SPHERICAL HECKE ALGEBRAS OF SL_2 OVER 2-DIMENSIONAL LOCAL FIELDS

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Abstract. In this paper, we study spherical Hecke algebras of SL_2 over two dimensional local fields. In order to define the convolution product, we make explicit use of coset decompositions. We also consider spherical Hecke algebras of the torus of SL_2 and construct the Satake isomorphism between two spherical Hecke algebras. In order to define the Satake isomorphism, we use the invariant measure on two-dimensional local fields with values in $\mathbb{R}((X))$ constructed by I. Fesenko.

Introduction. The Satake isomorphism (or Satake parameters) play an important role in the Langlands program. Especially, the local L-functions of spherical representations are defined using Satake parameters. More precisely, let $\mathbf{G}$ be a connected split reductive algebraic group defined over a local field F. Then the Satake isomorphism gives one-to-one correspondence between the set of equivalence classes of spherical representations of $\mathbf{G}(F)$ and the set of semi-simple conjugacy classes in the L-group $\hat{G}(\mathbb{C})$ (see, for example [Ca]). Let π be a spherical representation of $\mathbf{G}(F)$ with the semi-simple conjugacy class t_π in $\hat{G}(\mathbb{C})$, and let $r : \hat{G}(\mathbb{C}) \longrightarrow GL_N(\mathbb{C})$ be a finite-dimensional representation. Then the local L-function is defined by

$$L(s, \pi, r) = \det (I - r(t_\pi)q^{-s})^{-1},$$

where q is the cardinality of the residue field of F.

The study of spherical representations is exactly the study of the spherical Hecke algebras, the set of $\mathbb{C}$-valued compactly-supported functions which are bi-invariant under the action of the maximal compact subgroup. On the other hand, there is another important algebra, namely, the Iwahori Hecke algebra; the one attached to the Iwahori subgroup. It is well known by Bernstein's theorem that spherical Hecke algebras are the center of Iwahori Hecke algebras. This fact plays an important role in the representation theory of p-adic groups.

This paper arose in an attempt to define local L-functions for affine Kac-Moody groups over a local field, which are attached to extended Dynkin diagrams.

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Kac-Moody groups are infinite dimensional groups which are not locally compact. Hence there are no Haar measures. Affine Kac-Moody groups over a local field are closely related to split simple algebraic groups over a 2-dimensional local field. The class of such fields includes finite extensions of the fields $\mathbb{Q}_p((t))$, $\mathbb{F}_p((t_1))((t_2))$ and $\mathbb{Q}_p\{\{t\}\}$ (for the definition see [LF]). Kapranov [Ka] studied an analogue of Iwahori Hecke algebras for certain central extensions of split simple algebraic groups over 2-dimensional local fields. He used an analytic continuation technique and the residue construction of the Cherednik algebra which does not use generators and relations. He proved that indeed the algebra of Hecke operators associated with double cosets is isomorphic to the Cherednik algebra. By analogy with Bernstein's theorem, we could consider the center of the Cherednik algebra. The problem is that the center contains an infinite sum and so we run into a convergence problem.

In this paper, we study spherical Hecke algebras of $SL_2(F)$ with respect to the "maximal compact subgroup" $SL_2(O)$, where F is a 2-dimensional local field and O is the ring of integers with respect to the 2-dimensional valuation. There are several difficulties we need to overcome. First, since the group is not locally compact, we need to find an analogous notion of "functions with compact support" in the classical case. We will consider functions which are zero outside certain union of double cosets, introducing the concept of weight (See (3.6)). Similarly, we define spherical functions on the torus T. Second, since there is no Haar measure, we need to define the convolution product in a combinatorial way. This will be done by using an explicit coset decomposition of a double coset into right cosets. Because a double coset of $K = SL_2(O)$ is an uncountable union of right cosets of K, we need to make the convolution product to be zero outside certain union of double cosets (See (3.9)). Third, when we want to define the "Satake isomorphism", imitating the classical construction by using the integral on the unipotent radical $\mathbf{U}(F)$, we need measure on $\mathbf{U}(F)$. We use the invariant measure constructed by I. Fesenko. However, since it is $\mathbb{R}((X))$-valued, we have to define the integral to be zero outside certain double cosets to make the resulting function to be $\mathbb{C}$-valued (See (5.5)). We show that the spherical Hecke algebra $\mathcal{H}(G,K)$ is a commutative algebra, "generated" by three elements. (Here "generated" means that we must allow certain infinite sums.)

We explain briefly the contents of this paper. In Section 1, we briefly review the theory of an invariant $\mathbb{R}((X))$-valued measure on F constructed by I. Fesenko. We note that the additive group of a 2-dimensional local field is not locally compact with respect to its topology and therefore by Weil's theorem, there is no nontrivial Haar measure on it. In Section 2, we generalize the Cartan Decomposition to the case of groups over 2-dimensional local fields. In Section 3, we define the spherical Hecke algebra of $SL_2(F)$ with respect to the subgroup $SL_2(O)$, and its convolution product. Since we have not yet constructed an invariant measure on $SL_2(F)$, we need to define the convolution product in a combinatorial way. However, we hope to construct an invariant measure so that we may define the convolution product as an integral. In fact, our definition of the convolution product was motivated by such a formula (See Remark 5.13). In Section 4, we construct the spherical Hecke algebra of the torus of $SL_2(F)$. The final section is devoted to the construction of the "Satake isomorphism" between two spherical Hecke algebras in Sections 3 and 4. In the definition of this algebra isomorphism, we use Fesenko's measure and integration theory in Section 1.

There are many questions to be answered such as determining the structure of spherical Hecke algebras, the study of spherical representations of $\mathbf{G}(F)$, and an analogue of the semi-simple conjugacy classes. We hope to generalize our results such as spherical Hecke algebras and the Satake isomorphisms to any split semi-simple algebraic groups over 2-dimensional local fields in the near future.

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1. Invariant measure on 2-dimensional local fields. In this section, we briefly review the theory of the invariant measure on 2-dimensional local fields defined by I. Fesenko [Fe]. We refer the reader to [LF] for the definition and properties of a 2-dimensional local field.

