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ON THE CONVERGENCE OF KAC–MOODY EISENSTEIN SERIES

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Abstract. Let G be a representation-theoretic Kac–Moody group associated to a nonsingular symmetrizable generalized Cartan matrix. We first consider Kac–Moody analogs of Borel Eisenstein series (induced from quasicharacters on the Borel), and prove they converge almost everywhere inside the Tits cone for arbitrary spectral parameters in the Godement range. We then use this result to show the full absolute convergence everywhere inside the Tits cone (again for spectral parameters in the Godement range) for a class of Kac–Moody groups satisfying a certain combinatorial property, in particular for rank-2 hyperbolic groups.

1. Introduction. The fundamental theory of Eisenstein series on reductive groups has played pivotal roles in the formulation of Langlands' functoriality conjecture [La1, La2], and in the study of L-functions by means of the Langlands–Shahidi method (e.g., [KimSh, Kim]).

The extension of the theory of Eisenstein series to Kac–Moody groups is of great interest, due to its conjectural roles in some of the central problems in number theory, such as establishing important analytic properties of L-functions [BFH, Sh]. Recently, Eisenstein series on exceptional Lie groups have been shown to occur explicitly as coefficients of correction terms in certain maximally supersymmetric string theories [GRV, GMRV, GMV]. Conjectural Eisenstein series on Kac–Moody groups appear in recent developments in string theory [FK, FKP]. All of these potential applications require a proof of convergence (and in some cases, analytic continuation) of Kac–Moody Eisenstein series.

In his papers [G04, G06, GMS1, GMS2, GMS3, GMS4], Garland extended the classical theory of Eisenstein series to arithmetic quotients $G_{\mathbb{Z}}\backslash G_{\mathbb{R}}/K$ of affine Kac–Moody groups G. In particular, he established absolute convergence for spectral parameters in a Godement range, and then proved a meromorphic continuation beyond it. Absolute convergence has been generalized to affine Kac–Moody groups over number fields by Liu [Li]. Garland, Miller, and Patnaik [GMP] showed that affine Eisenstein series induced from cusp forms over $\mathbb{Q}$ are entire functions of the spectral parameter. It should be mentioned that earlier results were established in the function field setting by [BK]; related results also appear in [Ka, LL, P].

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Beyond the affine case, Carbone, Lee, and Liu [CLL] studied Eisenstein series on the rank 2 hyperbolic Kac–Moody groups with symmetric generalized Cartan matrices, and established almost-everywhere convergence of the series. However, they could not obtain everywhere convergence as the method in the affine case does not generalize directly to the hyperbolic case.

Borel Eisenstein series. In this paper we consider the problem of establishing the absolute convergence of Eisenstein series on arbitrary Kac–Moody groups, for spectral parameters λ in the traditionally-studied Godement range (i.e., $\mathrm{Re}(\lambda-\rho)$ is strictly dominant). We indeed establish their almost-everywhere convergence in this generality, and additionally show the absolute convergence for a wide class of groups.

More precisely, let G be a representation-theoretic Kac–Moody group, and let $\mathfrak{g}$ be the corresponding real Kac–Moody algebra with a fixed Cartan subalgebra $\mathfrak{h}$. We assume that $\mathfrak{g}$ is infinite-dimensional and non-affine, since the finite-dimensional and affine cases have been well studied. Let $r=\dim(\mathfrak{h})$ denote the rank of G, I the index set $\{1,\dots,r\}$, and Φ_+ (resp., Φ_-) the positive (resp., negative) roots of $\mathfrak{g}_{\mathbb{C}}$. Then $G_{\mathbb{R}}$ has the Iwasawa decomposition $G_{\mathbb{R}}=UA^+K$, where U is a maximal pro-unipotent subgroup, A^+ is the connected component of a maximal torus, and K is a subgroup of G playing the role of the maximal compact subgroup from the finite-dimensional theory. (See Section 2 for more details.)

We formally define the Borel Kac–Moody Eisenstein series $E_\lambda(g)$ for $g\in G_{\mathbb{R}}$ and $\lambda\in\mathfrak{h}^*_{\mathbb{C}}$ by

$$E_\lambda(g)=\sum_{\gamma\in(\Gamma\cap B)\backslash\Gamma} a(\gamma g)^{\lambda+\rho}, \tag{1.1}$$

where $a(g)$ is the A^+–component of the Iwasawa decomposition of g, ρ is the Weyl vector, $\Gamma=G_{\mathbb{Z}}$ is the arithmetic subgroup defined at the end of Section 2, and $B\supset UA^+$ is a Borel subgroup. The goal of this paper is to study the convergence and analyticity of this formal sum. Our main result can be stated as follows:

THEOREM 1.1. *Assume that $\lambda\in\mathfrak{h}^*_{\mathbb{C}}$ satisfies $\mathrm{Re}(\langle\lambda,\alpha_i^\vee\rangle)>1$ for each simple coroot $\alpha_i^\vee$, $i\in I$, and that Property 4.2 holds. Then the Kac–Moody Eisenstein series $E_\lambda(g)$ converges absolutely for $g\in\Gamma U A_{\mathfrak{C}}K$, where $A_{\mathfrak{C}}\subset A^+$ is the image of the Tits cone $\mathfrak{C}$ under the exponential map* $\exp:\mathfrak{h}\to A^+$.

The condition on λ is precisely the Godement range, and appears in the classical theory [La2]. Property 4.2 and hence the conclusions of Theorem 1.1 hold for all G of rank 2 (Proposition 4.4), as well when the Cartan matrix is symmetric and has sufficiently large entries (Proposition 4.3). However, Property 4.2 is not true for all Kac–Moody root systems—it even fails in the finite-dimensional Example 4.5.

Though a Godement condition is very natural on the spectral parameter, it is not clear what the full range of absolute convergence of $E_\lambda(g)$ is in the variable g.

Theorem 1.1 adopts a Tits cone constraint, one which naturally occurs in the literature when studying simpler sums, e.g., in the Lemma of Looijenga [Lo] (see also [K, Section 10.6]) quoted in Lemma 3.2, which corresponds to g in the set $UA_{\mathfrak{C}}K$. The extension to the domain $\Gamma UA_{\mathfrak{C}}K$—which Proposition 4.10 shows is in fact larger—comes from the Γ-invariance in (1.1).

In order to prove Theorem 1.1 we consider the constant term $E_\lambda^\sharp(g)$ of the series $E_\lambda(g)$, which is computed by the Gindikin–Karpelevich formula, and establish its absolute convergence in Theorem 3.3. This is achieved by using Looijenga's Lemma and the observation that $M^{\ell(w)}c(\lambda, w)$ is bounded for any fixed $M > 0$, as w varies over the Weyl group W. Here the function $c(\lambda, w)$ is the product of ratios of the Riemann ζ-function defined in (3.2). Unlike Theorem 1.1, which assumes Property 4.2, Theorem 3.3 and its almost-everywhere convergence Corollary 3.4 hold for *all* symmetrizable G.

In Section 4 we show that the Kac–Moody Eisenstein series $E_\lambda(g)$ is dominated by a sum over the Weyl group W, which is a close variant of the constant term $E_\lambda^\sharp(g)$. Having proved the convergence of $E_\lambda^\sharp(g)$ in Theorem 3.3, we use some consequences of Property 4.2 to establish Theorem 1.1.

Remark. Though Theorem 1.1 is stated only for Borel Eisenstein series, its conclusions (with modifications just as in the classical finite-dimensional setting) hold for *cuspidally* induced Eisenstein series as well, at least for parabolics with finite-dimensional Levi components. This is because cusp forms on finite-dimensional semisimple groups are bounded, hence bounded above by certain Borel Eisenstein series on the Levi component. A parabolic Eisenstein series for G induced from such an Eisenstein series is itself a Borel Eisenstein series for G, hence Theorem 1.1 implies a corresponding convergence statement for Eisenstein series induced from cusp forms. See [Bo, Proposition 12.6], [MW, Proposition II.1.5], and [G11, Section 3], where this argument is carried out in more detail in simpler settings. For the same reason, the weaker statements Theorem 3.3 and Corollary 3.4 also transfer to results for almost-everywhere convergence for cuspidally-induced Eisenstein series.

We expect Kac–Moody Eisenstein series to provide interesting applications. In analogy with $SL_2(\mathbb{Z})$, the group $E_{10}(\mathbb{Z})$ is conjectured to be the discrete invariance group for certain functions that arise in 11-dimensional supersymmetric string theory [DKN, Ga]. Automorphic forms on E_{10} and E_{11} are conjectured to encode higher derivative corrections in string theory and M-theory [DN, DHH+, FGKP, W].

