Algebras and Representation Theory
https://doi.org/10.1007/s10468-023-10225-x

RESEARCH

Auto-Correlation Functions for Unitary Groups

Kyu-Hwan Lee[1] **· Se-Jin Oh**[2]

Received: 24 October 2022 / Accepted: 28 July 2023

Abstract
We compute the auto-correlations functions of order $m \geq 1$ for the characteristic polynomials of random matrices from certain subgroups of the unitary groups $\mathrm{U}(2)$ and $\mathrm{U}(3)$ by establishing new branching rules. These subgroups can be understood as certain analogues of Sato–Tate groups of $\mathrm{USp}(4)$ in our previous paper. Our computation yields symmetric polynomial identities with m-variables involving irreducible characters of $\mathrm{U}(m)$ for all $m \geq 1$ in an explicit, uniform way.

Keywords Auto-correlation functions · Unitary groups · Symmetric functions

Mathematics Subject Classification (2010) 11M50 · 17B10 · 05E05

1 Introduction

1.1 Auto-Correlation Functions

The distribution of characteristic polynomials of random matrices has been of great interest for their applications in mathematical physics and number theory. Since Keating and Snaith [21, 22] computed averages of characteristic polynomials of random matrices in 2002 motivated in part by connections to number theory and in part by the importance of these averages in quantum chaos [1], it has become clear that averages of characteristic polynomials are fundamental for random matrix models [2–5, 15, 16, 28].

Presented by: Cristian Lenart

This work was partially supported by a grant from the Simons Foundation (#712100)
The research of S.-J. Oh was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2022R1A2C1004045).

✉ Se-Jin Oh
sejin092@gmail.com

Kyu-Hwan Lee
khlee@math.uconn.edu

1 Department of Mathematics, University of Connecticut, 06269 Storrs, CT, USA

2 Department of Mathematics, Sungkyunkwan University, 16419 Suwon, South Korea

Published online: 11 September 2023

On the way of these developments, the auto-correlation functions of the distributions of characteristic polynomials in the compact classical groups were computed by Conrey, Farmer, Keating, Rubinstein and Snaith [8, 9] and by Conrey, Farmer and Zirnbauer [10, 11]. Later, Bump and Gamburd [7] obtained different derivations of the formulas starting from (analogues of) the dual Cauchy identity and adopting a representation-theoretic method. Their results show that the auto-correlation functions are actually combinations of characters of classical groups.

For example, for the symplectic groups, an analogue of the dual Cauchy identity is due to Jimbo–Miwa [19] and Howe [18]:

$$\prod_{i=1}^{m}\prod_{j=1}^{g}(x_i+x_i^{-1}+t_j+t_j^{-1}) = \sum_{\lambda\trianglelefteq(g^m)} \chi_\lambda^{\mathrm{Sp}(2m)}(x_1^{\pm1},\ldots,x_m^{\pm1})\chi_{\tilde\lambda}^{\mathrm{Sp}(2g)}(t_1^{\pm1},\ldots,t_g^{\pm1}), \quad (1.1)$$

where $\chi_\lambda^{\mathrm{Sp}(2m)}$ is the irreducible character of $\mathrm{Sp}(2m,\mathbb{C})$ associated with the partition $\lambda \trianglelefteq (g^m)$ and we set $\tilde\lambda = (m-\lambda'_g,\ldots,m-\lambda'_1)$ with $\lambda' = (\lambda'_1,\ldots,\lambda'_g)$ the transpose of λ. This identity can be considered as a reflection of Howe duality. Using this identity, Bump and Gamburd computed the auto-correlation functions to obtain

$$\int_{\mathrm{USp}(2g)} \left(\prod_{j=1}^{m}\det(I+x_j\gamma)\right) d\gamma = (x_1\ldots x_m)^g\,\chi_{(g^m)}^{\mathrm{Sp}(2m)}(x_1^{\pm1},\ldots,x_m^{\pm1}). \quad (1.2)$$

