpha_1) + (\alpha_0) + (\alpha_1) + (\alpha_2) + (\delta_1) + (\delta_2) + (\alpha_0+\alpha_2)\\ &\qquad + (\alpha_1+\alpha_0) + (\alpha_2+\alpha_1)\\ &= 2(\alpha_1+\alpha_0) + (\alpha_0+\alpha_2) + (\alpha_2+\alpha_1) + 2(\delta_1) + (\delta_2) + (\alpha_0) + (\alpha_1) + (\alpha_2).\end{aligned}$$

Hence the number of distinct parts is 8. On the other hand, we get

$$S_1(Y) = \{3,0\}, \quad S_2(Y) = \{2,0\}, \quad S_3(Y) = \{9,6,0\}.$$

From $N_{3(m+1)}(Y) = 3^{p_m}q_m$, we obtain $(p_0,q_0) = (2,1)$, $(p_1,q_1) = (1,2)$ and $(p_m,q_m) = (0,0)$ for $m \geq 2$. Then $\mathscr{Q}(Y) = \{1,3,2,0\}$ and $\mathscr{P}(Y) = 2+1 = 3$. So

$$\mathscr{N}(Y) = 0+0+2+2\cdot 3 = 8.$$

Proof of Proposition 3.15.

Step 1: Assume that $p_m = 0$ for all $m \geq 0$. Then $\Psi(Y)$ has no $\boldsymbol{\delta}^{(k)}$, or equivalently, Y is such that $N_{(n+1)(m+1)}(Y) \neq (n+1)k$ for any $m \geq 0$ and $k \geq 1$. Then $(\phi\circ\Psi)(Y) = \Psi(Y)$ as $\Psi(Y)$ does not have a $\boldsymbol{\delta}^{(k)}$. On the other hand, since $p_m = 0$ for all $m \geq 0$, we have $\mathscr{Q}(Y) = \{0\}$ and $\mathscr{P}(Y) = 0$. Hence

$$\mathscr{N}(Y) = \sum_{j=1}^{n+1} \#\big(S_j(Y)\setminus\{0\}\big).$$

For each $1 \leq j \leq n+1$, define $R_j(Y)$ to be the collection of kth rows of Y with $k \equiv j \pmod{n+1}$. From (3.3), we see that two nonempty rows $y_1, y_2 \in R_j(Y)$ correspond to distinct parts in $\Psi(Y)$ if and only if the lengths of y_1 and y_2 are different. Since $\#\left(S_j(Y)\setminus\{0\}\right)$ is the number of distinct nonzero lengths of rows in $R_j(Y)$, it is equal to the number of distinct parts in $\Psi(Y)$ corresponding to $R_j(Y)$. Furthermore, if $j \neq j'$, then $y \in R_j(Y)$ and $y' \in R_{j'}(Y)$ correspond to distinct parts in $\Psi(Y)$. Thus $\mathscr{N}(Y)$ is the total number of distinct parts in $\Psi(Y) = (\phi\circ\Psi)(Y)$, as required.

Step 2: Now assume that $p_m \geq 1$ for some m and $p_{m'} = 0$ for all $m' \neq m$. From the definition $N_{(n+1)(m+1)}(Y) = (n+1)^{p_m}q_m$, we see that the row $(n+1)(m+1)$ has $(n+1)^{p_m}q_m$ boxes, and the corresponding part in $\Psi(Y)$ is $\boldsymbol{\delta}^{((n+1)^{p_m-1}q_m)}$. We obtain from Lemma 3.8

$$\phi\big(\boldsymbol{\delta}^{((n+1)^{p_m-1}q_m)}\big) = \sum_{j=1}^{n+1}(r_m\delta + \alpha_{j-1}^{(s_m)}) + \sum_{i=0}^{p_m-2}\Big(\sum_{j=1}^{n}\big((n+1)^i q_m\,\delta_j\big)\Big), \tag{3.6}$$

where we write $q_m = (n+1)r_m + s_m$, $1 \leq s_m \leq n$. Thus $\phi\big(\boldsymbol{\delta}^{((n+1)^{p_m-1}q_m)}\big)$ has np_m+1 distinct parts, some of which may be the same as other parts in $\Psi(Y)$. It follows from (3.3) that the part $(r_m\delta + \alpha_{j-1}^{(s_m)})$ corresponds to q_m boxes in a row $\equiv j \pmod{n+1}$ for $1 \le j \le n+1$. Similarly, the part $((n+1)^i q_m\, \delta_j)$ corresponds to $(n+1)^{i+1} q_m$ boxes in a row $\equiv j \pmod{n+1}$ for $1 \le j \le n$ and $0 \le i \le p_m - 2$. Then the number of distinct parts in $(\phi \circ \Psi)(Y)$ is

