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Gröbner–Shirshov Bases for Irreducible sl_{n+1}-Modules

Seok-Jin Kang[1] and Kyu-Hwan Lee[2]

Department of Mathematics, Seoul National University, Seoul 151-742, Korea
E-mail: sjkang@math.snu.ac.kr, khlee@math.snu.ac.kr

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We determine the Gröbner–Shirshov bases for finite-dimensional irreducible representations of the special linear Lie algebra sl_{n+1} and construct explicit monomial bases for these representations. We also show that each of these monomial bases is in 1–1 correspondence with the set of semistandard Young tableaux of a given shape.

0. INTRODUCTION

In [10], inspired by an idea of Gröbner, Buchberger discovered an effective algorithm for solving the reduction problem for commutative algebras, which is now called the *Gröbner Basis Theory*. It was generalized to associative algebras through Bergman's *Diamond Lemma* [2], and the parallel theory for Lie algebras was developed by Shirshov [21]. The key ingredient of Shirshov's theory is the *Composition Lemma*, which turned out to be valid for associative algebras as well (see [3]). For this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called *Gröbner–Shirshov Basis Theory*.

For finite-dimensional simple Lie Algebras, Bokut and Klein constructed the Gröbner–Shirshov bases explicitly [5–7]. In [4], Bokut, *et al.* unified the Gröbner–Shirshov basis theory for Lie superalgebras and their universal

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enveloping algebras and gave an explicit construction of Gröbner–Shirshov bases for classical Lie superalgebras.

The main idea of the Gröbner–Shirshov Basis Theory for Lie (super)-algebras may be summarized as follows: Let L be a Lie (super)algebra defined by generators and relations. Then the set of relations can be *completed* to a *basis* S for the relations which are closed under composition. We call S a *Gröbner–Shirshov basis* for the Lie (super)algebra L and the set of all *S-standard monomials* forms a monomial basis of L.

The next natural step is to develop the Gröbner–Shirshov Basis Theory for representations. In [15], we developed Gröbner–Shirshov basis theory for representations of associative algebras by introducing the notion of the *Gröbner–Shirshov pair*. More precisely, let (S, T) be a pair of subsets of free associative algebra $\mathscr{A}$, let J be the two-sided ideal of $\mathscr{A}$ generated by S, and let I be the left ideal of the algebra $A = \mathscr{A}/J$ generated by (the image of) T. Then the left A-module $M = A/I$ is said to be *defined by* the pair (S, T) and the pair (S, T) is called a *Gröbner–Shirshov pair* for M if it is closed under composition, or equivalently if the set of (S, T)-*standard* monomials forms a linear basis of M. In [15], we proved the generalized version of Shirshov's Composition Lemma for a *Gröbner–Shirshov pair* (S, T).

In this paper, we apply the Gröbner–Shirshov Basis Theory for representations developed in [15] to the reduction problem for finite-dimensional irreducible representations of the special linear Lie algebra sl_{n+1}. That is, we determine the Gröbner–Shirshov pairs for finite-dimensional irreducible sl_{n+1}-modules and construct explicit monomial bases for these modules. We also show that each of these monomial bases is in 1–1 correspondence with the set of semistandard Young tableaux of a given shape.

Let us describe our approach in more detail. Recall that the finite-dimensional irreducible sl_{n+1}-modules are parametrized by the partitions with at most n parts and that each of these partitions corresponds to a Young diagram with at most n rows. Let λ be a Young diagram corresponding to a partition with at most n parts and let $V(\lambda)$ denote the finite-dimensional irreducible representation of sl_{n+1} with highest weight λ. Then the sl_{n+1}-module $V(\lambda)$ is defined by the pair $(S_-,\ T_\lambda)$, where S_- is the set of (negative) Serre relations and T_λ is the set of annihilating relations for the highest weight vector of $V(\lambda)$. First, we derive sufficiently many relations in $V(\lambda)$, the set of which is denoted by $(\mathscr{S}_-,\ \mathscr{T}_\lambda)$, and determine the set $G(\lambda)$ of all $(\mathscr{S}_-, \mathscr{T}_\lambda)$-standard monomials. We then give a bijection between $G(\lambda)$ and the set of all semistandard Young tableaux of shape λ. Hence we conclude that $G(\lambda)$ is a monomial basis of the irreducible sl_{n+1}-module $V(\lambda)$ and that $(\mathscr{S}_-, \mathscr{T}_\lambda)$ is a Gröbner–Shirshov pair for $V(\lambda)$.

The monomial basis $G(\lambda)$ can be given the structure of a colored oriented graph, called the *Gröbner–Shirshov graph*, which reflects the internal structure of the representation $V(\lambda)$. However, the Gröbner–Shirshov graph is usually different from the *crystal graph* introduced by Kashiwara [17].

1. GRÖBNER–SHIRSHOV BASIS THEORY FOR REPRESENTATIONS

In this section, we briefly recall the Gröbner–Shirshov Basis Theory for the representation of associative algebras which was developed in [15]. We present the theory in a slightly different way so that it may fit into a more general setting.

Let X be a set and let X^* be the free monoid of associative monomials on X. We denote the empty monomial by 1 and the *length* of a monomial u by $l(u)$. Thus we have $l(1) = 0$.

DEFINITION 1.1. A well-ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$.

Fix a monomial order $\prec$ on X^* and let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial (called the *leading term*) appearing in p under the ordering $\prec$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$, and $w_i \prec \overline{p}$. If $\alpha = 1$, p is said to be *monic*.

Let (S, T) be a pair of subsets of monic elements of $\mathcal{A}_X$, let J be the two-sided ideal of $\mathcal{A}_X$ generated by S, and let I be the left ideal of the algebra $A = \mathcal{A}_X/J$ generated by (the image of) T. Then we say that the algebra $A = \mathcal{A}_X/J$ is *defined by* S and the left A-module $M = A/I$ is *defined by* the pair (S, T). The images of $p \in \mathcal{A}_X$ in A and in M under the canonical quotient maps will also be denoted by p.

DEFINITION 1.2. Given a pair (S, T) of subsets of monic elements of $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be *(S, T)-standard* if $u \neq a\overline{s}b$ and $u \neq c\overline{t}$ for any $s \in S$, $t \in T$, and $a, b, c \in X^*$. Otherwise, the monomial u is said to be *(S, T)-reducible*. If $T = \emptyset$, we will simply say that u is *S-standard* or *S-reducible*.

Using the same argument as that in the proof of Theorem 3.2 in [15], we can prove:

THEOREM 1.3. *Every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j + \sum \gamma_k u_k, \tag{1.1}$$

where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$; $a_i, b_i, c_j, u_k \in X^*$; $s_i \in S$; $t_j \in T$; $a_i\overline{s_i}b_i \preceq \overline{p}$; $c_j\overline{t_j} \preceq \overline{p}$; and $u_k \preceq \overline{p}$; and u_k are (S, T)-standard.