Eisenstein series on Kac–Moody groups $G_{\mathbb{R}}$ are invariant under translations by $G_{\mathbb{Z}}$ and hence have Fourier expansions. In [CLL], the authors defined and calculated the degenerate Fourier coefficients for Eisenstein series on rank 2 hyperbolic Kac–Moody groups over $\mathbb{R}$. Fleig [Fl] gave the Fourier integrals needed to obtain the constant term and higher order Fourier modes for Eisenstein series on E_9,

E_{10}, and E_{11}, and showed how the string scattering amplitude "collapse mechanism" of [FK] extends to the higher order Fourier modes. He also gave explicit expressions for the constant terms and Fourier modes of some Kac–Moody Eisenstein series. However, these calculations tacitly assume absolute convergence (and in some cases, meromorphic continuation) of the Kac–Moody Eisenstein series, which has yet to be accomplished for E_{10} and E_{11}; Theorem 1.1 does not cover these cases because of its restrictive assumption of Property 4.2.

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2. Kac–Moody groups. Let $I = \{1, 2, \dots, r\}$, $\mathsf{A} = (a_{ij})_{i,j\in I}$ be an $r \times r$ symmetrizable generalized Cartan matrix, and $(\mathfrak{h}, \Delta, \Delta^\vee)$ be a realization of A, where $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ and $\Delta^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\} \subset \mathfrak{h}$ are the set of simple roots and set of simple coroots, respectively (see [K, Section 1] for definitions).

Throughout this paper we shall make the simplifying assumption that A is nonsingular, which means that $\mathfrak{h}$ and $\mathfrak{h}^*$ are spanned by the simple roots α_i and simple coroots $\alpha_i^\vee$, respectively. In particular we intentionally exclude the affine case, which has been studied extensively in Garland's works and in [GMP] (and which has a somewhat different flavor anyhow).

Recall that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. Denote the fundamental weights by $\varpi_i \in \mathfrak{h}^*$, $i \in I$, which form the basis of $\mathfrak{h}^*$ dual to the $\alpha_i^\vee$. Their integral span is the weight lattice P.

Let $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{C}(\mathsf{A})$ be the Kac–Moody algebra associated to $(\mathfrak{h}, \Delta, \Delta^\vee)$. We denote by Φ the set of roots of $\mathfrak{g}_\mathbb{C}$ and have $\Phi = \Phi_+ \sqcup \Phi_-$, where Φ_+ (resp. Φ_-) is the set of positive (resp. negative) roots corresponding to the choice of Δ. Let $w_i := w_{\alpha_i}$ denote the simple Weyl reflection associated to the simple root α_i; the w_i for $i \in I$ generate the Weyl group W of $\mathfrak{g}_\mathbb{C}$. A root $\alpha \in \Phi$ is called a *real root* if there exists $w \in W$ such that $w\alpha$ is a simple root. A root α which is not real is called *imaginary*. For each real root α written as $w\alpha_i$ for some $w \in W$ and $i \in I$, its associated coroot is well defined by the formula $\alpha^\vee = w\alpha_i^\vee$ [Ku, 1.3.8].

For $i \in I$ let e_i and f_i be the Chevalley generators of $\mathfrak{g}_\mathbb{C}$; we denote by $\mathfrak{g}$ the real Lie subalgebra they generate, so that $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. Let $\mathcal{U}_\mathbb{C}$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Let $\Lambda \subseteq \mathfrak{h}^*$ be the integral linear span of the simple roots α_i, $i \in I$, and let $\Lambda^\vee \subseteq \mathfrak{h}$ be the integral span of the simple coroots $\alpha_i^\vee$, $i \in I$. Let $\mathcal{U}_\mathbb{Z} \subseteq \mathcal{U}_\mathbb{C}$ be the $\mathbb{Z}$–subalgebra generated by

$$\frac{e_i^m}{m!}, \quad \frac{f_i^m}{m!}, \quad \text{and} \quad \binom{h}{m} = \frac{h(h-1)\cdots(h-m+1)}{m!}$$

for $i \in I$, $h \in \Lambda^\vee$, and $m \geq 0$ (see [CG, Section 4]).

We will now review the representation theoretic Kac–Moody groups G_F associated to $\mathfrak{g}$ and a field F. The paper [CG] provides an explicit construction for arbitrary fields F, and we shall presently review the construction for the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Q}_p$. Keeping this in mind, we assume $F \supset \mathbb{Q}$.

Let (π, V) denote the unique irreducible highest weight module for $\mathfrak{g}$ corresponding to a choice of some dominant integral weight, and let $v \in V$ be a nonzero highest weight vector. We set

$$V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} \cdot v.$$

Then $V_{\mathbb{Z}}$ is a $\mathcal{U}_{\mathbb{Z}}$–module contained in $V_F = F \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$. Since V is integrable, e_i and f_i are locally nilpotent on V and $V_{\mathbb{Z}}$. It follows that they are locally nilpotent on V_F and hence elements of $\mathrm{End}(V_F)$. Thus for $s, t \in F$ and $i \in I$, their exponentials

$$u_{\alpha_i}(s) = \exp(\pi(se_i)) \quad \text{and} \quad u_{-\alpha_i}(t) = \exp(\pi(tf_i))$$

are actually locally finite sums (meaning their action on any fixed vector is given by a finite sum), and thus define elements of $\mathrm{Aut}(V_F)$.

For $t \in F^\times$ and $i \in I$ we set

$$w_{\alpha_i}(t) = u_{\alpha_i}(t) u_{-\alpha_i}(-t^{-1}) u_{\alpha_i}(t)$$

and define

$$h_{\alpha_i}(t) = w_{\alpha_i}(t) w_{\alpha_i}(1)^{-1}.$$

Each simple root α_j defines a character on $\{h_{\alpha_i}(t) | t \in F^\times\}$ by

$$h_{\alpha_i}(t)^{\alpha_j} = t^{\langle \alpha_j, \alpha_i^\vee \rangle}. \tag{2.1}$$

The subgroup $\langle w_{\alpha_i}(1) : i \in I \rangle$ of $\mathrm{Aut}(V_F)$ contains a full set of Weyl group representatives. For a real root α we let $u_\alpha(s)$, $s \in F$, denote a choice of corresponding one-parameter subgroup, chosen so that

$$u_\alpha(s) = w u_{\alpha_i}(\pm s) w^{-1} \in \mathrm{Aut}(V_F) \quad (s \in F) \tag{2.2}$$

for $w = w_{\beta_1}(1) \cdots w_{\beta_\ell}(1)$ and $\alpha = w_{\beta_1} \cdots w_{\beta_\ell} \alpha_i$, for some $i \in I$, where $\beta_1, \ldots, \beta_\ell \in \Delta$.

We let

$$G_F^0 = \langle u_{\alpha_i}(s), u_{-\alpha_i}(t) : s, t \in F,\ i \in I \rangle \subset \mathrm{Aut}(V_F).$$

Choose a coherently ordered basis (see [CG, Section 5]) $\Psi = \{v_1, v_2, \ldots\}$ of $V_{\mathbb{Z}}$, and denote by B_F^0 the subgroup of G_F^0 consisting of elements which act upper-triangularly with respect to Ψ. For $t \in \mathbb{Z}_{>0}$, we let U_t be the span of the $v_s \in \Psi$ for $s \leq t$. Then $B_F^0 U_t \subseteq U_t$ for each t. Let B_t be the image of B_F^0 in $\mathrm{Aut}(U_t)$. We then have surjective homomorphisms

$$\pi_{tt'} : B_{t'} \longrightarrow B_t, \quad t' \geq t,$$

which we use to define B_F as the projective limit of the projective family $\{B_t, \pi_{tt'}\}$. When the field F is unspecified, B is to be interpreted as $B_{\mathbb{R}}$.

We define a topology on G_F^0 by decreeing that a base of open neighborhoods of the identity is given by the sets

$$V_t = \{g \in G_F^0 : gv_i = v_i,\ i = 1, 2, \ldots, t\}.$$

Let G_F be the completion of G_F^0 with respect to this topology. (See [CLL] for more details.) If $R \supset \mathbb{Z}$ is a subring of F, the subgroup $G_R \subset G_F$ is defined as the stabilizer of $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$ in G_F. For future reference, we define the following subgroups of G_F which play an important role in the rest of the paper:

- $A_F = \langle h_{\alpha_i}(s) : s \in F^\times,\ i \in I\rangle$; and
- $U_F \subset B_F$ is defined exactly as B_F, but with the additional stipulation that elements act unipotently upper triangularly with respect to Ψ. It contains all subgroups parameterized by the $u_\alpha(\cdot)$, where $\alpha \in \Phi_+$ is a real root. Then $B_F = U_F A_F = A_F U_F$. When no subscript is given, U is to be interpreted as $U_{\mathbb{R}}$.