(3.7)

$$np_m+1+\sum_{j=1}^{n} \#\big(S_j(Y)\setminus\{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big)+\#\big(S_{n+1}(Y)\setminus\{0,q_m,(n+1)^{p_m}q_m\}\big)$$
$$= np_m + \sum_{j=1}^{n} \#\big(S_j(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) + \#\big(S_{n+1}(Y) \setminus \{0, q_m\}\big).$$

Since $S_{n+1}(Y)$ does not contain $(n+1)^i q_m$, $1 \le i \le p_m - 1$, by the assumption, the expression (3.7) is equal to

$$\begin{aligned} np_m &+ \sum_{j=1}^{n} \#\big(S_j(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) \\ &\qquad + \#\big(S_{n+1}(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) \\ &= np_m + \sum_{j=1}^{n+1} \#\big(S_j(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) \\ &= np_m + \sum_{j=1}^{n+1} \#\big(S_j(Y) \setminus \mathscr{Q}(Y)\big) \\ &= \mathscr{N}(Y). \end{aligned}$$

Thus the number of distinct parts in $(\phi \circ \Psi)(Y)$ is $\mathscr{N}(Y)$.

Step 3: Next we assume $p_m = \max\{p_{m'} : m' \ge 0\}$ and $q_m = q_{m'}$ for any $p_{m'} \ge 1$. We have $\boldsymbol{\delta}^{((n+1)^{p_{m'}-1} q_{m'})}$ in $\Psi(Y)$ for each $p_{m'} \ge 1$, and each $\phi(\boldsymbol{\delta}^{((n+1)^{p_{m'}-1} q_{m'})})$ yields $np_{m'}+1$ parts as in (3.6). However, we can see from (3.6) that $\phi(\boldsymbol{\delta}^{(n+1)^{p_m-1} q_m})$ with the maximal p_m generates all the distinct parts including those from other $p_{m'}$, since $q_m = q_{m'}$ for all $p_{m'} \ge 1$ by the assumption. Then the number of distinct parts in $(\phi \circ \Psi)(Y)$ is given by

$$\begin{aligned} np_m + 1 &+ \sum_{j=1}^{n} \#\big(S_j(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) \\ &\qquad + \#\big(S_{n+1}(Y) \setminus \{0, q_m, (n+1)^{p_{m'}} q_m\}_{1\le p_{m'}\le p_m}\big) \\ &= np_m + \sum_{j=1}^{n} \#\big(S_j(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) \\ &\qquad + \#\big(S_{n+1}(Y) \setminus \{0,(n+1)^i q_m\}_{0\le i\le p_m-1}\big) \\ &= np_m + \sum_{j=1}^{n+1} \#\big(S_j(Y) \setminus \mathscr{Q}(Y)\big) = \mathscr{N}(Y). \end{aligned}$$

Step 4: Finally we consider the general case. We group p_m's using the rule that p_m and $p_{m'}$ are in the same group if and only if $q_m = q_{m'}$. For each of such groups, we use the result in Step 3, and see that the number of distict parts in $(\phi \circ \Psi)(Y)$

is equal to

$$n\mathscr{P}(Y) + \sum_{j=1}^{n+1} \#\big(S_j(Y) \setminus \mathscr{Q}(Y)\big),$$

recalling the definitions

$$\mathscr{P}(Y) = \sum_{\substack{t \geq 1 \\ (n+1) \nmid t}} \max\left\{p_m : q_m = t,\ m \geq 0\right\},$$

$$\mathscr{Q}(Y) = \Big(\bigcup_{m \geq 0} \{(n+1)^s q_m : s = 0, 1, \dots, p_m - 1\}\Big) \cup \{0\}.$$

Hence the number of distinct parts in $(\phi \circ \Psi)(Y)$ is $\mathscr{N}(Y)$. ■