Let $F(=F_2)$ be a two dimensional local field with the first residue field F_1 and the last residue field $F_0(=\mathbb{F}_q)$ of q elements. We denote by v_{21} the discrete valuation of rank one of F and by v_1 the discrete valuation of F_1. Also fix a discrete valuation $v : F^\times \to \mathbb{Z}^2$ of rank two. Recall that $\mathbb{Z}^2$ is endowed with the lexicographic ordering from the right. Let t_1, t_2 be local parameters with respect to the valuation v. With respect to the 2-dimensional valuation v, we have the ring of integers O of F, its maximal ideal M and the group of units U. These do not depend on the choice of v. Similarly, for the field F_1, we use the notations O_1, M_1 and U_1, respectively. Also, we denote by O_{21} the ring of integers of F with respect to the discrete valuation v_{21} of rank one, so t_2 generates the maximal ideal M_{21} of O_{21}. There are natural projections

$$p : O \to O/M = F_0, \qquad p_{21} : O_{21} \to O_{21}/M_{21} = F_1 \quad \text{and}$$
$$p_1 : O_1 \to O_1/M_1 = F_0.$$

Example 1.1. Assume that $F = \mathbb{Q}_p((t))$ with local parameters $t_1 = p$ and $t_2 = t$. Then

$$\begin{array}{llll} F_1 = \mathbb{Q}_p, & F_0 = \mathbb{F}_p, & O = \mathbb{Z}_p + t\mathbb{Q}_p[[t]], & M = p\mathbb{Z}_p + t\mathbb{Q}_p[[t]], \\ U = \mathbb{Z}_p^\times + t\mathbb{Q}_p[[t]], & O_{21} = \mathbb{Q}_p[[t]], & M_{21} = t\mathbb{Q}_p[[t]], & O_1 = \mathbb{Z}_p, \\ M_1 = p\mathbb{Z}_p, & U_1 = \mathbb{Z}_p^\times. & & \end{array}$$

In the following definition, we specify a family of measurable sets.

Definition 1.2. A subset of F is called *distinguished* if the set is of the form

$$\alpha + t_1^i t_2^j O, \quad \alpha \in F, i, j \in \mathbb{Z}.$$

We denote by $\mathcal{A}$ the minimal ring containing all distinguished subsets of F.

Alternatively, distinguished sets are shifts of fractional principal O-ideals of F, and $\mathcal{A}$ is the minimal ring which contains sets $\alpha + t_2^j p_{21}^{-1}(S)$, $j \in \mathbb{Z}$, where S is a compact open subset in F_1.

A sequence $\{\sum_i s_i^{(m)} X^i\}_{m\in\mathbb{N}}$ in $\mathbb{C}((X))$ *converges to* 0 if $s_i^{(m)} \to 0$ for each i and there is i_0 such that $s_i^{(m)} = 0$ for all $i < i_0$ and for all m. We define a linear map $\mathrm{res}_i : \mathbb{C}((X)) \to \mathbb{C}$ by $\sum a_j X^j \mapsto a_i$. A series $\sum c_n$, $c_n \in \mathbb{C}((X))$, is called *absolutely convergent* if it converges and $\sum |\mathrm{res}_i(c_n)|$ converges for every i.

LEMMA 1.3. [Fe] *There is a unique measure μ on F with values in $\mathbb{R}((X))$ which is shift invariant and finitely additive on $\mathcal{A}$ such that $\mu(\emptyset) = 0$ and*

$$\mu(t_1^i t_2^j O) = q^{-i} X^j \qquad \text{for } i, j \in \mathbb{Z}.$$

If S is a compact open subset in F_1, and μ_1 is the normalized Haar measure on F_1 such that $\mu_1(O_1) = 1$, we get $\mu(t_2^j p_{21}^{-1}(S)) = X^j \mu_1(S)$. We define a two dimensional *module* $|\cdot|$ on F by

$$|0| = 0, \qquad |t_1^i t_2^j u| = q^{-i} X^j \qquad \text{for } u \in U.$$

Then we see that $\mu(\alpha A) = |\alpha|\mu(A)$ for $A \in \mathcal{A}$ and $\alpha \in F^\times$.

Let R_F be the vector space generated by functions $f = \sum c_n \, char_{A_n}$, with countably many disjoint distinguished sets A_n, and $c_n \in \mathbb{C}$ such that $\sum c_n \mu(A_n)$ converges absolutely, and by functions g which are zero outside finitely many points. Then we define the integrals

$$\int f \, d\mu = \sum c_n \mu(A_n) \qquad \text{and} \qquad \int g \, d\mu = 0.$$

See [Fe] for the well-definedness.

Remark 1.4. The measure μ is *countably additive* in the following sense. Assume that $A \in \mathcal{A}$ is a disjoint union of $A_i \in \mathcal{A}$ ($i = 1, 2, \ldots$). Then we have $\mu(A) = \sum \mu(A_i)$, whenever $\sum \mu(A_i)$ is absolutely convergent in $\mathbb{R}((X))$. For more properties of the measure μ on F, we refer the reader to [Fe], where one can also find the definition and discussion of local zeta integrals on topological K_2-groups of F. A generalization of the measure and harmonic analysis to higher dimensional local fields has been developed in [Fe1].

2. Cartan decomposition of G. We prove the Cartan Decomposition with respect to the 2-dimensional local field structure for a connected split semi-simple algebraic group.

Let $\mathbf{G}$ be a connected split semi-simple algebraic group defined over $\mathbb{Z}$. We fix a maximal torus $\mathbf{T}$ and a Borel subgroup $\mathbf{B}$ such that $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$, and then we have $\mathbf{U} = [\mathbf{B}, \mathbf{B}]$, the unipotent radical of $\mathbf{B}$ and $W_0 = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, the Weyl group of $\mathbf{G}$. We consider the groups

$$G = \mathbf{G}(F), \quad K = \mathbf{G}(O), \quad K_{21} = \mathbf{G}(O_{21}) \quad \text{and} \quad T = \mathbf{T}(F).$$

Let

$$N = N_G(T), \quad I = \{x \in K : p(x) \in \mathbf{B}(F_0)\} \quad \text{and} \quad W = N/\mathbf{T}(O).$$

We call I *double Iwahori subgroup* of G and W *double affine Weyl group* of G. We define the characters and cocharacters of $\mathbf{T}$ by

$$X^* = X^*(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m) \quad \text{and} \quad X_* = X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T}).$$

We denote the set of roots of $\mathbf{G}$ by $\Phi \subset X^*$, the set of positive roots by Φ^+ and the set of simple roots by $\Delta \subset \Phi^+$. The positive Weyl chamber P^+ in X_* is defined to be

$$P^+ = \{\lambda \in X_* : \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Recall that there is a partial ordering on X_*: $\lambda \geq \mu$ if $\lambda - \mu$ can be written as a nonnegative linear combination of simple coroots. We denote by ρ the half sum of positive roots. We put $\mathfrak{X}_* = X_* \oplus X_*$ and let $\mathfrak{P}^+$ be the set of all $(\lambda_1, \lambda_2) \in \mathfrak{X}_*$ such that $\lambda_2 \in P^+$ and $\langle \lambda_1, \alpha \rangle \geq 0$ whenever $\langle \lambda_2, \alpha \rangle = 0$ for $\alpha \in \Delta$. We define a partial ordering on $\mathfrak{X}_*$ to be the lexicographic ordering from the right, i.e.