1.2 Sato–Tate Groups

The celebrated Sato–Tate conjecture for elliptic curves (i.e. genus 1 curves) predicts that the distribution of Euler factors of an elliptic curve is the same as the distribution of characteristic polynomials of random matrices from $\mathrm{SU}(2)$, $\mathrm{U}(1)$ or $N(\mathrm{U}(1))$, where $N(\mathrm{U}(1))$ is the normalizer of $\mathrm{U}(1)$ in $\mathrm{SU}(2)$. The conjecture is proven (under some conditions) by the works of R. Taylor, jointly with L. Clozel, M. Harris, and N. Shepherd-Barron [12, 17, 30]. For curves of higher genera, J.-P. Serre, N. Katz and P. Sarnak [20, 29] proposed a generalized Sato–Tate conjecture. Pursuing this direction, K. S. Kedlaya and A. V. Sutherland [23] and later together with F. Fité and V. Rotger [13] made a list of 52 compact subgroups of $\mathrm{USp}(4)$ called *Sato–Tate groups* that would classify all the distributions of Euler factors for abelian surfaces. Recently, Fité, Kedlaya and Sutherland showed that there are 410 Sato–Tate groups for abelian threefolds [14].

1.3 Our Previous Work

Inspired by the approach of Bump and Gamburd, in a previous paper [26], the authors computed the auto-correlation functions of characteristic polynomials for Sato–Tate groups $H \le \mathrm{USp}(4)$, which appear in the generalized Sato–Tate conjecture for genus 2 curves. The result of [26] can be described as follows. Let $H \le \mathrm{USp}(4)$ be a Sato–Tate group. Then, for arbitrary $m \in \mathbb{Z}_{\ge 1}$, we have

$$\int_H \prod_{j=1}^{m}\det(I+x_j\gamma)d\gamma = (x_1\cdots x_m)^2\sum_{b=0}^{m}\sum_{z=0}^{\lfloor\frac{m-b}{2}\rfloor}\mathfrak{m}_{(b+2z,b)}\chi^{\mathrm{Sp}(2m)}_{(2^{m-b-2z},1^{2z})}, \quad (1.3)$$

where the coefficients $\mathfrak{m}_{(b+2z,b)}$ are the multiplicities of the trivial representation in the restrictions $\chi^{\mathrm{Sp}(4)}_{(b+2z,b)}\big|_H$ and are explicitly given in the paper [26] for all the Sato–Tate groups of abelian surfaces. Exploiting the representation-theoretic meaning of $\mathfrak{m}_{(b+2z,b)}$, the authors obtained this result by establishing branching rules for $\chi^{\mathrm{Sp}(4)}_{(b+2z,b)}\big|_H$.

Moreover, since most of the Sato–Tate groups are disconnected, we can decompose the integral in (1.3) according to coset decompositions, and find that the characteristic polynomials over some cosets are *independent* of the elements of the cosets. Combining this observation with the computations of branching rules, we obtain families of non-trivial identities of irreducible characters of $\mathrm{Sp}(2m,\mathbb{C})$ for all $m \in \mathbb{Z}_{\geq 1}$. For example, we have, for any $m \in \mathbb{Z}_{\geq 1}$,

$$\prod_{i=1}^{m}(x_i^2 + x_i^{-2}) = \sum_{b=0}^{m}\sum_{z=0}^{\lfloor \frac{m-b}{2} \rfloor} \psi_4(z,b)\chi^{\mathrm{Sp}(2m)}_{(2^{m-b-2z},1^{2z})}, \tag{1}$$

where $\psi_4(z,b)$ is defined on the congruence classes of z and b modulo 4 by the table

$z\backslash b$	0	1	2	3
0	1	−1	0	0
1	0	1	−1	0
2	−1	1	0	0
3	0	−1	1	0

Notice that the irreducible characters $\chi^{\mathrm{Sp}(2m)}_{\lambda}$ are symmetric functions with the number of terms growing very fast as m increases, but that the coefficients $\psi_4(z,b)$ are independent of m. In order to produce the left-hand side of the identities, there must be systematic cancelations in the right-hand side.