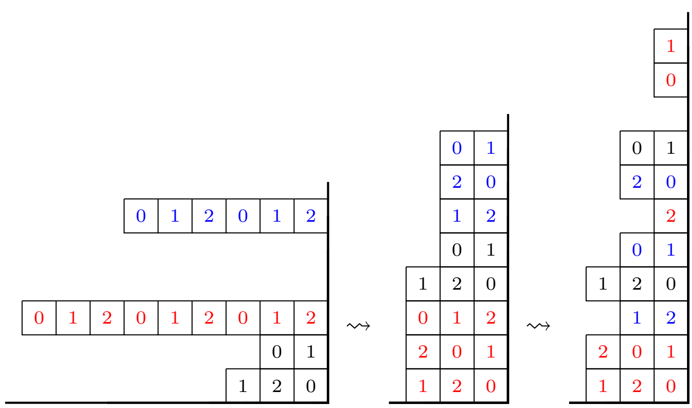
The rule for calculating the number $\mathscr{N}(Y)$, for $Y \in \mathcal{Y}(\infty)$, may be reinterpreted using the following algorithm. For this algorithm, we say that two rows in Y are *distinct* if their rightmost boxes are different or if their rightmost boxes are equal but they have an unequal number of boxes.

Algorithm 3.18. Define a map $\psi_{\mathcal{Y}}$ on $\mathcal{Y}(\infty)$ as follows.

(1) If Y has no row with rightmost box n and length $\equiv 0 \bmod n+1$, then $\psi_{\mathcal{Y}}(Y) := Y$.
(2) If Y has at least one row with rightmost box n and length $(n+1)\ell$, then replace any row with maximal such ℓ with $n+1$ distinct rows of length ℓ. Rearrange all rows (if necessary) so that it is proper. This gives $\psi_{\mathcal{Y}}^{(\ell)}(Y)$.
(3) Apply Step 2 with ℓ replaced by $\ell - 1$ and Y replaced by $\psi_{\mathcal{Y}}^{(\ell)}(Y)$. This gives $\psi_{\mathcal{Y}}^{(\ell-1)}(Y)$.
(4) Iterate this process until $\ell = 1$. Then $\psi_{\mathcal{Y}}(Y) = \psi_{\mathcal{Y}}^{(1)}(Y)$.

Note that $\psi_{\mathcal{Y}}(Y)$ is proper, but need not be reduced, so $\psi_{\mathcal{Y}}(Y) \notin \mathcal{Y}(\infty)$ in general. Then $\mathscr{N}(Y)$ is the number of distinct rows in $\psi_{\mathcal{Y}}(Y)$.

Example 3.19. Let $n = 2$ and let Y be as in Example 3.17. Then

$Y =$  $= \psi_{\mathcal{Y}}(Y)$.

Counting the number of distinct rows gives $8 = \mathscr{N}(Y)$.

Let W be the Weyl group of $\mathfrak{g}$ and s_i $(i \in I)$ be the simple reflections. We fix $\boldsymbol{h} = (\dots, i_{-1}, i_0, i_1, \dots)$ as in Section 3.1 in [**BN04**]. Then for any integers

$m < k$, the product $s_{i_m} s_{i_{m+1}} \cdots s_{i_k} \in W$ is a reduced expression, so is the product $s_{i_k} s_{i_{k-1}} \cdots s_{i_m} \in W$. We set

$$\beta_k = \begin{cases} s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \le 0, \\ s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0. \end{cases} \tag{3.8}$$

Let $T_i = T''_{i,1}$ be the automorphism of $U_v(\mathfrak{g})$ as in Section 37.1.3 of [**Lus93**], and let

$$\boldsymbol{c}_+ = (c_0, c_{-1}, c_{-2}, \dots) \in \mathbf{Z}_{\ge 0}^{\mathbf{Z}_{\le 0}} \quad \text{and} \quad \boldsymbol{c}_- = (c_1, c_2, \dots) \in \mathbf{Z}_{\ge 0}^{\mathbf{Z}_{>0}}$$

be functions (or sequences) that are almost everywhere zero. We denote by $\mathscr{C}_>$ (resp. by $\mathscr{C}_<$) the set of such functions $\boldsymbol{c}_+$ (resp. $\boldsymbol{c}_-$). For an element $\boldsymbol{c}_+ = (c_0, c_{-1}, \dots) \in \mathscr{C}_>$ (resp. $\boldsymbol{c}_- = (c_1, c_2, \dots) \in \mathscr{C}_>$), we define

$$E_{\boldsymbol{c}_+} = E_{i_0}^{(c_0)} T_{i_0}^{-1}(E_{i_{-1}}^{(c_{-1})}) T_{i_0}^{-1} T_{i_{-1}}^{-1}(E_{i_{-2}}^{(c_{-2})}) \cdots$$

and

$$E_{\boldsymbol{c}_-} = \cdots T_{i_1} T_{i_2}(E_{i_3}^{(c_3)}) T_{i_1}(E_{i_2}^{(c_2)}) E_{i_1}^{(c_1)}.$$

We also define $\mathscr{N}(\boldsymbol{c}_+)$ (resp. $\mathscr{N}(\boldsymbol{c}_-)$) to be the number of nonzero c_i's in $\boldsymbol{c}_+$ (resp. $\boldsymbol{c}_-$).