Additionally, the following subgroups are specific to the situation $F = \mathbb{R}$:

- K is the subgroup of $G_{\mathbb{R}}$ generated by all $\exp(t(e_i - f_i))$, $t \in \mathbb{R}$ and $i \in I$ [KP]; and
- $A^+ = \langle h_{\alpha_i}(s) : s \in \mathbb{R}_{>0},\ i \in I\rangle$. In fact, $(\mathbb{R}_{>0})^r$ can be identified with A^+ via the isomorphism $(x_1, \ldots, x_r) \mapsto h_{\alpha_1}(x_1) \cdots h_{\alpha_r}(x_r)$, under which A^+ has the Haar measure da corresponding to $\prod_{i=1}^r \frac{dx_i}{x_i}$.

THEOREM 2.1 ([DGH]). *We have the Iwasawa decomposition*

$$G_{\mathbb{R}} = UA^+K, \tag{2.3}$$

with uniqueness of expression.

We let $u(g)$, $a(g)$, and $k(g)$ denote the projections from $G_{\mathbb{R}}$ onto each of the respective factors in (2.3). We define the discrete group $\Gamma = G_{\mathbb{Z}}$ as $G_{\mathbb{R}} \cap \mathrm{Aut}(V_{\mathbb{Z}}) = \{\gamma \in G_{\mathbb{R}} : \gamma \cdot V_{\mathbb{Z}} = V_{\mathbb{Z}}\}$. As in [G04], it can be shown that $(\Gamma \cap U)\backslash U$ is the projective limit of a projective family of finite-dimensional compact nil-manifolds and thus admits a projective limit measure du, a right U-invariant probability measure.

3. Convergence of the constant term. Using the identification of A^+ with $(\mathbb{R}_{>0})^r$, each element of $\mathfrak{h}^*$ gives rise to a quasicharacter of A^+ by (2.1) and linearity. Let $\lambda \in \mathfrak{h}^*$ and let $\rho \in \mathfrak{h}^*$ be the Weyl vector, which is characterized by $\langle \rho, \alpha_i^\vee \rangle = 1$, $i \in I$. We set

$$\begin{aligned} &\Phi_\lambda : G_{\mathbb{R}} \to \mathbb{C}^\times \\ &\Phi_\lambda : g \mapsto a(g)^{\lambda+\rho}, \end{aligned} \tag{3.1}$$

which is well defined by the uniqueness of the Iwasawa decomposition. Clearly, Φ_λ is left U-invariant and right K-invariant.

Let B and Γ be as defined in Section 2. Define the Eisenstein series on $G_{\mathbb{R}}$ to be the infinite formal sum

$$E_\lambda(g) \quad = \sum_{\gamma \in (\Gamma \cap B)\backslash \Gamma} \Phi_\lambda(\gamma g).$$

Assume first that λ is a real linear combination of the α_i, $i \in I$, so that $\Phi_\lambda > 0$. Then we may interpret the infinite sum $E_\lambda(g)$ as a function taking values in $(0, \infty]$. Moreover, the function E_λ may be regarded as a function on

$$(\Gamma \cap U)\backslash G_{\mathbb{R}}/K \cong (\Gamma \cap U)\backslash U \times A^+$$

by the Iwasawa decomposition (2.3). We define for all $g \in G_{\mathbb{R}}$ the so-called "upper triangular" constant term

$$E_\lambda^\sharp(g) = \int_{(\Gamma \cap U)\backslash U} E_\lambda(ug)du,$$

which is left U-invariant and right K-invariant. (This is to be distinguished from the proposal in [BK] to consider constant terms in uncompleted, or so-called "lower triangular", parabolics.) In particular $E_\lambda^\sharp(g)$ is determined by the A^+-component $a(g)$ of g in the Iwasawa decomposition. Applying the Gindikin–Karpelevich formula, a formal calculation as in [G04] yields that

$$E_\lambda^\sharp(g) = \sum_{w \in W} a(g)^{w\lambda + \rho} c(\lambda, w) \text{ with } c(\lambda, w) = \prod_{\alpha \in \Phi_w} \frac{\xi(\langle \lambda, \alpha^\vee \rangle)}{\xi(1 + \langle \lambda, \alpha^\vee \rangle)}, \tag{3.2}$$

where

$$\Phi_w = \Phi_+ \cap w^{-1}\Phi_-, \quad w \in W, \tag{3.3}$$

and $\xi(s)$ is the completed Riemann ζ-function

$$\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s),$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. We shall often make use of the following explicit parametrization of Φ_w, where w is written as a reduced word (i.e., minimal length) $w = w_{i_1} w_{i_2} \cdots w_{i_\ell}$ in the generators $\{w_i : i \in I\}$ of W:

$$\Phi_w = \{\alpha_{i_\ell}, w_{i_\ell}\alpha_{i_{\ell-1}}, w_{i_\ell} w_{i_{\ell-1}} \alpha_{i_{\ell-2}}, \ldots, w_{i_\ell} \cdots w_{i_2}\alpha_{i_1}\} \tag{3.4}$$

(see [Ku, Lemma 1.3.14]).

Returning to (3.2), we now state an elementary estimate on the Riemann ζ-function from analytic number theory.

LEMMA 3.1. *One has*

$$\lim_{\sigma\to\infty}\max_{t\in\mathbb{R}}\left|\frac{\xi(\sigma+it)}{\xi(\sigma+1+it)}\right|=0.$$

Proof. It follows from the Euler product $\zeta(s)=\prod_p \frac{1}{1-p^{-s}}$ that

$$\log\zeta(s)=-\sum_p \log(1-p^{-s})=\sum_{n\geq 2} c_n n^{-s},$$

where

$$c_n=\begin{cases}1/k & \text{if } n=p^k \text{ for prime } p,\\ 0 & \text{otherwise.}\end{cases}$$

In particular $|c_n|\leq 1$, hence

$$|\log\zeta(\sigma+it)|\leq\sum_{n\geq 2}|c_n|n^{-\sigma}\leq\sum_{n\geq 2}n^{-\sigma}\longrightarrow 0,\quad \text{as } \sigma\to\infty,$$

e.g., by dominated convergence. Therefore $\lim_{\sigma\to\infty}\zeta(\sigma+it)=1$ uniformly in $t\in\mathbb{R}$. On the other hand, it is easy to see from standard properties of Γ-functions (or applying dominated convergence to the integral formula (4.2)) that

$$\lim_{\sigma\to\infty}\frac{\Gamma_{\mathbb{R}}(\sigma+it)}{\Gamma_{\mathbb{R}}(\sigma+it+1)}=0,$$

again uniformly in $t\in\mathbb{R}$. Thus the Lemma follows by multiplying these two estimates. □

Let $\mathcal{C}\subset\mathfrak{h}$ be the open fundamental chamber

$$\mathcal{C}=\{x\in\mathfrak{h}:\langle\alpha_i,x\rangle>0,\ i\in I\}.$$

Let $\mathfrak{C}$ denote the interior of the Tits cone $\bigcup_{w\in W} w\overline{\mathcal{C}}$ corresponding to $\mathcal{C}$ [K, Section 3.12]. Using the exponential map $\exp:\mathfrak{h}\to A^+$, set $A_{\mathcal{C}}=\exp\mathcal{C}$ and $A_{\mathfrak{C}}=\exp\mathfrak{C}$. Let

$$\mathcal{C}^*=\{\lambda\in\mathfrak{h}^*:\langle\lambda,\alpha_i^\vee\rangle>0,\ i\in I\}. \tag{3.5}$$

Let $\mathcal{K}$ be a compact subset of $\mathfrak{C}$ and $\mu\in P\cap\mathcal{C}^*$. We define $A_{\mathcal{K},\mu}(N)$ to be the number of μ' in the Weyl orbit $W\cdot\{\mu\}$ whose maximum on $\mathcal{K}$ is $\geq -N$. We now recall a lemma due to Looijenga [Lo] (which is contained in Lemma 3.2 and the beginning of the proof of Proposition 3.4 there—see also [K, Section 10.6]).

LEMMA 3.2. *We have $A_{\mathcal{K},\mu}(N)=O(N^r)$ as $N\to\infty$. Furthermore, for $\lambda\in\mathfrak{h}^*_{\mathbb{C}}$ with $\mathrm{Re}(\lambda)\in\mathcal{C}^*$, $\sum_{w\in W}a^{w\lambda}$ converges absolutely and uniformly for a in any fixed compact subset of $A_{\mathfrak{C}}$.*

We apply Lemmas 3.1 and 3.2 to show the following convergence result.

THEOREM 3.3. *If $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda-\rho) \in \mathcal{C}^*$, then $E_\lambda^\sharp(g)$ given in* (3.2) *converges absolutely for $g \in UA_{\mathfrak{C}}K$, and in fact uniformly for $a(g)$ lying in any fixed compact subset of $A_{\mathfrak{C}}$.*

Proof. By applying absolute values, we may assume without loss of generality that λ is real. Let $S \geq 1$ be a constant such that $|\frac{\xi(s)}{\xi(s+1)}| \leq 1$ when $\mathrm{Re}(s) > S$, which exists by Lemma 3.1. Since $\langle \lambda, \alpha_i^\vee \rangle > 1$ for all $i \in I$, we see that $\langle \lambda, \alpha^\vee \rangle > S$ for all but finitely many positive roots α. Therefore $c(\lambda, w)$ is bounded in $w \in W$, and the convergence follows from that of $\sum_{w\in W} a^{w\lambda}$ in Lemma 3.2. □

By applying Tonelli's theorem as in [G04, Section 9], we obtain the following result for arbitrary Kac–Moody groups.