The term $\sum \gamma_k u_k$ in the expression (1.1) is called a *normal form* (or a *remainder*) of p with respect to the pair (S, T) (and with respect to the monomial order $\prec$). As an immediate corollary of Theorem 1.3, we obtain:

PROPOSITION 1.4. *The set of (S, T)-standard monomials spans the left A-module $M = A/I$ defined by the pair (S, T).*

DEFINITION 1.5. A pair (S, T) of subsets of monic elements of $\mathscr{A}_X$ is a *Gröbner–Shirshov pair* if the set of (S, T)-standard monomials forms a linear basis of the left A-module $M = A/I$ defined by the pair (S, T). In this case, we say that (S, T) is a *Gröbner–Shirshov pair* for the module M defined by (S, T). If a pair $(S, \emptyset)$ is a Gröbner–Shirshov pair, then we also say that S is a *Gröbner–Shirshov basis* for the algebra $A = \mathscr{A}_X/J$ defined by S.

Let p and q be monic elements of $\mathscr{A}_X$ with leading terms $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

DEFINITION 1.6. (a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p, q)_w = pa - bq$. Furthermore, if $a = 1$, the composition $(p, q)_w$ is called *right-justified*.

(b) If there exist a and b in X^* such that $a \neq 1$, $a\overline{p}b = \overline{q} = w$, then the *composition of inclusion* is defined to be $(p, q)_w = apb - q$.

Remark. The role of w is important since there can be different choices of overlaps for given p and q, which does not occur in commutative case. Also, we do not consider a composition of the type $(p, q)_w = p - aqb$ with $b \neq 1$, $\overline{p} = a\overline{q}b = w$. This point will become critical when we consider the notion of *closedness* under composition for a pair (S, T).

Let $p, q \in \mathscr{A}_X$ and $w \in X^*$. We define a *congruence relation* on $\mathscr{A}_X$ as follows: $p \equiv q \bmod (S, T; w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i, \beta_j \in \mathbb{F}$; $a_i, b_i, c_j \in X^*$; $s_i \in S$; $t_j \in T$; $a_i\overline{s_i}b_i \prec w$; and $c_j\overline{t_j} \prec w$. When $T = \emptyset$, we simply write $p \equiv q \bmod (S; w)$.

DEFINITION 1.7. A pair (S, T) of subsets of monic elements in $\mathscr{A}_X$ is said to be *closed under composition* if

(i) $(p, q)_w \equiv 0 \bmod (S; w)$ for all $p, q \in S$, $w \in X^*$, whenever the composition $(p, q)_w$ is defined;

(ii) $(p, q)_w \equiv 0 \bmod (S, T; w)$ for all $p, q \in T$, $w \in X^*$, whenever the right-justified composition $(p, q)_w$ is defined;

(iii) $(p, q)_w \equiv 0 \bmod (S, T; w)$ for all $p \in S$, $q \in T$, $w \in X^*$ whenever the composition $(p, q)_w$ is defined.

If $T = \emptyset$, we will simply say that S is closed under composition.

In the following theorem, we recall the main result of [15] which is a generalized version of Shirshov's Composition Lemma for representations of associative algebras. The definition of a Gröbner–Shirshov pair in this paper is different from the one given in [15]. But, as we will see in Proposition 1.9, these two definitions coincide with each other.

THEOREM 1.8 [15]. *Let (S, T) be a pair of subsets of monic elements in the free associative algebra $\mathscr{A}_X$ generated by X; $A = \mathscr{A}_X/J$ the associative algebra defined by S; and $M = A/I$ the left A-module defined by (S, T). If (S, T) is closed under composition and the image of $p \in \mathscr{A}_X$ is trivial in M, then the word $\overline{p}$ is (S, T)-reducible.*

As a corollary, we obtain:

PROPOSITION 1.9. *Let (S, T) be a pair of subsets of monic elements in $\mathscr{A}_X$. Then the following are equivalent:*

(a) *(S, T) is a Gröbner–Shirshov pair.*

(b) *(S, T) is closed under composition.*

(c) *For each $p \in \mathscr{A}_X$, the normal form of p is unique.*

Proof. (a) $\Rightarrow$ (b). Consider a composition $(p, q)_w$. By Theorem 1.3, we obtain a normal form of $(p, q)_w$. Since $(p, q)_w$ is trivial in M, any normal form of $(p, q)_w$ must be zero. Moreover, $\overline{(p, q)_w} \prec w$, which implies $(p, q)_w \equiv 0 \bmod (S, T; w)$.

(b) $\Rightarrow$ (c). Given $p \in \mathscr{A}_X$, assume that we have two normal forms $\sum \gamma_k u_k$ and $\sum \gamma'_k u_k$ of p, where $\gamma_k, \gamma'_k \in \mathbb{F}$. Then $\sum(\gamma_k - \gamma'_k)u_k$ is trivial in M, and by Theorem 1.8 we must have $\gamma_k = \gamma'_k$ for all k.

(c) $\Rightarrow$ (a). By Proposition 1.4, we have only to check the linear independence of (S, T)-standard monomials, which follows immediately from the uniqueness of the normal form. ∎

Remark. Part (b) of Proposition 1.9 gives an analogue of *Buchberger's algorithm* as we have shown in [15]. However, in general, there is no guarantee that this process will terminate in finite steps. Still, such an algorithm works in many interesting cases as we can see in the following examples.

Before focusing on finite-dimensional irreducible sl_{n+1}-modules, we give a couple of examples of the general Gröbner–Shirshov basis theory for associative algebras and their representations. In the examples given below, we take the field $\mathbb{F}$ to be the complex number $\mathbb{C}$ and the monomial order $\prec$ to be the *degree–lexicographic order*. That is, we define $u \prec v$ if and only if $l(u) < l(v)$ or $l(u) = l(v)$ and $u \prec v$ in the lexicographic order (cf. [15]).

EXAMPLE 1.10. Let $X = \{d, u\}$ with $u \prec d$ and

$$S = \left\{ \begin{array}{c} d^2u - \alpha dud - \beta ud^2 - \gamma d, \\ du^2 - \alpha udu - \beta u^2 d - \gamma u \end{array} \right\},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. The algebra $A(\alpha, \beta, \gamma)$ defined by S is called the *down-up algebra* (cf. [1]). There is only one possible composition among the elements of S, which turns out to be trivial:

$$\begin{aligned} &(d^2u - \alpha dud - \beta ud^2 - \gamma d, du^2 - \alpha udu - \beta u^2 d - \gamma u)_{d^2u^2} \\ &\quad = -\beta ud^2u + \beta du^2 d \\ &\quad = -\beta u(d^2u - \alpha dud - \beta ud^2 - \gamma d) + \beta(du^2 - \alpha udu - \beta u^2 d - \gamma u)d \\ &\quad \equiv 0 \bmod (S; d^2u^2). \end{aligned}$$

