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Hecke algebras, Specht modules and Gröbner–Shirshov bases

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Abstract

In this paper, we study the structure of Specht modules over Hecke algebras using the Gröbner–Shirshov basis theory for the representations of associative algebras. The Gröbner–Shirshov basis theory enables us to construct Specht modules in terms of generators and relations. Given a Specht module S_q^λ, we determine the Gröbner–Shirshov pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$ and the monomial basis $G(\lambda)$ consisting of standard monomials. We show that the monomials in $G(\lambda)$ can be parameterized by the cozy tableaux. Using the division algorithm together with the monomial basis $G(\lambda)$, we obtain a recursive algorithm of computing the Gram matrices. We discuss its applications to several interesting examples including Temperley–Lieb algebras. © 2002 Elsevier Science (USA). All rights reserved.

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0. Introduction

The purpose of this paper is to study the structure of Specht modules over Hecke algebras of type A using the *Gröbner–Shirshov basis theory* for representations of associative algebras. The Gröbner basis theory has originated from Buchberger's algorithm of solving the reduction problem for commutative algebras [3]. In [1], it was generalized to associative algebras through the *Diamond Lemma*, and during the past three decades, a wide variety of interesting and significant developments has been made both in pure and applied algebra in connection with Gröbner basis theory.

On the other hand, in [11], Shirshov developed a parallel theory for Lie algebras by proving the *Composition Lemma*, and in [2], Bokut showed that Shirshov's method works for associative algebras as well. For this reason, Shirshov's theory for Lie algebras and their universal enveloping algebras is called the *Gröbner–Shirshov basis theory*.

The next natural step is to develop the Gröbner–Shirshov basis theory for representations. For commutative algebras, there is no difference between the Gröbner basis theory for algebras and the one for their representations because the two-sided ideals and the one-sided ideals coincide. But for general associative algebras, we need a generalized version of Shirshov's Composition Lemma that combines both two-sided ideals and one-sided ideals.

In [8], Kang and Lee developed the Gröbner–Shirshov basis theory for the representations of associative algebras by introducing the notion of *Gröbner–Shirshov pair*. More precisely, let $\mathcal{A}$ be a free associative algebra and let (S,T) be a pair of subsets of monic elements of $\mathcal{A}$. Let J be the two-sided ideal of $\mathcal{A}$ generated by S and $A=\mathcal{A}/J$ be the quotient algebra. We denote by I the right ideal of A generated by (the image of) T. Then the right A-module $M=A/I$ is said to be *defined by the pair* (S,T). The pair (S,T) is called a Gröbner–Shirshov pair for M if it is closed under composition. In this case, the set of (S,T)-*standard monomials* forms a linear basis of M.

In this paper, using the Gröbner–Shirshov basis theory, we construct the Specht modules over Hecke algebras in terms of generators and relations, and determine the Gröbner–Shirshov pairs and monomial bases for the Specht modules. The Specht modules are canonical indecomposable modules over Hecke algebras and are labeled by partitions. Our approach can be explained as follows.

Fix a positive integer n, let λ be a partition of n, and let S_q^λ be the Specht module over the Hecke algebra $\mathcal{H}_n(q)$ corresponding to λ. We denote by t^λ the unique standard tableau of shape λ such that $t^\lambda(i,j+1)=t^\lambda(i,j)+1$ for all nodes (i,j), and define

$$x_\lambda=\sum_{w\in W_\lambda} T_w,$$

where W_λ is the row-stabilizer of t^λ. We first construct an $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ defined by the pair (R_q, R_q^λ), where R_q is the set of defining relations for the Hecke algebra $\mathcal{H}_n(q)$ and R_q^λ is the set of annihilating relations of x_λ. Then we show that there is a surjective homomorphism $\Psi : \widehat{S}_q^\lambda \to S_q^\lambda$. Taking the composition of relations in R_q and R_q^λ and their extensions, we derive sufficiently many relations for $\widehat{S}_q^\lambda$ to obtain a pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$, and determine the set $G(\lambda)$ of $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$-standard monomials. Now we show that the set $G(\lambda)$ is in one-to-one correspondence with the set of *cozy tableaux* of shape λ. Since the number of cozy tableaux of shape λ is the same as the dimension of the Specht module S_q^λ, we conclude that the $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ is isomorphic to the Specht module S_q^λ, the pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$ is the Gröbner–Shirshov pair for $\widehat{S}_q^\lambda$, and the set $G(\lambda)$ forms a linear basis of $\widehat{S}_q^\lambda$. Actually, the monomial basis $G(\lambda)$ is mapped onto the *Murphy basis* of S_q^λ under the isomorphism Ψ (cf. [10]).

The Gröbner–Shirshov basis theory can be applied to find a recursive algorithm of computing the Gram matrices of the Specht modules. The *Gram matrix* Γ_λ of the Specht module S_q^λ is the matrix of the canonical bilinear form $B_\lambda : S_q^\lambda \times S_q^\lambda \to \mathbb{F}$ induced by the bilinear map

$$\mathcal{H}_n(q) \times \mathcal{H}_n(q) \to S_q^\lambda \quad \text{defined by} \quad (u, v) \mapsto uv^* x_\lambda.$$

If q is not a root of unity, the Specht modules are irreducible and the Gram matrices are nonsingular. If q is an eth root of unity, then the Specht modules are no longer irreducible and the irreducible modules arise as the simple quotients of the Specht modules corresponding to the *e-regular partitions*. However, in general, the dimensions of irreducible modules are not known explicitly. Since the rank of the Gram matrix Γ_λ is equal to the dimension of the irreducible module D_q^λ corresponding to an e-regular partition λ, it is an important problem to determine the entries and the rank of the Gram matrix Γ_λ.

We briefly explain our algorithm of computing the Gram matrices. Using the Gröbner–Shirshov pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$, we can compute $B_\lambda(u, 1)$ for all $u \in G(\lambda)$. Let $u, v \in G(\lambda)$ and assume that $B_\lambda(u, w)$ can be computed for all $u, w \in G(\lambda)$ with $l(w) < l(v)$. Observe that every $v \in G(\lambda)$ can be written as $v = v' T_i$ for some $v' \in G(\lambda)$ with $l(v') = l(v) - 1$. Then we have

$$B_\lambda(u, v) = B_\lambda(u, v' T_i) = B_\lambda(u T_i, v').$$

By the *division algorithm* given in Lemma 1.4, uT_i can be expressed as a linear combination of the elements in $G(\lambda)$. Hence, by induction, we can compute $B_\lambda(u, v) = B_\lambda(uT_i, v')$ for all $u, v \in G(\lambda)$.

At the end of this paper, we discuss the application of our algorithm to several interesting examples. Furthermore, viewing the Temperley–Lieb algebras as the quotients of Hecke algebras, we can determine the Gröbner–Shirshov pairs and the monomial bases for the Specht modules over Temperley–Lieb algebras.

Therefore, as in the case of Hecke algebras, the division algorithm in Lemma 1.4 gives a recursive algorithm of computing the Gram matrices of Specht modules over Temperley–Lieb algebras.

1. Gröbner–Shirshov pair

First, we briefly recall the Gröbner–Shirshov basis theory for the representations of associative algebras which was developed in [8,9]. In this paper, we will deal with right modules.

Let X be a set and let X^* be the free monoid of associative monomials on X. We denote the empty monomial by 1 and the *length* of a monomial u by $l(u)$. Thus we have $l(1) = 0$.

Definition 1.1. A well-ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$.

Example 1.2. Let $X = \{x_1, x_2, \ldots\}$ be the set of alphabets and let

$$u = x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad v = x_{j_1} x_{j_2} \cdots x_{j_l} \in X^*.$$

(a) We define $u \prec_{\text{deg-lex}} v$ if and only if $k < l$ or $k = l$ and $i_r < j_r$ for the first r such that $i_r \neq j_r$; it is a monomial order on X^* called the *degree lexicographic order*.
(b) We define $u \prec_{\text{deg-rlex}} v$ if and only if $k < l$ or $k = l$ and $i_r > j_r$ for the last r such that $i_r \neq j_r$; it is a monomial order on X^* called the *degree reverse lexicographic order*.

Fix a monomial order $\prec$ on X^* and let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\bar{p}$ the maximal monomial appearing in p under the ordering $\prec$. Thus $p = \alpha \bar{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \prec \bar{p}$. If $\alpha = 1$, p is said to be *monic*.

Let (S, T) be a pair of subsets of monic elements of $\mathcal{A}_X$, let J be the two-sided ideal of $\mathcal{A}_X$ generated by S, and let I be the right ideal of the algebra $A = \mathcal{A}_X / J$ generated by (the image of) T. Then we say that the algebra $A = \mathcal{A}_X / J$ is *defined by* S and that the right A-module $M = A/I$ is *defined by the pair* (S, T). The images of $p \in \mathcal{A}_X$ in A and in M under the canonical quotient maps will also be denoted by p.

Definition 1.3. Given a pair (S, T) of subsets of monic elements of $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be (S, T)*-standard* if $u \neq a\bar{s}b$ and $u \neq \bar{t}c$ for any $s \in S$, $t \in T$ and $a, b, c \in X^*$. Otherwise, the monomial u is said to be (S, T)*-reducible*. If $T = \emptyset$, we will simply say that u is S*-standard* or S*-reducible*.

Lemma 1.4 [8,9]. *Every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j t_j c_j + \sum \gamma_k u_k, \tag{1.1}$$

where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$, $a_i, b_i, c_j, u_k \in X^*$, $s_i \in S$, $t_j \in T$, $a_i \overline{s_i} b_i \preceq \bar{p}$, $\overline{t_j} c_j \preceq \bar{p}$, $u_k \preceq \bar{p}$ and u_k are (S,T)-standard.

Proof. We may assume $p \in X^*$. If p is (S,T)-standard, then there is nothing to prove. If $p = a\bar{s}b$, $s \in S$, then $p = asb + \sum \eta_i v_i$ with $v_i \prec p$. If $p = \bar{t}c$, $t \in T$, then $p = tc + \sum \eta'_i v'_i$ with $v'_i \prec p$. We now apply the induction to complete the proof. □

Remark. The proof of the above lemma actually gives an algorithm of writing an element p of $\mathcal{A}_X$ in the form (1.1). It may be considered as a *division algorithm*.

The term $\sum \gamma_k u_k$ in the expression (1.1) is called a *normal form* of p with respect to the pair (S,T) (and with respect to the monomial order $\prec$). In general, a normal form is not unique.

Example 1.5. Let $X = \{x_1, x_2, x_3\}$ and choose the monomial order $\prec_{\text{deg-lex}}$. If we set

$$S = \{x_1 x_2^2 - x_3, x_1 x_3 + x_3\} \quad \text{and} \quad T = \{x_3^2 + 1\},$$

then the element $x_1^2 x_2^2 x_3 x_1$ becomes

$$\begin{aligned} x_1^2 x_2^2 x_3 x_1 &= x_1(x_1 x_2^2 - x_3) x_3 x_1 + x_1 x_3^2 x_1 \\ &= x_1(x_1 x_2^2 - x_3) x_3 x_1 + (x_1 x_3 + x_3) x_3 x_1 - x_3^2 x_1 \\ &= x_1(x_1 x_2^2 - x_3) x_3 x_1 + (x_1 x_3 + x_3) x_3 x_1 - (x_3^2 + 1) x_1 + x_1. \end{aligned}$$

Thus a normal form of $x_1^2 x_2^2 x_3 x_1$ is x_1. □

As an immediate corollary of Lemma 1.4, we obtain the following proposition.

Proposition 1.6 [8,9]. *The set of (S,T)-standard monomials spans the right A-module $M = A/I$ defined by the pair (S,T).*

Definition 1.7. A pair (S,T) of subsets of monic elements of $\mathcal{A}_X$ is a *Gröbner–Shirshov pair* if the set of (S,T)-standard monomials forms a linear basis of the right A-module $M = A/I$ defined by the pair (S,T). In this case, we say that (S,T) is a *Gröbner–Shirshov pair* for the module M defined by (S,T). If a pair $(S,\emptyset)$ is a Gröbner–Shirshov pair, then we also say that S is a *Gröbner–Shirshov basis* for the algebra $A = \mathcal{A}_X/J$ defined by S.

Let p and q be monic elements of $\mathcal{A}_X$ with leading terms $\bar{p}$ and $\bar{q}$. We define the *composition* of p and q as follows.

Definition 1.8. (a) If there exist a and b in X^* such that $\bar{p}a = b\bar{q} = w$ with $l(\bar{p}) > l(b)$, then the *composition of intersection* is defined to be $(p,q)_w = pa - bq$. Furthermore, if $b = 1$, the composition $(p,q)_w$ is called *left-justified*.

(b) If there exist a and b in X^* such that $b \neq 1$, $p = a\bar{q}b = w$, then the *composition of inclusion* is defined to be $(p,q)_w = p - aqb$.

For $p, q \in \mathcal{A}_X$ and $w \in X^*$, we define a *congruence relation* on $\mathcal{A}_X$ as follows: $p \equiv q \bmod (S,T;w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i + \sum \beta_j t_j c_j$, where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, $t_j \in T$, $a_i \overline{s_i} b_i \prec w$, and $\overline{t_j} c_j \prec w$. When $T = \emptyset$, we simply write $p \equiv q \bmod (S;w)$.