1.4 Schur Functions

The Schur functions S_λ form the *distinguished* self-dual basis of the ring of symmetric functions. They appear naturally in representation theory, algebraic combinatorics, enumerative combinatorics, algebraic geometry and quantum physics. In particular, (i) every Schur function corresponds to an irreducible character of the unitary group, which implies the ring of symmetric functions form the Grothendieck ring for unitary groups, (ii) it has various combinatorial realizations in various aspects of algebraic combinatorics. Hence understanding the properties of Schur functions takes a center stage in these research areas. One of the key features of understanding Schur functions is how other symmetric functions can be expressed in the basis of Schur functions; that is, computing the coefficients of S_λ, called *Schur coefficients*, in the expansions. One of the well-known instances is the (inverse of) *Kostka matrix*, which can be understood as Schur coefficients for monomial (complete) symmetric functions in algebraic combinatorics, and as *composition multiplicities* of $V(\lambda)$ in the permutation representation $W(\lambda)$ in representation theory. Recall that the (inverse of) Kostka matrix is a uni-upper triangular matrix with integer coefficients and entries in the Kostka matrix has a description in terms of semistandard Young tableaux. However, the closed-form formulas for entries in the (inverse of) Kostka matrix are not available in general. Another well-known instance is *Littlewood–Richardson rule*, which can be understood as Schur coefficients for a product of two Schur functions in algebraic combinatorics, and as composition multiplicities of $V(\lambda)$'s in the tensor product $V(\mu) \otimes V(\eta)$ in representation theory.

1.5 Main Result

In what follows, we describe the main result of this paper and its application.

(**M**) We compute explicitly the auto-correlation functions for $H = \mathrm{U}(1) \le \mathrm{U}(g)$ ($g = 2, 3$) and for the subgroups $H \le \mathrm{U}(g)$ ($g = 2, 3$) defined in (1.6) below. Namely, for any $m \in \mathbb{Z}_{\ge 1}$, we obtain

$$\int_{H \le \mathrm{U}(g)} \prod_{i=1}^{m} \det(I + x_i \gamma)\, d\gamma = \sum_{\lambda \trianglelefteq (g^m)} \mathfrak{m}_{\lambda'}(H)\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}), \tag{1.4}$$

where the coefficients $\mathfrak{m}_{\lambda'}(H)$ are completely determined. Here $S_\lambda^{\mathrm{U}(m)}(\mathbf{x})$ denotes the character of the irreducible representation $V(\lambda)$ of the unitary group $\mathrm{U}(m)$, which are Schur functions.

(**A**) As an application of the main result, we give closed-form formulas of Schur coefficients $\mathfrak{c}^\lambda_{g_1,g_2}$ for special infinite families of symmetric functions $\mathfrak{t}^{(m)}_{g_1,g_2}(\mathbf{x})$, which also have simple expansions in terms of monomial symmetric functions $\{\mathsf{m}^{(m)}_\lambda(\mathbf{x})\}$. That is, for *any* $m \in \mathbb{Z}_{\ge 1}$ and $1 \le g_1 + g_2 \le 3$, we obtain

$$\mathfrak{t}^{(m)}_{g_1,g_2}(\mathbf{x}) := \prod_{i=1}^{m} (1 \pm x_i^{g_1})(1 \pm x_i^{g_2}) = \sum_{\lambda \trianglelefteq ((g_1+g_2)^m)} \mathfrak{c}^\lambda_{g_1,g_2} S_\lambda^{\mathrm{U}(m)}(\mathbf{x}), \tag{1.5}$$

where the Schur coefficients $\mathfrak{c}^\lambda_{g_1,g_2}$ are given in closed-form formulas. Note that $\mathfrak{t}^{(m)}_{g_1,g_2}(\mathbf{x})$ has an expansion in $\mathsf{m}^{(m)}_\lambda(\mathbf{x})$ with coefficients from $\{1, 0, -1\}$. Thus the explicitly determined $\mathfrak{c}^\lambda_{g_1,g_2}$ is a combination of entries in the inverse Kostka matrices for which explicit expressions are not known.