Hence S is a Gröbner–Shirshov basis for the down-up algebra $A(\alpha, \beta, \gamma)$ and the monomial basis of $A(\alpha, \beta, \gamma)$ consisting of S-standard monomials is given by

$$\{u^i (du)^j d^k | i, j, k \geq 0\}.$$

Let $T = \{du - \lambda, d\}$ with $\lambda \in \mathbb{C}$ and let $M(\lambda)$ be the left $A(\alpha, \beta, \gamma)$-module defined by the pair (S, T). Note that there is no right-justified composition among elements of T. Then the only possible composition between the elements of S and T is trivial:

$$\begin{aligned} &(d^2u - \alpha dud - \beta ud^2 - \gamma d, du - \lambda)_{d^2u} \\ &\quad = -\alpha dud - \beta ud^2 - \gamma d + \lambda d \equiv 0 \bmod (S, T; d^2u). \end{aligned}$$

Therefore (S, T) is closed under composition, and by Proposition 1.9 the pair (S, T) is a Gröbner–Shirshov pair for the $A(\alpha, \beta, \gamma)$-module $M(\lambda)$. Hence we obtain the monomial basis of $M(\lambda)$ consisting of (S, T)-standard monomials, $\{u^i | i \geq 0\}$.

EXAMPLE 1.11. Let $\mathscr{H}_q(S_n)$ be the *Iwahori–Hecke algebra of type A* with $q \in \mathbb{C}^\times$. Thus $\mathscr{H}_q(S_n)$ is the associative algebra over $\mathbb{C}$ generated by $X = \{T_1, T_2, \ldots, T_{n-1}\}$ with defining relations

$$S: \begin{array}{ll} T_iT_j - T_jT_i & \text{for } i > j+1, \\ T_i^2 - (q-1)T_i - q & \text{for } 1 \leq i \leq n-1, \\ T_{i+1}T_iT_{i+1} - T_iT_{i+1}T_i & \text{for } 1 \leq i \leq n-2. \end{array} \tag{1.2}$$

Define $T_i \prec T_j$ if $i < j$. We claim that S can be completed to a Gröbner–Shirshov basis $\mathscr{S}$ for $\mathscr{H}_q(S_n)$ given as

$$\mathscr{S}: \begin{array}{ll} T_iT_j - T_jT_i & \text{for } i > j+1, \\ T_i^2 - (q-1)T_i - q & \text{for } 1 \leq i \leq n-1, \\ T_{i+1,j}T_{i+1} - T_iT_{i+1,j} & \text{for } i \geq j, \end{array} \tag{1.3}$$

where $T_{i,j} = T_i T_{i-1} \cdots T_j$ for $i \geq j$ (hence $T_{i,i} = T_i$). Since T_{i+1} commutes with T_k for $j \leq k \leq i-1$, we have

$$\begin{aligned} T_{i+1,j} T_{i+1} &= T_{i+1} T_i T_{i-1,j} T_{i+1} = T_{i+1} T_i T_{i+1} T_{i-1,j} \\ &= T_i T_{i+1} T_i T_{i-1,j} = T_i T_{i+1,j}. \end{aligned}$$

Hence all the relations in $\mathcal{S}$ hold in $\mathcal{H}_q(S_n)$. Note that the set B of $\mathcal{S}$-standard monomials is given by

$$\begin{aligned} B = \{T_{1,j_1} T_{2,j_2} T_{3,j_3} \cdots T_{n-1,j_{n-1}} \mid 1 \leq j_k \leq k+1 \\ \text{for all } k = 1, 2, \ldots, n-1\}, \end{aligned}$$

where $T_{i,i+1} = 1$. Then the number of elements in B is $n!$ which is equal to the dimension of $\mathcal{H}_q(S_n)$. Therefore, by Proposition 1.4, the set B is a linear basis of $\mathcal{H}_q(S_n)$, and hence the set $\mathcal{S}$ is a Gröbner–Shirshov basis for the Iwahori–Hecke algebra $\mathcal{H}_q(S_n)$.

For the representations of $\mathcal{H}_q(S_n)$, we only consider one special example. The general Gröbner–Shirshov basis theory for Iwahori–Hecke algebras and their representations will be investigated elsewhere [16].

Let $n = 4$ and let M be the $\mathcal{H}_q(S_4)$-module defined by the pair $(\mathcal{S}, T)$, where

$$T = \{T_1 T_{2,1} + T_{2,1} + T_1 T_2 + T_2 + T_1 + 1\}.$$

It is clear that there is no right-justified composition among the elements of T. As for the compositions between the elements of $\mathcal{S}$ and T, there are three possibilities:

$$\begin{aligned} &(T_3 T_1 - T_1 T_3, T_1 T_{2,1} + T_{2,1} + T_1 T_2 + T_2 + T_1 + 1)_{T_3 T_1 T_{2,1}} \\ &\quad = -T_1 T_{3,1} - T_{3,1} - T_3 T_1 T_2 - T_{3,2} - T_{3,1} - T_3 \\ &\quad \equiv -T_1 T_{3,1} - T_{3,1} - T_1 T_{3,2} - T_{3,2} - T_1 T_3 - T_3 \bmod (\mathcal{S}, T; T_3 T_1 T_{2,1}); \\ &(T_1^2 - (q-1) T_1 - q, T_1 T_{2,1} + T_{2,1} + T_1 T_2 + T_2 + T_1 + 1)_{T_1 T_{2,1}} \\ &\quad \equiv -q(T_1 T_{2,1} + T_{2,1} + T_1 T_2 + T_2 + T_1 + 1) \equiv 0 \bmod (\mathcal{S}, T; T_1 T_{2,1}); \\ &(T_{2,1} T_2 - T_1 T_{2,1}, T_1 T_{2,1} + T_{2,1} + T_1 T_2 + T_2 + T_1 + 1)_{T_{2,1} T_{2,1}} \\ &\quad = -T_1 T_2 T_1^2 - T_2^2 T_1 - T_2 T_1 T_2 - T_2^2 - T_{2,1} - T_2 \\ &\quad \equiv -q(T_1 T_{2,1} + T_{2,1} + T_1 T_2 + T_2 + T_1 + 1) \equiv 0 \bmod (\mathcal{S}, T; T_{2,1} T_{2,1}). \end{aligned}$$

Thus T can be extended by adding the nontrivial composition

$$\mathcal{T} = T \cup \{T_1 T_{3,1} + T_{3,1} + T_1 T_{3,2} + T_{3,2} + T_1 T_3 + T_3\}.$$

Then it is straightforward to verify that there is no additional nontrivial composition between the elements of $\mathcal{S}$ and $\mathcal{T}$. Hence $(\mathcal{S}, \mathcal{T})$ is a

Gröbner–Shirshov pair for the left $\mathscr{H}_q(S_n)$-module M defined by the pair $(\mathscr{S}, T)$. It is now easy to see that the monomial basis of M consisting of $(\mathscr{S}, \mathscr{T})$-standard monomials is given by

$$B = \{T_{1,j}T_{2,k}T_{3,l} \mid 1 \le j \le 2,\ 1 \le k \le 3,\ 1 \le l \le 4\}$$
$$\setminus\{T_1T_{2,1},\ T_1T_{3,1},\ T_{2,1}T_{3,1}, T_1T_{2,1}T_{3,1}\}$$

and that $\dim M = 20$.