Definition 1.9. A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if

(i) $(p,q)_w \equiv 0 \bmod (S;w)$ for all $p, q \in S$, $w \in X^*$, whenever the composition $(p,q)_w$ is defined;
(ii) $(p,q)_w \equiv 0 \bmod (S,T;w)$ for all $p, q \in T$, $w \in X^*$, whenever the left-justified composition $(p,q)_w$ is defined;
(iii) $(p,q)_w \equiv 0 \bmod (S,T;w)$ for all $p \in T$, $q \in S$, $w \in X^*$, whenever the composition $(p,q)_w$ is defined.

If $T = \emptyset$, we will simply say that S is closed under composition.

In the following lemma, we recall the main result of [8], which is a generalization of Shirshov's Composition Lemma (for Lie algebras and associative algebras) to the representations of associative algebras.

Lemma 1.10 [8]. *Let (S,T) be a pair of subsets of monic elements in the free associative algebra $\mathcal{A}_X$ generated by X, let $A = \mathcal{A}_X/J$ be the associative algebra defined by S, and let $M = A/I$ be the right A-module defined by (S,T). If (S,T) is closed under composition and the image of $p \in \mathcal{A}_X$ is trivial in M, then the word $\bar{p}$ is (S,T)-reducible.*

As an immediate consequence, we obtain the following theorem.

Theorem 1.11 [9]. *Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$. Then the following conditions are equivalent*:

(a) *(S,T) is a Gröbner–Shirshov pair.*
(b) *(S,T) is closed under composition.*
(c) *For each $p \in \mathcal{A}_X$, the normal form of p is unique.*

2. Specht modules

In this section, we recall Murphy's construction [10] of Specht modules over Hecke algebras of type A. One of the nicest features of Murphy's construction is that it works over any integral domain. We first study our object over the ring $\mathbb{F}[v, v^{-1}]$ with v an indeterminate, and then specialize v to a nonzero scalar $q \in \mathbb{F}^{\times}$.

Let S_n be the symmetric group on n letters and let $\tau_i = (i, i+1)$ be the transposition of i and $i+1$. Then the *Hecke algebra* $\mathcal{H}_n(v)$ *of type* A is defined to be the associative algebra over $\mathbb{F}[v, v^{-1}]$ generated by $X = \{T_1, T_2, \ldots, T_{n-1}\}$ with defining relations

$$R_v\colon \quad \begin{cases} T_i T_j = T_j T_i & \text{for } i > j+1, \\ T_i^2 = (v - l)T_i + v & \text{for } 1 \leqslant i \leqslant n-1, \\ T_{i+1} T_i T_{i+1} = T_i T_{i+1} T_i & \text{for } 1 \leqslant i \leqslant n-2. \end{cases} \tag{2.1}$$

We write $T_{i,j} = T_i T_{i-1} \cdots T_j$ for $i \geqslant j$ and $T^{j,i} = T_j T_{j+1} \cdots T_i$ for $j \leqslant i$ (hence $T_{i,i} = T_i$ and $T^{i,i} = T_i$). We also set $T_{i,i+1} = 1$ $(i \geqslant 0)$ and $T^{i,i-1} = 1$ $(i \geqslant 1)$. We define for $T = T_{i_1} T_{i_2} \cdots T_{i_k} \in \mathcal{H}_n(v)$, $T^* = T_{i_k} T_{i_{k-1}} \cdots T_{i_1}$ and extend $*$ to an anti-automorphism of $\mathcal{H}_n(v)$ by linearity. Note that $T_{i,j}^* = T^{j,i}$. For a reduced expression $w = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_k} \in S_n$, we define $T_w \in \mathcal{H}_n(v)$ to be

$$T_w = T_{i_1} T_{i_2} \cdots T_{i_k}.$$

Then T_w is well-defined and $T_w^* = T_{w^{-1}}$. For any subset $W \subseteq S_n$, we define

$$\iota(W) = \sum_{w \in W} T_w.$$

A *composition* λ of n, denoted by $\lambda \models n$, is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of nonnegative integers whose sum is n. By convention we set $\lambda_0 = 0$. A *partition* λ of n, denoted by $\lambda \vdash n$, is a composition such that $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k)$. For a composition of n, the *diagram* $[\lambda]$ is the set of *nodes* $\{(i, j) \mid 1 \leqslant j \leqslant \lambda_i,\ i = 1, 2, \ldots, n\}$.

Definition 2.1. Suppose that λ is a composition of n.

(a) A λ-*tableau* is a map $t : [\lambda] \to (1, 2, \ldots, n)$. If a λ-tableau t is a bijection, t is said to be *bijective*.
(b) A tableau t is *row-standard* if it is bijective and $t(i, j) < t(i, j+1)$ for all i and j.
(c) A tableau t is *standard* if λ is a partition, t is row-standard and $t(i, j) < t(i+1, j)$ for all i and j.

The diagram corresponding to a tableau t will be denoted by $[t]$. If $m \leqslant n$ and t is a row-standard tableau with n nodes, then the restriction of t to the image set $\{1, 2, \ldots, m\}$ is a row-standard tableau denoted by $t \downarrow m$. The corresponding diagram will be denoted by $[t \downarrow m]$.

Let t^λ be the unique standard λ-tableau such that $t^\lambda(i, j+1) = t^\lambda(i, j) + 1$ for all nodes (i, j). For example, $t^{(5,3,2,1)}$ is the following tableau:

1	2	3	4	5
6	7	8		
9	10			
11				

The symmetric group S_n acts naturally from the right on the set of all bijective λ-tableaux. If t is row-standard, we denote by $d(t)$ the element of S_n for which $t = t^\lambda d(t)$. We denote by W_λ the group of row stabilizers of t^λ.

For compositions λ and μ of n, we write $\lambda \trianglerighteq \mu$ if $\sum_{i=1}^m \lambda_i \geqslant \sum_{i=1}^m \mu_i$ for each m. Let s, t be row-standard λ- and μ-tableaux, respectively. We write $s \trianglerighteq t$ if for each $m \leqslant n$, $[s \downarrow m] \trianglerighteq [t \downarrow m]$. Let λ be a composition of n. By a λ-*pair* we mean a pair (s, t) of row-standard λ-tableaux. A λ-pair is called *standard* if both s and t are standard.

For a composition λ of n and for any λ-pair (s, t), we define

$$x_\lambda = \sum_{w \in W_\lambda} T_w \quad \text{and} \quad x_{st} = T^*_{d(s)} x_\lambda T_{d(t)}. \tag{2.2}$$

Hence $x_{t^\lambda t^\lambda} = x_\lambda$. From now on, whenever the subscript is t^λ, we will abbreviate it to λ. For example, we will write $x_{t^\lambda t^\lambda} = x_{\lambda\lambda} = x_\lambda$ and $x_{t^\lambda s} = x_{\lambda s}$.

For a partition $\lambda \vdash n$, let N^λ (respectively $\overline{N}^\lambda$) be the $\mathbb{F}[v, v^{-1}]$-submodule of $\mathcal{H}_n(v)$ spanned by x_{rs}, where (r, s) runs over all standard μ-pair for a partition $\mu \vdash n$ with $\mu \trianglerighteq \lambda$ (respectively $\mu \triangleright \lambda$). Let $M^\lambda = x_\lambda \mathcal{H}_n(v)$ be the cyclic $\mathcal{H}_n(v)$-module generated by x_λ and set $\overline{M}^\lambda = M^\lambda \cap \overline{N}^\lambda$.

Definition 2.2. The $\mathcal{H}_n(v)$-module $S_v^\lambda = M^\lambda / \overline{M}^\lambda$ is called the *Specht module* over $\mathcal{H}_n(v)$ corresponding to the partition λ.

Proposition 2.3 [10]. *The Specht module S_v^λ is a free $\mathbb{F}[v, v^{-1}]$-module with a basis consisting of the vectors $x_{\lambda s} + \overline{M}^\lambda$, where s runs over all standard λ-tableaux.*

The basis of S_v^λ in the above proposition is called the *Murphy basis* of the Specht module S_v^λ.

3. Some natural relations in S_v^λ

In this section, we will derive some natural relations in S_v^λ, which will serve as the defining relations and will be extended to form a Gröbner–Shirshov pair in the next section. One can observe that the basic relations are the relations in Lemma 3.2 along with the defining relations (2.1) for the Hecke algebra $\mathcal{H}_n(v)$. All the other relations will be derived from these basic relations. This indicates what would be the set of defining relations for the Specht modules, and leads us to the abstract definition of the Specht modules by generators and relations in the next section.

Lemma 3.1. *The relations*

$$T_{a,b}T_{c,d} = T_{c-1,d-1}T_{a,b} \quad (a \geqslant c \geqslant d \geqslant b) \tag{3.1}$$

hold in $\mathcal{H}_n(v)$.

Proof. If $b = a-1$, then (3.1) becomes one of the relations in (2.1), so assume that $b \leqslant a-2$. Suppose further that $a = c = d$. Since T_a commutes with T_k for $b \leqslant k \leqslant a-2$, we have

$$\begin{aligned} T_{a,b}T_a &= T_{a,a-1}T_{a-2,b}T_a = T_{a,a-1}T_aT_{a-2,b} = T_{a-1}T_{a,a-1}T_{a-2,b} \\ &= T_{a-1}T_{a,b}, \end{aligned}$$

which is the desired relation.

If $a \geqslant d \geqslant b$, we use induction on c. If $c = d$ then

$$T_{a,b}T_c = T_{a,c+1}T_{c,b}T_c = T_{a,c+1}T_{c-1}T_{c,b} = T_{c-1}T_{a,b},$$

as desired. If $c > d$, then by induction

$$\begin{aligned} T_{a,b}T_{c,d} &= T_{a,c+1}T_{c,b}T_{c,d} = T_{a,c+1}T_{c-1}T_{c,b}T_{c-1,d} = T_{c-1}T_{a,b}T_{c-1,d} \\ &= T_{c-1}T_{c-2,d-1}T_{a,b} = T_{c-1,d-1}T_{a,b}, \end{aligned}$$

which completes the induction argument. □

For any $i \leqslant j$, we denote by $S_{i,j}$ the subgroup of S_n which permutes only $i, i+1, \ldots, j$ $(1 \leqslant i \leqslant j \leqslant n)$. Considering the coset representatives of $S_{1,n-1}$ and $S_{2,n}$, respectively, we have

$$\iota(S_n) = \iota(S_{1,n-1})\left(\sum_{l=1}^{n} T_{n-1,l}\right) = \left(\sum_{l=0}^{n-1} T_{l,1}\right)\iota(S_{2,n}). \tag{3.2}$$

Fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n. We introduce some notations. For $i = 1, 2, \ldots, k$, we define

$$0_\lambda = 0 \quad \text{and} \quad i_\lambda = \sum_{l=1}^{i} \lambda_l.$$

That is, $\{i_\lambda\}_{i=1}^{k}$ is the sequence of partial sums of the partition λ. For example, if $\lambda = (4,3,1) \vdash 8$, then we have $0_\lambda = 0$, $1_\lambda = 4$, $2_\lambda = 7$, and $3_\lambda = 8$.

With this notation, we define

$$x_{\lambda_i} = \iota(S_{(i-1)_\lambda+1,i_\lambda}). \tag{3.3}$$

Then we have the following statement.

Lemma 3.2. *The following relations hold in* S_v^λ:

(a) *For* $i = 1,\ldots,n-1$ *with* $i \neq l_\lambda$ $(l = 1,\ldots,k-1)$, *we have*

$$x_\lambda(T_i - v) = 0.$$

(b) *For* $i = 1,\ldots,k-1$, *we have*

$$x_\lambda\left(\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l}\right) = 0.$$

Proof. By [10, Lemma 4.1], x_λ has a right factor $T_i + 1$ for $i \neq l_\lambda$. Hence the relation (2.1) implies $x_\lambda(T_i - v) = 0$ for $i \neq l_\lambda$.