Let us explain the main result and its application in more detail. We note that more than half of the Sato–Tate groups for genus 2 curves considered in [13, 26] are subgroups of $\mathrm{USp}(4)$ having $\mathrm{U}(1)$ as the connected component of the identity. They are generated by $\mathrm{U}(1)$ and some twisting elements, resulting in disconnected groups, and those twists bring about interesting structures. We make a search of analogous structures inside $\mathrm{U}(2)$ and $\mathrm{U}(3)$ and find the following six groups:

$$\begin{aligned} &H_2 := \langle \mathrm{U}(1), J_2 \rangle, \quad H'_{2,4} := \langle \mathrm{U}(1), \boldsymbol{\zeta}_{2,4} \rangle \le H_{2,4} := \langle \mathrm{U}(1), J_2, \boldsymbol{\zeta}_{2,4} \rangle \le \mathrm{U}(2), \\ &H_3 := \langle \mathrm{U}(1), J_3 \rangle, \quad H'_{3,4} := \langle \mathrm{U}(1), \boldsymbol{\zeta}_{3,4} \rangle \le H_{3,4} := \langle \mathrm{U}(1), J_3, \boldsymbol{\zeta}_{3,4} \rangle \le \mathrm{U}(3), \end{aligned} \tag{1.6}$$

where

$$J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \boldsymbol{\zeta}_{2,4} := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\zeta}_{3,4} := \begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & \sqrt{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Indeed, the groups in (1.6) are generated by $\mathrm{U}(1)$ and some twisting elements. One can see that these groups also satisfy direct analogues of the axioms (ST1)–(ST3) for the Sato–Tate groups in [13, Definition 3.1] or [14, Definition 3.12], since each G of these groups satisfies the following[1]:

[1] (AST1) and (AST2) follow from the definitions. For (AST3), it will be shown in later sections that the integral in (1.4) with H replaced by G^i has integer coefficients as a polynomial of $x_1, x_2, \dots, x_m$. This implies that the expected value of $\chi(\gamma)$ is an integer when χ is the character of any tensor product of fundamental representations. Since an irreducible representation appears as the highest component in such a tensor product, we can use induction on weights with the lexicographic order to obtain an integer expected value for an irreducible character χ.

(AST1) The group G is a closed subgroup of $\mathrm{U}(2)$ or $\mathrm{U}(3)$.
(AST2) There exists a homomorphism $\theta : \mathrm{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues u or u^{-1} and the image of θ generates a dense subgroup of G^0, where G^0 is the identity component of G.
(AST3) For each component G^i of G ($i = 0, 1, \ldots, k-1$) and each irreducible character χ of $\mathrm{GL}_d(\mathbb{C})$, the expected value of $\chi(\gamma)$ over $\gamma \in G^i$ is an integer, where k is the number of connected components of G.

Moreover, these groups bring about interesting identities in **Application** below as some Sato–Tate groups do in [26]. In that sense, we consider them as analogues of the Sato–Tate groups. However, in this paper, we do not attempt to classify subgroups of $\mathrm{U}(2)$ and $\mathrm{U}(3)$ that satisfy axioms (AST1)–(AST3).

Now we present the main result more precisely.

