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Machine-learning number fields

Yang-Hui He*, Kyu-Hwan Lee†§, and Thomas Oliver‡

We show that standard machine-learning algorithms may be trained to predict certain invariants of algebraic number fields to high accuracy. A random-forest classifier that is trained on finitely many Dedekind zeta coefficients is able to distinguish between real quadratic fields with class number 1 and 2, to 0.96 precision. Furthermore, the classifier is able to extrapolate to fields with discriminant outside the range of the training data. When trained on the coefficients of defining polynomials for Galois extensions of degrees 2, 6, and 8, a logistic regression classifier can distinguish between Galois groups and predict the ranks of unit groups with precision > 0.97.

1. Introduction & summary

Algebraic number fields are characterized by various invariants. One such invariant is the class number, which encodes how far the ring of integers in the number field is from being a unique factorisation domain. To this day, there remain central open questions regarding class numbers of algebraic number fields. For example, whilst there is a well-known list of imaginary quadratic number fields with class number 1, it is not known whether or not there are infinitely many real quadratic fields with class number 1. Actually, Gauss conjectured in his famous *Disquisitiones Arithmeticae* of 1801 that there are infinitely many. Furthermore, it is conjectured that there are infinitely many real quadratic fields of the form $\mathbb{Q}(\sqrt{p})$, for a prime $p \equiv 1 \bmod 4$, with class number 1. In fact, experimental evidence, and the Cohen–Lenstra heuristics, predict that around 76% of such fields have class number 1 [6]. In this paper, we show that a machine-learning algorithm may be trained to predict certain invariants, including the class number of real quadratic fields.

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§Corresponding author.

For a broad introduction to machine-learning, see [7, 8]. The machine-learning of mathematical structures is a relatively recent enterprise. Interesting early neural-network experiments exploring the non-trivial zeros of the Riemann zeta function were documented in [22] (a more recent work is [17]). Building on work in superstring theory, more precisely the computation of topological invariants for Calabi–Yau compactifications [10, 16, 21, 4] (q.v., [11] for a summary), a programme developing the applications of machine-learning to abstract mathematics was proposed in [9, 10]. Since then, machine-learning has been applied to various branches within the discipline with the intention of pattern-recognition and conjecture-raising. To name a few: representation theory [13], graph theory [15], metric geometry [2], dessins d'enfants [12], and quiver mutations [3].

Recently, the present authors demonstrated that techniques from machine-learning could be used to resolve a classification problem in arithmetic geometry [14]. To be precise, we showed that a Bayesian classifier can distinguish between Sato–Tate groups given a small number of Euler factors for the L-function with over 99% accuracy. Given the efficient nature of the machine-learning approach, [Loc. cit.] suggested a machine can be trained to learn the Sato–Tate distributions and may be able to classify curves much more efficiently than the methods available in the literature.

This paper is a continuation of our observation that machine-learning can be used in number theory. In particular, we will apply logistic regression and random forest classifiers—these are reviewed in [8, Sections 4.4 & 15]. Our experiments are concerned with predicting the following invariants: degree, signature, Galois group, and class number. We utilise three training data sets associated to algebraic number fields: (1) coefficients of their defining polynomials, (2) finitely many coefficients of their Dedekind zeta functions, and (3) binary vectors encoding finitely many completely-split rational primes. Each training dataset has its own strengths and weaknesses. We review the utility of datasets (1), (2) and (3) below.

1. Using both defining polynomial and zeta coefficient training, we observe high-accuracy predictions for number field signatures and Galois groups. The signature of a number field determines the rank of its unit group, which is equal to the vanishing order of the associated Dedekind zeta function at $s = 0$. Elsewhere it has been demonstrated that a machine cannot be efficiently trained to predict the ranks of elliptic curves from their minimal Weierstrass equation [1], which is in contrast to our observations for number fields.
2. The Dedekind zeta function of an algebraic number field has a simple pole at $s = 1$ with residue given by the analytic class number formula.

We train a random forest classifier through 1000 zeta coefficients of real quadratic fields with class number 1 or 2 with discriminant less than one million which are available at [18, Number Fields], and the resulting classifier can distinguish between class numbers 1 and 2 with accuracy 0.96. When we apply the same classifier to real quadratic fields with discriminants between 1 million and 3 million, we find that it distinguish between class numbers 1 and 2 with accuracy 0.92.