COROLLARY 3.4. *For $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ with $\mathrm{Re}(\lambda-\rho) \in \mathcal{C}^*$ and any compact subset $\mathfrak{S}$ of $A_{\mathfrak{C}}$, there exists a measure zero subset S_0 of $(\Gamma \cap U)\backslash U\mathfrak{S}$ such that the series $E_\lambda(g)$ converges absolutely for $g \in U\mathfrak{S}K$ off the set S_0K.*

For later use, we further strengthen Lemma 3.2 and prove that

THEOREM 3.5. *Assume that $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ with $\mathrm{Re}(\lambda) \in \mathcal{C}^*$. Then for any $M > 0$,*

$$\sum_{w\in W} M^{\ell(w)} a^{w\lambda}$$

converges absolutely and uniformly for a in any fixed compact subset of $A_{\mathfrak{C}}$.

Proof. We take advantage of the fact that the constraints on both λ and a are preserved under small perturbations. Let $s > 0$ be sufficiently small so that $\lambda' := \lambda - s\rho \in \mathcal{C}^*$. Then

$$M^{\ell(w)} a^{w\lambda} = M^{\ell(w)} a^{ws\rho + w\lambda'} \leq \frac{1}{2}\big(M^{2\ell(w)} a^{2ws\rho} + a^{2w\lambda'}\big).$$

Since the convergence of $\sum_{w\in W} a^{2w\lambda'}$ is handled by Lemma 3.2, it suffices to prove the absolute and uniform convergence of

$$\sum_{w\in W} M^{2\ell(w)} a^{2ws\rho} = \sum_{w\in W} T^{\ell(w)} a^{2ws\rho}, \quad T = M^2,$$

for a in any fixed compact subset of $A_{\mathfrak{C}}$, for any $T, s > 0$.

Using the fact that

$$\rho - w\rho = \sum_{\alpha \in \Phi_{w^{-1}}} \alpha \tag{3.6}$$

is a sum of $\ell(w)$ positive roots (see (3.4) and [Ku, 1.3.22 (3)]), we may write

$$T^{\ell(w)}a^{2ws\rho} = a^{2s\rho}\prod_{\alpha\in\Phi_{w^{-1}}} Ta^{-2s\alpha} = a^{2s\rho}\prod_{\alpha\in\Phi_{w^{-1}}} (Ta^{-s\alpha})a^{-s\alpha}.$$

We shall prove that $a^{s\alpha} > T$ for all but finitely many positive roots α (in particular, those in $\Phi_{w^{-1}}$). Assuming this momentarily, the parenthetical term $Ta^{-s\alpha}$ is bounded, and can only be greater than 1 for finitely many positive roots α. Hence there exists a constant $D > 0$ depending continuously on a, s and T such that

$$T^{\ell(w)}a^{2ws\rho} \le Da^{2s\rho}\prod_{\alpha\in\Phi_{w^{-1}}} a^{-s\alpha} = Da^{2s\rho}a^{s(w\rho-\rho)} = Da^{s(w\rho+\rho)}.$$

Hence

$$\sum_{w\in W} T^{\ell(w)}a^{2ws\rho} \le Da^{s\rho}\sum_{w\in W} a^{w(s\rho)},$$

and the Theorem follows from a second application of Lemma 3.2.

Finally, we return to the claim that $a^{s\alpha} > T$ for all but finitely many positive roots α. Let H_ρ denote the unique element of $\mathfrak{h}$ such that $\langle\alpha_i, H_\rho\rangle = \langle\rho, \alpha_i^\vee\rangle = 1$ for all $i \in I$ (it exists by the assumed nondegeneracy of the Cartan matrix). Since a is assumed to lie in the open set $A_{\mathfrak{C}}$, there exists some $\varepsilon > 0$ such that $a = a_1a_2$, with a_1 also an element of $A_{\mathfrak{C}}$ and $a_2 = e^{\varepsilon H_\rho}$. Then

$$a^{s\alpha} = a_1^{s\alpha}\,a_2^{s\alpha} = a_1^{s\alpha}\,e^{\varepsilon s\langle\alpha,H_\rho\rangle}. \tag{3.7}$$

Since $a_1 \in A_{\mathfrak{C}}$, the values of a_1^α are at least 1 for all but finitely many α [K, Prop. 3.12(c)]. At the same time, $\langle\alpha, H_\rho\rangle$ is the sum of the (nonnegative) coefficients of α when expanded as a linear combination of simple roots, and thus tends to infinity as α varies. The claim now follows from (3.7). □

4. Everywhere convergence of Eisenstein series. In the previous section we demonstrated the almost-everywhere absolute convergence of Eisenstein series. In this section we will prove the absolute convergence of the Eisenstein series on $\Gamma U A_{\mathfrak{C}} K$ under Godement's criterion on the spectral parameter λ, subject to a condition (Property 4.2) on the root system of the Kac–Moody group. The key idea is that the Eisenstein series can be nearly bounded by its constant term.

We start with some calculations for the group $SL(2,\mathbb{R})$. It follows from direct computation that

$$a\left(\begin{pmatrix}0 & -1\\ 1 & u\end{pmatrix}\right)^\alpha = \begin{pmatrix}\frac{1}{\sqrt{1+u^2}} & 0\\ 0 & \sqrt{1+u^2}\end{pmatrix}^\alpha = \frac{1}{1+u^2}, \tag{4.1}$$

where α is the positive simple root of the diagonal Cartan. Define

$$c_\infty(s) := \int_{\mathbb{R}} a\left(\begin{pmatrix} 0 & -1 \\ 1 & u \end{pmatrix}\right)^{(s+1)\alpha/2} du \tag{4.2}$$
$$= \int_{\mathbb{R}} (1+u^2)^{-\frac{s+1}{2}} du \;=\; \frac{\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(s+1)}, \quad \mathrm{Re}(s) > 0.$$

Returning to the setting of a general Kac–Moody group $G_{\mathbb{R}}$, we shall now assume that $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ is actually real, i.e., $\lambda \in \mathfrak{h}^*$, since this entails no loss of generality in considering absolute convergence. Recall the notation $u_\alpha(x)$ from (2.2).

LEMMA 4.1. *Assume that α is a positive simple root such that $\langle \lambda, \alpha^\vee \rangle > 0$ (equivalently, $\langle \lambda + \rho, \alpha^\vee \rangle > 1$). For any $x \in \mathbb{R}$ and $g = uak \in G_{\mathbb{R}}$, with $u \in U$, $a \in A^+$, and $k \in K$, we have*

$$\sum_{m \in \mathbb{Z}} a(w_\alpha u_\alpha(x+m)g)^{\lambda+\rho} \le a^{w_\alpha(\lambda+\rho)}\big(2 + a^\alpha c_\infty(\langle \lambda, \alpha^\vee \rangle)\big).$$

In particular, for any $\epsilon > 0$ there exists a constant $M = M_\epsilon > 2$ such that

$$\sum_{m \in \mathbb{Z}} a(w_\alpha u_\alpha(x+m)g)^{\lambda+\rho} \le M a^{w_\alpha(\lambda+\rho)}(1+a^\alpha) \tag{4.3}$$

whenever $\langle \lambda, \alpha^\vee \rangle \ge \epsilon$, uniformly over $x \in \mathbb{R}$.