Main Theorem. Let $H \leq \mathrm{U}(g)$ ($g = 2, 3$) be $\mathrm{U}(1)$ or a group in (1.6). Then, for any $m \in \mathbb{Z}_{\geq 1}$, we have

$$\int_{H \leq \mathrm{U}(g)} \prod_{i=1}^{m} \det(I + x_i \gamma)\, d\gamma = \sum_{\lambda \trianglelefteq (g^m)} \mathfrak{m}_{\lambda'}(H)\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x})$$

where the coefficient $\mathfrak{m}_{\lambda'}(H)$ are the *composition multiplicities* of the trivial representation in the restriction $\chi_{\lambda'}^{\mathrm{U}(g)}|_H$ and are explicitly given in Theorems 3.1, 3.4, 4.6 and 4.14.

This theorem can be interpreted as a result on branching rules from $\mathrm{U}(g)$ to H in representation theory. Since H are disconnected except $\mathrm{U}(1)$, the branching rules do not follow from classical results. Actually, the proof of the theorem requires concrete realization of the representation structure of $V(\lambda')$ over $\mathrm{U}(g)$ with respect to H and various combinatorial consideration to determine the cardinalities of certain linearly independent subsets in $V(\lambda')$.

For an application of **Main Theorem**, we observe (i) H's are decomposed into disconnected cosets

$$\begin{aligned} H_2 &= \mathrm{U}(1) \sqcup \underline{J_2\, \mathrm{U}(1)}, & H_{2,4} &= H'_{2,4} \sqcup \underline{J_2\, \mathrm{U}(1)} \sqcup \underline{\boldsymbol{\zeta}_{2,4} J\, \mathrm{U}(1)}, \\ H_3 &= \mathrm{U}(1) \sqcup \underline{J_3\, \mathrm{U}(1)}, & H_{3,4} &= H'_{3,4} \sqcup \underline{J_3\, \mathrm{U}(1)} \sqcup \underline{\boldsymbol{\zeta}_{3,4} J\, \mathrm{U}(1)}, \end{aligned}$$

and (ii) the characteristic polynomials in the underlined cosets of H are independent of elements of the coset. These observations together with some manipulations of the integral in **Main Theorem** enable us to obtain closed-form identities involving Schur functions.

Application. For arbitrary $m \in \mathbb{Z}_{\geq 1}$, we have the following identities:

(A) $$\prod_{i=1}^{m}(1+x_i^2) = \sum_{b=0}^{m} \sum_{j=0}^{\lfloor \frac{m-b}{2} \rfloor} (-1)^j\, S_{(2^b,1^{2j})}^{\mathrm{U}(m)}(\mathbf{x})$$ (derived from H_2, Theorem 3.1),

(B) $$\frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i^2) + \prod_{i=1}^{m}(1-x_i^2)\right) = \sum_{\substack{(b+2j,b) \trianglelefteq (m^2) \\ b+j \equiv_2 0}} (-1)^j\, S_{(2^b,1^{2j})}^{\mathrm{U}(m)}(\mathbf{x})$$ (derived from $H_{2,4}$ and $H'_{2,4}$, Theorem 3.4),

(C) $$\prod_{i=1}^{m}(1+x_i)(1+x_i^2) = \sum_{\substack{\lambda \trianglelefteq (3^m) \\ \lambda=(3^k,2^b,1^z)}} \tau(z,b)\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x})$$ (derived from H_3, Theorem 4.6),

(D) $$\frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i)(1+x_i^2)+\prod_{i=1}^{m}(1+x_i)(1-x_i^2)\right)=\sum_{\lambda\trianglelefteq(3^m)}\omega_\epsilon(z,b')\,S_\lambda^{\mathrm{U}(m)}(\mathbf{x}),$$

(derived from $H_{3,4}$ and $H'_{3,4}$, Theorem 4.14),

where

(i) $\tau(z,b)\in\{-1,0,1\}$ is defined on the congruence of z and b modulo 4 as in the following table

$\tau(z,b) =$

$z\backslash b$	0	1	2	3
0	1	1	0	0
1	1	0	−1	0
2	0	−1	−1	0
3	0	0	0	0

(ii) $\omega_\epsilon(z,b')$ depends on the congruence of z and b modulo 2, 4 and given in Corollary 4.13.