3. It is well-known that the set of split primes uniquely characterizes a Galois extension over $\mathbb{Q}$, cf. [20, VII, §13]. Motivated by this, we train classifiers using binary data recording split primes and apply the classifiers to various invariants of number fields. However, the classifiers perform poorly except for detecting degrees of the extensions.

An outline of the contents of this paper is as follows. In Section 2, we recall basic terminology and establish the notation used in the sequel. In Section 3, we define our three forms of training data, and explain the experimental set-up. In Section 4, it is shown that, when trained on zeta coefficients, a random forest classifier is able to distinguish between extension degrees and signatures. Furthermore, we apply logistic regression to the defining polynomial dataset. In Section 5, we outline our experiments with Galois groups of order 8. In this case, zeta coefficients and defining polynomial coefficients perform equally well. In Section 6, it is observed that, when trained on zeta coefficients, a random forest classifier is able to distinguish between real quadratic fields of class number 1 and class number 2. The classifier is trained using quadratic fields with discriminant less than one million, but is able to extrapolate to ranges far beyond the training data.

2. Nomenclature

We will use the following notation throughout:

Algebraic number field denoted by F. We will assume that the extension $F/\mathbb{Q}$ is Galois;

Extension degree of F over $\mathbb{Q}$ is denoted $[F : \mathbb{Q}]$;

Signature of F is the pair (r_1, r_2), in which r_1 (resp. r_2) denotes the number of real embeddings (resp. conjugate pairs of complex embeddings) of F. If (r_1, r_2) is the signature of F, then $[F : \mathbb{Q}] = r_1 + 2r_2$. If $r_2 = 0$ (resp. $r_1 = 0$) then we refer to F as totally real (resp. imaginary);

Ring of integers denoted by $\mathcal{O}_F$;

Rank of the unit group $\mathcal{O}_F^\times$ is equal to $r := r_1 + r_2 - 1$ by Dirichlet's unit theorem;

Discriminant of F denoted by Δ_F; it is known that $\mathrm{sgn}(\Delta_F) = (-1)^{r_2}$;
Ramification A rational prime p **ramifies** in F if and only if p divides Δ_F; an unramified prime p **splits completely** in F if $p\mathcal{O}_F$ is a product of $[F : \mathbb{Q}]$-many distinct prime ideals in $\mathcal{O}_F$, and p is **inert** in F if $p\mathcal{O}_F$ is itself a prime ideal;
Class number of F denoted by h_F. That is, the size of the ideal class group (the quotient group of the fractional ideals by the principal ideals);
Norm of an ideal I in $\mathcal{O}_F$ is denoted by $N(I)$;
Prime ideal denoted by $\mathfrak{p}$. A prime ideal ideal in $\mathcal{O}_F$ lies above a rational prime p if $\mathfrak{p}$ divides the ideal generated by p; we denote this situation by $\mathfrak{p}|p$;
Quadratic number field has the form $\mathbb{Q}(\sqrt{d})$ with d a square-free integer. If $d < 0$ (resp. $d > 0$) then we call the field imaginary quadratic (resp. real quadratic). The discriminant of $F = \mathbb{Q}(\sqrt{d})$ is d (resp. $4d$) if $d \equiv 1 \bmod 4$ (resp. $d \equiv 2, 3 \bmod 4$). In particular, a real quadratic number field has positive discriminant;
Galois group associated to the Galois extension $F/\mathbb{Q}$ is denoted by $\mathrm{Gal}(F/\mathbb{Q})$;
Cyclic group of order n denoted by C_n;
Dihedral group of order $2n$ denoted by D_n.

Remark 1. The invariants that we have sought to machine-learn in this article are discrete, whereas the machine-learning algorithms used may involve modelling functions of continuous variables. For the classifiers that will appear below, the resolution is rather concrete: the logistic regression classifier simply rounds a fitted logistic sigmoid model to the nearest integer, and the random forest classifier models a function that is locally constant on high-dimensional Euclidean space away from the model's decision boundaries. More philosophically, we note that the discrete invariants under discussion all appear in the class number formula for the Dedekind zeta function, which is a meromorphic function of a complex variable.

3. Establishing the datasets

In this section, we explain our training datasets, and outline the basic experimental strategy.