Let $\mu = (\lambda_1,\ldots,\lambda_{i-1},\lambda_i+1,\lambda_{i+1}-1,\lambda_{i+2},\ldots,\lambda_k)$. It follows from (3.2) that

$$\begin{aligned} x_{\lambda_i}x_{\lambda_{i+1}}\left(\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l}\right) &= x_{\lambda_{i+1}}x_{\lambda_i}\left(\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l}\right) \\ &= \left(\sum_{l=1}^{\mu_{i+1}+1} T_{(i+1)_\mu-l,i_\mu}\right)x_{\mu_{i+1}}x_{\mu_i} \\ &= \left(\sum_{l=1}^{\mu_{i+1}+1} T_{(i+1)_\mu-l,i_\mu}\right)x_{\mu_i}x_{\mu_{i+1}}. \end{aligned}$$

Thus we have

$$x_\lambda\left(\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l}\right) = \left(\sum_{l=1}^{\mu_{i+1}+1} T_{(i+1)_\mu-l,i_\mu}\right)x_\mu.$$

Note that

$$\left(\sum_{l=1}^{\mu_{i+1}+1} T_{(i+1)_\mu-l,i_\mu}\right)x_\mu = \sum_{l=1}^{\mu_{i+1}+1} \left(T^{i_\mu,(i+1)_\mu-l}\right)^* x_\mu = \sum_\tau x_{\tau,\mu},$$

where each τ is a row-standard μ-tableau. Since $\lambda \lhd \mu$, by [10, Theorem 4.18] and the definition of S_v^λ, we conclude that

$$x_\lambda\left(\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l}\right) = 0. \qquad \square$$

For $s \in \mathbb{Z}_{\geqslant 0}$ and a sequence $\boldsymbol{a} = (a_l)_{l=1}^{j} = (a_1, a_2, \ldots, a_j)$ of positive integers, define

$$s + \boldsymbol{a} = s + (a_1, a_2, \ldots, a_j) = (s + a_1, s + a_2, \ldots, s + a_j).$$

If $\boldsymbol{a} = (a_1, a_2, \ldots, a_j)$ satisfies $1 \leqslant a_l \leqslant N + l$ for some nonnegative integer N, we define

$$\langle \boldsymbol{a} \rangle_N = \langle a_1, a_2, \ldots, a_j \rangle_N = T_{N,a_1} T_{N+1,a_2} \cdots T_{N+j-1,a_j}.$$

Thus we have $\langle s + \boldsymbol{a} \rangle_N = T_{N,s+a_1} T_{N+1,s+a_2} \cdots T_{N+j-1,s+a_j}$.

Lemma 3.3. *For p $(0 \leqslant p \leqslant k-1)$ and j $(1 \leqslant j < \lambda_{p+1})$, let*

$$\boldsymbol{a}_i = (a_l^i) = \left(a_1^i, \ldots, a_{\lambda_i+1}^i\right) \quad (i = 0, 1, \ldots, p-1)$$

and

$$\boldsymbol{b} = (b_l) = (b_1, \ldots, b_j, b_{j+1})$$

be sequences of positive integers such that

$$1 \leqslant a_l^i \leqslant i_\lambda + l, \qquad 1 \leqslant b_l \leqslant p_\lambda + l, \quad \text{and} \quad b_j \geqslant b_{j+1}.$$

Then the relation

$$x_\lambda \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle \boldsymbol{b} \rangle_{p_\lambda} = v x_\lambda \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle \boldsymbol{b}' \rangle_{p_\lambda} \tag{3.4}$$

holds in S_v^λ, where $\boldsymbol{b}' = (b_1, b_2, \ldots, b_{j-1}, b_{j+1}, b_j + 1)$.

Proof. From the first relations in Lemma 3.2, we have $x_\lambda T_{p_\lambda + j} = v x_\lambda$. Since $(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda}) \langle b_1, \ldots, b_{j-1} \rangle_{p_\lambda}$ commutes with $T_{p_\lambda + j}$, we have the following relation in S_v^λ:

$$\begin{aligned} &x_\lambda \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle b_1, b_2, \ldots, b_{j-1}, p_\lambda + j, p_\lambda + j \rangle_{p_\lambda} \\ &\quad = v x_\lambda \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle b_1, b_2, \ldots, b_{j-1}, p_\lambda + j, p_\lambda + j + 1 \rangle_{p_\lambda}. \end{aligned}$$

Thus the relations in (3.4) hold for $b_j = b_{j+1} = p_\lambda + j$.

Assume that for $p_\lambda + j > b_j = b_{j+1}$, we have

$$\begin{aligned} &x_\lambda \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle b_1, b_2, \ldots, b_{j-1}, b_j, b_j \rangle_{p_\lambda} \\ &\quad = v x_\lambda \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle b_1, b_2, \ldots, b_{j-1}, b_j, b_j + 1 \rangle_{p_\lambda}. \end{aligned}$$

Multiplying the above relation by $T_{b_j-1}T_{b_j}$ from the right, the left-hand side yields

$$\begin{aligned}
&x_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,\dots,b_{j-1},b_j,b_j\rangle_{p_\lambda}T_{b_j-1}T_{b_j}\\
&\quad=x_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,\dots,b_{j-1},b_j,b_j+1\rangle_{p_\lambda}T_{b_j-1}T_{b_j,b_j-1}\\
&\quad=x_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,\dots,b_j-1,b_j-1\rangle_{p_\lambda},
\end{aligned}$$

and the right-hand side yields

$$\begin{aligned}
&vx_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,b_2,\dots,b_{j-1},b_j,b_j+1\rangle_{p_\lambda}T_{b_j-1}T_{b_j}\\
&\quad=vx_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,\dots,b_{j-1},b_j-1,b_j\rangle_{p_\lambda}\\
&\quad=vx_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,\dots,b_{j-1},b_j-1,b_j\rangle_{p_\lambda}.
\end{aligned}$$

By induction, we obtain the relations in (3.4) for all $b_j=b_{j+1}$.

Now assume that for $b_j\geqslant b_{j+1}$, we have

$$\begin{aligned}
&x_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,b_2,\dots,b_{j-1},b_j,b_{j+1}\rangle_{p_\lambda}\\
&\quad=vx_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,b_2,\dots,b_{j-1},b_{j+1},b_j+1\rangle_{p_\lambda}.
\end{aligned}$$

Multiplying the relation $T_{b_{j+1}-1}$ from the right, we obtain

$$\begin{aligned}
&x_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,b_2,\dots,b_{j-1},b_j,b_{j+1}-1\rangle_{p_\lambda}\\
&\quad=vx_\lambda\left(\prod_{i=0}^{p-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle b_1,b_2,\dots,b_{j-1},b_{j+1}-1,b_j+1\rangle_{p_\lambda}.
\end{aligned}$$

By induction on b_{j+1}, the relations in (3.4) hold for all $b_j\geqslant b_{j+1}$. □

For any natural number i, let

$$[i]_v = 1 + v + v^2 + \cdots + v^{i-1}, \qquad \{i\}_v = \prod_{j=1}^{i} [j]_v,$$

and $[0]_v = 1$, $\{0\}_v = 1$. For each i $(1 \leqslant i \leqslant k-1)$ and j $(1 \leqslant j \leqslant \lambda_{i+1})$, we define

$$C_{j,i,\lambda} = \left\{ (a_1, a_2, \ldots, a_j) \in \mathbb{Z}_{>0}^{\oplus n} \;\middle|\; j \leqslant a_1 < a_2 < \cdots < a_j \leqslant \lambda_i + j \right\}.$$

Lemma 3.4. *For each i $(1 \leqslant i \leqslant k-1)$ and j $(1 \leqslant j \leqslant \lambda_{i+1})$, the relation*

$$x_\lambda \left(\sum_{\boldsymbol{a} \in C_{j,i,\lambda}} \langle (i-1)_\lambda + \boldsymbol{a} \rangle_{i_\lambda} \right) = 0 \tag{3.5}$$

holds in S_v^λ.

Proof. At first, for each i $(1 \leqslant i \leqslant k-1)$ and j $(1 \leqslant j \leqslant \lambda_{i+1})$, we define

$$C^1_{j,i,\lambda} = C_{j,i,\lambda} \setminus \left\{ (j, j+1, \ldots, 2j-1) \right\},$$
$$C^2_{j,i,\lambda} = C_{j,i,\lambda} \setminus \left\{ (j, j+1, \ldots, 2j-2, a_j) \;\middle|\; 2j-1 \leqslant a_j \leqslant \lambda_i + j \right\},$$
$$C^3_{j,i,\lambda} = \left\{ (a_1, a_2, \ldots, a_j) \in \mathbb{Z}_{>0}^{\oplus n} \;\middle|\; j+1 \leqslant a_1 < a_2 < \cdots < a_j \leqslant \lambda_i + j \right\}.$$

If $j = 1$ for a fixed i, then the relation (3.5) becomes the relation in Lemma 3.2. Assume that for $j-1$ we have the relation (3.5):

$$x_\lambda \left(\sum_{\boldsymbol{a} \in C_{j-1,i,\lambda}} \langle (i-1)_\lambda + \boldsymbol{a} \rangle_{i_\lambda} \right) = 0.$$

Note that

$$x_\lambda \bigl\langle (i-1)_\lambda + (j-1, j, \ldots, 2j-3) \bigr\rangle_{i_\lambda} = -x_\lambda \left(\sum_{\boldsymbol{a} \in C_{j-1,i,\lambda}} \langle (i-1)_\lambda + \boldsymbol{a} \rangle_{i_\lambda} \right). \tag{3.6}$$

Set

$$A = [j-1]_v x_\lambda \bigl\langle (i-1)_\lambda + (j-1, j, \ldots, 2j-3, j-1) \bigr\rangle_{i_\lambda},$$
$$B = -v^{j-1} x_\lambda \bigl\langle (i-1)_\lambda + (j-1, j, \ldots, 2j-3, 2j-2) \bigr\rangle_{i_\lambda},$$

and

$$C_l = -v^{j-1} x_\lambda \bigl\langle (i-1)_\lambda + (j-1, j, \ldots, 2j-3, j-1+l) \bigr\rangle_{i_\lambda}$$

for $1 \leqslant l \leqslant j-2$.

By Lemma 3.3, we have

$$\begin{aligned}&\left(A+B+\sum_{l=1}^{j-2} C_l\right)\\&\quad= v^{j-1}[j-1]_v\, x_\lambda\big\langle(i-1)_\lambda+(j-1,j,\ldots,2j-2)\big\rangle_{i_\lambda}\\&\qquad - v^{j-1}x_\lambda\big\langle(i-1)_\lambda+(j-1,j,\ldots,2j-2)\big\rangle_{i_\lambda}\\&\qquad - v^{j-1}\sum_{l=1}^{j-2} v^{j-1-l}x_\lambda\big\langle(i-1)_\lambda+(j-1,j,\ldots,2j-2)\big\rangle_{i_\lambda}\\&\quad=0.\end{aligned}$$

On the other hand, by Lemma 3.3 and the relation (3.6), we get

$$\begin{aligned}A &= [j-1]_v\, x_\lambda\big\langle(i-1)_\lambda+(j-1,j,\ldots,2j-3,j-1)\big\rangle_{i_\lambda}\\&= -[j-1]_v \sum_{\boldsymbol{a}\in C^1_{j-1,i,\lambda}} x_\lambda\big\langle(i-1)_\lambda+(\boldsymbol{a},j-1)\big\rangle_{i_\lambda}\\&= -v^{j-1}[j-1]_v \sum_{\boldsymbol{a}\in C^1_{j-1,i,\lambda}} x_\lambda\big\langle(i-1)_\lambda+(j-1,1+\boldsymbol{a})\big\rangle_{i_\lambda}\\&= -v^{j-1}[j-1]_v \sum_{\boldsymbol{a}\in C^2_{j-1,i,\lambda}} x_\lambda\big\langle(i-1)_\lambda+(j-1,1+\boldsymbol{a})\big\rangle_{i_\lambda}\\&\quad - v^{j-1}[j-1]_v \sum_{a_{j-1}=2j-2}^{\lambda_i+j-1} x_\lambda\big\langle(i-1)_\lambda\\&\qquad\qquad + (j-1,j,\ldots,2j-3,a_{j-1}+1)\big\rangle_{i_\lambda}.\end{aligned}$$

Applying the relation (3.6) to B, we obtain

$$\begin{aligned}B &= -v^{j-1}x_\lambda\big\langle(i-1)_\lambda+(j-1,j,\ldots,2j-3,j-2)\big\rangle_{i_\lambda}\\&= v^{j-1}\sum_{\boldsymbol{a}\in C^1_{j-1,i,\lambda}} x_\lambda\big\langle(i-1)_\lambda+(\boldsymbol{a},2j-2)\big\rangle_{i_\lambda}.\end{aligned}$$

For $1\leqslant l\leqslant j-2$, Lemma 3.3 and the relation (3.6) yield

$$\begin{aligned}C_l &= -v^{j-1}x_\lambda\big\langle(i-1)_\lambda+(j-1,j,\ldots,2j-3,j-1+l)\big\rangle_{i_\lambda}\\&= v^{j-1}\sum_{\boldsymbol{a}\in C^1_{j-1,i,\lambda}} x_\lambda\big\langle(i-1)_\lambda+(\boldsymbol{a},j-1+l)\big\rangle_{i_\lambda}\\&= v^{j-1}\sum_{\boldsymbol{a}\in C^2_{j-1,i,\lambda}} x_\lambda\big\langle(i-1)_\lambda+(\boldsymbol{a},j-1+l)\big\rangle_{i_\lambda}\end{aligned}$$

$$+ v^{2j-2-l} \sum_{a_{j-1}=2j-2}^{\lambda_i+j-1} x_\lambda \big\langle (i-1)_\lambda + (j-1, j, \dots, 2j-3, a_{j-1}+1) \big\rangle_{i_\lambda}.$$