2. GRÖBNER–SHIRSHOV PAIRS FOR IRREDUCIBLE sl_{n+1}-MODULES

We now turn to finite-dimensional irreducible representations of the special linear Lie algebra sl_{n+1}, the Lie algebra of $(n+1)\times(n+1)$ matrices with trace 0. In this section, the base field will be the complex field $\mathbb{C}$ and our monomial order will be the degree-lexicographic order.

Recall that the Lie algebra sl_{n+1} is generated by $\{e_i, h_i, f_i \mid 1 \le i \le n\}$ with the defining relations

$$\begin{array}{rlll}
W: & [h_ih_j]\ (i>j), & [e_if_j]-\delta_{ij}h_i, & \\
 & [e_ih_j]+a_{ji}e_i, & [h_if_j]+a_{ij}f_j, & \\
S_+: & [e_{i+1}[e_{i+1}e_i]],\ [[e_{i+1}e_i]e_i] & (1 \le i \le n-1), & \\
 & [e_ie_j] & (i>j+1), & \\
S_-: & [f_{i+1}[f_{i+1}f_i]],\ [[f_{i+1}f_i]f_i] & (1 \le i \le n-1), & \\
 & [f_if_j] & (i>j+1), &
\end{array} \tag{2.1}$$

where the Cartan matrix $(a_{ij})_{1\le i,j\le n}$ is given by

$$a_{ii}=2,\ a_{i+1,i}=a_{i,i+1}=-1,\ a_{ij}=0 \quad \text{for } |i-j|>1. \tag{2.2}$$

Let U be the universal enveloping algebra of sl_{n+1} and let U_+ (resp. U_-) be the subalgebra of U generated by $E=\{e_1,\dots,e_n\}$ (resp. $F=\{f_1,\dots,f_n\}$). Then the algebra U_+ (resp. U_-) is the associative algebra defined by the set S_+ (resp., S_-) of relations in the free associative algebra $\mathscr{A}_E$ on E (resp. $\mathscr{A}_F$ on F).

For $i \ge j$, we define

$$\begin{array}{ll}
[e_{ij}] = [e_i[e_{i-1}[\cdots[e_{j+1}e_j]]]], & e_{ij} = e_ie_{i-1}\cdots e_j, \\
[f_{ij}] = [f_i[f_{i-1}[\cdots[f_{j+1}f_j]]]], & f_{ij} = f_if_{i-1}\cdots f_j.
\end{array} \tag{2.3}$$

(Hence $[e_{ii}]=e_{ii}=e_i$ and $[f_{ii}]=f_{ii}=f_i$.) We also define $e_i \prec e_j$, $f_i \prec f_j$ if and only if $i<j$, and $(i,j)>(k,l)$ if and only if $i>k$ or $i=k$, $j>l$.

In [5], Bokut and Klein extended the set $S_\pm$ to obtain a Gröbner–Shirshov basis $\mathscr{S}_\pm$ for the algebra $U_\pm$ as given in the following proposition.

PROPOSITION 2.1 ([5, 19]) *Let*

$$\begin{aligned}\mathscr{S}_+ &= \{[[e_{ij}],\,[e_{kl}]] \mid (i,j) > (k,l),\; k \neq j-1\},\\ \mathscr{S}_- &= \{[[f_{ij}],\,[f_{kl}]] \mid (i,j) > (k,l),\; k \neq j-1\}.\end{aligned} \tag{2.4}$$

Then $\mathscr{S}_\pm$ is a Gröbner–Shirshov basis for the algebra $U_\pm$. In addition, in $U_\pm$,

$$[[e_{ij}],\,[e_{j-1,k}]] = [e_{ik}] \qquad \textit{and} \qquad [[f_{ij}],\,[f_{j-1,k}]] = [f_{ik}]. \tag{2.5}$$

Remark. The Gröbner–Shirshov bases for classical Lie algebras were completely determined in [5–7]. There is another answer to this problem given by Lalonde and Ram [19]. (See also [13].) For classical Lie superalgebras, the Gröbner–Shirshov bases were determined in [4]. ∎

Recall that the finite-dimensional irreducible representations of sl_{n+1} are indexed by the partitions with at most n parts and that each of these partitions corresponds to a Young diagram with at most n rows. Thus we will identify a partition with the corresponding Young diagram.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n parts and let $V(\lambda)$ denote the finite-dimensional irreducible representation of sl_{n+1} with highest weight λ. Set

$$m_i = \lambda_i - \lambda_{i+1} \qquad \text{for } i = 1, 2, \ldots, n. \tag{2.6}$$

(Here, $\lambda_{n+1} = 0$.) Then it is well-known (see [14], for example) that the sl_{n+1}-module $V(\lambda)$ can be regarded as a U_--module defined by the pair $(S_-,\, T_\lambda)$, where

$$T_\lambda = \{f_i^{m_i+1} \mid i = 1,\, \ldots,\, n\}. \tag{2.7}$$

For convenience, we define

$$H_{i+1,\,k} = \prod_{s=i+1}^{k}\left(\prod_{t=1}^{i-1}[f_{s,\,t}]^{a_{s,\,t}}\right)[f_{s,\,i}]^{a_{s,\,i}-r_s}[f_{s,\,i+1}]^{a_{s,\,i+1}+r_s}\left(\prod_{t=i+2}^{s}[f_{s,\,t}]^{a_{s,\,t}}\right),$$

for i, k and $a_{s,\,t}, r_s \in \mathbb{Z}_{\geq 0}$.

We will say that *a relation $R = 0$ holds in U_-* whenever R belongs to the two-sided ideal of $\mathscr{A}_F$ generated by S_-. Similarly, we will say that *a relation $R = 0$ holds in $V(\lambda)$* whenever R is contained in the left ideal of U_- generated by T_λ.

We now can state the main theorem of this paper.

THEOREM 2.2. *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ be a partition with at most n parts and let $V(\lambda)$ denote the finite-dimensional irreducible representation of sl_{n+1} with highest weight λ. Then the relations*

$$\sum_{r_n=0}^{a_{n,\,j}} \cdots \sum_{r_{j+1}=0}^{a_{j+1,\,j}} \frac{b_{j,\,j}!}{(b_{j,\,j} + |r|_{j+1})!} B_{j,\,j+1} f_j^{b_{j,\,j}+|r|_{j+1}} H_{j+1,\,n} = 0, \tag{2.8}$$

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{i+1}=0}^{a_{i+1,j}} \sum_{r_i=-k}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} \binom{b_{i,j}+k}{r_i+k} f_j^{|r|_i} \times [f_{i,j}]^{b_{i,j}-r_i}[f_{i,j+1}]^{r_i+k}\left(\prod_{t=j+2}^{i}[f_{i,t}]^{a_{i,t}}\right)H_{i+1,n}=0, \tag{2.9}$$

where $b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n}(a_{s,j+1} - a_{s,j})$, $B_{j,i+1} = \prod_{s=i+1}^{n}\binom{a_{s,j}}{r_s}$, $|r|_i = \sum_{s=i}^{n} r_s$, *and the summand is* 0 *whenever* $|r|_i < 0$, *hold in* $V(\lambda)$.