Proof. Factor $u = u_\alpha(y)u^{(\alpha)}$, where $y \in \mathbb{R}$ and $u^{(\alpha)}$ lies in the unipotent radical $U^{(\alpha)}$ of the parabolic subgroup whose Levi component is generated by A and the one parameter subgroups $u_{\pm\alpha}(\cdot)$. Then since $w_\alpha u_\alpha(x+m+y)$ lies inside this Levi, it normalizes $U^{(\alpha)}$. Hence we may write

$$\begin{aligned} a(w_\alpha u_\alpha(x+m)g) &= a(w_\alpha u_\alpha(x+m+y)u^{(\alpha)}ak) \\ &= a(w_\alpha u_\alpha(x+m+y)a), \end{aligned}$$

since $U^{(\alpha)} \subset U$. As x is arbitrary and plays no role in the asserted upper bounds, we may replace x by $x-y$ so that we are instead considering the sum

$$\sum_{m \in \mathbb{Z}} a(w_\alpha u_\alpha(x+m)a)^{\lambda+\rho} = a^{w_\alpha(\lambda+\rho)} \sum_{m \in \mathbb{Z}} a(w_\alpha u_\alpha(a^{-\alpha}(x+m)))^{\lambda+\rho}. \tag{4.4}$$

Recalling (4.1), the sum (4.4) becomes

$$a^{w_\alpha(\lambda+\rho)} \sum_{m \in \mathbb{Z}} (1 + a^{-2\alpha}(x+m)^2)^{-\frac{1}{2}\langle \lambda+\rho, \alpha^\vee \rangle}.$$

Since $\langle \lambda+\rho, \alpha^\vee\rangle = \langle \lambda, \alpha^\vee\rangle + 1$, comparing the sum to its corresponding integral gives the estimate

$$\begin{aligned}\sum_{m\in\mathbb{Z}} (1+a^{-2\alpha}(x+m)^2)^{-\frac{s+1}{2}} &\le 2+\int_{\mathbb{R}} (1+a^{-2\alpha}x^2)^{-\frac{s+1}{2}}\,dx \\ &= 2+a^{\alpha}\,c_\infty(s),\end{aligned}$$

for $s=\langle \lambda, \alpha^\vee\rangle > 0$. The second statement (4.3) for $M=\max(2, c_\infty(\epsilon))$ now follows from the bound of $c_\infty(\langle \lambda, \alpha^\vee\rangle) \le c_\infty(\epsilon)$. □

We wish to generalize (4.3) to arbitrary $w \in W$ by induction on $\ell(w)$, which is the most complicated part of the argument. For technical reasons we need to introduce the following property. Recall the notation $\Phi_w = \Phi_+ \cap w^{-1}\Phi_-$ from (3.3)–(3.4).

PROPERTY 4.2. *Every nontrivial $w \in W$ can be written as $w = vw_\beta$, where β is a positive simple root, $\ell(v) < \ell(w)$, and $\alpha - \beta$ is never a real root for any $\alpha \in \Phi_v$.*

It is not hard to see (e.g., see [T, Section 3.2]) that Property 4.2 is unchanged if the word "real" is omitted. This is because if $\alpha - \beta = \alpha + w_\beta\beta$ is a (necessarily, positive) root, then $v(\alpha-\beta) = v\alpha + w\beta$ is a negative root (cf. (3.4)), and all elements of Φ_v are real roots (again appealing to (3.4)).

Although Property 4.2 does not hold for arbitrary Kac–Moody root systems (see Example 4.5 for counterexamples), we shall nevertheless demonstrate that it holds in infinitely many examples.

PROPOSITION 4.3. *Property 4.2 holds if the Cartan matrix $\mathsf{A} = (a_{ij})$ of $\mathfrak{g}$ is symmetric and $|a_{ij}| \ge 2$ for all $i, j \in I$.*

Proof. First we claim that if $w_{i_1}\cdots w_{i_k}$ is a reduced word and the (positive, by (3.4)) root $w_{i_k}\cdots w_{i_2}\alpha_{i_1}$ is expanded as $\sum_{j\in I} m_j\alpha_j$, then $m_{i_k} > m_j$ for $j \ne i_k$. Indeed, this can be seen as follows by induction, the case $k=2$ being a consequence of our assumption. Assume that the claim is true for k. Then we write $w_{i_{k+1}}w_{i_k}\cdots w_{i_2}\alpha_{i_1} = \sum_{j\in I} m'_j\alpha_j$ and obtain

$$\begin{aligned}\sum_{j\in I} m'_j\alpha_j &= w_{i_{k+1}}\sum_{j\in I} m_j\alpha_j \\ &= m_{i_k}(\alpha_{i_k} - a_{i_{k+1}i_k}\alpha_{i_{k+1}}) - m_{i_{k+1}}\alpha_{i_{k+1}} \\ &\quad + \sum_{j\ne i_k, i_{k+1}} m_j(\alpha_j - a_{i_{k+1}j}\alpha_{i_{k+1}}).\end{aligned}$$

We have $m'_j = m_j < m_{i_k} = m'_{i_k}$ for $j \neq i_k, i_{k+1}$ and

$$m'_{i_{k+1}} = -m_{i_k} a_{i_{k+1} i_k} - m_{i_{k+1}} - \sum_{j \neq i_k, i_{k+1}} m_j a_{i_{k+1} j} > m_{i_k} = m'_{i_k},$$

since $-(1 + a_{i_{k+1} i_k}) m_{i_k} - m_{i_{k+1}} \geq m_{i_k} - m_{i_{k+1}} > 0$. This proves the claim.

We now claim that for any reduced word $w = w_{i_1} \cdots w_{i_\ell}$, one has $\langle \alpha, \alpha_{i_\ell}^\vee \rangle < 0$ for each $\alpha \in \Phi_v$, where $v = w_{i_1} \cdots w_{i_{\ell-1}}$. By (3.4) the elements of Φ_v can be written as $w_{i_{\ell-1}} \cdots w_{i_{k+1}} \alpha_{i_k}$, $k = 1, \ldots, \ell - 1$. By the above claim, if $w_{i_{\ell-1}} \cdots w_{i_{k+1}} \alpha_{i_k} = \sum_{j \in I} m_j \alpha_j$, then $m_{i_{\ell-1}} > m_j$ for $j \neq i_{\ell-1}$. In particular, $m_{i_{\ell-1}} > m_{i_\ell}$ since $i_\ell \neq i_{\ell-1}$ for a reduced word. Therefore

$$\begin{aligned} \langle w_{i_{\ell-1}} \cdots w_{i_{k+1}} \alpha_{i_k}, \alpha_{i_\ell}^\vee \rangle &= \Big\langle \sum_{j \in I} m_j \alpha_j, \alpha_{i_\ell}^\vee \Big\rangle = \sum_{j \neq i_\ell} m_j a_{i_\ell j} + 2 m_{i_\ell} \\ &\leq m_{i_{\ell-1}} a_{i_\ell i_{\ell-1}} + 2 m_{i_\ell} < 0 \end{aligned} \tag{4.5}$$

after dropping all but the $j = i_{\ell-1}$ term from the sum.

Let $(\cdot|\cdot)$ denote the symmetric bilinear form on $\mathfrak{h}^*$ defined in [K, Section 2.3], which in general involves a choice of scaling by a positive constant; since A is symmetric it can be written as $(\alpha_i|\alpha_j) = a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$. The bilinear form $(\cdot|\cdot)$ is Weyl invariant [K, Prop. 3.9], and so one has $(\beta|\beta) = (\alpha_i|\alpha_i)$ for any root of the form $\beta = w\alpha_i$, for some $w \in W$ and $i \in I$. On the other hand, [K, Prop. 5.2(c)] shows that $(\beta|\beta) \leq 0$ for any imaginary root (that is, a root which is not a Weyl translate of a simple root). Thus in any event one has

$$(\beta|\beta) \leq \max_{i \in I} (\alpha_i|\alpha_i) \tag{4.6}$$

for any root β. For later reference we remark that even though A here is assumed to be symmetric, the conclusion (4.6) nevertheless holds even if A is merely symmetrizable (with $(\cdot|\cdot)$ as defined in [K, Section 2.3], or any positive scalar multiple of it).

Let $\alpha = w_{i_{\ell-1}} \cdots w_{i_{k+1}} \alpha_{i_k}$ be an element of Φ_v, where $1 \leq k \leq \ell - 1$. Since

$$\Phi_w = w_{i_\ell} \Phi_v \cup \{\alpha_{i_\ell}\} \tag{4.7}$$

by (3.4), α cannot equal α_{i_ℓ}, for if it did then $-\alpha_{i_\ell} = w_{i_\ell} \alpha \in \Phi_w$ would be a positive root. We compute

$$\begin{aligned} (\alpha - \alpha_{i_\ell}|\alpha - \alpha_{i_\ell}) &= (\alpha_{i_k}|\alpha_{i_k}) + (\alpha_{i_\ell}|\alpha_{i_\ell}) - 2(\alpha|\alpha_{i_\ell}) \\ &= 4 - 2\langle \alpha, \alpha_{i_\ell}^\vee \rangle > 4 \end{aligned}$$

using the second claim above. Thus $\alpha - \alpha_{i_\ell}$ is not a root for $\alpha \in \Phi_v$. □

Unfortunately Proposition 4.3's restrictive assumption on $|a_{ij}|$ rules out many cases of interest. In fact, only one hyperbolic Kac–Moody algebra of rank ≥ 3 satisfies its hypotheses, namely the one having $\mathsf{A} = \left(\begin{smallmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{smallmatrix}\right)$. In contrast, however, the assumptions of Proposition 4.3 hold for many rank 2 hyperbolic root systems, even ones with non-symmetric Cartan matrices:

PROPOSITION 4.4. *Let* $\mathsf{A} = \left(\begin{smallmatrix} 2 & -b \\ -a & 2 \end{smallmatrix}\right)$, $a, b \geq 2$ *and* $ab \geq 5$. *Then Property 4.2 holds for the Kac–Moody root system associated to* A.