3.1. Defining polynomials

Recall from Section 2 that we assume the extension $F/\mathbb{Q}$ to be Galois. A defining polynomial for F is an irreducible polynomial $P(x) \in \mathbb{Q}[x]$ such

that $F = \mathbb{Q}(\alpha)$ for a root α of $P(x)$. We choose $P(x)$ as in [18, Normalization of defining polynomials for number fields]. In particular, $P(x)$ is monic with integer coefficients, and, if $\alpha_1, \ldots, \alpha_n$ are the complex roots of $P(x)$, then the sum $\sum_{i=1}^{n} |\alpha_i|^2$ is minimized. We write:

$$P(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0, \quad c_i \in \mathbb{Z}, \quad n = [F : \mathbb{Q}]. \tag{1}$$

Using the coefficients of $P(x)$, we define the vector:

$$v_P(F) = (c_0, \ldots, c_{n-1}) \in \mathbb{Z}^n. \tag{2}$$

Let $\mathcal{F}$ denote a finite set of number fields, and, for all $F \in \mathcal{F}$, let $c(F)$ be an invariant of interest. For example, $\mathcal{F}$ could be the set of all real quadratic fields with discriminant less than one million and, for $F \in \mathcal{F}$, the invariant $c(F)$ could be the class number of F. We introduce the following labeled dataset:

$$\mathcal{D}_P = \{v_P(F) \to c(F) : F \in \mathcal{F}\}. \tag{3}$$

Example 1. In Section 5.1, we will take $\mathcal{F}$ to contain certain degree 8 number fields with Galois group isomorphic to either C_8 or D_4. For $F \in \mathcal{F}$ we will let $c(F)$ be 0 (resp. 1) corresponding to $\mathrm{Gal}(F/\mathbb{Q}) \cong C_8$ (resp. $\mathrm{Gal}(F/\mathbb{Q}) \cong D_4$). A large database of such fields can be downloaded from [18, Number Fields], including around 6200 such that $c(F) = 0$. The set $\mathcal{F}$ consists of these fields, and a random sample of around 6200 (out of around 28000) fields such that $c(F) = 1$. An instance of $v_P(F)$ such that $c(F) = 0$ is

$$(4096, -512, 320, 136, -46, 17, 5, -1).$$

The $v_P(F)$ with the largest c_0 such that $c(F) = 0$ is

$$(153220409851123184, 812631532526484, 13364221512257, \\ -78983668469, -234643970, -7256689, 11478, -1).$$

3.2. Dedekind zeta functions

The Dedekind zeta function of a number field F is given by the following formulas:

$$\zeta_F(s) = \prod_{\mathfrak{p}} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1} = \sum_{I \leq \mathcal{O}_F} N(I)^{-s} = \sum_{n=1}^{\infty} a_n n^{-s},$$

where $\mathfrak{p}$ varies over prime ideals in $\mathcal{O}_F$, I varies over the non-zero ideals in $\mathcal{O}_F$, and, for a positive integer n,

$$a_n = \#\{N(I) = n : I \leq \mathcal{O}_F\}, \quad n \in \mathbb{Z}_{\geq 1}. \tag{4}$$

Since we assume that F is Galois over $\mathbb{Q}$, the zeta function $\zeta_F(s)$ uniquely determines F. However, we caution that in general a number field is not determined by its Dedekind zeta function.[1] Using SageMath [23], we may compute a large amount of a_n quickly. We introduce the vector:

$$v_Z(F) = (a_1, \ldots, a_{1000}) \in \mathbb{Z}^{1000}. \tag{5}$$

Example 2. For the dataset of 6206 number fields with Galois group C_8 mentioned in Example 1, the largest absolute value in the 1000×6206 $v_Z(F)$-entries a_i is 109824.

Given a finite set $\mathcal{F}$ of number fields F and an invariant $c(F)$ for each $F \in \mathcal{F}$, we associate the following labeled dataset:

$$\mathcal{D}_Z = \{v_Z(F) \to c(F) : F \in \mathcal{F}\}. \tag{6}$$

We may write $\zeta_F(s)$ as a product indexed by rational primes:

$$\zeta_F(s) = \prod_p E_p(s)^{-1}, \qquad E_p(s) := \prod_{\mathfrak{p}|p} (1 - N(\mathfrak{p})^{-s}). \tag{7}$$

If $\mathfrak{p}|p$ then $N(\mathfrak{p})$ has the form p^a for $a \in \mathbb{Z}_{>0}$. Thus the product $E_p(s)$ is a polynomial in p^{-s}.