$$(2.1) \quad (\lambda_1, \lambda_2) \geq (\mu_1, \mu_2) \quad \Leftrightarrow \quad \lambda_2 > \mu_2, \quad \text{or} \quad \lambda_2 = \mu_2 \text{ and } \lambda_1 \geq \mu_1.$$

LEMMA 2.2. *The set $\mathfrak{P}^+$ is a fundamental domain for W_0-action on $\mathfrak{X}_*$, where the action is given by $w(\lambda_1, \lambda_2) = (w\lambda_1, w\lambda_2)$ for $w \in W_0$ and $(\lambda_1, \lambda_2) \in \mathfrak{X}_*$.*

Proof. Given $(\lambda_1, \lambda_2) \in \mathfrak{X}_*$, we can find $w_0 \in W_0$ such that $w_0\lambda_2 \in P^+$. Assume that $\langle w_0\lambda_2, \alpha \rangle = 0$ for some $\alpha \in \Delta$. If $\langle w_0\lambda_1, \alpha \rangle \geq 0$, then we put $w_1 = w_0$. Otherwise, let σ_α be the simple reflection corresponding to α. In this case, we put $w_1 = \sigma_\alpha w_0$, and then

$$\begin{aligned} &\langle w_1\lambda_2, \alpha \rangle = \langle \sigma_\alpha(w_0\lambda_2), \alpha \rangle = \langle w_0\lambda_2, \alpha \rangle = 0, \\ \text{and} \quad &\langle w_1\lambda_1, \alpha \rangle = \langle \sigma_\alpha(w_0\lambda_1), \alpha \rangle = -\langle w_0\lambda_1, \alpha \rangle > 0. \end{aligned}$$

Continuing this process, we can find an element w of W_0 such that $(w\lambda_1, w\lambda_2) \in \mathfrak{P}^+$.

Now assume that $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathfrak{P}^+$ and $(w\lambda_1, w\lambda_2) = (\mu_1, \mu_2)$ for some $w \in W_0$. Since $\lambda_2, \mu_2 \in P^+$ and $w\lambda_2 = \mu_2$, we see that $\lambda_2 = \mu_2$ and w is a product of simple reflections fixing λ_2, say $w = \sigma_{\alpha_1} \cdots \sigma_{\alpha_s}$. Then $\Delta_I := \{\alpha_1, \dots, \alpha_s\}$ forms a root system with the Weyl group W_I generated by the simple reflections $\sigma_{\alpha_1}, \dots, \sigma_{\alpha_s}$. Furthermore, $\langle \lambda_1, \alpha_i \rangle \geq 0$ and $\langle \mu_1, \alpha_i \rangle \geq 0$ for all $1 \leq i \leq s$. Now a standard argument tells us that $w\lambda_1 = \mu_1$ actually implies $\lambda_1 = \mu_1$. See, for example, [Hu]. □

Given $(\lambda_1, \lambda_2) \in \mathfrak{X}_*$, there is a $w \in W_0$ such that $(w\lambda_1, w\lambda_2) \in \mathfrak{P}^+$ by Lemma 2.2, and we define

$$D(\lambda_1, \lambda_2) = \{\mu \in X_* : (w\mu, w\lambda_2) \in \mathfrak{P}^+ \text{ and } w\mu \leq w\lambda_1\}.$$

PROPOSITION 2.3. *The group G has the following decompositions.*

(1) (Bruhat Decomposition) *[Ka]*

$$G = \coprod_{w \in W} IwI,$$

and the resulting identification $I \backslash G / I \to W$ is independent of the choice of representatives of elements of W.

(2) (Cartan Decomposition I)

$$G = \coprod_{\lambda \in P^+} K_{21} \lambda(t_2) K_{21}.$$

(3) (Cartan Decomposition II)

$$G = \coprod_{(\lambda_1, \lambda_2) \in \mathfrak{P}^+} K \lambda_1(t_1) \lambda_2(t_2) K.$$

Especially, we will use the fact that

$$K_{21} \lambda_2(t_2) K_{21} = \coprod_{\substack{\lambda_1 \\ (\lambda_1, \lambda_2) \in \mathfrak{P}^+}} K \lambda_1(t_1) \lambda_2(t_2) K$$

Proof. For the proof of part (1), see [Ka]. The part (2) is a classical result.

We can choose representatives of W_0 to be elements of K, and see that $W \simeq W_0 \ltimes (T/\mathbf{T}(O))$. Thus, given $w \in W$, we can write

$$w = w' \lambda_1(t_1) \lambda_2(t_2)$$

with $(\lambda_1, \lambda_2) \in \mathfrak{X}_*$ making a suitable choice of a representative of $w' \in W_0$ such that $w' \in K$. By Lemma 2.2, each $(\lambda_1, \lambda_2) \in \mathfrak{X}_*$ has one and only one W_0-conjugate in $\mathfrak{P}^+$. Combining these results with part (1), we obtain part (3). □

Remark 2.4. The Cartan Decomposition II is proved by A. N. Parshin in [Pa] for $\mathbf{G} = SL_n$. He also proved Bruhat and Iwasawa decompositions for PGL_2.

3. Spherical Hecke algebras of SL_2. We define spherical Hecke algebras of SL_2 and its convolution product after investigating the decomposition of a double coset into right cosets.

In the rest of this paper, we assume that $\mathbf{G} = SL_2$. We make the identification $\mathfrak{X}_* = \mathbb{Z} \oplus \mathbb{Z}$ and the partial ordering on $\mathfrak{X}_*$ corresponds to the lexicographic ordering from the right on $\mathbb{Z} \oplus \mathbb{Z}$. We will denote by the same notation $\leq$ the corresponding ordering on $\mathbb{Z} \oplus \mathbb{Z}$. Then we have the identification

$$\mathfrak{P}^+ = \{(i,j) \in \mathbb{Z} \oplus \mathbb{Z} : (i,j) \geq (0,0)\}$$

and we have

$$D(i,j) = \begin{cases} \{k \in \mathbb{Z} : k \leq i, (k,j) \geq (0,0)\} & \text{if } (i,j) \geq (0,0), \\ \{k \in \mathbb{Z} : k \geq i, (k,j) \leq (0,0)\} & \text{if } (i,j) < (0,0). \end{cases}$$

In the following lemma, we present an explicit formula of the Cartan Decomposition II for $SL_2(F)$.