Let $\boldsymbol{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)})$ be a multi-partition with n components; *i.e.*, each component $\rho^{(i)}$ is a partition. We denote by $\mathcal{P}(n)$ the set of all multi-partitions with n components. Let $S_{\boldsymbol{c}_0}$ be defined as in [**BN04**, p. 352] for $\boldsymbol{c}_0 \in \mathcal{P}(n)$. For a partition $\boldsymbol{p} = (1^{m_1} 2^{m_2} \cdots r^{m_r} \cdots)$, we define

$$\mathscr{N}(\boldsymbol{p}) = \#\{r : m_r \ne 0\} \quad \text{and} \quad |\boldsymbol{p}| = m_1 + 2m_2 + 3m_3 + \cdots . \tag{3.9}$$

Then for a multi-partition $\boldsymbol{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we set

$$\mathscr{N}(\boldsymbol{c}_0) = \mathscr{N}(\rho^{(1)}) + \mathscr{N}(\rho^{(2)}) + \cdots + \mathscr{N}(\rho^{(n)}).$$

Let $\mathscr{C} = \mathscr{C}_> \times \mathcal{P}(n) \times \mathscr{C}_<$. We denote by $\mathscr{B}$ the Kashiwara-Lusztig canonical basis for $U_v^+(\mathfrak{g})$, the positive part of the quantum affine algebra.

THEOREM 3.20 ([**BCP99**, **BN04**]). *There is a bijection $\eta\colon \mathscr{B} \longrightarrow \mathscr{C}$ such that for each $\boldsymbol{c} = (\boldsymbol{c}_+, \boldsymbol{c}_0, \boldsymbol{c}_-) \in \mathscr{C}$, there exists a unique $b = \eta^{-1}(\boldsymbol{c}) \in \mathscr{B}$ satisfying*

$$b \equiv E_{\boldsymbol{c}_+} S_{\boldsymbol{c}_0} E_{\boldsymbol{c}_-} \mod v^{-1}\mathbf{Z}[v^{-1}]. \tag{3.10}$$

Now the number $\mathscr{N}(\boldsymbol{c})$ is defined by $\mathscr{N}(\boldsymbol{c}) = \mathscr{N}(\boldsymbol{c}_+) + \mathscr{N}(\boldsymbol{c}_0) + \mathscr{N}(\boldsymbol{c}_-)$ for each $\boldsymbol{c} \in \mathscr{C}$. Using the canonical basis $\mathscr{B}$, H. Kim and K.-H. Lee expanded the product side of the Gindikin-Karpelevich formula as a sum, and obtained the following theorem.

THEOREM 3.21 ([**KL12**]). *We have*

$$\prod_{\alpha \in \Delta^+} \left(\frac{1 - q^{-1}\boldsymbol{z}^\alpha}{1 - \boldsymbol{z}^\alpha} \right)^{\operatorname{mult}(\alpha)} = \sum_{b \in \mathscr{B}} (1 - q^{-1})^{\mathscr{N}(\eta(b))} \boldsymbol{z}^{\operatorname{wt}(b)}. \tag{3.11}$$

In the rest of this section, we will prove a combinatorial description of the formula (3.11) using the set $\mathcal{Y}(\infty)$ of reduced proper generalized Young walls.

We define a map $\theta\colon \mathcal{P}(n) \longrightarrow \mathcal{K}$ as follows. For $\boldsymbol{c}_0 = (\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(n)}) \in \mathcal{P}(n)$, we define

$$\theta(\boldsymbol{c}_0) = \sum_{i=1}^{n} m_{1,i}(\delta_i) + m_{2,i}(2\delta_i) + \cdots + m_{r,i}(r\delta_i) + \cdots ,$$

where $\rho^{(i)} = (1^{m_{1,i}} 2^{m_{2,i}} \cdots r^{m_{r,i}} \cdots)$ for $i = 1, 2, \ldots, n$. Then we define a map $\Theta \colon \mathscr{C} \longrightarrow \mathcal{K}$ by

$$\Theta(\boldsymbol{c}) = \theta(\boldsymbol{c}_0) + \sum_{i \in \mathbf{Z}} c_i(\beta_i),$$

where $\boldsymbol{c} = (\boldsymbol{c}_+, \boldsymbol{c}_0, \boldsymbol{c}_-)$, $\boldsymbol{c}_+ = (c_0, c_{-1}, c_{-2}, \ldots)$, $\boldsymbol{c}_- = (c_1, c_2, \ldots)$ and β_i is given by (3.8) with $(\beta_i) \in \mathcal{K}$.