Example 3. Assume that F is a quadratic extension of $\mathbb{Q}$. For a rational prime p, we have

$$E_p(s) = \begin{cases} 1 - 2p^{-2} + p^{-2s}, & \text{if } p \text{ is split,} \\ 1 - p^{-2s}, & \text{if } p \text{ is inert,} \\ 1 - p^{-s}, & \text{if } p \text{ is ramified.} \end{cases}$$

The splitting property of an unramified prime p in F is determined by the Legendre symbol and the quadratic reciprocity law. See [20, I, §8] for more details.

[1] In fact, a given Dedekind zeta function only determines the product of the class number and the regulator.

3.3. Split primes

For a number field F that is Galois over $\mathbb{Q}$, write $\mathrm{Spl}(F)$ for the set of rational primes that split completely in F. The Chebotarev density theorem implies that the set $\mathrm{Spl}(F)$ has density $1/[F:\mathbb{Q}]$, and it can be shown that

$$F \subset K \qquad \Longleftrightarrow \qquad \mathrm{Spl}(F) \supset \mathrm{Spl}(K)$$

for finite Galois extensions F and K over $\mathbb{Q}$ [20, VII, §13]. This shows that the set $\mathrm{Spl}(F)$ characterizes a Galois extension completely.

For $i \in \mathbb{Z}_{\geq 1}$, let p_i denote the i^{th} rational prime. Given a number field F, we write

$$\delta_i = \begin{cases} 1, & \text{if } p_i \in \mathrm{Spl}(F), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Except for finitely many primes, in order to calculate δ_i it suffices to reduce the defining polynomial $P(x)$ modulo p_i. If the reduction splits into a product of distinct linear factors then $\delta_i = 1$; otherwise, we have $\delta_i = 0$. Using SageMath [23], it is possible to calculate a large number of δ_i quickly. Associated to F, we introduce the following binary vector:

$$v_B(F) = (\delta_1, \ldots, \delta_{500}) \in \{0,1\}^{500}. \tag{9}$$

Given a finite set $\mathcal{F}$ of number fields and an invariant $c(F)$ for each $F \in \mathcal{F}$, we associate the following dataset:

$$\mathcal{D}_B = \{v_B(F) \to c(F) : F \in \mathcal{F}\}. \tag{10}$$

Remark 2. We emphasize that the ith component in equation (9) corresponds to the ith prime, whereas the ith component in equation (5) corresponds to the ith positive integer. The latter may be much smaller, for instance the 500^{th} prime is 3571. The binary vectors in equation (9) have smaller dimension than the zeta-coefficient vectors in (5), but nevertheless incorporate some information about the behaviour of larger primes in the underlying number field. The zeta coefficient vectors in equation (5) are of higher dimension, but only pertain to primes < 1000. Before the experiments below were undertaken, it was unclear whether basic knowledge of larger primes would be more important than more detailed knowledge of smaller integers, or vice versa. In what follows, we will see various examples where classifiers trained on zeta-coefficient data outperform those trained on binary vector data. Of course, the costs and benefits of this trade-off could be analyzed by varying the choices of dimensions made.

3.4. Experimental strategy

1. Let $\mathcal{F}$ be a finite set of number fields. The choice of $\mathcal{F}$ depends on the experiment. For example, $\mathcal{F}$ could be a random sample of degree 8 extensions with discriminant less than some bound.
2. For a number field $F \in \mathcal{F}$, let $c(F)$ denote a certain invariant of interest. For example, $c(F)$ could be a binary digit (category) corresponding to whether or not $\mathrm{Gal}(F/\mathbb{Q})$ is abelian.
3. Generate datasets of the form $\mathcal{D} = \{v(F) \to c(F) : F \in \mathcal{F}\}$, where $\mathcal{D}$ is as in (3), (6), or (10).
4. Decompose $\mathcal{D}$ as a disjoint union $\mathcal{T} \sqcup \mathcal{V}$, where $\mathcal{T}$ is a training set and $\mathcal{V}$ is a validation set. We use various ratios for splits of $\mathcal{T}$ and $\mathcal{V}$ such as 80-20, 70-30 or 20-80 percentage-wise. As there is no significant difference in the results, we will not specify ratios for individual experiments.
5. Train a classifier on the set $\mathcal{T}$. In this paper, we will use random forests and logistic regression, which we implement using MATHEMATICA [24].
6. For all unseen number fields $F \in \mathcal{V}$, ask the classifier to determine $c(F)$. We record the precision and confidence. Here, precision is defined to be the percentage agreement of the actual value with the one predicted by the classifier. As an extra check to minimize false positives and false negatives, the confidence in the form of Matthews' correlation coefficient [19] is computed. Both precision and confidence are desired to be close to 1.