LEMMA 3.1. *Assume that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ *and let* $(-k, -l) = \min\{v(a), v(b), v(c), v(d)\}$. *Then* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \begin{pmatrix} t_1^k t_2^l & 0 \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix} K, \quad (k,l) \geq (0,0)$. *Namely,*

$$SL_2(F) = \coprod_{(k,l)\geq(0,0)} K \begin{pmatrix} t_1^k t_2^l & 0 \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix} K.$$

Proof. We can check the assertion of the Lemma using column operations and row operations of matrices, and we omit the detail. □

LEMMA 3.2. *For* $(i,j) \geq (0,0)$*, we have*

$$K \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} K = \coprod_g Kg,$$

where the disjoint union is over g in the following list.

(i) $\begin{pmatrix} t_1^{-i}t_2^{-j} & 0 \\ 0 & t_1^i t_2^j \end{pmatrix}$, (ii) $\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}$,

(iii) $\begin{pmatrix} t_1^{-i}t_2^{-j} & t_1^k t_2^l u \\ 0 & t_1^i t_2^j \end{pmatrix}$ *for* $(-i,-j) \le (k,l) < (i,j)$*, where* $u \in U$ *are units belonging to a fixed set of representatives of* $O/t_1^{i-k}t_2^{j-l}O$*,*

(iv) $\begin{pmatrix} t_1^k t_2^l & t_1^{-i}t_2^{-j}u \\ 0 & t_1^{-k}t_2^{-l} \end{pmatrix}$ *for* $(-i,-j) < (k,l) < (i,j)$*, where* $u \in U$ *are units belonging to a fixed set of representatives of* $O/t_1^{i-k}t_2^{j-l}O$*.*

Proof. Consider elements $g, g' \in \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix} K$ and write

$$g = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g' = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in K$. We see that the condition $Kg = Kg'$ is equivalent to

$$c'd - d'c \in t_1^{2i}t_2^{2j}O. \tag{3.3}$$

We write $(c,d) \sim (c',d')$ if $c'd - d'c \in t_1^{2i}t_2^{2j}O$. Note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ then either c or d is a unit. Let C be the set of pairs $(c,d) \in O^2$ such that either c or d is a unit. Then it is easy to check that $\sim$ is an equivalence relation on C. Thus in order to determine different cosets, we need only to determine a set of representatives of the equivalence relations $\sim$.

Assume that $(c,d) \in C$. If c is a unit then $(c,d) \sim (1, d/c)$. We write $d/c = t_1^{i+k}t_2^{j+l}u$ for some $(k,l) \ge (-i,-j)$ and $u \in U \cup \{0\}$. If $(k,l) \ge (i,j)$, then $(c,d) \sim (1, t_1^{i+k}t_2^{j+l}u) \sim (1,0)$. Assume that $(-i,-j) \le (k,l) < (i,j)$. If $(k',l') \ne (k,l)$ and $(-i,-j) \le (k',l') < (i,j)$, then $(1, t_1^{i+k}t_2^{j+l}u) \not\sim (1, t_1^{i+k'}t_2^{j+l'}u')$ for any $u' \in U \cup \{0\}$. If $(k',l') = (k,l)$, then $(1, t_1^{i+k}t_2^{j+l}u) \sim (1, t_1^{i+k'}t_2^{j+l'}u')$ if and only if $u - u' \in t_1^{i-k}t_2^{j-l}O$. Thus if c is a unit, a set of representatives of the equivalence relation is given by

$$(1,0) \quad \text{and} \quad (1, t_1^{i+k}t_2^{j+l}u),$$

where $(-i,-j) \le (k,l) < (i,j)$ and u are units belonging to a fixed set of representatives of $O/t_1^{i-k}t_2^{j-l}O$. The representative $(1,0)$ corresponds to the matrix of the part (i). We consider the other representatives. Through elementary operations,

$$K\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & t_1^{i+k}t_2^{j+l}u \end{pmatrix} = K\begin{pmatrix} 0 & -t_1^i t_2^j \\ t_1^{-i}t_2^{-j} & t_1^k t_2^l u \end{pmatrix} = K\begin{pmatrix} t_1^{-i}t_2^{-j} & t_1^k t_2^l u \\ 0 & t_1^i t_2^j \end{pmatrix}.$$

So we obtain the matrices in the part (iii).

If c is not a unit, then d is a unit. In this case, a set of representatives of the equivalence relation is given by

$$(0,1) \quad \text{and} \quad (t_1^{i+k}t_2^{j+l}, u)$$

where $(-i,-j) < (k,l) < (i,j)$ and u are units belonging to a fixed set of representatives of $O/t_1^{i-k}t_2^{j-l}O$. The representative $(0,1)$ corresponds to the matrix of the part (ii). For the other representatives, through elementary operations, we obtain

$$K\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ t_1^{i+k}t_2^{j+l} & u \end{pmatrix} = K\begin{pmatrix} t_1^i t_2^j u^{-1} & 0 \\ t_1^k t_2^l & t_1^{-i}t_2^{-j}u \end{pmatrix} = K\begin{pmatrix} t_1^k t_2^l & t_1^{-i}t_2^{-j}u \\ 0 & t_1^{-k}t_2^{-l} \end{pmatrix}.$$

So we get the matrices in the part (iv). □

Remark 3.4. If $j \neq l$, then the cardinality of $O/t_1^{i-k}t_2^{j-l}O$ is uncountable. Hence in general, the double coset $K\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}K$ is an uncountable union of right cosets of K. This is very different from the classical p-adic case of $SL_2(\mathbb{Q}_p)$ where any double coset of $SL_2(\mathbb{Z}_p)$ is a finite union of right cosets of $SL_2(\mathbb{Z}_p)$.

Now we begin our construction of Hecke algebras.

Definition 3.5. A $\mathbb{C}$-valued function f on G is called *spherical* if f satisfies the following properties:

(1) $f(k_1 x k_2) = f(x)$ for $k_1, k_2 \in K$ and $x \in G$,

(2) there exists $(i,j) \in \mathfrak{P}^+$ such that

$$f(x) = 0 \quad \text{if } x \notin \coprod_{m \in D(i,j)} K\begin{pmatrix} t_1^m t_2^j & 0 \\ 0 & t_1^{-m}t_2^{-j} \end{pmatrix}K. \tag{3.6}$$

If f is spherical, then we can always find the minimal $(i,j) \in \mathfrak{P}^+$ satisfying (3.6) with respect to the ordering (2.1) on $\mathfrak{X}_*$. The minimal (i,j) will be called the *weight* of f.