The proof of Theorem 2.2 will be given in the next section. In the rest of the section, assuming that Theorem 2.2 is proved, we will determine the Gröbner–Shirshov pair for the irreducible representation $V(\lambda)$ of sl_{n+1} with highest weight λ.

Let $\mathcal{T}_\lambda$ be the subset of $\mathcal{A}_X$ consisting of the elements from (2.8) and (2.9),

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{j+1}=0}^{a_{j+1,j}} \frac{b_{j,j}!}{(b_{j,j}+|r|_{j+1})!} B_{j,j+1} f_j^{b_{j,j}+|r|_{j+1}} H_{j+1,n},$$

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{i+1}=0}^{a_{i+1,j}} \sum_{r_i=-k}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} \binom{b_{i,j}+k}{r_i+k} f_j^{|r|_i} \times [f_{i,j}]^{b_{i,j}-r_i}[f_{i,j+1}]^{r_i+k}\left(\prod_{t=j+2}^{i}[f_{i,t}]^{a_{i,t}}\right)H_{i+1,n}.$$

Note that the maximal monomials in the relations (2.8) are of the form

$$f_j^{b_{j,j}} \prod_{s=j+1}^{n} \prod_{t=1}^{s} f_{s,t}^{a_{s,t}}$$

with $r_n = \cdots = r_{j+1} = 0$. Similarly, the maximal monomials in the relations (2.9) are of the form

$$f_{i,j}^{b_{i,j}} f_{i,j+1}^{k} \prod_{t=j+2}^{i} f_{i,t}^{a_{i,t}} \prod_{s=i+1}^{n} \prod_{t=1}^{s} f_{s,t}^{a_{s,t}}$$

with $r_n = \cdots = r_{i+1} = r_i = 0$.

Our observation yields the first half of the following proposition.

PROPOSITION 2.3. (a) *The set* $G(\lambda)$ *of* $(\mathcal{S}_-, \mathcal{T}_\lambda)$*-standard monomials is given by*

$$G(\lambda) = \left\{ \prod_{i=1}^{n} \prod_{j=1}^{i} f_{i,j}^{a_{i,j}} \mid 0 \le a_{i,j} \le b_{i,j} - 1 \right\}, \tag{2.10}$$

where $b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n}(a_{s,j+1} - a_{s,j})$. *Note that, since* $b_{i,j}$ *involves those* $a_{s,t}$*'s with* $s \ge i+1$ *only, the* $a_{i,j}$*'s can be determined recursively.*

(b) *The set* $G(\lambda)$ *is in* 1–1 *correspondence with the set of all semistandard Young tableaux of shape* λ.

Proof. We need only to prove part (b). Let $Y^\lambda = \{(i, j) \mid 1 \le i \le n,\ 1 \le j \le \lambda_i\}$ be the Young diagram corresponding to λ. (Note that $\lambda_i = m_i + m_{i+1} + \cdots + m_n$.) A *semistandard Young tableau of shape* λ is a function τ from the set Y^λ into the set $\{1, 2, \ldots, n, n+1\}$ such that

$$\tau(i, j) \le \tau(i, j+1) \qquad \text{and} \qquad \tau(i, j) < \tau(i+1, j) \qquad \text{for all } i \text{ and } j.$$

As usual, we can present a semistandard Young tableau by an array of colored boxes. The following are examples of semistandard Young tableaux of shape $2\Lambda_1$, $3\Lambda_2$, and $\Lambda_1 + 2\Lambda_3$, respectively.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} \end{array}$$

Let $\mathcal{Y}(\lambda)$ denote the set of all semistandard tableaux of shape λ. We define a map $\Psi: G(\lambda) \to \mathcal{Y}(\lambda)$ as follows:

(a) Let $\Psi(1)$ be the semistandard Young tableau τ^λ defined by $\tau^\lambda(i, j) = i$ for all i and j.

(b) Let $w = \prod_{i=1}^n \prod_{j=1}^i f_{i,j}^{a_{i,j}}$ be an $(\mathcal{S}_-, \mathcal{T}_\lambda)$-standard monomial in $G(\lambda)$. We define $\Psi(w)$ to be the semistandard Young tableau τ obtained from τ^λ by applying the words $f_{i,j}$ successively (as a left U_--action) in the following way:

The word $f_{i,j}$ *changes the rightmost occurrence of the box* $\boxed{j}$ *in the jth row of* τ^λ *to the box* $\boxed{i+1}$.

For example, for the word $w = f_1 f_2^2 f_{3,1} f_3^2$ in $G(\Lambda_1 + \Lambda_2 + 2\Lambda_3)$, $\Psi(w)$ is the semistandard Young tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2} \end{array} \quad .$$

It is now straightforward to verify that Ψ is a bijection between $G(\lambda)$ and $\mathcal{Y}(\lambda)$. ∎

By Proposition 1.4, $G(\lambda)$ is a spanning set of $V(\lambda)$, and since $\dim V(\lambda) = \#\mathcal{Y}(\lambda)$ (see, for example, [18]), it is actually a linear basis of $V(\lambda)$. Therefore, we conclude:

THEOREM 2.4. *The pair $(\mathcal{S}_-, \mathcal{T}_\lambda)$ is a Gröbner–Shirshov pair for the irreducible sl_{n+1}-module $V(\lambda)$ with highest weight λ, and the set $G(\lambda)$ is a monomial basis for $V(\lambda)$.*

3. THE RELATIONS IN $V(\lambda)$

In this section, we will derive sufficiently many relations in $V(\lambda)$ in a series of lemmas so that we would get a proof of Theorem 2.2. Recall that we say that a *relation* $R = 0$ *holds in* U_- whenever R belongs to the two-sided ideal of $\mathcal{A}_F$ generated by S_-, and that *a relation* $R = 0$ *holds in* $V(\lambda)$ whenever R is contained in the left ideal of U_- generated by T_λ.

We start with some relations in U_- which will play an important role in deriving the other relations in $V(\lambda)$.