4. Degree, signature, and rank

We recall that the extension degree $[F : \mathbb{Q}]$ is equal to $n = r_1 + 2r_2$, and that the rank of the unit group $\mathcal{O}_F^\times$ is $r = r_1 + r_2 - 1$. The Dedekind zeta function vanishes to order r at $s = 0$. To perform the experiments in this section, we downloaded datasets from [18, Number fields]. The completeness of this data is documented at [18, Completeness of number field data].

4.1. Experiment I: Extension degree

Whilst the defining polynomial of a Galois extension clearly encodes the extension degree (as the degree of the polynomial), the same is not obviously true for zeta coefficients or split prime data. Datasets consisting of Galois extensions of $\mathbb{Q}$ with Galois group C_4, C_6 and C_8 are obtained from [18,

Number Fields] and thus a 3-category label can be established in the form of (10):

$$\mathcal{D}_B = \{(\delta_1, \ldots, \delta_{500}) \to c\}, \tag{11}$$

where $\delta_i \in \{0, 1\}$ and $c = 0, 1, 2$, say, according to which of the 3 Galois groups the number field F corresponds. As with all cases below, in order to balance the data, we sample around 6200 in each category. We find that, when trained on split prime data, a *logistic regression classifier* is able to perform this 3-way classification with precision 0.976 and confidence 0.968. Even better, when trained on zeta coefficient data, a *random forest classifier* performs the same classification with precision 0.999 and confidence 0.998.

4.2. Experiment II: Rank of unit group

Recall that the signature (r_1, r_2) determines the rank r of $\mathcal{O}_F^\times$ through $r = r_1 + r_2 - 1$, and $\operatorname{sgn}(\Delta_F) = (-1)^{r_2}$.

Example 4. If $[F : \mathbb{Q}] = 2$, then the unit group $\mathcal{O}_F^\times$ has rank 1 (resp. 0) if F is totally real (resp. imaginary), that is, that the signature of F is $(2, 0)$ (resp. $(0, 1)$). The number field F has rank 1 (resp. 0) if and only if $\Delta_F > 0$ (resp. $\Delta_F < 0$). More generally, if $[F : \mathbb{Q}]$ has even degree, then the unit group has odd (resp. even) rank if and only if $\Delta_F > 0$ (resp. $\Delta_F < 0$). We note that if $[F : \mathbb{Q}] = 6$ (resp. 8), then the rank of $\mathcal{O}_F^\times$ is an integer in the set $\{2, 3, 4, 5\}$ (resp. $\{3, 4, 5, 6, 7\}$).

We obtain, from [18, Number Fields], the datasets consisting of Galois extensions of $\mathbb{Q}$ with cyclic Galois group and signatures: $(2, 0)$ and $(0, 1)$ for C_2, $(6, 0)$ and $(0, 3)$ for C_6, $(8, 0)$ and $(0, 4)$ for C_8. These signatures correspond to ranks $1, 0, 5, 2, 7, 3$ respectively. Furthermore, we downloaded those with Galois group D_4 and signatures $(8, 0)$ and $(0, 4)$. This establishes datasets in the form of (3) and (6), *for each* choice of the 4 Galois groups, as

$$\begin{aligned} \mathcal{D}_P &= \{(c_0, \ldots, c_{n-1}) \to r\}, \\ \mathcal{D}_Z &= \{(a_1, \ldots, a_{1000}) \to r\}. \end{aligned} \tag{12}$$

Here, $c_i \in \mathbb{Z}$ are the coefficients of the (monic) defining polynomial and a_i are the first 1000 coefficients of the Dedekind zeta function. The rank r, conveniently, takes values with the binary categories for each Galois group, as indicated in Table 1.