For each $(i,j) \geq (0,0)$, we let $\chi_{i,j}$ be the characteristic function of the double coset

$$K \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} K.$$

Then a spherical function of weight (i,j) can be written as

$$\sum_{k \leq i} a_k \chi_{k,j} \qquad \text{for some } (i,j) \geq (0,0) \text{ and } a_k \in \mathbb{C}. \tag{3.7}$$

Definition 3.8. We define the convolution product of two characteristic functions by

$$(\chi_{i,j} * \chi_{k,l})(x) = \begin{cases} \sum_z \chi_{i,j}(\tau(x) z^{-1}) & \text{if } x \in \coprod_{m \in D(i+k,j+l)} K \begin{pmatrix} t_1^m t_2^{j+l} & 0 \\ 0 & t_1^{-m} t_2^{-j-l} \end{pmatrix} K, \\ 0 & \text{otherwise}, \end{cases} \tag{3.9}$$

where $\tau(x) = \begin{pmatrix} t_1^m t_2^{j+l} & 0 \\ 0 & t_1^{-m} t_2^{-j-l} \end{pmatrix}$ for $x \in K \begin{pmatrix} t_1^m t_2^{j+l} & 0 \\ 0 & t_1^{-m} t_2^{-j-l} \end{pmatrix} K$, and the sum is over the representatives z of the decomposition

$$K \begin{pmatrix} t_1^k t_2^l & 0 \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix} K = \coprod_z Kz.$$

Even though a double coset of K is generally an uncountable union of right cosets of K, we show in the following lemma that given x, there are only finitely many nonzero terms in (3.9). Hence the convolution product is well-defined. Now we prove:

PROPOSITION 3.10. *The convolution product $\chi_{i,j} * \chi_{k,l}$ is a spherical function. Moreover, we have the following explicit formulas:*

(1) *If $i > 0$, then*

$$\chi_{i,0} * \chi_{i,0} = \chi_{2i,0} + (1 - \frac{1}{q}) \sum_{0<r<2i} q^r \chi_{2i-r,0} + q^{2i} \left(1 + \frac{1}{q}\right) \chi_{0,0}.$$

(2) *If $(i,0) < (k,l)$, then*

$$\chi_{i,0} * \chi_{k,l} = \chi_{k,l} * \chi_{i,0} = \chi_{i+k,l} + \left(1 - \frac{1}{q}\right) \sum_{0<r<2i} q^r \chi_{i+k-r,l} + q^{2i} \chi_{-i+k,l}.$$

(3) *If $j > 0$ and $l > 0$, then*

$$\chi_{i,j} * \chi_{k,l} = \chi_{i+k,j+l} + \left(1 - \frac{1}{q}\right) \sum_{r>0} q^r \chi_{i+k-r,j+l}.$$

Proof. We may let

$$x = \tau(x) = \begin{pmatrix} t_1^m t_2^{j+l} & 0 \\ 0 & t_1^{-m} t_2^{-j-l} \end{pmatrix}, \quad m \in D(i+k, j+l).$$

We consider

$$K \begin{pmatrix} t_1^k t_2^l & 0 \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix} K = \coprod_z Kz,$$

where the disjoint union is over z in the list (i)-(iv) of Lemma 3.2. We need to determine the conditions under which $\chi_{i,j}(xz^{-1}) = 1$.

Since the other cases are similar, we prove only the part (3) of the lemma. So we assume that $j > 0$ and $l > 0$. In this case, $m \in D(i+k, j+l)$ if and only if $m \le i+k$.

(i) If $z = \begin{pmatrix} t_1^{-k} t_2^{-l} & 0 \\ 0 & t_1^k t_2^l \end{pmatrix}$ then we have

$$\chi_{i,j}(xz^{-1}) = \chi_{i,j} \begin{pmatrix} t_1^{m+k} t_2^{j+2l} & 0 \\ 0 & t_1^{-m-k} t_2^{-j-2l} \end{pmatrix} = 0$$

for all $m \le i+k$.

(ii) If $z = \begin{pmatrix} t_1^k t_2^l & 0 \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix}$ then we have

$$\chi_{i,j}(xz^{-1}) = \chi_{i,j} \begin{pmatrix} t_1^{m-k} t_2^{j} & 0 \\ 0 & t_1^{-m+k} t_2^{-j} \end{pmatrix} = 1 \quad \Leftrightarrow \quad m = i+k.$$

(iii) If $z = \begin{pmatrix} t_1^{-k} t_2^{-l} & t_1^{k'} t_2^{l'} u \\ 0 & t_1^k t_2^l \end{pmatrix}$ for $(-k, -l) \le (k', l') < (k, l)$ and $u \in U$ belonging to a fixed set of representatives of $O/t_1^{k-k'} t_2^{l-l'} O$, then we get

$$\chi_{i,j}(xz^{-1}) = \chi_{i,j} \begin{pmatrix} t_1^{m+k} t_2^{j+2l} & -t_1^{m+k'} t_2^{j+l+l'} u \\ 0 & t_1^{-m-k} t_2^{-j-2l} \end{pmatrix} = 0$$

for all $m \le i+k$.

(iv) $z = \begin{pmatrix} t_1^{k'} t_2^{l'} & t_1^{-k} t_2^{-l} u \\ 0 & t_1^{-k'} t_2^{-l'} \end{pmatrix}$ for $(-k, -l) < (k', l') < (k, l)$ and $u \in U$ belonging to a fixed set of representatives of $O/t_1^{k-k'} t_2^{l-l'} O$, then

$$\chi_{i,j}(xz^{-1}) = \chi_{i,j} \begin{pmatrix} t_1^{m-k'} t_2^{j+l-l'} & -t_1^{m-k} t_2^{j} u \\ 0 & t_1^{-m+k'} t_2^{-j-l+l'} \end{pmatrix} = 1$$
$$\Leftrightarrow \quad l = l', \ m = i + k' \ \text{ and } \ k' < k.$$

Note that for each $k' < k$ the number of different u's is $(1 - \frac{1}{q}) q^{k-k'}$.