Proof. The Weyl group of the root system is the infinite dihedral group generated by the simple reflections w_1 and w_2. In terms of the basis $\{\alpha_1, \alpha_2\}$ of $\mathfrak{h}^*$, w_1 acts by the matrix $\left(\begin{smallmatrix} -1 & b \\ 0 & 1 \end{smallmatrix}\right)$ and w_2 acts by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ a & -1 \end{smallmatrix}\right)$. Since w_1 and w_2 play symmetric roles, it suffices to establish Property 4.2 in the case that $\beta = \alpha_2$. Thus $w = vw_\beta$ is represented by a reduced word ending in w_2, and hence v is represented by a reduced word ending in w_1. In particular, v must have the form $(w_2w_1)^m$ or $w_1(w_2w_1)^m$ for some $m \geq 0$. The roots $\alpha \in \Phi_v$ either have the form

$$(w_1w_2)^n\alpha_1 \quad \text{or} \quad (w_1w_2)^n w_1\alpha_2,$$

for some $n \geq 0$ by (3.4). To establish Property 4.2, we show that none of

$$(w_1w_2)^n\alpha_1 - \alpha_2 \quad \text{or} \quad (w_1w_2)^n w_1\alpha_2 - \alpha_2, \quad n \geq 0, \tag{4.8}$$

are roots. Following the strategy in the proof of Proposition 4.3, we will use the symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ associated to the symmetrized matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\mathsf{A} = \left(\begin{smallmatrix} 2a & -ab \\ -ab & 2b \end{smallmatrix}\right)$, and appeal to condition (4.6), which on our context states that

$$(\beta|\beta) \leq \max\{2a, 2b\} \tag{4.9}$$

for any root β.

Let $\mu = \frac{\sqrt{ab}+\sqrt{ab-4}}{2} > 1$ and $h_n = \frac{1}{\mu-\mu^{-1}}(\mu^n - \mu^{-n})$, which is a monotonically increasing function of n satisfying

$$h_{n+2} = h_n + \mu^{n+1} + \mu^{-n-1}.$$

Since $\sqrt{ab} > 2$ we have for any $n \geq 1$ that

$$\begin{aligned} \sqrt{ab}\,h_{n+2} - 2h_{n+1} &= \sqrt{ab}\,(\mu^{n+1} + \mu^{-n-1}) - 2(\mu^n + \mu^{-n}) + \sqrt{ab}\,h_n - 2h_{n-1} \\ &> \sqrt{ab}\,h_n - 2h_{n-1}. \end{aligned}$$

Repeating, we obtain

$$\begin{aligned} \sqrt{ab}\,h_{2n+2} - 2h_{2n+1} &\geq \sqrt{ab}\,h_2 - 2h_1 = ab - 2, \\ \sqrt{ab}\,h_{2n+1} - 2h_{2n} &\geq \sqrt{ab}\,h_1 - 2h_0 = \sqrt{ab} > 2 \end{aligned} \tag{4.10}$$

for any $n \geq 0$.

It follows from the formulas for the Weyl action that

$$(w_1w_2)^n\alpha_1 = h_{2n+1}\alpha_1 + \frac{\sqrt{a}}{\sqrt{b}}\,h_{2n}\alpha_2,$$
$$(w_1w_2)^n w_1\alpha_2 = \frac{\sqrt{b}}{\sqrt{a}}\,h_{2n+2}\alpha_1 + h_{2n+1}\alpha_2$$

(the case $n = 0$ is obvious, and both sides satisfy the same recurrence relations in n). Since $(\cdot|\cdot)$ is W-invariant and $\sqrt{ab} > 2$, we compute

$$\begin{aligned}
&\big((w_1w_2)^n\alpha_1 - \alpha_2|(w_1w_2)^n\alpha_1 - \alpha_2\big)\\
&\quad = (\alpha_1|\alpha_1) + (\alpha_2|\alpha_2) - 2\big((w_1w_2)^n\alpha_1|\alpha_2\big)\\
&\quad = 2a + 2b - 2\big(h_{2n+1}\alpha_1 + \tfrac{\sqrt{a}}{\sqrt{b}}\,h_{2n}\alpha_2|\alpha_2\big)\\
&\quad = 2a + 2b + 2ab\,h_{2n+1} - 4\sqrt{ab}\,h_{2n}\\
&\quad = 2a + 2b + 2\sqrt{ab}(\sqrt{ab}\,h_{2n+1} - 2\,h_{2n})\\
&\quad > 2a + 2b > \max\{2a, 2b\},
\end{aligned}$$

so that $(w_1w_2)^n\alpha_1 - \alpha_2$ cannot be a root according to (4.9).

Similarly,

$$\begin{aligned}
&\big((w_1w_2)^n w_1\alpha_2 - \alpha_2|(w_1w_2)^n w_1\alpha_2 - \alpha_2\big)\\
&\quad = 2(\alpha_2|\alpha_2) - 2\big((w_1w_2)^n w_1\alpha_2|\alpha_2\big)\\
&\quad = 4b - 2\big(\tfrac{\sqrt{b}}{\sqrt{a}}\,h_{2n+2}\alpha_1 + h_{2n+1}\alpha_2|\alpha_2\big)\\
&\quad = 4b + 2\tfrac{\sqrt{b}}{\sqrt{a}}ab\,h_{2n+2} - 4b\,h_{2n+1}\\
&\quad = 4b + 2b(\sqrt{ab}\,h_{2n+2} - 2h_{2n+1})\\
&\quad \geq 4b + 2b(ab - 2) = 2ab^2 > \max\{2a, 2b\},
\end{aligned}$$

where we have used the assumption $a, b \geq 2$. Thus $(w_1w_2)^n w_1\alpha_2 - \alpha_2$ is not a root either. □

Example 4.5. (1) Let Φ be the root system for the (non-symmetric generalized) Cartan matrix $\mathsf{A} = \left(\begin{smallmatrix} 2 & -1 \\ -a & 2 \end{smallmatrix}\right)$, $a \geq 5$ (so that $\det \mathsf{A} < 0$). Consider $w = w_{\alpha_2}w_{\alpha_1}w_{\alpha_2}$, which is the unique reduced for w of minimal length. Letting $v = w_{\alpha_2}w_{\alpha_1}$ and $\beta = \alpha_2$, we see $w_{\alpha_1}\alpha_2 \in \Phi_v$ and $w_{\alpha_1}\alpha_2 - \alpha_2 = \alpha_1$, hence Property 4.2 does not hold for this Kac–Moody root system.

(2) Let Φ be the A_2 root system and consider $w = w_{\alpha_2}w_{\alpha_1}w_{\alpha_2} = w_{\alpha_1}w_{\alpha_2}w_{\alpha_1}$. Then Property 4.2 does not hold for w, regardless of how it is written as a reduced word.

In the rest of this section we will establish our main result on convergence (which assumes Property 4.2).

Definition 4.6. Assume that Property 4.2 holds. Then by recursion any non-trivial $w \in W$ can be expressed as a reduced word $w = w_{\beta_1} \cdots w_{\beta_\ell}$ such that

$$\alpha - \beta_{i+1} \text{ is never a root for any } \alpha \in \Phi_{v_i}, \quad i = 1, 2, \ldots, \ell - 1,$$

where $v_i = w_{\beta_1} \cdots w_{\beta_i}$. Such a reduced word will be called an *admissible word* for w.

LEMMA 4.7. *Assume Property 4.2 and that $\lambda \in \mathcal{C}^*$. Let $w = w_{\beta_1} \cdots w_{\beta_\ell}$ be an admissible word for $w \in W$ and again set $v_i = w_{\beta_1} \cdots w_{\beta_i}$. Then*

$$\left\langle v_i^{-1}(\lambda + \rho) + \sum_{\alpha \in S_i} \alpha, \beta_{i+1}^\vee \right\rangle > 1, \quad i = 1, 2, \ldots, \ell - 1, \tag{4.11}$$

for any subset S_i of Φ_{v_i}.

Proof. For $\alpha \in \Phi_{v_i}$, consider the root string $\alpha + m\beta_{i+1}$, $m \in \mathbb{Z}$. Since α and β_i are real roots and $\alpha - \beta_{i+1}$ is not a root by Property 4.2, we have by [K, Prop. 5.1(c)] that

$$\langle \alpha, \beta_{i+1}^\vee \rangle = -\max\{m : \alpha + m\beta_{i+1} \text{ is a root}\} \leq 0.$$

Note that (4.11) is equivalent to

$$\langle v_i^{-1}\lambda - \sum_{\alpha \in \Phi_{v_i} \setminus S_i} \alpha, \beta_{i+1}^\vee \rangle > 0$$

by (3.6). Since $\lambda \in \mathcal{C}^*$ and the fact that $v_i \beta_{i+1}^\vee$ is a positive root (cf. (3.4)), we have

$$\left\langle v_i^{-1}\lambda - \sum_{\alpha \in \Phi_{v_i} \setminus S_i} \alpha, \beta_{i+1}^\vee \right\rangle \geq \langle v_i^{-1}\lambda, \beta_{i+1}^\vee \rangle = \langle \lambda, v_i \beta_{i+1}^\vee \rangle > 0,$$

and the claim follows. □

Let $U_w \subset U$ be the subgroup generated by the one-parameter subgroups for the roots in Φ_w.