Combining the identities in **Application** and replacing x_i with $-x_i$, we obtain all the other identities in (1.5) (see (3.6), Corollary 4.17, Corollary 4.18 and Remark 4.19).

Note that the identities obtained involves *negative* Schur coefficients and the number of terms in Schur functions grows enormously as m increases. However, our result implies that such combinations of Schur functions have miraculous cancellations and yield symmetric functions with *positive* coefficients. Furthermore, the identities state that the Schur coefficients do *not* depend on m (see Example 3.3, Example 4.8 and Example 4.16). These identities seem intriguing from the viewpoint of representation theory and algebraic combinatorics. It might have been difficult for us to expect that such identities exist, without regard to the auto-correlation functions and branching rules of the newly introduced groups in (1.6).

1.6 Organization of the Paper

In Section 2, we review the necessary backgrounds for auto-correlation functions and the dual Cauchy identity. In Section 3, we compute the auto-correlation functions of H's of U(2) and establish the corresponding identities involving Schur functions. In Section 4, we present the auto-correlation functions of H's of U(3) and consider the corresponding identities by analyzing the representation structure of $V(\lambda)$ with respect to H's. But we postpone a part of the proof to Section 5, which is devoted to determine the composition multiplicities of trivial representations in $\chi_{\lambda'}^{\mathrm{U}(g)}|_H$'s. This amounts to the proof for U(3). We convert this problem into counting the pairs of integers encoding certain information from representation theory. By expressing the cardinalities as closed-form formulas, we complete the proof.

Convention 1.1 *Throughout this paper, we keep the following conventions.*

(i) *For a statement P, the notation $\delta(P)$ is equal to 1 or 0 according to whether P is true or not.*
(ii) *For $m, m' \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$, we write $m \equiv_k m'$ if k divides $m-m'$, and $m \not\equiv_k m'$ otherwise.*

2 Dual Cauchy Identity and Auto-Correlation Functions

In this section, we fix notations and review the dual Cauchy identity and establish a general formula for the auto-correlation functions of characteristic polynomials.

A *partition* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ is a non-increasing sequence of non-negative integers λ_i. Define $|\lambda| = \sum_{i=1}^{k} \lambda_i$ and $\ell(\lambda) = k$. We write $\lambda = (m^k)$ when $m = \lambda_1 = \cdots = \lambda_k$. More generally, a partition is written as $(m_1^{k_1}, m_2^{k_2}, \ldots, m_s^{k_s})$ for $m_1 > m_2 > \cdots > m_s$ and $k_i \geq 1$. For two partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ and $\mu = (\mu_1 \geq \cdots \geq \mu_l)$, we define a partial order $\lambda \trianglelefteq \mu$ if $k \leq l$ and $\lambda_i \leq \mu_i$ for all $i = 1, 2, \ldots, k$. A partition λ corresponds to a Young diagram, and the *transpose* λ' is defined to be the partition corresponding to the transpose of the Young diagram of λ.

Let $\mathrm{U}(g)$ be the unitary group for $g \geq 1$. For a partition λ with at most g parts, let $S_\lambda^{\mathrm{U}(g)}$ be the Schur function associated with λ. It is well-known that $S_\lambda^{\mathrm{U}(g)}$ is the irreducible character of $\mathrm{U}(g)$ with highest weight λ. Denote by $V_g(\lambda)$ the representation space of $S_\lambda^{\mathrm{U}(g)}$. When g is clear from the context, we will simply write $V(\lambda)$.

Definition 2.1 Let H be a closed subgroup of $\mathrm{U}(g)$. Define $\mathfrak{m}_\lambda(H)$ to be the multiplicity of the trivial representation 1_H in the restriction of $V(\lambda)$ to H.