Since there is only finite number of z's for which $\chi_{i,j}(xz^{-1}) \neq 0$, we have

$$(\chi_{i,j} * \chi_{k,l})(x) = \sum_z \chi_{i,j}(xz^{-1}) = \begin{cases} 1 & \text{if } m = i + k, \\ (1 - \frac{1}{q}) q^{k-k'} & \text{if } m = i + k' \text{ for each } k' < k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} \chi_{i,j} * \chi_{k,l} &= \chi_{i+k,j+l} + \left(1 - \frac{1}{q}\right) \sum_{k' < k} q^{k-k'} \chi_{i+k',j+l} \\ &= \chi_{i+k,j+l} + \left(1 - \frac{1}{q}\right) \sum_{r>0} q^r \chi_{i+k-r,j+l}. \end{aligned}$$ □

For example, we have the following identities:

$$\begin{aligned} \chi_{1,0} * \chi_{1,0} &= \chi_{2,0} + (q-1)\chi_{1,0} + (q^2+q)\chi_{0,0}, \\ \chi_{1,0} * \chi_{0,1} &= \chi_{1,1} + (q-1)\chi_{0,1} + q^2 \chi_{-1,1}, \\ \chi_{0,1} * \chi_{0,1} &= \chi_{0,2} + (1 - \tfrac{1}{q}) \textstyle\sum_{k>0} q^k \chi_{-k,2}. \end{aligned} \tag{3.11}$$

Assume that $f = \sum_{p \le i} a_p \chi_{p,j}$ and $g = \sum_{q \le k} b_q \chi_{q,l}$ are spherical functions. We define the convolution product $f * g$ by

$$f * g = \left(\sum_{p \le i} a_p \chi_{p,j} \right) * \left(\sum_{q \le k} b_q \chi_{q,l} \right) = \sum_{p \le i, \ q \le k} a_p b_q \chi_{p,j} * \chi_{q,l}. \tag{3.12}$$

It follows from Proposition 3.10 that we can write

$$f * g = \sum_{r \le i+k} c_r \chi_{r,j+l},$$

and note that $\chi_{r,j+l}$ appears in the expansion of $\chi_{p,j} * \chi_{q,l}$ only if $p + q \ge r$. Since the number of such pairs (p, q) is finite, the coefficient c_r is well defined for each

$r \leq i+k$. Therefore the convolution product $f * g$ is a well defined spherical function of weight $(i+k, j+l)$.

THEOREM 3.13. *The convolution product is commutative. Namely, $f * g = g * f$ for spherical functions f and g on G.*

Proof. Note that we have $\chi_{i,j} * \chi_{k,l} = \chi_{k,l} * \chi_{i,j}$ in the part (3) of Proposition 3.10. With this observation, the proposition follows from Proposition 3.10 and definitions. □

Now we are ready to introduce the main object of this paper.

Definition 3.14. We define the *spherical Hecke algebra* $\mathcal{H}(G,K)$ of G (relative to K) to be the $\mathbb{C}$-algebra whose elements are finite linear combinations of spherical functions on G with the multiplication given by the convolution product (3.9) and (3.12).

PROPOSITION 3.15. (a) *The elements $\chi_{i,0}, i > 0$ and $\chi_{i,1}, i \in \mathbb{Z}$ are contained in the subalgebra generated by $\chi_{1,0}$, $\chi_{0,1}$ and $\chi_{-1,1}$.*

(b) *The element $\chi_{i,j}$ for each (i,j), $j > 1$ is given by the formula*

$$\chi_{i,j} = \left(\chi_{i,j-1} - (q-1) \sum_{r>0} \chi_{i-r,j-1} \right) * \chi_{0,1}.$$

Proof. Using Proposition 3.10, one can check the assertions and we omit the detail. □

Remark 3.16. If we allow infinite sums of the form (3.7), an induction argument using the above proposition shows that the algebra $\mathcal{H}(G,K)$ is "generated" by three elements $\chi_{1,0}$, $\chi_{0,1}$ and $\chi_{-1,1}$.

4. Spherical Hecke algebra of T. In this section, we construct the spherical Hecke algebra of T relative to $\mathbf{T}(O)$ for $G = SL_2(F)$.

Definition 4.1. A $\mathbb{C}$-valued functions f on T is called *spherical* if f satisfies the following conditions:

(1) $f(xz) = f(x)$ for $z \in \mathbf{T}(O)$ and $x \in T$,

(2) there exist $(i,j) \in \mathfrak{X}_*$ such that

$$f(x) = 0 \quad \text{if } x \notin \coprod_{m \in D(i,j)} \begin{pmatrix} t_1^m t_2^j & 0 \\ 0 & t_1^{-m} t_2^{-j} \end{pmatrix} \mathbf{T}(O). \tag{4.2}$$

We can find $(i,j) \in \mathfrak{X}_*$ satisfying (4.2), which is minimal in the sense that if another (k,j) satisfies the condition then $i \in D(k,j)$. A minimal (i,j) is uniquely determined, and will be called the *weight* of f. For each $(i,j) \in \mathbb{Z} \oplus \mathbb{Z}$, we define $c_{i,j}$ to be the characteristic function of the (double) coset

$$\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} \mathbf{T}(O).$$

Then a spherical function on T can be written as one of the following:

$$\sum_{k \le i} a_k c_{k,j}, \quad \sum_{k \le i} a_k c_{-k,-j}, \quad \sum_{0 \le k \le i} a_k c_{k,0}, \quad \sum_{0 \le k \le i} a_k c_{-k,0} \qquad (a_k \in \mathbb{C}, j > 0). \tag{4.3}$$

Definition 4.4. We define the convolution product $c_{i,j} * c_{k,l}$ by

$$c_{i,j} * c_{k,l} = \begin{cases} c_{i+k,j+l} & \text{if } jl \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

Assume that $f = \sum_{p \le i} a_p c_{p,j}$ and $g = \sum_{q \le k} b_q c_{q,l}$ are spherical functions of T of weights $(i,j), (k,l)$, resp. where $j, l > 0$. We define the convolution product $f * g$ by

$$f * g = \left(\sum_{p \le i} a_p c_{p,j} \right) * \left(\sum_{q \le k} b_q c_{q,l} \right) = \sum_{p \le i,\ q \le k} a_p b_q c_{p,j} * c_{q,l}. \tag{4.6}$$

Then

$$f * g = \sum_{p \le i,\ q \le k} a_p b_q c_{p+q,j+l} = \sum_{r \le i+k} \left(\sum_{\substack{p \le i, q \le k \\ r = p+q}} a_p b_q \right) c_{r,j+l}.$$

Hence $f * g$ is a well-defined spherical function of weight $(i+k, j+l)$. If f, g are of the forms $\sum_{k \le i} a_k c_{-k,-j}$, $\sum_{0 \le k \le i} a_k c_{k,0}$, or $\sum_{0 \le k \le i} a_k c_{-k,0}$, then $f * g$ is defined in a similar way. Note that if f, g have weights $(i,j), (k,l)$ resp. where $jl < 0$, then by (4.5) the convolution product $f * g$ is identically zero.