LEMMA 4.8. *Assume Property 4.2 and let $w = w_{\beta_1} \cdots w_{\beta_\ell}$ be an admissible word for an element $w \in W$ (see Definition 4.6). Suppose that*

$$u = u_{\nu_1}(x_1) \cdots u_{\nu_\ell}(x_\ell) \in U_w \quad \textit{and} \quad \gamma = u_{\nu_1}(m_1) \cdots u_{\nu_\ell}(m_\ell) \in \Gamma \cap U_w$$

for $x_1, \ldots, x_\ell \in \mathbb{R}$ and $m_1, \ldots, m_\ell \in \mathbb{Z}$, where $\nu_i = w_{\beta_\ell} \cdots w_{\beta_{i+1}} \beta_i \in \Phi_w$, $i = 1, \ldots, \ell$ (cf. (3.4)*). Then*

$$wu\gamma = w_{\beta_1} u_{\beta_1}(\epsilon_1(x_1 + m_1)) \cdots w_{\beta_\ell} u_{\beta_\ell}(\epsilon_\ell(x_\ell + m_\ell))$$

for some signs $\epsilon_1, \dots, \epsilon_\ell \in \{-1, 1\}$ *depending only on* $\beta_1, \dots, \beta_\ell$ *and the choice of Weyl group representatives in* $\mathrm{Aut}(V_F)$.

Proof. We will show that the sum of the positive real roots ν_i and ν_j is not a root, which implies that their root spaces and hence one-parameter subgroups $u_{\nu_i}(\cdot)$ and $u_{\nu_j}(\cdot)$ commute. The Lemma then follows immediately from this commutativity, since

$$u\gamma = u_{\nu_1}(x_1 + m_1) \cdots u_{\nu_\ell}(x_\ell + m_\ell)$$

and one can apply (2.2).

After multiplying on the left by $w_{\beta_j} \cdots w_{\beta_\ell}$, the claim that $\nu_i + \nu_j$ is not a root is equivalent to the statement that

$$w_{\beta_{j-1}} \cdots w_{\beta_{i+1}} \beta_i - \beta_j, \quad 1 \le i < j \le \ell,$$

is never a root. This follows from Definition 4.6 since

$$w_{\beta_{j-1}} \cdots w_{\beta_{i+1}} \beta_i \in \Phi_{v_{j-1}},$$

where $v_{j-1} = w_{\beta_1} \cdots w_{\beta_{j-1}}$ satisfies

$$v_{j-1} w_{\beta_{j-1}} \cdots w_{\beta_{i+1}} \beta_i = w_{\beta_1} \cdots w_{\beta_{i-1}} (w_{\beta_i} \beta_i) = -w_{\beta_1} \cdots w_{\beta_{i-1}} \beta_i$$

and we have used (3.4). □

We can now prove the convergence of Eisenstein series on Kac–Moody groups satisfying Property 4.2, after establishing one more lemma.

LEMMA 4.9. *Assume that* $\lambda \in \mathcal{C}^*$ *and Property 4.2. Then there exists a constant* $M > 0$ *depending continuously on* λ *such that, for an admissible word* $w = w_{\beta_1} \dots w_{\beta_\ell}$,

$$\sum_{m_1, \dots, m_\ell \in \mathbb{Z}} a(w_{\beta_1} u_{\beta_1}(x_1 + m_1) \cdots w_{\beta_\ell} u_{\beta_\ell}(x_\ell + m_\ell) g)^{\lambda + \rho} \le M^\ell a^{w^{-1}(\lambda+\rho)} \prod_{\alpha \in \Phi_w} (1 + a^\alpha),$$

uniformly for $x_1, \dots, x_\ell \in \mathbb{R}$, $g \in G_{\mathbb{R}}$, *and* $a = a(g)$.

Proof. Since λ is in the open chamber $\mathcal{C}^*$, there exists a positive constant $\epsilon > 0$ such that $\lambda - \epsilon\rho \in \mathcal{C}^*$. In particular, $\langle \lambda, \alpha^\vee \rangle \ge \epsilon$ for any positive root α. We will prove the lemma with the value $M = \max(2, c_\infty(\epsilon))$ from the proof of Lemma 4.1, using an induction on $\ell = \ell(w)$. Indeed, the case $\ell = 1$ is precisely (4.3).

Assume the Lemma is known for $v = w_{\beta_1} \cdots w_{\beta_{\ell-1}}$. Recall from (3.4) that

$$\Phi_v = \Phi_+ \cap v^{-1} \Phi_- = \{\beta_{\ell-1}, w_{\beta_{\ell-1}} \beta_{\ell-2}, \dots, w_{\beta_{\ell-1}} \cdots w_{\beta_2} \beta_1\}.$$

and $\Phi_w = w_{\beta_\ell}\Phi_v \cup \{\beta_\ell\}$ (cf. (4.7)). By induction we have

$$\begin{aligned}(4.12)\qquad &\sum_{m_1,\dots,m_\ell\in\mathbb{Z}} a(w_{\beta_1}u_{\beta_1}(x_1+m_1)\cdots w_{\beta_\ell}u_{\beta_\ell}(x_\ell+m_\ell)g)^{\lambda+\rho}\\ &\le M^{\ell-1}\sum_{m_\ell\in\mathbb{Z}} a(w_{\beta_\ell}u_{\beta_\ell}(x_\ell+m_\ell)g)^{v^{-1}(\lambda+\rho)}\prod_{\alpha\in\Phi_v}\left(1+a(w_{\beta_\ell}u_{\beta_\ell}(x_\ell+m_\ell)g)^{\alpha}\right)\\ &= M^{\ell-1}\sum_{S\subset\Phi_v}\sum_{m_\ell\in\mathbb{Z}} a(w_{\beta_\ell}u_{\beta_\ell}(x_\ell+m_\ell)g)^{v^{-1}(\lambda+\rho)+\sum_{\alpha\in S}\alpha}.\end{aligned}$$

The $i=\ell-1$ case of Lemma 4.7 applied to $\lambda-\epsilon\rho$ gives that

$$1<\left\langle v^{-1}(\lambda-\epsilon\rho+\rho)+\sum_{\alpha\in S}\alpha,\beta_\ell^\vee\right\rangle=\left\langle v^{-1}(\lambda+\rho)+\sum_{\alpha\in S}\alpha,\beta_\ell^\vee\right\rangle-\epsilon\langle\rho,v\beta_\ell^\vee\rangle$$

for any $S\subset\Phi_v$. As $v\beta_\ell$ is a positive root by (3.4), $\langle\rho,v\beta_\ell^\vee\rangle$ at least 1 and hence $\langle v^{-1}(\lambda+\rho)+\sum_{\alpha\in S}\alpha,\beta_\ell^\vee\rangle$ must be at least $1+\epsilon$. This shows that the assumptions of Lemma 4.1 (with $\alpha=\beta_\ell$) apply to the m_ℓ-sum, therefore showing (4.12) is bounded by

$$M^\ell\sum_{S\subset\Phi_v} a^{w_{\beta_\ell}v^{-1}(\lambda+\rho)+w_{\beta_\ell}\sum_{\alpha\in S}\alpha}(1+a^{\beta_\ell})=M^\ell a^{w^{-1}(\lambda+\rho)}\prod_{\alpha\in\Phi_w}(1+a^\alpha).\quad\square$$

Proof of Theorem 1.1. We must show that the series $E_\lambda(g)$ converges absolutely for $g\in\Gamma UA_{\mathfrak{C}}K$, whenever $\lambda\in\mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda)-\rho\in\mathcal{C}^*$ and Property 4.2 holds. It is equivalent to show the same assertion for $g\in UA_{\mathfrak{C}}K$ and $\lambda\in\mathfrak{h}^*$, since the sums defining $E_\lambda(\gamma g)$ and $E_\lambda(g)$ contain the same terms and are term-by-term bounded by $E_{\mathrm{Re}(\lambda)}(g)$.