LEMMA 3.1. *The following relations hold in U_-:*

$$[f_{ij}][f_{j-1,k}]^m = m[f_{j-1,k}]^{m-1}[f_{ik}] + [f_{j-1,k}]^m[f_{ij}] \qquad (m \geq 1). \tag{3.1}$$

Proof. If $m = 1$, then there is nothing to prove. Assume that the relations in (3.1) hold for some fixed m. Since $[f_{ik}][f_{j-1,k}] = [f_{j-1,k}][f_{ik}]$ in U_-, Proposition 2.1 yields

$$\begin{aligned}[f_{ij}][f_{j-1,k}]^{m+1} &= m[f_{j-1,k}]^{m-1}[f_{ik}][f_{j-1,k}] + [f_{j-1,k}]^m[f_{ij}][f_{j-1,k}] \\ &= m[f_{j-1,k}]^m[f_{ik}] + [f_{j-1,k}]^m([f_{ik}] + [f_{j-1,k}][f_{ij}]) \\ &= (m+1)[f_{j-1,k}]^m[f_{ik}] + [f_{j-1,k}]^{m+1}[f_{ij}],\end{aligned}$$

as desired. ∎

For $i, j, k \geq 0$ and $a_{s,t}, r_s \in \mathbb{Z}_{\geq 0}$, we define

$$F_{i+1,k} = \prod_{s=i+1}^{k} \left(\prod_{t=1}^{i-1} [f_{s,t}]^{a_{s,t}} \right) \left(\prod_{t=i+2}^{s} [f_{s,t}]^{a_{s,t}} \right),$$

$$G_{i+1,k} = \prod_{s=i+1}^{k} \left(\prod_{t=1}^{i-1} [f_{s,t}]^{a_{s,t}} \right) \left(\prod_{t=i+1}^{s} [f_{s,t}]^{a_{s,t}} \right),$$

$$B_{j,i} = \prod_{s=i}^{n} \binom{a_{s,j}}{r_s}, \qquad |r|_i = \sum_{s=i}^{n} r_s,$$

$$b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n} (a_{s,j+1} - a_{s,j}).$$

Then we can derive the following set of relations in $V(\lambda)$:

LEMMA 3.2. *In $V(\lambda)$, we have*

$$f_i^{c_i} \prod_{s=i+1}^{n} \left(\prod_{t=1}^{i-1} [f_{s,t}]^{a_{s,t}} \right) \left(\prod_{t=i+1}^{s} [f_{s,t}]^{a_{s,t}} \right) = f_i^{c_i} G_{i+1,n} = 0, \tag{3.2}$$

where $a_{s,t} \in \mathbb{Z}_{\geq 0}$ and $c_i = m_i + 1 + \sum_{s=i+1}^{n} a_{s,i+1}$.

Proof. Note that $F_{i+1,n}$ involves only $[f_{s,t}]$ such that $(s,t) > (i,i)$ and $t \neq i+1$. Hence, by Proposition 2.1, $F_{i+1,n}$ commutes with f_i. Since $f_i^{m_i+1}$ belongs to T_λ, we have

$$f_i^{m_i+1} F_{i+1,n} = 0.$$

Thus the relation (3.2) holds when $a_{s,i+1} = 0$ for all $i+1 \leq s \leq n$. Assume that the relation (3.2) holds when $a_{s,i+1} = 0$ for $i+1 \leq s \leq k$ and $a_{s,i+1} \in \mathbb{Z}_{\geq 0}$ are arbitrary for $k+1 \leq s \leq n$ with some fixed k. Then we have

$$f_i^{c_i} F_{i+1,k} G_{k+1,n} = 0.$$

Furthermore, assume that, for some fixed $a_{k,i+1} = l > 0$, we have

$$f_i^{c_i+l} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i+1}]^l \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n} = 0. \tag{3.3}$$

Multiplying by $[f_{k,i+1}]$ from the left and using Lemma 3.1, we obtain

$$\begin{aligned} 0 &= [f_{k,i+1}] f_i^{c_i+l} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i+1}]^l \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n} \\ &= (c_i+l) f_i^{c_i+l-1} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i}][f_{k,i+1}]^l \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n} \\ &\quad + f_i^{c_i+l} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i+1}]^{l+1} \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n}. \end{aligned}$$

It follows that

$$\begin{aligned} &f_i^{c_i+l-1} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i}][f_{k,i+1}]^l \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n} \\ &\quad = -\frac{1}{c_i+l} f_i^{c_i+l} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i+1}]^{l+1} \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n}. \end{aligned} \tag{3.4}$$

Now, by multiplying the left-hand side of the relation (3.3) by $[f_{k,i}]$ and using the relation (3.4) obtained in the above, we get

$$\begin{aligned}
0 &= [f_{k,i}] f_i^{c_i+l} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i+1}]^l \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n} \\
&= f_i^{c_i+l} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i}][f_{k,i+1}]^l \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n} \\
&= -\frac{1}{c_i+l} f_i^{c_i+l+1} F_{i+1,k-1} \left(\prod_{t=1}^{i-1} [f_{k,t}]^{a_{k,t}} \right) [f_{k,i+1}]^{l+1} \left(\prod_{t=i+2}^{k} [f_{k,t}]^{a_{k,t}} \right) G_{k+1,n},
\end{aligned}$$

which is the relation (3.3) for $a_{k,i+1} = l+1$. Hence, by induction, we obtain the relation (3.2) when $a_{s,i+1} = 0$ for $i+1 \le s \le k-1$ and $a_{s,i+1}$ are arbitrary for $k \le s \le n$. By applying the induction once more, we obtain the desired relations. ∎

We now derive the relations in (2.8).

LEMMA 3.3. *In $V(\lambda)$, we have*

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{j+1}=0}^{a_{j+1,j}} \frac{b_{j,j}!}{(b_{j,j}+|r|_{j+1})!} B_{j,j+1} f_j^{b_{j,j}+|r|_{j+1}} H_{j+1,n} = 0. \tag{3.5}$$

Proof. If $a_{s,j} = 0$ for $j+1 \le s \le n$, then the above relations are just the relations proved in Lemma 3.2. Assume that the relations hold when $a_{s,j} = 0$ for $j+1 \le s \le k$ and $a_{s,j}$ are arbitrary for $k+1 \le s \le n$ with k fixed. Then we have

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{k+1}=0}^{a_{k+1,j}} \frac{b_{j,j}!}{(b_{j,j}+|r|_{k+1})!} B_{j,k+1} f_j^{b_{j,j}+|r|_{k+1}} G_{j+1,k} H_{k+1,n} = 0.$$

Furthermore, for some fixed $a_{k,j} = l > 0$, suppose that we have the relations

$$\begin{aligned}
&\sum_{r_n,\dots,r_{k+1}} \sum_{r_k=0}^{l} \frac{b_{j,j}! B_{j,k}}{(b_{j,j}+|r|_k)!} f_j^{b_{j,j}+|r|_k} G_{j+1,k-1} \left(\prod_{t=1}^{j-1} [f_{k,t}]^{a_{k,t}} \right) \\
&\quad \times [f_{k,j}]^{l-r_k} [f_{k,j+1}]^{a_{k,j+1}+r_k} \left(\prod_{t=j+2}^{k} [f_{k,t}]^{a_{k,t}} \right) H_{k+1,n} = 0.
\end{aligned}$$