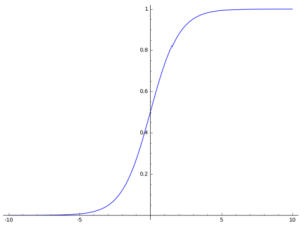
Table 1: Random forest classifier results on distinguishing ranks of the unit group for field extensions F over $\mathbb{Q}$ with fixed Galois groups

Galois group	signature(F)	rank($\mathcal{O}_F^\times$)	$\mathcal{D}_P$ precision	$\mathcal{D}_P$ confidence
C_2	(2,0) (0,1)	1 0	> 0.99	> 0.99
C_6	(6,0) (0,3)	5 2	0.97	0.93
C_8	(8,0) (0,4)	7 3	> 0.99	> 0.99
D_4	(8,0) (0,4)	7 3	0.98	0.95

When trained on zeta coefficients using $\mathcal{D}_Z$ we found that all the standard classifiers, including neural-classifiers with convolutional networks, performed quite poorly. In all the cases of Galois groups, the precision was around 0.6 or less. It is interesting that this particular case requires so much effort without success whilst in the majority of experiments zeta coefficient training is superior. On the other hand, when trained on the defining polynomial coefficients using $\mathcal{D}_P$, the random forest classifier consistently performed the best and the precisions were > 0.99, 0.97, > 0.99 and 0.98 for the Galois groups as summarized in Table 1 with the corresponding confidences.

In fact, we can do more. Note that for our data, in each Galois group, the rank takes one of 2 possible values, which we can take to be 0 or 1 by appropriate labeling. This naturally makes one think of the logistic sigmoid function:

$$\sigma(z) = \frac{1}{1+\exp(-z)} \tag{13}$$



which has range $[0, 1]$ as shown in the graph above.

When trained on defining polynomial coefficients, we found that logistic regression performed well in predicting the rank of $\mathcal{O}_F^\times$, giving us an explicit and interpretable model. As with all regression, we are interested in finding a best fit to a function, here a sigmoid of the form:

$$\sigma(c_0 w_0 + \cdots + c_{n-1} w_{n-1} + w_n), \quad (w_0, \ldots, w_n) \in \mathbb{R}^{n+1}, \tag{14}$$

where c_i, we recall, are the coefficients of defining minimal polynomials for F as in equation (1). The parameters w_i are to be optimized (fitted) by minimizing squared-mean-error. By rounding the above to the nearest integer, the function in equation (14) gives the value 0 or 1 corresponding to the two possibilities for the rank.

Example 5. In the case that $\mathrm{Gal}(F/\mathbb{Q}) = C_6$, regression by the logistic function yields around 94% precision with best fit:

$$c_0w_0+\cdots+c_5w_5+w_6 = -0.000169037c_0-0.0000689721c_1-0.000120625c_2 \\ - 0.00196535c_3 - 0.058735c_4 + 0.917924c_5.$$

The accuracy of this model varies with ranks and Δ_F. More precisely, the model predicts rank 2 with accuracy > 0.91 for almost all ranges of $|\Delta_F|$ occurring in the dataset with overall precision 0.987. On the other hand, the model above performs poorly for rank 5 fields with $|\Delta_F| < 1.80 \times 10^9$ (around 77% accuracy) and $|\Delta_F| > 2.55 \times 10^{14}$ (around 60% accuracy); the overall precision for the classification of rank 5 fields is 0.892.

We point out that what we did above should, strictly speaking, be called "non-linear regression by the integer round of the logistic function". The terminology *logistic regression*, though similar, is different in the probabilistic nature of the interpretation. Ordinarily, for discrete classification models such as our binary category problem, we fit the probability p of the output being 0 or 1 to the logistic function.

5. Galois groups of order 8

Traditional algorithms for computing Galois groups are presented in [5, Section 6.3]. In this section, we will see that a classifier can distinguish between various Galois groups of order 8, in a very efficient way. There are five possibilities for such groups, namely: C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_4, and Q_8 (the quaternion group). The last of these (Q_8) has about 1100 occurrences on the LMFDB, so we do not use it for experimentation; the others range from around 6200 to 28000 cases. All of these can be obtained from [18, Number Fields].

5.1. Experiment III: Abelian vs. non-Abelian groups

Let us consider Galois extensions of $\mathbb{Q}$ with Galois group C_8 (resp. D_4). Note that C_8 is Abelian but D_4 is not. We establish a dataset of the form

of (3) as

$$\mathcal{D}_P = \{(c_0, \ldots, c_7) \to a\}, \tag{15}$$

where the input is the list of the 8 non-trivial coefficients c_i of the minimal polynomial (the leading coefficient is always 1) and the output a is 0 or 1 according to whether the Galois group is C_8 or D_4. We find that a random forest classifier was able to distinguish between these groups with precision 0.971 and confidence 0.941. Similarly, if we use the zeta coefficients $(a_1, \ldots, a_{1000})$ as input along the lines of (6), the random forest classifier achieves precision 0.973 and confidence 0.947.