Corollary 3.22. *The map $\Theta \colon \mathscr{C} \longrightarrow \mathcal{K}$ is a bijection, and for $\boldsymbol{c} \in \mathscr{C}$, the number of distinct parts in $\mathfrak{p} = \Theta(\boldsymbol{c})$ is the same as $\mathscr{N}(\boldsymbol{c})$; i.e., $\mathscr{N}(\Theta(\boldsymbol{c})) = \mathscr{N}(\boldsymbol{c})$.*

Proof. By Theorem 3.20, the set $\mathscr{C}$ parametrizes a PBW type basis of $U_v^+(\mathfrak{g})$. Thus the set $\mathscr{C}$ also parametrizes a PBW basis of the universal enveloping algebra $U^+(\mathfrak{g})$. Now the first assertion follows from the fact that the Kostant partitions parametrize the elements in a PBW basis of $U^+(\mathfrak{g})$ and that the function Θ is defined according to these correspondences. The second assertion follows from the definitions of $\mathscr{N}$ for $\mathscr{C}$ and $\mathcal{K}$, respectively. ∎

Theorem 3.23. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $A_n^{(1)}$. Then*

$$\prod_{\alpha \in \Delta^+} \left(\frac{1 - q^{-1}\boldsymbol{z}^{\alpha}}{1 - \boldsymbol{z}^{\alpha}} \right)^{\mathrm{mult}(\alpha)} = \sum_{Y \in \mathcal{Y}(\infty)} (1 - q^{-1})^{\mathscr{N}(Y)} \boldsymbol{z}^{-\mathrm{wt}(Y)}, \tag{3.12}$$

where $\mathscr{N}(Y)$ is defined in (3.5).

Proof. By Lemma 3.7, Proposition 3.13, Theorem 3.20 and Corollary 3.22, we have bijections

$$\mathscr{B} \xrightarrow{\eta} \mathscr{C} \xrightarrow{\Theta} \mathcal{K} \xrightarrow{\psi} \mathcal{K}(\infty) \xrightarrow{\Phi} \mathcal{Y}(\infty).$$

For $b \in \mathscr{B}$, we write $Y = (\Phi \circ \psi \circ \Theta \circ \eta)(b) \in \mathcal{Y}(\infty)$. Then, by Proposition 3.15 and Corollary 3.22, we have $\mathscr{N}(\eta(b)) = \mathscr{N}(Y)$. We also see from the constructions that $\mathrm{wt}(b) = -\mathrm{wt}(Y)$. Now the equality (3.12) follows from Theorem 3.21. ∎

4. Connection to Braverman-Finkelberg-Kazhdan's formula

We briefly recall the framework of the paper [**BFK12**]. Let G (resp. $\widehat{G}$) be the minimal (resp. formal) Kac-Moody group functor attached to a symmetrizable Kac-Moody root datum and let $\mathfrak{g}$ be the corresponding Lie algebra. There is a natural imbedding $G \hookrightarrow \widehat{G}$. The group G has the closed subgroup functors $U \subset B$, $U_- \subset B_-$ such that the quotients B/U and B_-/U_- are naturally isomorphic to the Cartan subgroup H of G. We denote by $\widehat{B}$ and $\widehat{U}$ the closures of B and U in $\widehat{G}$, respectively. We will denote the coroot lattice of G by Λ and the set of positive coroots by $R^+ \subset \Lambda$. The subsemigroup of Λ generated by R^+ will be denoted by Λ^+. For an element $\gamma = \sum a_i \alpha_i^\vee \in \Lambda^+$ with simple coroots $\alpha_i^\vee$, we write $|\gamma| = \sum a_i$. We assume that G is "simply connected"; *i.e.*, the lattice Λ is equal to the cocharacter lattice of H.