Multiplying by $[f_{k,j+1}]$ from the left, we get

$$\begin{aligned}
0 = &\sum_{r_n,\dots,r_{k+1}} \sum_{r_k=0}^{l} \frac{b_{j,j}! B_{j,k}}{(b_{j,j}+|r|_k)!} [f_{k,j+1}] f_j^{b_{j,j}+|r|_k} G_{j+1,k-1} \left(\prod_{t=1}^{j-1} [f_{k,t}]^{a_{k,t}} \right) \\
&\quad \times [f_{k,j}]^{l-r_k} [f_{k,j+1}]^{a_{k,j+1}+r_k} \left(\prod_{t=j+2}^{k} [f_{k,t}]^{a_{k,t}} \right) H_{k+1,n}
\end{aligned}$$

$$= \sum_{r_n, \dots, r_{k+1}} \sum_{r_k=0}^{l} \frac{b_{j,j}! B_{j,k}}{(b_{j,j} + |r|_k - 1)!} f_j^{b_{j,j}+|r|_k-1} G_{j+1,k-1}\left(\prod_{t=1}^{j-1} [f_{k,t}]^{a_{k,t}}\right)$$

$$\times [f_{k,j}]^{l+1-r_k} [f_{k,j+1}]^{a_{k,j+1}+r_k} \left(\prod_{t=j+2}^{k} [f_{k,t}]^{a_{k,t}}\right) H_{k+1,n}$$

$$+ \sum_{r_n, \dots, r_{k+1}} \sum_{r_k=0}^{l} \frac{b_{j,j}! B_{j,k}}{(b_{j,j} + |r|_k)!} f_j^{b_{j,j}+|r|_k} G_{j+1,k-1}\left(\prod_{t=1}^{j-1} [f_{k,t}]^{a_{k,t}}\right)$$

$$\times [f_{k,j}]^{l-r_k} [f_{k,j+1}]^{a_{k,j+1}+r_k+1} \left(\prod_{t=j+2}^{k} [f_{k,t}]^{a_{k,t}}\right) H_{k+1,n}$$

$$= \sum_{r_n, \dots, r_{k+1}} \sum_{r_k=0}^{l} \frac{b_{j,j}!(B_{j,k} + B'_{j,k})}{(b_{j,j} + |r|_k - 1)!} f_j^{b_{j,j}+|r|_k-1} G_{j+1,k-1}\left(\prod_{t=1}^{j-1} [f_{k,t}]^{a_{k,t}}\right)$$

$$\times \quad [f_{k,j}]^{l+1-r_k} [f_{k,j+1}]^{a_{k,j+1}+r_k} \left(\prod_{t=j+2}^{k} [f_{k,t}]^{a_{k,t}}\right) H_{k+1,n},$$

where

$$B_{j,k} = B_{j,k+1}\binom{l}{r_k}, B'_{j,k} = B_{j,k+1}\binom{l}{r_k - 1} \qquad \text{for } 1 \le r_k \le l,$$

$$B_{j,k} = 0 \text{ if } r_k = l+1, \qquad \text{and} \qquad B'_{j,k} = 0 \text{ if } r_k = 0.$$

Dividing out by the leading coefficient $b_{j,j}$, we get the relation (3.5) for $a_{k,j} = l+1$. Using the induction twice as in Lemma 3.2, we obtain the desired relations. ∎

Finally, we can derive the relations in (2.9), which completes the proof of Theorem 2.2.

LEMMA 3.4. *The relations*

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{i+1}=0}^{a_{i+1,j}} \sum_{r_i=-k}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} \binom{b_{i,j}+k}{r_i+k} f_j^{|r|_i}$$

$$\times [f_{i,j}]^{b_{i,j}-r_i} [f_{i,j+1}]^{r_i+k} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{a_{i,t}}\right) H_{i+1,n} = 0, \qquad (3.6)$$

where the summand is 0 *whenever* $|r|_i < 0$, *hold in* $V(\lambda)$.

Proof. In Lemma 3.3, set $a_{i,j} = b_{i,j} = m_j + 1 + \sum_{s=i+1}^{n}(a_{s,j+1} - a_{s,j})$, $a_{i,j+1} = 0$, and $a_{s,j+1} = a_{s,j} = 0$ for $j+1 \le s \le i-1$. Then $b_{j,j} = 0$ and we have

$$\sum_{r_n=0}^{a_{n,j}} \cdots \sum_{r_{i+1}=0}^{a_{i+1,j}} \sum_{r_i=0}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} \binom{b_{i,j}}{r_i} f_j^{|r|_i} \times [f_{i,j}]^{b_{i,j}-r_i} [f_{i,j+1}]^{r_i} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{a_{i,t}} \right) H_{i+1,n} = 0,$$

which is just a scalar multiple of the relation (3.6) for $k = 0$. Assume that the relations in (3.6) hold for some fixed k. Multiplying by $[f_{i,j+1}]$ from the left, we get

$$\begin{aligned}
0 &= \sum_{r_n,\ldots,r_{i+1}} \sum_{r_i=-k}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} \binom{a_{i,j}+k}{r_i+k} [f_{i,j+1}] f_j^{|r|_i} \\
&\quad \times [f_{i,j}]^{b_{i,j}-r_i} [f_{i,j+1}]^{r_i+k} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{a_{i,t}} \right) H_{i+1,n} \\
&= \sum_{r_n,\ldots,r_{i+1}} \sum_{r_i=-k}^{b_{i,j}} \frac{B_{j,i+1}}{(|r|_i-1)!} \binom{a_{i,j}+k}{r_i+k} f_j^{|r|_i-1} \\
&\quad \times [f_{i,j}]^{b_{i,j}-r_i+1} [f_{i,j+1}]^{r_i+k} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{a_{i,t}} \right) H_{i+1,n} \\
&\quad + \sum_{r_n,\ldots,r_{i+1}} \sum_{r_i=-k}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} \binom{a_{i,j}+k}{r_i+k} f_j^{|r|_i} \\
&\quad \times [f_{i,j}]^{b_{i,j}-r_i} [f_{i,j+1}]^{r_i+k+1} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{a_{i,t}} \right) H_{i+1,n} \\
&= \sum_{r_n,\ldots,r_{i+1}} \sum_{r_i=-(k+1)}^{b_{i,j}} \frac{B_{j,i+1}}{|r|_i!} B'' f_j^{|r|_i} \\
&\quad \times [f_{i,j}]^{b_{i,j}-r_i} [f_{i,j+1}]^{r_i+k+1} \left(\prod_{t=j+2}^{i} [f_{i,t}]^{a_{i,t}} \right) H_{i+1,n},
\end{aligned}$$

where

$$B'' = \binom{a_{i,j}+k}{r_i+k+1} + \binom{a_{i,j}+k}{r_i+k} = \binom{a_{i,j}+k+1}{r_i+k+1} \qquad \text{for } r_i \geq -k$$