Therefore, by Lemma 3.3 and the relation (3.6), we obtain

$$\begin{aligned}
0 &= \left(A + B + \sum_{l=1}^{j-2} C_l \right) \\
&= -v^{j-1}[j-1]_v \sum_{\boldsymbol{a} \in C^2_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (j-1, 1+\boldsymbol{a}) \big\rangle_{i_\lambda} \\
&\quad - v^{j-1} \sum_{a_{j-1}=2j-2}^{\lambda_i+j-1} x_\lambda \big\langle (i-1)_\lambda + (j-1, j, \dots, 2j-3, a_{j-1}+1) \big\rangle_{i_\lambda} \\
&\quad + v^{j-1} \sum_{\boldsymbol{a} \in C^1_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (\boldsymbol{a}, 2j-2) \big\rangle_{i_\lambda} \\
&\quad + v^{j-1} \sum_{l=1}^{j-2} \sum_{\boldsymbol{a} \in C^2_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (\boldsymbol{a}, j-1+l) \big\rangle_{i_\lambda} \\
&= -v^{j-1}[j-1]_v \sum_{\boldsymbol{a} \in C^2_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (j-1, 1+\boldsymbol{a}) \big\rangle_{i_\lambda} \\
&\quad + v^{j-1} \sum_{l=2j-2}^{\lambda_i+2} \sum_{\boldsymbol{a} \in C^1_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (\boldsymbol{a}, l) \big\rangle_{i_\lambda} \\
&\quad + v^{j-1} \sum_{l=j}^{2j-3} \sum_{\boldsymbol{a} \in C^2_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (\boldsymbol{a}, l) \big\rangle_{i_\lambda} \\
&= v^{j-1} \sum_{l=j}^{\lambda_i+j} \sum_{\boldsymbol{a} \in C^3_{j-1,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + (\boldsymbol{a}, l) \big\rangle_{i_\lambda} \\
&= v^{j-1}[j]_v \sum_{\boldsymbol{a} \in C_{j,i,\lambda}} x_\lambda \big\langle (i-1)_\lambda + \boldsymbol{a} \big\rangle_{i_\lambda}.
\end{aligned}$$

Since S_v^λ is free over $\mathbb{F}[v, v^{-1}]$, we get the desired relation (3.5) for j, which completes our induction argument. □

Remark. As we mentioned in the beginning of this section, all the relations have been derived from the defining relations (2.1) of Hecke algebra and those in Lemma 3.2. In the next section, we will specialize v to an invertible $q \in \mathbb{F}$, and consider the Specht module over $\mathbb{F}$. We will also construct a module by generators and relations and show that it is isomorphic to the Specht module over $\mathbb{F}$ using the relations derived in this section. Actually, if we define a module over $\mathbb{F}$ by the pair consisting of the relations (2.1) and those in Lemma 3.2 identifying x_λ with 1, all the relations in this section (other than the relations in Lemma 3.4) are valid for any invertible $q \in \mathbb{F}$. If q is not a roof of unity, then the relations in Lemma 3.4 still remain valid. But if q is a primitive eth root of unity, then the leth induction step ($l \geqslant 1$) in the proof of Lemma 3.4 does not work, as one can see in the proof, since it involves $[le]_v$ which is 0 for such q. Thus when q is a primitive eth root of unity, we are led to add more relations to the set of defining relations of the Specht modules over $\mathbb{F}$ as we can see in the next section.

4. The Gröbner–Shirshov pair for S_q^λ

In this section, we specialize the Specht module S_v^λ over $\mathbb{F}[v, v^{-1}]$ to the module S_q^λ over $\mathbb{F}$ for an invertible $q \in \mathbb{F}$ and define an $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ by generators and relations. The defining relations for $\widehat{S}_q^\lambda$ will be taken from the natural relations derived in the previous section identifying x_λ with 1. Thus there exists a surjective homomorphism of $\widehat{S}_q^\lambda$ onto S_q^λ. We will derive sufficiently many relations in $\widehat{S}_q^\lambda$ and show that the number of standard monomials with respect to these relations is equal to the dimension of S_q^λ. Hence we conclude that the $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ is isomorphic to the Specht module S_q^λ and obtain a Gröbner–Shirshov pair for $\widehat{S}_q^\lambda$. The standard monomials for this Gröbner–Shirshov pair will be indexed by the set of *cozy* tableaux.

Let $X = \{T_1, T_2, \ldots, T_{n-1}\}$ be the set of generators as before. Fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n. Let q be an invertible element of $\mathbb{F}$ and denote by R_q the set of relations in (2.1) with v replaced by q. The Hecke algebra defined by R_q over $\mathbb{F}$ will be denoted by $\mathcal{H}_n(q)$. Recall that there is an isomorphism of fields

$$\mathbb{F}\big[v, v^{-1}\big]/(v-q) \cong \mathbb{F} \quad \text{given by} \quad v \mapsto q,$$

where $(v-q)$ is the maximal ideal of $\mathbb{F}[v, v^{-1}]$ generated by $v-q$. The $\mathcal{H}_n(q)$-module S_q^λ is defined to be

$$S_q^\lambda = \mathbb{F} \otimes_{\mathbb{F}[v,v^{-1}]} S_v^\lambda.$$

We call S_q^λ the *Specht module* over $\mathcal{H}_n(q)$ corresponding to the partition λ. From the basis of S_v^λ given in Proposition 2.3, we naturally obtain a basis of S_q^λ and call it the *Murphy basis* of the Specht module S_q^λ. Recall that the elements of the Murphy basis are parameterized by the standard λ-tableaux.

We now define the $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ by generators and relations.

Definition 4.1. (1) If $q \in \mathbb{F}^\times$ is not a root of unity, then $\widehat{S}_q^\lambda$ is the $\mathcal{H}_n(q)$-module defined by the pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$, where R_q^λ is the set of elements:

$$T_i - q \quad (i \neq l_\lambda,\ l = 1, 2, \ldots, k-1),$$
$$\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l} \quad (1 \leqslant i \leqslant k-1). \tag{4.1}$$

(2) If q is a primitive eth root of unity, then $\widehat{S}_q^\lambda$ is the $\mathcal{H}_n(q)$-module defined by the pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$, where R_q^λ is the set of elements:

$$T_i - q \quad (i \neq l_\lambda,\ l = 1, 2, \ldots, k-1),$$
$$\sum_{l=1}^{\lambda_i+1} T_{i_\lambda,(i-1)_\lambda+l} \quad (1 \leqslant i \leqslant k-1),$$
$$\sum_{\boldsymbol{a} \in C_{le,i,\lambda}} \big\langle (i-1)_\lambda + \boldsymbol{a} \big\rangle_{i_\lambda} \quad (1 \leqslant i \leqslant k-1,\ l \geqslant 1). \tag{4.2}$$

From the construction, we see that all the relations in S_v^λ derived in the previous section still hold in S_q^λ. In particular, by Lemma 3.2 and Lemma 3.4, there exists a surjective $\mathcal{H}_n(q)$-module homomorphism

$$\Psi : \widehat{S}_q^\lambda \to S_q^\lambda \quad \text{given by} \quad 1 \mapsto x_\lambda. \tag{4.3}$$

We claim that the map Ψ is actually an isomorphism. In other words, we claim that the module $\widehat{S}_q^\lambda$ defined by the pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$ is isomorphic to the Specht module S_q^λ. The rest of this section will be devoted to proving this claim.

As was noticed in the remark at the end of the previous section, we can derive in $\widehat{S}_q^\lambda$ the same relations (identifying x_λ with 1) as in the previous section for any invertible $q \in \mathbb{F}^\times$. Note that even if q is a primitive eth root of unity, the induction argument in the proof of Lemma 3.4 works well from the defining relations (4.2) of $\widehat{S}_q^\lambda$. Since it is important to our argument, we make it into a lemma.

Lemma 4.2. *Let q be an invertible element of $\mathbb{F}$.*

(1) *The relations*

$$T_{a,b}T_{c,d} = T_{c-1,d-1}T_{a,b} \quad (a \geqslant c \geqslant d > b)$$

hold in $\mathcal{H}_n(q)$.

(2) *For p $(0 \leqslant p \leqslant k-1)$ and j $(1 \leqslant j < \lambda_{p+1})$, let*

$$\boldsymbol{a}_i = \big(a_l^i\big) = \big(a_1^i, \ldots, a_{\lambda_{i+1}}^i\big) \quad (i = 0, 1, \ldots, p-1)$$

and

$$\boldsymbol{b} = (b_l) = (b_1, \ldots, b_j, b_{j+1})$$

be the sequences of positive integers such that

$$1 \leqslant a_l^i \leqslant i_\lambda + l, \qquad 1 \leqslant b_l \leqslant p_\lambda + l, \quad \text{and} \quad b_j \geqslant b_{j+1}.$$

Then the relation

$$x_\lambda \left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle \boldsymbol{b} \rangle_{p_\lambda} = q x_\lambda \left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \right) \langle \boldsymbol{b}' \rangle_{p_\lambda}$$

holds in $\widehat{S}_q^\lambda$, *where* $\boldsymbol{b}' = (b_1, b_2, \ldots, b_{j-1}, b_{j+1}, b_j + 1)$.

(3) *For each* i $(1 \leqslant i \leqslant k-1)$ *and* j $(1 \leqslant j \leqslant \lambda_{i+1})$, *the relation*

$$\sum_{\boldsymbol{a} \in C_{j,i,\lambda}} \big\langle (i-1)_\lambda + \boldsymbol{a} \big\rangle_{i_\lambda} = 0$$

holds in $\widehat{S}_q^\lambda$.

In the following proposition, we determine a Gröbner–Shirshov basis for $\mathcal{H}_n(q)$ with respect to each of the monomial orders introduced in Example 1.2.

Proposition 4.3. (a) *The following relations form a Gröbner–Shirshov basis for* $\mathcal{H}_n(q)$ *with respect to the monomial order* $\prec_{\text{deg-iex}}$:

$$\mathcal{R}_{q,\text{deg-lex}}: \quad \begin{cases} T_i T_j - T_j T_i & \text{for } i > j+1, \\ T_i^2 - (q-1) T_i - q & \text{for } 1 \leqslant i \leqslant n-1, \\ T_{i+1,j} T_{i+1} - T_i T_{i+1,j} & \text{for } i \geqslant j. \end{cases} \tag{4.4}$$

Hence the set of $\mathcal{R}_{q,\text{deg-lex}}$*-standard monomials is given by*

$$\mathcal{B}_{\text{deg-lex}} = \{ T_{1,j_1} T_{2,j_2} \cdots T_{n-1,j_{n-1}} \mid 1 \leqslant j_k \leqslant k+1,\ k = 1, 2, \ldots, n-1 \}.$$

(b) *The following relations form a Gröbner–Shirshov basis for* $\mathcal{H}_n(q)$ *with respect to the monomial order* $\prec_{\text{deg-rlex}}$:

$$\mathcal{R}_{q,\text{deg-rlex}}: \quad \begin{cases} T_i T_j - T_j T_i & \text{for } i > j+1, \\ T_i^2 - (q-1) T_i - q & \text{for } 1 \leqslant i \leqslant n-1, \\ T_j T_{i,j} - T_{i,j} T_{j+1} & \text{for } i > j. \end{cases} \tag{4.5}$$

Hence the set of $\mathcal{R}_{q,\text{deg-rlex}}$*-standard monomials is given by*

$$\mathcal{B}_{\text{deg-rlex}} = \{ T_{j_1,1} T_{j_2,2} \cdots T_{j_{n-1},n-1} \mid k-1 \leqslant j_k \leqslant n-1, \\ k = 1, 2, \ldots, n-1 \}.$$

Proof. By Lemma 4.2(1), all the relations in $\mathcal{R}_{q,\text{deg-lex}}$ and in $\mathcal{R}_{q,\text{deg-rlex}}$ hold in $\mathcal{H}_n(q)$. It can be easily checked that $\mathcal{B}_{\text{deg-lex}}$ (respectively $\mathcal{B}_{\text{deg-rlex}}$) is the set of $\mathcal{R}_{q,\text{deg-lex}}$ (respectively $\mathcal{R}_{q,\text{deg-rlex}}$)-standard monomials. Observe that the number of elements in $\mathcal{B}_{\text{deg-lex}}$ (and in $\mathcal{B}_{\text{deg-rlex}}$) is $n!$. Since it is exactly the dimension of $\mathcal{H}_n(q)$, it follows from Proposition 1.6 that $\mathcal{R}_{q,\text{deg-lex}}$ and $\mathcal{R}_{q,\text{deg-rlex}}$

are Gröbner–Shirshov bases for the Hecke algebra $\mathcal{H}_n(q)$ with respect to $\prec_{\text{deg-lex}}$ and $\prec_{\text{deg-rlex}}$, respectively. □

Now let us consider how to complete the set R_q^λ to obtain a Gröbner–Shirshov pair. In general, even if we extend the set R_q^λ by adding the relations in Lemma 4.2, (2) and (3), the pair of the set $\mathcal{R}_{q,\text{deg-lex}}$ in the above lemma and the extended set of R_q^λ is not big enough to be a Gröbner–Shirshov pair for $\widehat{S}_q^\lambda$. Hence, we need more relations in $\widehat{S}_q^\lambda$, which we will derive in the next lemma. We set $\mathcal{R}_q = \mathcal{R}_{q,\text{deg-lex}}$. In the rest of this paper, we will only consider the monomial order $\prec_{\text{deg-lex}}$.