Definition 4.7. We define the spherical Hecke algebra $\mathcal{H}(T, \mathbf{T}(O))$ of T to be the $\mathbb{C}$-algebra whose elements are finite linear combinations of spherical functions on T with the multiplication given by the convolution product (4.5) and (4.6).

We need the notion of W_0-invariant subalgebra of $\mathcal{H}(T, \mathbf{T}(O))$; the algebra $\mathcal{H}(T, \mathbf{T}(O))^{W_0}$ is defined to be the subalgebra of $\mathcal{H}(T, \mathbf{T}(O))$ consisting of the elements f satisfying the condition

$$f(wxw^{-1}) = f(x) \qquad \text{for } w \in W_0.$$

Remark 4.8. If we allow infinite sums of the forms in (4.3), the W_0-invariant subalgebra $\mathcal{H}(T,\mathbf{T}(O))^{W_0}$ is "generated" by three elements $c_{1,0}+c_{-1,0}$, $c_{0,1}+c_{0,-1}$ and $c_{-1,1}+c_{1,-1}$.

5. Satake isomorphism. In this section, we define the "Satake isomorphism" for SL_2, namely, the isomorphism between $\mathcal{H}(G,K)$ and $\mathcal{H}(T,\mathbf{T}(O))^{W_0}$. For the classical p-adic groups, I. Satake [Sa] constructed such isomorphism. We follow his construction, using the measure and integration in Section 1. We hope to generalize our results to arbitrary split semi-simple algebraic groups.

Definition 5.1. A group homomorphism $\delta : T \to \mathbb{R}((X))^\times$ is defined to be

$$\delta(x) = q^{-2i}X^{2j} \quad \text{for } x \in \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix} \mathbf{T}(O). \tag{5.2}$$

We identify F with $\mathbf{U}(F)$ through

$$u : F \to \mathbf{U}(F), \quad u(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \tag{5.3}$$

Let $d\mu(a)$ be the invariant measure on $\mathbf{U}(F)$, given by the invariant measure on F as in section 1 and the isomorphism u.

Definition 5.4. We define a map $\mathcal{S} : \mathcal{H}(G,K) \to \mathcal{H}(T,\mathbf{T}(O))$ by

$$\mathcal{S}(f)(x) = \begin{cases} \delta(x)^{\frac{1}{2}} X^j \int_F f(xu(a))d\mu(a), & \text{if } x \in \coprod_{k\in\mathbb{Z}} \begin{pmatrix} t_1^k t_2^{\pm j} & 0 \\ 0 & t_1^{-k}t_2^{\mp j} \end{pmatrix} \mathbf{T}(O), \\ 0 & \text{otherwise}, \end{cases} \tag{5.5}$$

for f whose weight is $(i,j) \in \mathfrak{P}^+$, and by extending it to the whole $\mathcal{H}(G,K)$ through linearity.

It is clear that $\mathcal{S}(f)$ is $\mathbf{T}(O)$-invariant. However, it is not clear a priori that $\mathcal{S}(f)$ is a $\mathbb{C}$-valued function since the measure takes value in $\mathbb{R}((X))$. We show in the next two propositions that $\mathcal{S}(f)$ is a $\mathbb{C}$-valued function and the map $\mathcal{S}$ is a well-defined map.

PROPOSITION 5.6.

$$\mathcal{S}(\chi_{i,j}) = q^i \left\{ c_{i,j} + c_{-i,-j} + (1-\tfrac{1}{q}) \sum_{k<i} (c_{k,j} + c_{-k,-j}) \right\} \quad \textit{for } (i,j) > (0,0),$$

$$\textit{and} \quad \mathcal{S}(\chi_{i,0}) = q^i \left\{ c_{i,0} + c_{-i,0} + (1-\tfrac{1}{q}) \sum_{-i<k<i} c_{k,0} \right\} \quad \textit{for } i > 0. \tag{5.7}$$

Proof. From the definition, we need only to consider

$$x \in \coprod_{k\in\mathbb{Z}} \begin{pmatrix} t_1^k t_2^j & 0 \\ 0 & t_1^{-k} t_2^{-j} \end{pmatrix} \mathbf{T}(O) \cup \begin{pmatrix} t_1^k t_2^{-j} & 0 \\ 0 & t_1^{-k} t_2^{j} \end{pmatrix} \mathbf{T}(O).$$

Since $\mathcal{S}(f)$ is $\mathbf{T}(O)$-invariant, it is enough to consider $x = \begin{pmatrix} t_1^k t_2^l & 0 \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix}$, $k \in \mathbb{Z}$ and $l = \pm j$. From the definition, we get $\delta(x)^{\frac{1}{2}} = q^{-k} X^l$. If $u(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathbf{U}(F)$, then $xu(a) = \begin{pmatrix} t_1^k t_2^l & a t_1^k t_2^l \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix}$.

We first consider the case $(k,l) \geq (0,0)$, i.e., $l = j$. If $v(a) \geq (-2k, -2j)$, then it follows from Lemma 3.1 that $xu(a) \in K \begin{pmatrix} t_1^k t_2^j & 0 \\ 0 & t_1^{-k} t_2^{-j} \end{pmatrix} K$; otherwise, $xu(a) \in K \begin{pmatrix} a^{-1} t_1^{-k} t_2^{-j} & 0 \\ 0 & a t_1^k t_2^j \end{pmatrix} K$. We obtain

$$\begin{aligned} \mathcal{S}(\chi_{i,j})(x) &= q^{-k} X^{2j} \int_F \chi_{i,j}(xu(a)) d\mu(a) \\ &= q^{-k} X^{2j} \int_{v(a)\geq(-2k,-2j)} \chi_{i,j} \begin{pmatrix} t_1^k t_2^j & 0 \\ 0 & t_1^{-k} t_2^{-j} \end{pmatrix} d\mu(a) \\ &\quad + q^{-k} X^{2j} \int_{v(a)<(-2k,-2j)} \chi_{i,j} \begin{pmatrix} a^{-1} t_1^{-k} t_2^{-j} & 0 \\ 0 & a t_1^k t_2^j \end{pmatrix} d\mu(a). \end{aligned}$$