and

$$B'' = 1 \qquad \text{if } r_i = -(k+1).$$

By induction, we obtain the desired relations. ∎

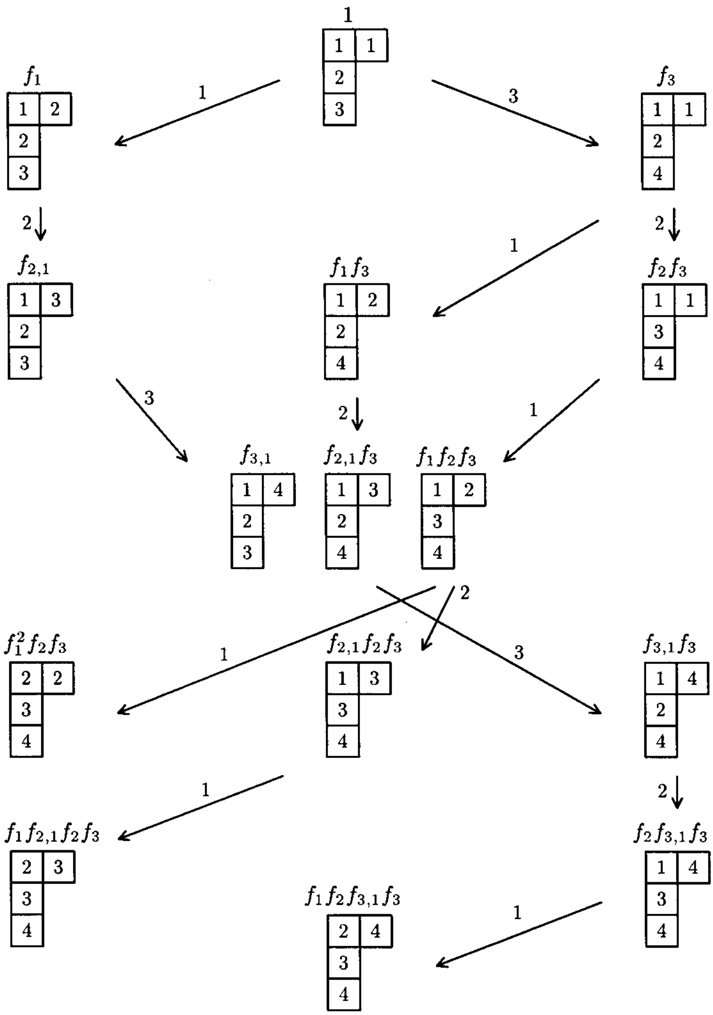

FIG. 1. The Gröbner–Shirshov graph $G(\Lambda_1 + \Lambda_3)$.

4. GRÖBNER–SHIRSHOV GRAPH

Let $G(\lambda)$ be the monomial basis of the irreducible sl_{n+1}-module $V(\lambda)$ consisting of $(\mathscr{S}_-, \mathscr{T}_\lambda)$-standard monomials. We define a colored oriented graph structure on the set $G(\lambda)$ (and hence on the set of semistandard Young tableaux of shape λ) as follows: for each $i = 1, 2, \ldots, n$, we define $w \xrightarrow{i} w'$ if and only if $w' = f_i w$.

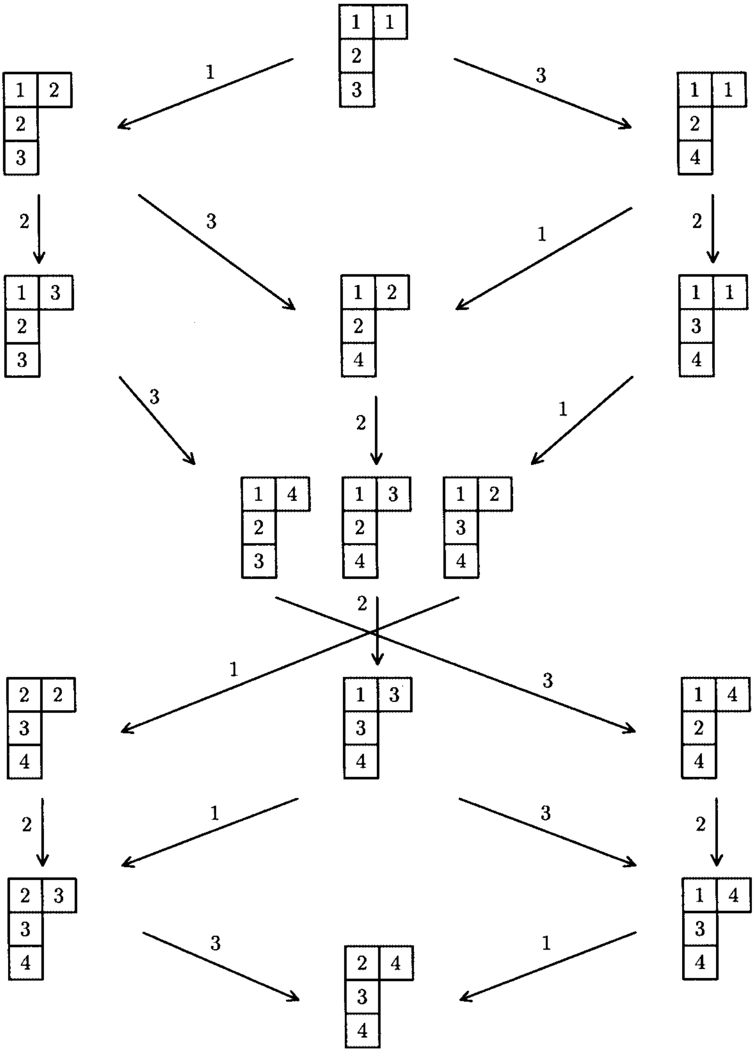

FIG. 2. The crystal graph $B(\Lambda_1 + \Lambda_3)$.

The resulting graph will be called the *Gröbner–Shirshov graph* for the irreducible sl_{n+1}-module $V(\lambda)$ (with respect to the monomial order $\prec$). However, the Gröbner–Shirshov graph $G(\lambda)$ is usually different from the *crystal graph* introduced by Kashiwara [17, 18]. In most cases, the crystal graph $B(\lambda)$ has a vertex receiving more than one arrow, whereas the Gröbner–Shirshov graph $G(\lambda)$ (with respect to any monomial order) cannot have such a vertex. It would be an interesting problem to investigate the behavior of a Gröbner–Shirshov pair and a Gröbner–Shirshov graph with respect to various monomial orders.

In Fig. 1 we give an example of the Gröbner–Shirshov graph $G(\lambda)$ of the irreducible sl_4-module $V(\lambda)$ with highest weight $\lambda = \Lambda_1 + \Lambda_3 = (2, 1, 1)$, where Λ_i are the fundamental weights. We draw the graph in such a way that it may reveal the weight structure of $V(\lambda)$. If we avoid the crossings in drawing the Gröbner–Shirshov graph, we will always obtain a tree as one can see easily from the definition. For comparison, we also give the crystal graph $B(\lambda)$ of $V(\lambda)$ in Fig. 2.

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