Lemma 4.4. *For p $(1 \leqslant p \leqslant k-1)$ and j $(1 \leqslant j \leqslant \lambda_{p+1})$, let $\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p-2}, \boldsymbol{b}$ and $\boldsymbol{c}$ be the sequences of positive integers satisfying the following conditions*:

$$\begin{aligned}
&\boldsymbol{a}_i = \left(a_1^i, a_2^i, \ldots, a_{\lambda_{i+1}}^i\right), \quad 1 \leqslant a_l^i \leqslant i_\lambda + l, \\
&\boldsymbol{b} = (b_1, b_2, \ldots, b_{\lambda_p}), \quad 1 \leqslant b_l \leqslant (p-1)_\lambda + l, \\
&\boldsymbol{c} = (c_1, c_2, \ldots, c_j), \quad 1 \leqslant c_l \leqslant p_\lambda + l, \\
&b_j < b_{j+1} < \cdots < b_{\lambda_p}, \qquad c_1 < c_2 < \cdots < c_j, \quad \textit{and} \quad c_j \leqslant b_j + j - 1.
\end{aligned}$$

Then the relation

$$\begin{aligned}
&\left(\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \langle \boldsymbol{b} \rangle_{(p-1)_\lambda} \langle \boldsymbol{c} \rangle_{p_\lambda} \\
&\quad + \left(\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \left(\sum_{\boldsymbol{a} \in C_{j,p,\lambda}^1} \big\langle (p-1)_\lambda + \boldsymbol{a} \big\rangle_{p_\lambda} \right) \langle \boldsymbol{b}_1, \boldsymbol{c}, j + \boldsymbol{b}_2 \rangle_{(p-1)_\lambda} = 0
\end{aligned} \tag{4.6}$$

holds in $\widehat{S}_q^\lambda$, where $\boldsymbol{b}_1 = (b_1, b_2, \ldots, b_{j-1})$ and $\boldsymbol{b}_2 = (b_j, b_{j+1}, \ldots, b_{\lambda_p})$.

Proof. From Lemma 4.2(3), we have the relation

$$\sum_{\boldsymbol{a} \in C_{j,p,\lambda}} \big\langle (p-1)_\lambda + \boldsymbol{a} \big\rangle_{p_\lambda} = 0$$

for any nonzero $q \in \mathbb{F}^\times$. Note that $\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}$ contains only T_l with $l \leqslant (p-1)_\lambda - 1$ and that $\sum_{\boldsymbol{a} \in C_{j,p,\lambda}} \langle (p-1)_\lambda + \boldsymbol{a} \rangle_{p_\lambda}$ contains only T_l with $l \geqslant (p-1)_\lambda + j$. Hence $\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}$ commutes with $\sum_{\boldsymbol{a} \in C_{j,p,\lambda}} \langle (p-1)_\lambda + \boldsymbol{a} \rangle_{p_\lambda}$, and we obtain

$$\left(\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \sum_{\boldsymbol{a} \in C_{j,p,\lambda}} \big\langle (p-1)_\lambda + \boldsymbol{a} \big\rangle_{p_\lambda} = 0.$$

By separating the leading term, we can write

$$\left(\prod_{i=0}^{p-2}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\big\langle (p-1)_\lambda+(j,j+1,\ldots,2j-1)\big\rangle_{p_\lambda} + \left(\prod_{i=0}^{p-2}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\left(\sum_{\boldsymbol{a}\in C^1_{j,p,\lambda}}\big\langle (p-1)_\lambda+\boldsymbol{a}\big\rangle_{p_\lambda}\right)=0.$$

Multiplying it by $\langle \boldsymbol{b}_1,\boldsymbol{c}\rangle_{(p-1)_\lambda}=\langle \boldsymbol{b}_1\rangle_{(p-1)_\lambda}\langle \boldsymbol{c}\rangle_{(p-1)_\lambda+j-1}$ from the right, we obtain

$$\left(\prod_{i=0}^{p-2}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle \boldsymbol{b}_1\rangle_{(p-1)_\lambda}\langle \boldsymbol{c}\rangle_{p_\lambda} + \left(\prod_{i=0}^{p-2}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\left(\sum_{\boldsymbol{a}\in C^1_{j,p,\lambda}}\big\langle (p-1)_\lambda+\boldsymbol{a}\big\rangle_{p_\lambda}\right)\langle \boldsymbol{b}_1,\boldsymbol{c}\rangle_{(p-1)_\lambda}=0. \tag{4.7}$$

Since $c_1<c_2<\cdots<c_j$, $c_j<b_j+j$ and $p_\lambda+j-1\geqslant (p-1)_\lambda+j-1+l$ for $j\leqslant l<\lambda_p$, it follows from Lemma 4.2(1) that

$$\langle \boldsymbol{b}_1\rangle_{(p-1)_\lambda}\langle \boldsymbol{c}\rangle_{p_\lambda}\langle j+\boldsymbol{b}_2\rangle_{(p-1)_\lambda+2j-1}=\langle \boldsymbol{b}\rangle_{(p-1)_\lambda}\langle \boldsymbol{c}\rangle_{p_\lambda}.$$

Hence multiplying (4.7) by $\langle j+\boldsymbol{b}_2\rangle_{(p-1)_\lambda+2j-1}$ from the right, we get

$$\left(\prod_{i=0}^{p-2}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\langle \boldsymbol{b}\rangle_{(p-1)_\lambda}\langle \boldsymbol{c}\rangle_{p_\lambda} + \left(\prod_{i=0}^{p-2}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)\left(\sum_{\boldsymbol{a}\in C^1_{j,p,\lambda}}\big\langle (p-1)_\lambda+\boldsymbol{a}\big\rangle_{p_\lambda}\right)\langle \boldsymbol{b}_1,\boldsymbol{c},j+\boldsymbol{b}_2\rangle_{(p-1)_\lambda}=0,$$

as desired. □

Recall that a λ-tableau is a map $t:[\lambda]\to\{1,2,\ldots,n\}$ and that t^λ is the unique standard λ-tableau such that $t^\lambda(i,j+1)=t^\lambda(i,j)+1$ for all nodes (i,j).

Definition 4.5. A λ-tableau is said to be *cozy* if it satisfies the following conditions:

(i) $t(i,j)\leqslant t^\lambda(i,j)$,
(ii) $t(i,j)<t(i,j+1)$,
(iii) $t(i,j)+j\leqslant t(i+1,j)$.

Note that if t is a cozy λ-tableau, then we always have $t(1, j) = j$. For example, the tableau in the left is cozy, while the one in the right is not:

1	2	3	4	5	6	7
2	4	6	8	10		
3	6	9				
4	8	12				

1	2	3	4	5	6	7
3	4	6	9	10		
4	7	9				
7	10	11				

Taking some of the relations from Lemmas 4.2(2) and 4.4, we define the set $\mathcal{R}_q^\lambda$ to be

$$\mathcal{R}_q^\lambda: \quad \begin{cases} \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \langle \boldsymbol{b} \rangle_{p_\lambda} - q \left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \langle \boldsymbol{b}' \rangle_{p_\lambda} \quad \text{with } b_j = b_{j+1}, \\ \left(\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \langle \boldsymbol{b} \rangle_{(p-1)_\lambda} \langle \boldsymbol{c} \rangle_{p_\lambda} \\ \quad + \left(\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \left(\sum_{\boldsymbol{a} \in C^1_{j,p,\lambda}} \big\langle (p-1)_\lambda + \boldsymbol{a} \big\rangle_{p_\lambda} \right) \langle \boldsymbol{b}_1, \boldsymbol{c}, j + \boldsymbol{b}_2 \rangle_{(p-1)_\lambda} \\ \hfill \text{with } c_j = b_j + j - 1, \end{cases}$$

where we keep the notation in Lemmas 4.2(2) and 4.4. Note that the leading terms of the relations in $\mathcal{R}_q^\lambda$ with respect to $\prec_{\text{deg-lex}}$ are

$$\left(\prod_{i=0}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \langle \boldsymbol{b} \rangle_{p_\lambda} \quad \text{with } b_j = b_{j+1},$$

$$\left(\prod_{i=0}^{p-2} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right) \langle \boldsymbol{b} \rangle_{(p-1)_\lambda} \langle \boldsymbol{c} \rangle_{p_\lambda} \quad \text{with } c_j = b_j + j - 1.$$

Thus the set of $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$-standard monomials with respect to $\prec_{\text{deg-lex}}$ is

$$G(\lambda) = \left\{ \prod_{i=1}^{k-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \;\middle|\; \begin{array}{l} \boldsymbol{a}_i = \left(a^i_1, a^i_2, \ldots, a^i_{\lambda_{i+1}}\right) \\ a^i_j \leqslant i_\lambda + j,\ a^i_j < a^i_{j+1},\ a^i_j + j \leqslant a^{i+1}_j \end{array} \right\}.$$

Let $H(\lambda)$ be the set of all cozy λ-tableaux and let $I(\lambda)$ be the set of all standard λ-tableaux. We define a map $\kappa : G(\lambda) \to H(\lambda)$ by

$$\kappa\left(\prod_{i=1}^{k-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda}\right)(t, s) = a^{t-1}_s, \qquad a^0_j = j.$$

Recall that the symmetric group S_n acts naturally on the set of bijective λ-tableaux. We define a map $\zeta : G(\lambda) \to I(\lambda)$ by

$$\zeta(T_{s_1} T_{s_2} \cdots T_{s_l}) = t^\lambda \tau_{s_1} \tau_{s_2} \cdots \tau_{s_l},$$

where τ_{s_t} is the transposition $(s_t, s_t + 1) \in S_n$.

Proposition 4.6. *Let κ and ζ be the maps defined above. Then the map κ (respectively the map ζ) is a well-defined bijection between $G(\lambda)$ and $H(\lambda)$ (respectively $G(\lambda)$ and $I(\lambda)$). Hence we have* $\dim S_q^\lambda = \#(I(\lambda)) = \#(G(\lambda)) = \#(H(\lambda))$.

Proof. Clearly, κ is a well-defined bijection. Observe that the $\tau_i \tau_{i-1} \cdots \tau_j$ $(i \geqslant j)$ action on a tableau changes the boxes

$$\boxed{j}\,\boxed{j+1}\cdots\boxed{i}\,\boxed{i+1} \quad \text{into} \quad \boxed{j+1}\,\boxed{j+2}\cdots\boxed{i+1}\,\boxed{j},$$

respectively. Consider an element $\prod_{i=1}^{k-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \in G(\lambda)$. Since $2 \leqslant a_1^1 \leqslant 1_\lambda + 1$, the tableau $\zeta(T_{1_\lambda, a_1^1})$ is standard.

Assume that the tableau

$$\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p \rangle_{p_\lambda}\right)$$

is standard for some $p \geqslant 1$ and $j \geqslant 1$. Note that

$$\begin{aligned} &\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p, a_{j+1}^p \rangle_{p_\lambda}\right) \\ &\quad = \zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p \rangle_{p_\lambda}\right) \tau_{p_\lambda + j} \tau_{p_\lambda + j - 1} \cdots \tau_{a_{j+1}^p}. \end{aligned}$$

Since $a_j^p < a_{j+1}^p$ and $a_{j+1}^{p-1} + j + 1 \leqslant a_{j+1}^p$, we have

$$\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p, a_{j+1}^p \rangle_{p_\lambda}\right)(p+1, j) = a_j^p,$$

$$\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p, a_{j+1}^p \rangle_{p_\lambda}\right)(p+1, j+1) = a_{j+1}^p,$$

$$\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p, a_{j+1}^p \rangle_{p_\lambda}\right)(p, j+1) \leqslant a_{j+1}^{p-1} + j,$$

$$\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p, a_{j+1}^p \rangle_{p_\lambda}\right)(p+1, j+1) = a_{j+1}^p,$$

which implies that the tableau

$$\zeta\left(\prod_{i=1}^{p-1} \langle \boldsymbol{a}_i \rangle_{i_\lambda} \langle a_1^p, \ldots, a_j^p, a_{j+1}^p \rangle_{p_\lambda}\right)$$

is standard. Hence, by induction the tableau

$$\zeta\left(\prod_{i=1}^{k-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right)$$

is standard. Therefore, the map ζ is well-defined.

Conversely, if t is a standard tableau, then by reversing the induction steps in the above argument, we can obtain an element $\prod_{i=1}^{k-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}$ of $G(\lambda)$ such that

$$\zeta\left(\prod_{i=1}^{k-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda}\right) = t.$$

Therefore, the map ζ is a bijection. □

Recall that there is a surjective homomorphism $\Psi : \widehat{S}_q^\lambda \to S_q^\lambda$ given by (4.3), which implies $\dim \widehat{S}_q^\lambda \geqslant \dim S_q^\lambda$. However, by Proposition 1.6, we have

$$\dim \widehat{S}_q^\lambda \leqslant \#\big(G(\lambda)\big) = \dim S_q^\lambda.$$

Therefore, we conclude that the $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ is isomorphic to the Specht module S_q^λ, the set $G(\lambda)$ is a linear basis of $\widehat{S}_q^\lambda$, and the pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$ is a Gröbner–Shirshov pair for $\widehat{S}_q^\lambda$.

In the following theorem, we summarize the main results proved in this section.