We set $\mathcal{F} = \mathbf{F}_q((t))$ and $\mathcal{O} = \mathbf{F}_q[[t]]$, where $\mathbf{F}_q$ is the finite field with q elements. We let $\mathrm{Gr} = \widehat{G}(\mathcal{F})/\widehat{G}(\mathcal{O})$. Each $\lambda \in \Lambda$ defines a homomorphism $\mathcal{F}^* \longrightarrow H(\mathcal{F})$. We will denote the image of t under this homomorphism by t^λ, and its image in Gr will also be denoted by t^λ. We set

$$S^\lambda = \widehat{U}(\mathcal{F}) \cdot t^\lambda \subset \mathrm{Gr} \quad \text{and} \quad T^\lambda = U_-(\mathcal{F}) \cdot t^\lambda \subset \mathrm{Gr}.$$

In a recent paper [**BFK12**], Braverman, Finkelberg and Kazhdan defined the generating function

$$I_{\mathfrak{g}}(q) = \sum_{\gamma\in\Lambda^+} \#(T^{-\gamma}\cap S^0)q^{-|\gamma|}\boldsymbol{z}^{\gamma}, \tag{4.1}$$

and computed this sum as a product using a geometric method. Now we assume that the set of positive coroots R^+ forms a root system of type $A_n^{(1)}$, and we identify R^+ with the set of positive roots Δ^+ in the previous sections of this paper. In this case, the resulting product in [**BFK12**] is

$$I_{\mathfrak{g}}(q) = \prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1-q^{-i}\boldsymbol{z}^{j\delta}}{1-q^{-(i+1)}\boldsymbol{z}^{j\delta}} \prod_{\alpha\in\Delta^+}\left(\frac{1-q^{-1}\boldsymbol{z}^{\alpha}}{1-\boldsymbol{z}^{\alpha}}\right)^{\mathrm{mult}(\alpha)}.$$

We separate the factor

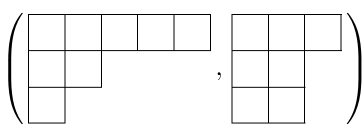
$$\prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1-q^{-i}\boldsymbol{z}^{j\delta}}{1-q^{-(i+1)}\boldsymbol{z}^{j\delta}},$$

and call it the *correction factor*. Our goal of this section is to write this correction factor and the function $I_{\mathfrak{g}}(q)$ as sums over reduced proper generalized Young walls.

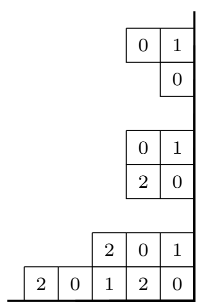
Let $\mathcal{Y}_0$ denote the subset of $\mathcal{Y}(\infty)$ consisting of the reduced proper generalized Young walls with empty rows in positions $\equiv 0 \bmod n+1$. We define a map $\xi\colon \mathcal{P}(n) \longrightarrow \mathcal{Y}_0$ by the following assignment. If $\boldsymbol{p} = (\rho^{(1)},\dots,\rho^{(n)})$ is a multi-partition, then the parts of the partition $\rho^{(j)}$ give the lengths of the rows $\equiv j \bmod n+1$ in a reduced proper generalized Young wall $\xi(\boldsymbol{p}) = Y \in \mathcal{Y}_0$. The following is clear from the definition.

Lemma 4.1. *The map $\xi\colon \mathcal{P}(n)\longrightarrow\mathcal{Y}_0$ defined above is a bijection.*

Example 4.2. Let $n=2$. If

$\boldsymbol{p} =$ ,

then the corresponding element in $\mathcal{Y}_0$ is

$Y = \xi(\boldsymbol{p}) =$ 

For $Y \in \mathcal{Y}_0$, define

$$\mathscr{M}(Y) = \sum_{i=1}^{n}(i+1)M_i(Y),$$

where $M_i(Y)$ is the number of nonempty rows $\equiv i \bmod n+1$ in Y. Moreover, we define $|Y|$ to be the total number of blocks in Y.

EXAMPLE 4.3. Let Y be as in Example 4.2. Then

$$\mathscr{M}(Y) = 2 \cdot 3 + 3 \cdot 3 = 15 \quad \text{and} \quad |Y| = 15.$$

Let us consider $\mathscr{N}(Y)$ for $Y \in \mathcal{Y}_0$, where $\mathscr{N}(Y)$ is defined in (3.5). Since Y has empty rows in positions $\equiv 0 \bmod n+1$, we have $(p_m, q_m) = (0,0)$ for all $m \geq 0$, and obtain $\mathscr{Q}(Y) = \{0\}$ and $\mathscr{P}(Y) = 0$. Hence we have

$$\mathscr{N}(Y) = \sum_{j=1}^{n} \#(S_j(Y) \setminus \{0\}) \qquad \text{for } Y \in \mathcal{Y}_0. \tag{4.2}$$