On the other hand, when we use the split primes data $\mathcal{D}_B$ in (10), the random forest classifier yields precision 0.736.

5.2. Experiment IV: Distinguishing between Abelian groups

The above experiment showed that a classifier could distinguish between Abelian versus non-Abelian groups. We now ask: can a similar classifier perform the more refined distinction between different Abelian groups? Here, we have 3 abelain Galois groups of order 8: C_8, $C_4 \times C_2$, and $C_2 \times C_2 \times C_2$. Using the zeta coefficient data with the output being one of the 3 categories, we find that a random forest classifier was able to distinguish between these groups with precision 0.95472 and confidence 0.932148.

6. Class numbers

The Dedekind zeta function of an algebraic number field has a simple pole at $s = 1$. At this pole, the residue is computed by the analytic class number formula which involves various arithmetic invariants including the class number.[2] Algorithms for computing the class numbers of general number fields are given in [5, Section 6.5]. For the special case of quadratic extensions, see [5, Sections 5.2, 5.6].

[2]Specifically, the class number formula dictates that the Dedekind zeta function has a simple pole at 1 and

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}\mathrm{Reg}_F h_F}{w_F\sqrt{|\Delta_F|}},$$

where, in addition to the nomenclature in §2, Reg_F is the regulator, and w_F is the number of roots of unity in F.

6.1. Experiment V: Real quadratic fields

In the discriminant range $0 < |\Delta_F| \leq 10^6$, one finds 83464 real quadratic number fields with class number 1, and 83324 real quadratic number fields with class number 2 [18, Number Fields], and the list is complete for this discriminant range.

Whilst there are many thousands of examples of real quadratic number fields with larger class number, the sample size varies from case to case. In order to avoid biases in our datasets, we simply focus on the binary classification problem of distinguishing real quadratic fields with class number 1 from those with class number 2. Thus, we have datasets, using (3), (6) and (10),

$$\mathcal{D}_P = \{(c_0, c_1) \to c\}, \quad \mathcal{D}_Z = \{(a_1, \ldots, a_{1000}) \to c\},$$
$$\mathcal{D}_B = \{(\delta_1, \ldots, \delta_{500}) \to c\},$$

where c_i are the coefficients of the minimal polynomial, a_i are the first 1000 zeta coefficients, and δ_i are 0 or 1 according to whether the ith rational prime splits completely or not. Here $c = 1$ or 2 is the class number. Note that we have a fairly balanced dataset with around $80,000$ each of class number 1 and 2. When trained on zeta coefficient data, the random forest classifier yielded the best precision of 0.96 with confidence 0.92. This experiment is summarized in the table below. On the other hand, when trained on defining polynomial data or split primes data, no standard classifier was able to distinguish between class numbers 1 and 2 with precision greater than 0.60.

Next, we try something more drastic. Consider the discriminant range $10^6 < \Delta_F < 2 \times 10^6$, in which we find 75202 real quadratic number fields with class number 1 and 80217 with class number 2. According to [18, Number Fields], the list is complete for this discriminant range.

Can a classifier be trained within $|\Delta_F|$ of a certain range and *extrapolate* to a larger $|\Delta_F|$ range? If so, this would strengthen even further our notion that machine-learning has found some underlying pattern. We applied the classifier trained on the previous datasets, i.e., real quadratic fields with discriminant less than one million, to this new discriminant range. The result was precision 0.92 with confidence 0.86. It seems that the classifier is able to extrapolate from data of smaller discriminant. We tried the same for discriminants between 2 million and 3 million, again with the same classifier trained on the data of discriminants smaller than one million. There are 18383 with class number 1, and 19827 with class number 2. The result was precision 0.91 with confidence 0.84. The results are summarized in Table 2.