Theorem 4.7. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n.*

(a) *The Specht module S_q^λ is isomorphic to the $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ defined by the pair (R_q, R_q^λ). Hence we obtain a presentation of the Specht module S_q^λ by generators and relations.*
(b) *The pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$ is a Gröbner–Shirshov pair for $\widehat{S}_q^\lambda$ with respect to the monomial order $\prec_{\text{deg-lex}}$.*
(c) *The set of $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$-standard monomials is given by*

$$G(\lambda) = \left\{ \prod_{i=1}^{k-1}\langle \boldsymbol{a}_i\rangle_{i_\lambda} \;\middle|\; \begin{array}{l} \boldsymbol{a}_i = \big(a_1^i, a_2^i, \ldots, a_{\lambda_{i+1}}^i\big) \\ a_j^i \leqslant i_\lambda + j,\ a_j^i < a_{j+1}^i,\ a_j^i + j \leqslant a_j^{i+1} \end{array} \right\}.$$

Hence the set $G(\lambda)$ is a linear basis of $\widehat{S}_q^\lambda$.

Remark. It is easy to see that the monomial basis $G(\lambda)$ of $\widehat{S}_q^\lambda$ is mapped onto the Murphy basis of S_q^λ under the isomorphism Ψ.

5. Gram matrix of S_q^λ

It is well known that if q is not a root of unity, the Specht modules are irreducible. If q is a root of unity, the irreducible modules are the simple quotients of the Specht modules, and in general, their dimensions are not known explicitly. In this section, we define a canonical bilinear form on the Specht module S_q^λ, whose matrix with respect to the monomial basis $G(\lambda)$ will be called the *Gram matrix*. As was shown in [10], the rank of the Gram matrix is equal to the dimension of the irreducible module D_q^λ. In this section, using the monomial basis $G(\lambda)$ and the division algorithm given in Lemma 1.4, we give a new recursive algorithm of computing the Gram matrix of the Specht module S_q^λ. At the end of this section, we will discuss the application of our algorithm with several interesting examples.

To begin with, we identify the Specht module S_q^λ with the $\mathcal{H}_n(q)$-module $\widehat{S}_q^\lambda$ defined by the pair (R_q, R_q^λ) and consider the Gröbner–Shirshov pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$.

Let $\lambda \vdash n$ be a partition and define a bilinear map

$$\mathcal{H}_n(q) \times \mathcal{H}_n(q) \to S_q^\lambda \quad \text{by} \quad (u, v) \mapsto uv^* x_\lambda.$$

Then the image of the map is actually in the field $\mathbb{F}$, and the map induces a symmetric bilinear form on S_q^λ

$$B_\lambda : S_q^\lambda \times S_q^\lambda \to \mathbb{F} \subset S_q^\lambda.$$

It was shown in [10] that B_λ satisfies

$$B_\lambda(u, vh) = B_\lambda(uh^*, v) \quad \text{for all } u, v \in S_q^\lambda,\ h \in \mathcal{H}_n(q).$$

Definition 5.1. The *Gram matrix* of the Specht module S_q^λ is the matrix Γ_λ of the symmetric bilinear form B_λ with respect to the monomial basis $G(\lambda)$.

We say that a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is *e-regular* if $\lambda_i - \lambda_{j+1} < e$ $(\lambda_{k+1} = 0)$ for all $i = 1, \ldots, k$. Otherwise, λ is called *e-singular*.

Proposition 5.2 [4,6,10]. *For each e-regular partition λ of n, the quotient $D_q^\lambda = S_q^\lambda / \mathrm{rad}(B_\lambda)$ of the Specht module by the radical of B_λ is an irreducible module over $\mathcal{H}_n(q)$. They form a complete set of inequivalent irreducible $\mathcal{H}_n(q)$-modules. In particular, the rank of the Gram matrix Γ_λ of S_q^λ is equal to the dimension of the irreducible module D_q^λ.*

Remark. In [6], Graham and Lehrer introduced the notion of *cellular algebras*. One can verify that the Hecke algebra $\mathcal{H}_n(q)$ is a cellular algebra with the cellular basis

$$\{x_{st} \mid (s, t) \text{ is a standard pair } \lambda \vdash n\},$$

where the elements x_{st} is defined in Section 2. Also it can be easily checked that the Specht module S_q^λ is the *cell representation* corresponding to λ (see [6, Theorem 3.4]).

Now we describe our algorithm of computing the Gram matrix of the Specht module S_q^λ. First, observe that using the Gröbner–Shirshov pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$, we can determine $B_\lambda(u, 1)$ for all $u \in G(\lambda)$. Hence the first column of the Gram matrix can be determined. Next, let $u, v \in G(\lambda)$ and assume that $B_\lambda(u, w)$ can be computed for all $u, w \in G(\lambda)$ with $l(w) \leqslant l(v) - 1$. Then, since every $v \in G(\lambda)$ can be written as $v = v'T_i$ for some i $(1 \leqslant i \leqslant n-1)$ and $v' \in G(\lambda)$ with $l(v') = l(v) - 1$, we have

$$B_\lambda(u, v) = B_\lambda(u, v'T_i) = B_\lambda(uT_i, v').$$

By Theorem 1.11 and the division algorithm given in Lemma 1.4, every uT_i $(u \in G(\lambda),\ i = 1, \ldots, n-1)$ can be uniquely expressed as a linear combination of the elements in $G(\lambda)$. Since $l(v') = l(v) - 1$, by induction, we can compute $B_\lambda(u, v) = B_\lambda(uT_i, v')$. Note that, since B_λ is symmetric, we have only to compute $B_\lambda(u, v)$ for all $u \succeq v$.

Hence our algorithm of computing the Gram matrix Γ_λ can be summarized as follows:

(1) Using the Gröbner–Shirshov pair, compute $B_\lambda(u, 1)$ for all $u \in G(\lambda)$.
(2) Using the division algorithm, write uT_i as a linear combination of the elements of $G(\lambda)$ for all $u \in G(\lambda)$, $i = 1, \ldots, n-1$.
(3) For any $u, v \in G(\lambda)$ with $u \succeq v$, write $v = v'T_i$ for some $v' \in G(\lambda)$ and i $(1 \leqslant i \leqslant n-1)$, and compute $B_\lambda(u, v) = B_\lambda(uT_i, v')$.

In the rest of this section, we present several examples illustrating how to carry out our algorithm.

Example 5.3. Let $\lambda = (2, 2, 1) \vdash 5$ and consider the Specht module S_q^λ. Then the cozy tableaux of shape λ and the corresponding standard monomials are given by

$$\begin{array}{ccccc}
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline 5 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \\
1 & T_2 & T_4 & T_2T_4 & T_2T_{4,3}
\end{array}$$

Hence $\dim S_q^\lambda = 5$ and $G(\lambda) = \{1, T_2, T_4, T_2T_4, T_2T_{4,3}\}$. By Theorem 4.7, the Gröbner–Shirshov pair $(\mathcal{R}_q, \mathcal{R}_q^\lambda)$ is given as follows:

$$\mathcal{R}_q = \begin{cases} T_i^2 - (q-1)T_i - 1 \quad (1 \leqslant i \leqslant 4), \\ T_4T_1 - T_1T_4, \quad T_3T_1 - T_1T_3, \quad T_4T_2 - T_2T_4, \\ T_{4,1}T_4 - T_3T_{4,1}, \quad T_{4,2}T_4 - T_3T_{4,2}, \quad T_{4,3}T_4 - T_3T_{4,3}, \\ T_{3,1}T_3 - T_2T_{3,1}, \quad T_{3,2}T_3 - T_2T_{3,2}, \quad T_{2,1}T_2 - T_1T_{2,1} \end{cases}$$

and

$$\mathcal{R}_q^\lambda = \begin{cases} T_1 - q, \quad T_3 - q, \quad T_{2,1} + T_2 + 1, \quad T_{4,3} + T_4 + 1, \\ T_2T_3 + T_2 + 1, \quad T_2T_{4,2} + T_2T_{4,3} + T_2T_3. \end{cases}$$

We now carry out our recursive algorithm.

(1) The values of $B_\lambda(u, 1)$ $(u \in G(\lambda))$ are:

$$\begin{aligned} B_\lambda(1,1) &= x_\lambda = (T_1+1)(T_3+1) = (q+1)^2, \\ B_\lambda(T_2,1) &= T_2(T_1+1)(T_3+1) = (T_{2,1}+T_2)(T_3+1) = -(q+1), \\ B_\lambda(T_4,1) &= T_4(T_1+1)(T_3+1) = (T_1+1)T_4(T_3+1) \\ &= (q+1)(T_{4,3}+T_4) = -(q+1), \\ B_\lambda(T_2T_4,1) &= T_2T_4(T_1+1)(T_3+1) = T_2(T_1+1)T_4(T_3+1) = 1, \\ B_\lambda(T_2T_{4,3},1) &= T_2T_{4,3}(T_1+1)(T_3+1) = T_2(T_1+1)T_{4,3}(T_3+1) \\ &= (T_4+1)(T_3+1) = T_3 = q. \end{aligned}$$

(2) Write uT_i $(u \in G(\lambda),\ i = 1, \ldots, 4)$ as a linear combination of the elements in $G(\lambda)$:

$$\begin{aligned} &T_1 = q, \qquad T_2 = T_2, \qquad T_3 = q, \qquad T_4 = T_4, \\ &T_2T_1 = -T_2 - 1, \qquad T_2T_2 = (q-1)T_2 + q, \\ &T_2T_3 = -T_2 - 1, \qquad T_2T_4 = T_2T_4, \\ &T_4T_1 = qT_4, \qquad T_4T_2 = T_2T_4, \\ &T_4T_3 = -T_4 - 1, \qquad T_4T_4 = (q-1)T_4 + q, \\ &T_2T_4T_1 = -T_2T_4 - T_4, \qquad T_2T_4T_2 = (q-1)T_2T_4 + qT_4, \\ &T_2T_4T_3 = T_2T_{4,3}, \qquad T_2T_4T_4 = (q-1)T_2T_4 + qT_2, \\ &T_2T_{4,3}T_1 = -T_2T_{4,3} + T_4 + 1, \qquad T_2T_{4,3}T_2 = -T_2T_{4,3} + T_2 + 1, \\ &T_2T_{4,3}T_3 = (q-1)T_2T_{4,3} + qT_2T_4, \qquad T_2T_{4,3}T_4 = -T_2T_{4,3} + T_4 + 1. \end{aligned}$$

(3) Compute all the values $B_\lambda(u, v)$ $(u, v \in G(\lambda))$ inductively.

(a) $B_\lambda(u, T_2)$ $(u \succeq T_2)$:

$$\begin{aligned}
B_\lambda(T_2,T_2) &= B_\lambda\big(T_2,1\big) = (q-1)B_\lambda(T_2,1) + qB_\lambda(1,1) = [4]_q,\\
B_\lambda(T_4,T_2) &= B_\lambda(T_4T_2,1) = B_\lambda(T_2T_4,1) = 1,\\
B_\lambda(T_2T_4,T_2) &= B_\lambda(T_2T_4T_2,1)\\
&= (q-1)B_\lambda(T_2T_4,1) + qB_\lambda(T_4,1) = -\big(q^2+1\big),\\
B_\lambda(T_2T_{4,3},T_2) &= B_\lambda(T_2T_{4,3}T_2,1)\\
&= -B_\lambda(T_2T_{4,3},1) + B_\lambda(T_2,1) + B_\lambda(1,1) = q^2.
\end{aligned}$$

(b) $B_\lambda(u,T_4)$ $(u \succeq T_4)$:

$$B_\lambda(T_4,T_4) = [4]_q, \qquad B_\lambda(T_2T_4,T_4) = -\big(q^2+1\big),$$
$$B_\lambda(T_2T_{4,3},T_4) = q^2.$$

(c) $B_\lambda(u,T_2T_4)$ $(u \succeq T_2T_4)$:

$$\begin{aligned}
B_\lambda(T_2T_4,T_2T_4) &= B_\lambda\big(T_2T_4^2,T_2\big)\\
&= (q-1)B_\lambda(T_2T_4,T_2) + qB_\lambda(T_2,T_2) = \big(q^2+1\big)^2,\\
B_\lambda(T_2T_{4,3},T_2T_4) &= B_\lambda(T_2T_{4,3}T_4,T_2)\\
&= -B_\lambda(T_2T_{4,3},T_2) + B_\lambda(T_4,T_2) + B_\lambda(1,T_2)\\
&= -q(q+1).
\end{aligned}$$

(d) $B_\lambda(u,T_2T_{4,3})$ $(u \succeq T_2T_{4,3})$:

$$\begin{aligned}
B_\lambda(T_2T_{4,3},T_2T_{4,3}) &= B_\lambda(T_2T_{4,3}T_3,T_2T_4)\\
&= (q-1)B_\lambda(T_2T_{4,3},T_2T_4) + qB_\lambda(T_2T_4,T_2T_4)\\
&= q^5+q^3+2q.
\end{aligned}$$

Hence the Gram matrix Γ_λ is

$$\Gamma_\lambda = \begin{pmatrix}
(q+1)^2 & -(q+1) & -(q+1) & 1 & q\\
-(q+1) & [4]_q & 1 & -\big(q^2+1\big) & q^2\\
-(q+1) & 1 & [4]_q & -\big(q^2+1\big) & q^2\\
1 & -\big(q^2+1\big) & -\big(q^2+1\big) & \big(q^2+1\big)^2 & -q(q+1)\\
q & q^2 & q^2 & -q(q+1) & q^5+q^3+2q
\end{pmatrix}.$$

If q is a primitive 3rd root of unity, then the Gram matrix is reduced to

$$\Gamma_\lambda = \begin{pmatrix}
q & q^2 & q^2 & 1 & q\\
q^2 & 1 & 1 & q & q^2\\
q^2 & 1 & 1 & q & q^2\\
1 & q & q & q^2 & 1\\
q & q^2 & q^2 & 1 & q
\end{pmatrix},$$

and it is easy to see that its rank is 1. Hence $\dim D_q^\lambda = 1$.