PROPOSITION 4.4. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $A_n^{(1)}$. Then*

$$\prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1 - q^{-i}\boldsymbol{z}^{j\delta}}{1 - q^{-(i+1)}\boldsymbol{z}^{j\delta}} = \sum_{Y \in \mathcal{Y}_0} (1-q)^{\mathscr{N}(Y)} q^{-\mathscr{M}(Y)} \boldsymbol{z}^{|Y|\delta}.$$

PROOF. We have

$$\begin{aligned}
\prod_{j=1}^{\infty} \frac{1 - q^{-i}\boldsymbol{z}^{j\delta}}{1 - q^{-(i+1)}\boldsymbol{z}^{j\delta}} &= \prod_{j=1}^{\infty}\left(1 + \sum_{k=1}^{\infty}(1-q)q^{-k(i+1)}\boldsymbol{z}^{kj\delta}\right) \\
&= \sum_{\rho^{(i)} \in \mathcal{P}(1)} (1-q)^{\mathscr{N}(\rho^{(i)})} q^{-(i+1)M(\rho^{(i)})} \boldsymbol{z}^{|\rho^{(i)}|\delta},
\end{aligned}$$

where $\mathscr{N}(\rho^{(i)}) = \#\{r : m_r \neq 0\}$ and $|\rho^{(i)}| = m_1 + 2m_2 + \cdots$ are defined in (3.9) and we set $M(\rho^{(i)}) = m_1 + m_2 + \cdots$ for $\rho^{(i)} = (1^{m_1}2^{m_2}\cdots) \in \mathcal{P}(1)$. For a multi-partition $\rho = (\rho^{(1)}, \ldots, \rho^{(n)}) \in \mathcal{P}(n)$, define

$$\mathscr{N}(\rho) = \sum_{i=1}^{n} \mathscr{N}(\rho^{(i)}), \quad |\rho| = \sum_{i=1}^{n} |\rho^{(i)}| \quad \text{and} \quad \mathscr{M}(\rho) = \sum_{i=1}^{n} (i+1) M(\rho^{(i)}).$$

Then we have

$$\begin{aligned}
\prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1 - q^{-i}\boldsymbol{z}^{j\delta}}{1 - q^{-(i+1)}\boldsymbol{z}^{j\delta}} &= \prod_{i=1}^{n} \sum_{\rho^{(i)} \in \mathcal{P}(1)} (1-q)^{\mathscr{N}(\rho^{(i)})} q^{-(i+1)M(\rho^{(i)})} \boldsymbol{z}^{|\rho^{(i)}|\delta} \\
&= \sum_{\rho \in \mathcal{P}(n)} (1-q)^{\mathscr{N}(\rho)} q^{-\mathscr{M}(\rho)} \boldsymbol{z}^{|\rho|\delta}. \qquad (4.3)
\end{aligned}$$

Using the map ξ in Lemma 4.1, one can see that $\mathscr{N}(\rho) = \mathscr{N}(\xi(\rho))$, $\mathscr{M}(\rho) = \mathscr{M}(\xi(\rho))$ and $|\rho| = |\xi(\rho)|$ for $\rho \in \mathcal{P}(n)$, and the proposition follows from (4.3). ■

The following formula provides a combinatorial description of the affine Gindikin-Karpelevich formula proved by Braverman, Finkelberg and Kazhdan.

COROLLARY 4.5. *When $\mathfrak{g}$ is an affine Kac-Moody algebra of type $A_n^{(1)}$, we have*

$$\begin{aligned}
I_{\mathfrak{g}}(q) &= \prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{1 - q^{-i}\boldsymbol{z}^{j\delta}}{1 - q^{-(i+1)}\boldsymbol{z}^{j\delta}} \prod_{\alpha \in \Delta^+} \left(\frac{1 - q^{-1}\boldsymbol{z}^{\alpha}}{1 - \boldsymbol{z}^{\alpha}}\right)^{\mathrm{mult}(\alpha)} \qquad (4.4) \\
&= \sum_{(Y_1, Y_2) \in \mathcal{Y}(\infty) \times \mathcal{Y}_0} (1-q^{-1})^{\mathscr{N}(Y_1)} (1-q)^{\mathscr{N}(Y_2)} q^{-\mathscr{M}(Y_2)} \boldsymbol{z}^{-\mathrm{wt}(Y_1) + |Y_2|\delta}.
\end{aligned}$$