Table 2: A summary of the precision and confidence of the random forest classifier trained on zeta coefficients of real quadratic fields with discriminant between one and one million with class number 1 or 2. The number $\#\{F\}$ is the cardinality of the set containing real quadratic fields with discriminant and class number as specified

Discriminant range	h_F	$\#\{F\}$	Precision	Confidence
$[1, 1\times 10^6]$	1 2	83464 83324	0.96	0.92
$[1\times 10^6, 2\times 10^6]$	1 2	75202 80217	0.92	0.86
$[2\times 10^6, 3\times 10^6]$	1 2	18383 19827	0.91	0.84

Remark 3. It is known that there is a finite set of imaginary quadratic fields with class number 1, viz., this is the list of $\mathbb{Q}[\sqrt{d}]$ for the Heegner numbers $d = -1, -2, -3, -7, -11, -19, -43, -67, -163$. This set has far too few examples for a machine to learn. Motivated by [6], we tested to see if any classifier could distinguish between class numbers divisible by 3 and class numbers not divisible by 3: The methods of zeta coefficients and polynomial coefficients gave both precision around 0.51, which is as good as randomly guessing. As always, divisibility and other patterns in primes seem very difficult to be machine-learned (cf. [11]).

6.2. Experiment VI: Quartic and sextic fields

We say a degree 4 number field is *bi-quadratic* if it has Galois group $C_2 \times C_2$. From [18, Number Fields] we downloaded the dataset of bi-quadratic Galois extensions of $\mathbb{Q}$ with class number 1 and 2. To get a balanced dataset, we randomly chose 6100 number fields for each class number. When trained on zeta coefficient data, we found that a logistic regression classifier could distinguish class number 1 from class number 2 with precision 0.819 with confidence 0.640. We suspect that the performance could have been better with a larger set of data.

In degree 6, the generic Galois group is S_3. From [18, Number Fields], we downloaded the dataset consisting of degree 6 Galois extensions of $\mathbb{Q}$ with Galois group S_3 and class number in the set $\{1, 2, 3, 4, 6, 8, 9\}$. The class numbers 5 and 7 were excluded on the grounds that there are too few data points on the LMFDB (less than 400 each), whereas the others have at least 1150 points; precise counts are given in Table 3.

Table 3: Frequency of S_3-extensions F of $\mathbb{Q}$ with class number h_F

h_F	1	2	3	4	6	8	9
$\#\{F\}$	7436	8680	1917	8165	1158	4230	2700

When trained on zeta coefficient data consisting of randomly chosen 1150 data points from each class number, we found no machine-learning approach was able to perform the corresponding 7-way classification with precision more than 0.38. Furthermore, when trained on a dataset consisting of 7400 data points from each of the class numbers $1, 2, 4$, the best precision given by a random forest classifier was 0.605 with confidence 0.415.

There are several factors that might contribute to the differing levels of success between distinguishing class numbers of real quadratic fields and S_3-extensions of $\mathbb{Q}$. For example, for $n > 2$, the n-ary classifications attempted for S_3-extensions are more complicated than the binary classifications considered for real quadratic fields. Furthermore, from a number-theoretic perspective, the zeta coefficients for S_3-extensions are somewhat more complicated. This complexity seems to be reflected in the poor performance of the ternary classification of class numbers 1, 2, 4. It would be interesting to see whether a larger dataset would bring a better performance.

7. Outlook

We conclude with a brief discussion of future experimental and mathematical projects.

As mentioned in the introduction, it is unknown whether or not there are infinitely many real quadratic fields of class number 1. It would be very interesting to investigate how a machine is able to distinguish such fields. If the criteria under which the classifier predicts class number 1 are satisfied infinitely often, then there could be scope for developing a new heuristic for or a new approach to this open problem. Furthermore, we note that the machine continues to make accurate predictions for real quadratic fields with discriminant outside the range of the training data. Perhaps this extrapolation offers a clue towards future progress.

The class number is subject to the analytic class number formula, which computes the residue of the Dedekind zeta function at its pole. The analytic class number formula can be compared to the famous BSD conjecture in the sense that both concern the leading terms of zeta functions at special points. In a forthcoming paper, we will examine whether or not a machine can be trained to predict the vanishing orders of elliptic L-functions and other invariants appearing in their Taylor expansions.

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Yang-Hui He
Department of Mathematics
City, University of London
EC1V 0HB
UK
Merton College
University of Oxford
OX14JD
UK
School of Physics
NanKai University
Tianjin, 300071
P.R. China
E-mail address: hey@maths.ox.ac.uk

Kyu-Hwan Lee
Department of Mathematics
University of Connecticut
Storrs, CT 06269-1009
USA
E-mail address: khlee@math.uconn.edu

Thomas Oliver
SCEDT
Teesside University
Middlesbrough, TS1 3BX
UK
E-mail address: T.Oliver@tees.ac.uk

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