Example 5.4. Consider $\lambda = (n)$. Then S_q^λ is one-dimensional with basis $G(\lambda) = \{1\}$ and $\mathcal{R}_q^\lambda$ $(= R_q^\lambda)$ is given by $\{T_i - q \mid 1 \leqslant i \leqslant n-1\}$. Thus

$$B_\lambda(1,1) = x_\lambda = \sum_{w \in S_n} T_w = \sum_{w \in S_n} q^{l(w)} = \{i\}_q.$$

(For the last equality, see, e.g., [7].)

If $\lambda = (1^n)$, then the Specht module S_q^λ is also one-dimensional with basis $G(\lambda) = \{1\}$ and $\mathcal{R}_q^\lambda = R_q^\lambda = \{T_i + 1 \mid 1 \leqslant i \leqslant n-1\}$. Since $x_\lambda = 1$ in S_q^λ, we have

$$B_\lambda(1,1) = x_\lambda = 1.$$

Example 5.5. Let $\lambda = (n-1, 1)$. Then the dimension of S_q^λ is $n-1$ with basis

$$G(\lambda) = \{T_{n-1,i} \mid 2 \leqslant i \leqslant n\}$$

and

$$\mathcal{R}_q^\lambda = R_q^\lambda = \left\{ \sum_{i=1}^n T_{n-1,i},\ T_i - q\ (1 \leqslant i \leqslant n-2) \right\}.$$

(1) The first column of the Gram matrix Γ_λ is

$$B_\lambda(1,1) = \{n-1\}_q,$$
$$B_\lambda(T_{n-1,i}, 1) = -q^{n-i-1}\{n-2\}_q \quad (2 \leqslant i \leqslant n-1).$$

(2) In S_q^λ, we have

$$T_{n-1,i}T_j = \begin{cases} T_j T_{n-1,i} = q T_{n-1,i} & (1 \leqslant j \leqslant i-2), \\ T_{n-1,i-1} & (j = i-1), \\ (q-1)T_{n-1,i} + q T_{n-1,i+1} & (j = i), \\ T_{j-1}T_{n-1,i} = q T_{n-1,i} & (i+1 \leqslant j \leqslant n-1). \end{cases}$$

(3) By induction, we have

$$B_\lambda(T_{n-1,i}, T_{n-1,j}) = \begin{cases} q^{n-i}\{n-1\}_q - q^{n-i-1}(q^{n-i}-1)\{n-2\}_q \\ \quad (i = j), \\ -q^{2n-i-j-1}\{n-2\}_q \\ \quad (i < j). \end{cases}$$

Thus we have determined all the entries of the Gram matrix Γ_λ.

Example 5.6. Let $\lambda = (\lambda_1, \lambda_2)$. The monomial basis $G(\lambda)$ is given by

$$\left\{ \langle a_1, a_2, \ldots, a_{\lambda_2} \rangle_{\lambda_1} \;\middle|\; 2i \leqslant a_i \leqslant \lambda_1 + i,\ a_i < a_{i+1} \text{ for each } i \right\},$$

and the dimension of S_q^λ is

$$\frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1} \binom{n}{\lambda_2}.$$

In S_q^λ, we have

$$\begin{aligned}
&\langle a_1,\ldots,a_{\lambda_2}\rangle_{\lambda_1}\iota(S_{1,\lambda_1})\\
&\quad= q^{\lambda_1\lambda_2-\frac{1}{2}\lambda_2(\lambda_2-1)-\sum_{j=1}^{\lambda_2}a_j}\iota(S_{1,\lambda_1-\lambda_2})\\
&\qquad\times\sum_{1\leqslant b_i\leqslant i+1 \text{ for each } i}\langle b_{\lambda_1-\lambda_2},\ldots,b_{\lambda_1-1}\rangle_{\lambda_1}\\
&\quad= q^{\lambda_1\lambda_2-\frac{1}{2}\lambda_2(\lambda_2-1)-\sum_{j=1}^{\lambda_2}a_j}\{\lambda_1-\lambda_2\}_q(-1)^{\lambda_2}q^{\frac{1}{2}\lambda_2(\lambda_2-1)}\{\lambda_2\}_q\\
&\quad= (-1)^{\lambda_2}q^{\lambda_1\lambda_2-\sum_{j=1}^{\lambda_2}a_j}\{\lambda_1-\lambda_2\}_q\{\lambda_2\}_q.
\end{aligned}$$

Given $\langle a_1,\ldots,a_{\lambda_2}\rangle_{\lambda_1}$, let $s\in[0,\lambda_2]$ be the least integer such that $\lambda_1<a_{s+1}$ and let $t\in[0,\lambda_2]$ be the least integer such that $a_{t+1}=\lambda_1+t+1$, where we set $a_{\lambda_2+1}=\lambda_1+\lambda_2+1$. Then

$$\begin{aligned}
&B_\lambda\bigl(\langle a_1,\ldots,a_{\lambda_2}\rangle_{\lambda_1},1\bigr)\\
&\quad= \langle a_1,\ldots,a_s,\ldots,a_t,\ldots,a_{\lambda_2}\rangle_{\lambda_1}\iota(S_{1,\lambda_1})\iota(S_{\lambda_1+1,\lambda_2})\\
&\quad= \langle a_1,\ldots,a_s,\ldots,a_t\rangle_{\lambda_1}\iota(S_{1,\lambda_1})\iota(S_{\lambda_1+1,\lambda_2})\\
&\quad= \langle a_1,\ldots,a_s\rangle_{\lambda_1}\iota(S_{1,\lambda_1})\langle a_{s+1},\ldots,a_t\rangle_{\lambda_1+s}\iota(S_{\lambda_1+1,\lambda_2})\\
&\quad= (-1)^s q^{\lambda_1 s-\sum_{j=1}^{s}a_j}\{\lambda_1-s\}_q\{s\}_q q^{\sum_{j=s+1}^{t}(\lambda_1+j-a_j)}\{\lambda_2\}_q\\
&\quad= (-1)^s q^{\lambda_1 t+\frac{1}{2}(t-s)(t+s+1)-\sum_{j=1}^{t}a_j}\{\lambda_1-s\}_q\{s\}_q\{\lambda_2\}_q.
\end{aligned}$$

Thus we completed the first step of our algorithm. The remaining steps can be worked out by direct calculation.

6. Gram matrices of Temperley–Lieb algebras

In this section, we apply the Gröbner–Shirshov basis theory to the Temperley–Lieb algebras. By modifying the parameter, the Temperley–Lieb algebra can be viewed as the quotient of Hecke algebra, and the kernel of this quotient map acts trivially on the Specht modules. In this way the Specht modules over Hecke algebras corresponding to the Young diagrams with at most two columns will naturally become modules over Temperley–Lieb algebras, which will also be called the *Specht modules* over Temperley–Lieb algebras. Using the Gröbner–Shirshov pairs $(\mathcal{R}_q,\mathcal{R}_q^\lambda)$ for the Specht modules over Hecke algebras, we can easily determine the Gröbner–Shirshov pairs and the monomial bases for the Specht modules over Temperley–Lieb algebras. Therefore, as in the case of Hecke algebras, we obtain a recursive algorithm of computing the Gram matrices of Specht modules over Temperley–Lieb algebras.

Definition 6.1. Let η be an invertible element of $\mathbb{F}$. The *Temperley–Lieb algebra* $TL_n(\eta)$ is the associative algebra over $\mathbb{F}$ generated by $X = \{l_1, l_2, \ldots, l_{n-1}\}$ with defining relations

$$L_\eta\colon \quad \begin{cases} l_i l_j = l_j l_i & \text{for } i > j+1, \\ l_i^2 = l_i & \text{for } 1 \leqslant i \leqslant n-1, \\ l_i l_j l_i = \eta l_i & \text{for } j = i \pm 1. \end{cases} \tag{6.1}$$

For the generators l_i, we use the same shorthand as we did for the generators T_i: we write $l_{i,j} = l_i l_{i-1} \cdots l_j$ for $i \geqslant j$ and set $l_{i,i+1} = 1$ $(i \geqslant 0)$. We define $(l_{i_1} l_{i_2} \cdots l_{i_j})^* = l_{i_j} l_{i_{j-1}} \cdots l_1$ and extend $*$ to an anti-automorphism of $TL_n(\eta)$ by linearity.

Now we can immediately determine a Gröbner–Shirshov basis and the corresponding monomial basis for the Temperley–Lieb algebra $TL_n(\eta)$.

Proposition 6.2. (a) *The following relations form a Gröbner–Shirshov basis for the Temperley–Lieb algebra $TL_n(\eta)$ with respect to the monomial order $\prec_{\text{deg-lex}}$:*

$$\mathcal{L}_\eta\colon \quad \begin{cases} l_i l_j - l_j l_i & \text{for } i > j+1, \\ l_i^2 - l_i & \text{for } 1 \leqslant i \leqslant n-1, \\ l_{i,j} l_i - \eta l_{i-2,j} l_i & \text{for } i > j, \\ l_i l_{j,i} - \eta l_i l_{j,i+2} & \text{for } i < j. \end{cases} \tag{6.2}$$

(b) *Let $\mathcal{B}_n^{TL}$ be the set of all monomials of the form*

$$l_{1,j_1} l_{2,j_2} \cdots l_{n-1,j_{n-1}},$$

where $1 \leqslant j_k \leqslant k+1$ and $j_k \neq k+1$ implies $j_k < j_l$ for all $l > k$.

Then the set $\mathcal{B}_n^{TL}$ forms a linear basis of the Temperley–Lieb algebra $TL_n(\eta)$ consisting of $\mathcal{L}_\eta$-standard monomials. In particular,

$$\dim TL_n(\eta) = \#\big(\mathcal{B}_n^{TL}\big) = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. It can be easily checked that the relations in (6.2) hold in $TL_n(\eta)$, and that the set $\mathcal{B}_n^{TL}$ is the set of $\mathcal{L}_\eta$-standard monomials. In [5], it was shown that the set $\mathcal{B}_n^{TL}$ form a linear basis of the algebra $TL_n(\eta)$. Hence, by definition, the set of relations $\mathcal{L}_\eta$ is a Gröbner–Shirshov basis for $TL_n(\eta)$. As for the number of such monomials, the readers may also refer to [5]. □

Remark. One can see that our description of the standard monomials is the same as in [5]. If we ignore the factor $l_{k,k+1}$, then each element of $\mathcal{B}_n^{TL}$ can be written as

$$l_{i_1,j_1} l_{i_2,j_2} \cdots l_{i_p,j_p} \quad (0 \leqslant p \leqslant n-1),$$

where

$$1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n-1, \qquad 1 \leqslant j_1 < j_2 < \cdots < j_p \leqslant n-1,$$
$$i_1 \geqslant j_1, \qquad i_2 \geqslant j_2, \quad \ldots, \quad i_p \geqslant j_p,$$

and if $p = 0$ the monomial corresponds to 1.

Take a number $q \neq 0, -1$ in $\mathbb{F}$ or in a quadratic extension of $\mathbb{F}$ such that $\eta = q(q+1)^{-2}$. In the latter case, we replace our base field with the quadratic extension of $\mathbb{F}$ and abuse the notation to denote it also by $\mathbb{F}$. The following proposition asserts that the Temperley–Lieb algebra $TL_n(\eta)$ is a quotient of the Hecke algebra $\mathcal{H}_n(q)$.

Proposition 6.3 [5]. *There is a surjective algebra homomorphism*

$$\Phi : \mathcal{H}_n(q) \to TL_n(\eta) \quad \textit{defined by} \quad \Phi(T_i) = (q+1)l_i - 1.$$

If $n \geqslant 3$, then $\mathfrak{a} = \ker \Phi$ is the two-sided ideal of $\mathcal{H}_n(q)$ generated by

$$(T_1 + 1)(T_{2,1} + T_2 + 1).$$

Moreover, the ideal $\mathfrak{a}$ contains the elements

$$(T_i + 1)(T_{i+1,i} + T_{i+1} + 1)$$

for $1 \leqslant i \leqslant n-2$.

In the rest of this section, we fix a partition $\lambda = (2^k, 1^{n-2k})$ $(k \geqslant 0)$ of n whose diagram has at most two columns.