Furthermore, comparing (4.4) with (4.1), we obtain a combinatorial formula for the number of points in the intersection $T^{-\gamma} \cap S^0$:

Corollary 4.6. *We have*

$$\#(T^{-\gamma} \cap S^0) = \sum_{\substack{(Y_1,Y_2)\in\mathcal{Y}(\infty)\times\mathcal{Y}_0 \\ -\mathrm{wt}(Y_1)+|Y_2|\delta=\gamma}} (1-q^{-1})^{\mathcal{N}(Y_1)}(1-q)^{\mathcal{N}(Y_2)} q^{|\gamma|-\mathcal{M}(Y_2)},$$

where $\gamma \in \Lambda^+$ is identified with the corresponding element of the root lattice of $\mathfrak{g}$.

Example 4.7. Assume $n = 1$ and $\gamma = \delta$. Then we have

$$(Y_1, Y_2) = \left(\varnothing\ ,\ \boxed{0}\right), \quad \left(\boxed{1}\boxed{0}\ ,\varnothing\right), \quad \text{or} \quad \left(\boxed{0}\boxed{1}\ ,\varnothing\right).$$

From the first pair, we get $(1-q^{-1})^0(1-q)^1q^{2-2} = 1-q$. The second yields $(1-q^{-1})^1(1-q)^0q^{2-0} = q^2-q$, and the third $(1-q^{-1})^2(1-q)^0q^{2-0} = (q-1)^2$. Thus we have

$$\#(T^{-\gamma} \cap S^0) = 1 - q + q^2 - q + (q-1)^2 = 2(q-1)^2.$$

Appendix A. Implementation in Sage

Together with Lucas Roesler and Travis Scrimshaw, the fourth named author has implemented generalized Young walls and the statistics developed here in the open-source mathematical software Sage [**SCc08**, **S**$^+$**14**]. We conclude with some examples using our package.

First we may verify examples given above. To verify Example 3.16, we have the following, where Y.number_of_parts() refers to $\mathcal{N}(Y)$.

```
sage: Yinf = crystals.infinity.GeneralizedYoungWalls(2)
sage: Y = Yinf([[0,2,1,0,2],[1,0,2],[2],[],[1]])
sage: Y.pp()
          1|
           |
          2|
      2|0|1|
  2|0|1|2|0|
sage: Y.number_of_parts()
4
```

Similarly, to see Examples 3.17 and 3.19 using Sage, use the following commands.

```
sage: Yinf = crystals.infinity.GeneralizedYoungWalls(2)
sage: row1 = [0,2,1]
sage: row2 = [1,0]
sage: row3 = [2,1,0,2,1,0,2,1,0]
sage: row6 = [2,1,0,2,1,0]
sage: Y = Yinf([row1,row2,row3,[],[],row6])
```

```
sage: Y.pp()
          0|1|2|0|1|2|
                     |
                     |
 0|1|2|0|1|2|0|1|2|
                 0|1|
               1|2|0|
sage: Y.number_of_parts()
8
```

Note that the remaining crystal structure pertaining to generalized Young walls has also been implemented. We continue using the Y from the previous example.

```
sage: Y.weight(root_lattice=True)
-7*alpha[0] - 7*alpha[1] - 6*alpha[2]
sage: Y.f(1).pp()
          0|1|2|0|1|2|
                    1|
                     |
 0|1|2|0|1|2|0|1|2|
                 0|1|
               1|2|0|
sage: Y.e(0).pp()
            1|2|0|1|2|
                     |
                     |
 0|1|2|0|1|2|0|1|2|
                 0|1|
               1|2|0|
sage: Y.content()
20
```

One may also generate the top part of the crystal graph.

```
sage: Yinf = crystals.infinity.GeneralizedYoungWalls(2)
sage: S = Yinf.subcrystal(max_depth=4)
sage: G = Yinf.digraph(subset=S)
sage: view(G, tightpage=True)
```

We conclude by mentioning that highest weight crystals realized by generalized Young walls have also been implemented in Sage, following Theorem 4.1 of [**KS10**].

```
sage: Delta = RootSystem(['A',3,1])
sage: P = Delta.weight_lattice(extended=True)
sage: La = P.fundamental_weights()
sage: YLa = crystals.GeneralizedYoungWalls(3,La[0])
sage: S = YLa.subcrystal(max_depth=6)
sage: G = YLa.digraph(subset=S)
sage: view(G, tightpage=True)
```

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