Let $\Gamma_w=\Gamma\cap BwB=\Gamma\cap BwU_w$. Then Γ_w is left-invariant under $\Gamma\cap B$ and right-invariant under $\Gamma\cap U_w$. Group the terms in the definition of Eisenstein series

$$E_\lambda(g)=\sum_{\gamma\in(\Gamma\cap B)\backslash\Gamma}\Phi_\lambda(\gamma g)=\sum_{\gamma\in(\Gamma\cap B)\backslash\Gamma}a(\gamma g)^{\lambda+\rho}$$

as

$$(4.13)\qquad E_\lambda(g)=\sum_{w\in W}\sum_{\gamma_1\in(\Gamma\cap B)\backslash\Gamma_w/(\Gamma\cap U_w)}\sum_{\gamma_2\in\Gamma\cap U_w}a(\gamma_1\gamma_2 g)^{\lambda+\rho}.$$

Write $\gamma_1=utwu'$, where $u\in U$, $t\in A$, $u'\in U_w$, so that

$$(4.14)\qquad a(\gamma_1\gamma_2 g)=a(utwu'\gamma_2 g)=t\cdot a(wu'\gamma_2 g).$$

Writing $w=w_{\beta_1}\cdots w_{\beta_\ell}$ as an admissible word (Definition 4.6) and using Lemma 4.8, we can express $wu'\gamma_2$ in the form

$$w_{\beta_1}u_{\beta_1}(x_1+m_1)\cdots w_{\beta_\ell}u_{\beta_\ell}(x_\ell+m_\ell)$$

for some $x_1,\dots,x_\ell \in \mathbb{R}$ (which parameterize u') and $m_1,\dots,m_\ell \in \mathbb{Z}$ (which parameterize the sum over γ_2).

Lemma 4.9 bounds $\sum_{\gamma_2\in\Gamma\cap U_w} a(\gamma_1\gamma_2 g)^{\lambda+\rho}$ by

$$t^{\lambda+\rho} M^{\ell(w)} a^{w^{-1}(\lambda+\rho)} \prod_{\alpha\in\Phi_w} (1+a^\alpha), \tag{4.15}$$

where M is the constant in Lemma 4.9. In particular, the sum over γ_2 converges absolutely. Since $\mathfrak{C}$ is contained in the Tits cone $\bigcup_{w\in W} w\overline{\mathcal{C}}$, we may write $a = w_0 a_0 w_0^{-1}$ for some $w_0 \in W$ and a_0 such that $a_0^\alpha \geq 1$ for any $\alpha > 0$. Since $w_0^{-1}\alpha > 0$ for all but finitely many $\alpha > 0$, we see that $a^\alpha = a_0^{w_0^{-1}\alpha} \geq 1$ for all but finitely many $\alpha > 0$. Hence there exists a constant $C_a > 0$ which depends continuously on a such that

$$\prod_{\alpha\in\Phi_w} (1+a^\alpha) \leq C_a 2^{\ell(w)} \prod_{\alpha\in\Phi_w} a^\alpha = C_a 2^{\ell(w)} a^{\rho - w^{-1}\rho}, \tag{4.16}$$

since the product has $2^{\ell(w)}$ terms and a^α is bounded below by a constant depending only on a (cf. (3.6)).

The only remaining manifestation of γ_1 (aside from w) is $t = t(\gamma_1)$. Recall the notation $\Phi_\lambda(g) = a(g)^{\lambda+\rho}$ and the Gindikin–Karpelevich integral (see (3.2))

$$\int_{(\Gamma\cap U_w)\backslash U_w} \sum_{\gamma_1\in(\Gamma\cap B)\backslash\Gamma_w/(\Gamma\cap U_w)} \sum_{\gamma_2\in\Gamma\cap U_w} \Phi_\lambda(\gamma_1\gamma_2 n)\,dn = \prod_{\alpha\in\Phi_{w^{-1}}} \frac{\xi(\langle\lambda,\alpha^\vee\rangle)}{\xi(1+\langle\lambda,\alpha^\vee\rangle)}. \tag{4.17}$$

We recall from (4.14) that $\Phi_\lambda(\gamma_1\gamma_2 n) = \Phi_\lambda(utwu'\gamma_2 n) = t^{\lambda+\rho}\Phi_\lambda(wu'\gamma_2 n)$, with $t = t(\gamma_1)$, and formally obtain

$$\begin{aligned}
&\int_{(\Gamma\cap U_w)\backslash U_w} \sum_{\gamma_1\in(\Gamma\cap B)\backslash\Gamma_w/(\Gamma\cap U_w)} \sum_{\gamma_2\in\Gamma\cap U_w} \Phi_\lambda(\gamma_1\gamma_2 n)\,dn \\
&\quad= \sum_{\gamma_1\in(\Gamma\cap B)\backslash\Gamma_w/(\Gamma\cap U_w)} \int_{U_w} t(\gamma_1)^{\lambda+\rho}\Phi_\lambda(wu'n')\,dn' \\
&\quad= \sum_{\gamma_1\in(\Gamma\cap B)\backslash\Gamma_w/(\Gamma\cap U_w)} t(\gamma_1)^{\lambda+\rho} \int_{U_w} \Phi_\lambda(wn')\,dn'.
\end{aligned}$$

The last integral is the absolutely-convergent Gindikin–Karpelevich integral

$$\int_{U_w} \Phi_\lambda(wn')dn' = \prod_{\alpha>0,\, w^{-1}\alpha<0} \frac{\Gamma_{\mathbb{R}}(\langle\lambda,\alpha^\vee\rangle)}{\Gamma_{\mathbb{R}}(1+\langle\lambda,\alpha^\vee\rangle)}$$

(cf. (3.2)), and so the unfolding step above is justified by the Fubini Theorem. We thus obtain

$$\sum_{\gamma_1 \in (\Gamma \cap B)\backslash \Gamma_w / (\Gamma \cap U_w)} t(\gamma_1)^{\lambda+\rho} = \prod_{\alpha>0, w^{-1}\alpha<0} \frac{\zeta(\langle \lambda, \alpha^\vee \rangle)}{\zeta(1+\langle \lambda, \alpha^\vee \rangle)},$$

which is bounded for $w \in W$, say, by a constant C_λ. Combining the above results with (4.15) and (4.16), we find that

$$E_\lambda(g) \le C_\lambda C_a \sum_{w \in W} (2M)^{\ell(w)} a^{w^{-1}\lambda+\rho},$$

which converges absolutely by Theorem 3.5. □

We end this paper with the observation that in our situation, $\Gamma U A_{\mathfrak{C}} K$ is strictly larger than $U A_{\mathfrak{C}} K$ (see the comments in the second paragraph following Theorem 1.1).

PROPOSITION 4.10. *Let* A *be a generalized Cartan matrix of indefinite type as in* [K, Theorem 4.3(Ind)]*, and further assume that* A *is nonsingular, as before. Let $G_{\mathbb{R}}$ be the complete Kac–Moody group associated with* A *as in Section 2. Then $U A_{\mathfrak{C}} K$ is not invariant under multiplication by Γ on the left.*

Proof. Recall that $\mathfrak{C}$ is the interior of the Tits cone $X := \bigcup_{w \in W} w\overline{\mathcal{C}}$. Since G is of indefinite type, by [K, Proposition 5.8 (c)] the closure of X is given by

$$\overline{X} = \{h \in \mathfrak{h} \colon \langle \alpha, h \rangle \ge 0 \text{ for all } \alpha \in \Phi_+^{\mathrm{im}}\},$$

where Φ_+^{im} is the set of positive imaginary roots. By [K, Theorem 5.6 (c)], there exists some $\alpha \in \Phi_+^{\mathrm{im}}$ such that $\langle \alpha, \alpha_i^\vee \rangle < 0$ for all $i \in I$. It follows that $\alpha_i^\vee \notin \overline{X}$ for any $i \in I$.

Fix a simple coroot $\alpha_i^\vee \notin \overline{X}$. Since $0 \in \overline{X} = \overline{\mathfrak{C}}$, $A_{\mathfrak{C}}$ contains elements h arbitrarily close to the identity. Recall that $u_{-\alpha_i}(1) \in \Gamma$. For $h \in A_{\mathfrak{C}}$, we have that

$$u_{-\alpha_i}(1) h = h u_{-\alpha_i}(t), \quad \text{where } t = h^{\alpha_i}.$$

The $\mathrm{SL}(2,\mathbb{R})$ calculation

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\delta(t)} & \frac{t}{\delta(t)} \\ 0 & \delta(t) \end{pmatrix} \begin{pmatrix} \frac{1}{\delta(t)} & -\frac{t}{\delta(t)} \\ \frac{t}{\delta(t)} & \frac{1}{\delta(t)} \end{pmatrix}, \quad \delta(t) = \sqrt{t^2+1},$$

shows that the Iwasawa A^+-component of $u_{-\alpha_i}(1)h \in \Gamma A_{\mathfrak{C}}$ is equal to $h \cdot h_{\alpha_i}(1/\delta(t))$. This cannot lie in $A_{\mathfrak{C}}$ for h near the identity, since $\delta(t) \to \sqrt{2}$ as $t \to 1$ and $\alpha_i^\vee \notin \overline{X}$. □

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