Lemma 6.4. *Let $\mathfrak{a}$ be the kernel of the surjective homomorphism $\Phi : \mathcal{H}_n(q) \to TL_n(\eta)$. Then the ideal $\mathfrak{a}$ acts trivially on the Specht module S_q^λ.*

Proof. The monomial basis $G(\lambda)$ of S_q^λ consists of the monomials of the form

$$\begin{cases} T_{2,a_1} T_{4,a_2} \cdots T_{2k-2,a_{k-1}} & \text{if } n = 2k, \\ T_{2,a_1} T_{4,a_2} \cdots T_{2k,a_k} T_{2k+1,a_{k+1}} T_{2k+2,a_{k+2}} \cdots T_{n-1,a_{n-k-1}} & \text{otherwise,} \end{cases}$$

where

$$a_i < a_{i+1} \quad \text{for } 1 \leqslant i \leqslant n-k-2,$$
$$i + 1 \leqslant a_i \leqslant 2i + 1 \quad \text{for } 1 \leqslant i \leqslant k,$$
$$i + 1 \leqslant a_i \leqslant k + i + 1 \quad \text{for } k + 1 \leqslant i \leqslant n-k-1.$$

Let $x_1 = (T_1 + 1)(T_{2,1} + T_2 + 1)$. It suffices to show that $ux_1 = 0$ for all $u \in G(\lambda)$. Since T_i with $i \geqslant 4$ commutes with x_1, we have only to show that

$$x_1 = 0, \qquad T_2 x_1 = 0, \qquad T_2 T_3 x_1 = 0, \qquad T_2 T_{4,3} x_1 = 0.$$

All of these can be checked easily by straightforward calculation. □

Therefore, by Lemma 6.4, the Specht module S_q^λ is given a $TL_n(\eta)$-module structure via the surjective homomorphism Φ.

Let Z_η^λ be the $TL_n(\eta)$-module defined by the pair (L_η, L_η^λ), where

$$L_\eta^\lambda: \quad \begin{cases} l_{2i-1} - 1 & \text{for } 1 \leqslant i \leqslant k, \\ l_{2i,2i-1} - \eta & \text{for } 1 \leqslant i \leqslant k, \\ l_j & \text{for } 2k+1 \leqslant j \leqslant n-1. \end{cases}$$

Lemma 6.5. *As a $TL_n(\eta)$-module, the Specht module S_q^λ is isomorphic to the $TL_n(\eta)$-module Z_η^λ.*

Proof. Recall that the Specht module S_q^λ is defined by the pair (R^λ, R_q^λ), and note that the set R_q^λ is mapped onto L_η^λ under Φ. Thus the map Φ induces a surjective $TL(\eta)$-module homomorphism $\overline{\Phi}$ from S_q^λ onto Z_η^λ as is shown in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}_n(q) & \xrightarrow{\Phi} & TL_n(\eta) \\ \downarrow & & \downarrow \\ S_q^\lambda & \xrightarrow{\overline{\Phi}} & Z_\eta^\lambda. \end{array}$$

By Lemma 6.4, we conclude that $\overline{\Phi}$ is an isomorphism. □

The $TL_n(\eta)$-module Z_η^λ will also be called the *Specht module* over $TL_n(\eta)$ corresponding to λ.

Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_j)$ be a sequence of positive integers satisfying $1 \leqslant a_i \leqslant N + i$ $(i = 1, 2, \ldots, j)$ for some positive integer N. We define

$$\mathfrak{l}\langle \boldsymbol{a} \rangle_N = l_{N,a_1} l_{N+1,a_2} \cdots l_{N+j-1,a_j}.$$

Theorem 6.6. (a) *Let $\mathcal{L}_\eta^\lambda$ be the set of the following relations*:

(i) $\mathfrak{l}\langle a_1, a_2, \ldots, a_{2p-1} \rangle_1 (l_{2p+1} - 1)$ *for* $1 \leqslant p \leqslant k-1$,
(ii) $\mathfrak{l}\langle a_1, a_2, \ldots, a_{2p-3} \rangle_1 (l_{2p,2p-1} - \eta)$ *for* $1 \leqslant p \leqslant k$,
(iii) $\mathfrak{l}\langle a_1, a_2, \ldots, a_{2k+p-2} \rangle_1 l_{2k+p}$ *for* $1 \leqslant p \leqslant n-2k-1$,
(iv) $\mathfrak{l}\langle a_1, a_2, \ldots, a_{2p-2} \rangle_1 (l_{2p,b} l_{2p+1} - \eta l_{2p-1,b})$ *for* $1 \leqslant p \leqslant k-1$, $1 \leqslant b \leqslant 2p$,

where $1 \leqslant a_i \leqslant i+1$ *for each* i.

Then the pair $(\mathcal{L}_\eta, \mathcal{L}_\eta^\lambda)$ is a Gröbner–Shirshov pair for the module Z_η^λ with respect to the monomial order $\prec_{\text{deg-lex}}$.

(b) *Let $G^{TL}(\lambda)$ be the set of monomials consisting of the monomials of the form*

$$\begin{cases} l_{2,a_1} l_{4,a_2} \cdots l_{2k-2,a_{k-1}} & \textit{if } n = 2k, \\ l_{2,a_1} l_{4,a_2} \cdots l_{2k,a_k} l_{2k+1,a_{k+1}} l_{2k+2,a_{k+2}} \cdots l_{n-1,a_{n-k-1}} & \textit{otherwise}, \end{cases}$$

where

$$a_i < a_{i+1} \quad \textit{for } 1 \leqslant i \leqslant n-k-2,$$
$$i+1 \leqslant a_i \leqslant 2i+1 \quad \textit{for } 1 \leqslant i \leqslant k,$$
$$i+1 \leqslant a_i \leqslant k+i+1 \quad \textit{for } k+1 \leqslant i \leqslant n-k-1.$$

Then the set $G^{TL}(\lambda)$ forms a linear basis of Z_η^λ consisting of $(\mathcal{L}_\eta, \mathcal{L}_\eta^\lambda)$-standard monomials.

Proof. The relations in (i)–(iii) are trivially derived from L_η^λ. Note that the relation

$$T_{2p}T_{2p+1} + T_{2p} + 1 \quad (1 \leqslant p \leqslant k)$$

is in $\mathcal{R}_q^\lambda$, and that the image of the relation under the map Φ is

$$l_{2p}l_{2p+1} - \eta \quad (1 \leqslant p \leqslant k)$$

(up to nonzero scalar). If we multiply it by $\mathfrak{l}\langle a_1, a_2, \ldots, a_{2p-2}\rangle_1 l_{2p-1,b}$ from the right, then we have the relations in (iv). Thus all the relations of $\mathcal{L}_\eta^\lambda$ hold in Z_η^λ. We can immediately check that the set $G^{TL}(\lambda)$ is the set of $(\mathcal{L}_\eta, \mathcal{L}_\eta^\lambda)$-standard monomials. If we replace l_i with T_i, then the set $G^{TL}(\lambda)$ yields the set $G(\lambda)$ (see the proof of Lemma 6.4). In particular,

$$\#\bigl(G^{TL}(\lambda)\bigr) = \dim S_q^\lambda = \dim Z_\eta^\lambda = \frac{n-2k+1}{n-k+1}\binom{n}{k}.$$

By Proposition 1.6, we conclude that $(\mathcal{L}_\eta, \mathcal{L}_\eta^\lambda)$ is a Gröbner–Shirshov pair for the module Z_η^λ. □

Define a bilinear map

$$TL_n(\eta) \times TL_n(\eta) \to Z_\eta^\lambda \quad \text{by} \quad (u, v) \mapsto uv^* l_1 l_3 \cdots l_{2k-1}.$$

As in the case of Hecke algebras, the map induces a symmetric bilinear form on Z_η^λ

$$B_\lambda^{TL} : Z_\eta^\lambda \times Z_\eta^\lambda \to \mathbb{F} \subset Z_\eta^\lambda$$

satisfying

$$B_\lambda^{TL}(u, vx) = B_\lambda^{TL}(ux^*, v) \quad \text{for all} \quad u, v \in Z_\eta^\lambda,\ x \in TL_n(\eta).$$

Actually, since S_q^λ is isomorphic to Z_η^λ via Φ and $\Phi(T_i) = (q+1)l_i - 1$, we see that

$$B_\lambda^{TL}(u, v) = (q+1)^{-k} B_\lambda(u', v') \quad \text{for } u, v \in Z_\eta^\lambda,\ u', v' \in S_q^\lambda,$$

where u' and v' are the inverse images of u and v, respectively, under the isomorphism $S_q^\lambda \xrightarrow{\sim} Z_\eta^\lambda$. In particular, the symmetric bilinear form B_λ^{TL} is well-defined. The matrix Γ_λ^{TL} of the symmetric bilinear form B_λ^{TL} with respect to the monomial basis $G^{TL}(\lambda)$ is called the *Gram matrix* of Z_η^λ.

As for the irreducible representation and the rank of the Gram matrix, we have the following proposition similar to Proposition 5.2.

Proposition 6.7 [6]. *For any partition $\lambda = (2^k, 1^{n-2k})$ $(k \geqslant 0)$ of n whose diagram has at most two columns, the quotient $E_\eta^\lambda = Z_\eta^\lambda / \mathrm{rad}(B_\lambda^{TL})$ of the Specht module by the radical of B_λ^{TL} is an irreducible module over $TL_n(\eta)$. Furthermore, they form a complete set of inequivalent irreducible $TL_n(\eta)$-modules. In particular, the rank of the Gram matrix Γ_λ^{TL} of Z_η^λ is equal to the dimension of the irreducible module E_η^λ.*

Remark. From Proposition 6.3 and the cellular algebra structure of Hecke algebra described in the remark of Proposition 5.2, we see that the Temperley–Lieb algebra $TL_n(\eta)$ is also a cellular algebra, and that the Specht module Z_η^λ over $TL_n(\eta)$ is the cell representation corresponding to λ (see [6]).

In the following proposition, we work out the first step of our algorithm of computing the Gram matrix. The remaining steps can be carried out by direct calculation.

Proposition 6.8. *If $l_{\boldsymbol{a}} = l_{2,a_1} l_{4,a_2} \cdots l_{n-1,a_{n-k-1}}$ is in $G^{TL}(\lambda)$, then we have*

$$B_\lambda^{TL}(l_{\boldsymbol{a}}, 1) = \begin{cases} 0 & \text{if } a_{k+j} \leqslant 2k+j \text{ for some } j \ (1 \leqslant j \leqslant n-2k-1), \\ \eta^\alpha & \text{otherwise} \end{cases}$$

where

$$\alpha = \sum_{a_i \leqslant 2k} \left[\frac{2i - a_i + 2}{2} \right].$$

Proof. Since $a_1 < a_2 < \cdots < a_{n-k-1}$, we can write the monomial $l_{\boldsymbol{a}}$ as $l_{b_1,1} l_{b_2,2} \cdots l_{b_n,n}$ after inserting $l_{i,i+1}$ in appropriate places. Then using the relations in $\mathcal{L}_\eta^\lambda$, we obtain

$$\begin{aligned} B_\lambda^{TL}(l_{\boldsymbol{a}}, 1) &= l_{b_1,1} l_{b_2,2} \cdots l_{b_n,n} l_1 l_3 \cdots l_{2k-1} \\ &= \left(\prod_{i=1}^{k} l_{b_{2i-1},2i-1} l_{b_{2i},2i-1} \right) \prod_{j=1}^{n-2k} l_{b_{2k+j},2k+j} \\ &= \begin{cases} 0 & \text{if } b_i \geqslant 2k+1 \text{ and } b_i \geqslant i \text{ for some } i \ (1 \leqslant i \leqslant n), \\ \eta^\alpha & \text{otherwise} \end{cases} \end{aligned}$$

$$= \begin{cases} 0 & \text{if } a_{k+j} \leqslant 2k + j \text{ for some } j \ (1 \leqslant j \leqslant n - 2k - 1), \\ \eta^{\eta} & \text{otherwise,} \end{cases}$$

where

$$\alpha = \sum_{i=1}^{2k} \left\lceil \frac{b_i - i + 1}{2} \right\rceil = \sum_{a_i \leqslant 2k} \left\lceil \frac{2i - a_i + 2}{2} \right\rceil. \qquad \square$$

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References

[1] G.M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178–218.
[2] L.A. Bokut, Imbedding into simple associative algebras, Algebra and Logic 15 (1976) 117–142.
[3] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional ideal, PhD Thesis, University of Innsbruck, 1965.
[4] R. Dipper, G.D. James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. 52 (3) (1986) 20–52.
[5] F.M. Goodman, P. de la Harpe, V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Math. Sci. Res. Inst. Publ., Vol. 14, Springer-Verlag, New York, 1989.
[6] J.J. Graham, G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996) 1–34.
[7] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, New York, 1990.
[8] S.-J. Kang, K.-H. Lee, Gröbner–Shirshov bases for representation theory, J. Korean Math. Soc. 37 (1) (2000) 55–72.
[9] S.-J. Kang, K.-H. Lee, Gröbner–Shirshov basis theory for irreducible sl_{n+1}-modules, J. Algebra 232 (2000) 1–20.
[10] G.E. Murphy, The representations of Hecke algebras of type A_n, J. Algebra 173 (1995) 97–121.
[11] A.I. Shirshov, Some algorithmic problems for Lie algebras, Siberian Math. J. 3 (1962) 292–296.