In the second integral, note that $v(a^{-1} t_1^{-k} t_2^{-j}) > (k,j)$. So if $(i,j) = (k,j)$, then the second integral is zero. Also $\{a \in F \mid v(a) \geq (-2k, -2j)\} = t_1^{-2k} t_2^{-2j} O$ in the notation of Definition 1.2. Hence $\int_{v(a)\geq(-2k,-2j)} d\mu(a) = q^{2k} X^{-2j}$. Therefore if $(i,j) = (k,j)$,

$$\mathcal{S}(\chi_{i,j})(x) = q^{-k} X^{2j} q^{2k} X^{-2j} = q^i.$$

If $(i,j) > (k,j)$, i.e., $k < i$, then the first integral is zero. The second integral is nonzero only when $v(a) = (-i-k, -2j)$. Here $\{a \in F \mid v(a) = (-i-k, -2j)\} = t_1^{-i-k} t_2^{-2j} U$ and the measure of U is $1 - \frac{1}{q}$. Hence $\int_{v(a)=(-i-k,-2j)} d\mu(a) = q^{i+k} X^{-2j} (1 - \frac{1}{q})$. Therefore, if $(i,j) > (k,j)$,

$$\mathcal{S}(\chi_{i,j})(x) = q^{-k} X^{2j} q^{i+k} X^{-2j} \left(1 - \frac{1}{q}\right) = q^i \left(1 - \frac{1}{q}\right).$$

Next we consider the case $(k,l) < (0,0)$. A similar calculation as in the above gives us

$$\mathcal{S}(\chi_{i,j})(x) = \begin{cases} q^i & \text{if } (i,j) = (-k,-l), \\ q^i(1-\frac{1}{q}) & \text{if } (i,j) > (-k,-l). \end{cases}$$

Combining these results we obtain the equalities in (5.7). □

PROPOSITION 5.8. *For each* $(i,j) \geq (0,0)$*,*

$$\mathcal{S}(\sum_{k\leq i} a_k\chi_{k,j}) = \sum_{k\leq i} a_k\mathcal{S}(\chi_{k,j}).$$

Proof. It is enough to prove that the right hand side is well defined. From (5.7), we only need to check the well definedness of the double sum

$$\sum_{k\leq i} a_k q^k \sum_{l<k} (c_{l,j} + c_{-l,-j}).$$

It is equal to

$$\sum_{l\leq i} (c_{l,j} + c_{-l,-j}) \sum_{k=l}^{i} a_k q^k.$$

Here $\sum_{k=l}^{i} a_k q^k$ is a finite sum and well defined. □

Hence, by Proposition 5.6 and Proposition 5.8, the map $\mathcal{S}$ is well defined and the image of the map $\mathcal{S}$ is contained in the W_0-invariant subalgebra $\mathcal{H}(T,\mathbf{T}(O))^{W_0}$.

LEMMA 5.9. *We have*

$$\mathcal{S}(\chi_{i,j} * \chi_{k,l}) = \mathcal{S}(\chi_{i,j}) * \mathcal{S}(\chi_{k,l})$$

for $(i,j) \geq (0,0)$ *and* $(k,l) \geq (0,0)$.

Proof. Using Proposition 3.10, Proposition 5.6 and Proposition 5.8, one can straightforwardly check the identity and we omit the detail. □

THEOREM 5.10. *Assume that* $\mathbf{G} = SL_2$. *Then the map* $\mathcal{S} : \mathcal{H}(G,K) \to \mathcal{H}(T,\mathbf{T}(O))^{W_0}$ *is an algebra isomorphism.*

Proof. In the formula (5.7) we see that the maximal weight of the terms in the right-hand side is the same as the weight of $\chi_{i,j}$. Thus $\mathcal{S}$ is weight-preserving and so injective. Since an element of $\mathcal{H}(T,\mathbf{T}(O))^{W_0}$ is a finite linear combination of the elements of the form

(5.11) $$g = \sum_{k\leq i} a_k(c_{k,j} + c_{-k,-j}), \qquad a_k \in \mathbb{C}, (i,j) \geq 0,$$

we need only to find an element f of $\mathcal{H}(G, K)$ such that $\mathcal{S}(f) = g$. If $j = 0$, then (5.11) is a finite sum and it is easy to find such an f using (5.7). So we assume that $j > 0$. By Proposition 5.8 and (5.7), we obtain

$$\mathcal{S}\left(\chi_{i,j} - (q-1)\sum_{r>0}\chi_{i-r,j}\right) = q^i(c_{i,j} + c_{-i,-j}).$$

Thus we put

$$f = \sum_{k\leq i} q^{-k}a_k\left(\chi_{k,j} - (q-1)\sum_{r>0}\chi_{k-r,j}\right). \tag{5.12}$$

If we rewrite (5.12) as $f = \sum_{k\leq i} b_k\chi_{k,j}$, then each $b_k \in \mathbb{C}$, $k \leq i$, is well defined. Now Proposition 5.8 enables us to have $\mathcal{S}(f) = g$. Hence, $\mathcal{S}$ is surjective. It follows from Proposition 5.8 and Lemma 5.9 that $\mathcal{S}$ is an algebra homomorphism. Therefore, $\mathcal{S}$ is an algebra isomorphism. □

Remark 5.13. We expect in the near future (See [K-L]) to construct an invariant $\mathbb{R}((X))$-valued measure $d\gamma$ on $\mathbf{G}(F)$ for $\mathbf{G}$ a connected split semi-simple algebraic group. (We follow [Go], namely, we define an additive invariant measure on the product space F^n and obtain the transformation rule. Then using the big cell decomposition, we can extend it to an invariant measure on $\mathbf{G}(F)$.) Then we can define the convolution product of two spherical functions f and g on $\mathbf{G}(F)$ by

$$(f*g)(x) = \begin{cases} \int_G f(xy^{-1})g(y)d\gamma(y) & \text{if } x \in \coprod_{\mu\in D(\lambda_1+\mu_1,\lambda_2+\mu_2)} K\mu(t_1)(\lambda_2+\mu_2)(t_2)K, \\ 0 & \text{otherwise.}\end{cases}$$

The definition of the convolution product (3.9) was motivated by this formula. We need to define the convolution product in this way so that given x, the function $y \longmapsto f(xy^{-1})g(y)$ is a finite linear combination of characteristic functions of right cosets of K. Then $f*g$ becomes a $\mathbb{C}$-valued function, even though the measure $d\gamma$ takes values in $\mathbb{R}((X))$.

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