2.1 Dual Cauchy Identity

Let us recall the dual Cauchy identity (see, e.g., [7, (9)]) :

Lemma 2.2 *For any $m \geq 1$ and $g \geq 1$, we have*

$$\prod_{i=1}^{m} \prod_{j=1}^{g} (1 + x_i t_j) = \sum_{\lambda \trianglelefteq (g^m)} S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) S_{\lambda'}^{\mathrm{U}(g)}(\mathbf{t}),$$

where $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{t} = (t_1, \ldots, t_g)$. We also have

$$\prod_{i=1}^{m} \prod_{j=1}^{g} (1 - x_i t_j) = \sum_{\lambda \trianglelefteq (g^m)} (-1)^{|\lambda|} S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) S_{\lambda'}^{\mathrm{U}(g)}(\mathbf{t}),$$

by replacing x_i to $-x_i$.

Following Proposition 3.3 in [26], we obtain a formula for the auto-correlation functions.

Proposition 2.3 *Let H be a subgroup of $\mathrm{U}(g)$ and $d\gamma$ be the probability Haar measure on H. Then, for each $m \geq 1$, the auto-correlation function for the distribution of characteristic polynomials of H is given by*

$$\int_H \prod_{i=1}^{m} \det(I + x_i \gamma)\, d\gamma = \sum_{\lambda \trianglelefteq (g^m)} \mathfrak{m}_{\lambda'}(H)\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}), \tag{2.1}$$

$$\int_H \prod_{i=1}^{m} \det(I - x_i \gamma)\, d\gamma = \sum_{\lambda \trianglelefteq (g^m)} (-1)^{|\lambda|} \mathfrak{m}_{\lambda'}(H)\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}). \tag{2.2}$$

Proof Let $t_1, \ldots, t_g$ be the eigenvalues of $\gamma \in H$. Since we have

$$\det(I + x_i \gamma) = \prod_{j=1}^{g} (1 + x_i t_j),$$

it follows from Lemma 2.2 that

$$\int_H \prod_{j=1}^{m} \det(I + x_j\gamma)d\gamma = \int_H \prod_{i=1}^{m}\prod_{j=1}^{g}(1 + x_i t_j)d\gamma$$
$$= \int_H \sum_{\lambda \trianglelefteq (g^m)} S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) S_{\lambda'}^{\mathrm{U}(g)}(\mathbf{t})d\gamma = \sum_{\lambda \trianglelefteq (g^m)} S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) \int_H S_{\lambda'}^{\mathrm{U}(g)}(\mathbf{t})d\gamma.$$

From Schur orthogonality (for example, [6]), the integral $\int_H S_{\lambda'}^{\mathrm{U}(g)}(\mathbf{t})d\gamma$ is equal to the multiplicity of the trivial representation 1_H of H in the restriction of $S_{\lambda'}^{\mathrm{U}(g)}$ to H, which is equal to $\mathfrak{m}_{\lambda'}(H)$ by Definition 2.1. This establishes the first identity. The second identity follows from replacing x_i with $-x_i$ in the first identity. □

Corollary 2.4 *Suppose that $-I \in H$. Then $\mathfrak{m}_{\lambda'}(H) = 0$ whenever $|\lambda'|$ is odd.*

Proof If $-I \in H$, we have

$$\int_H \prod_{j=1}^{m} \det(I + x_j\gamma)d\gamma = \int_H \prod_{j=1}^{m} \det(I - x_j\gamma)d\gamma.$$

Our assertion follows from (2.1) and (2.2) by comparing the right-hand sides, since the Schur functions are linearly independent. □

We recall the classical branching rule from $\mathrm{U}(g)$ to $\mathrm{U}(g-1)$ for $g \geq 2$.

Proposition 2.5 *Let $V_g(\lambda)$ be the irreducible representation of $\mathrm{U}(g)$ with highest weight λ. Then we have*

$$[V_g(\lambda) : V_{g-1}(\mu)] \leq 1$$

for any partition μ with at most $g-1$ parts. Furthermore, $[V_g(\lambda) : V_{g-1}(\mu)] = 1$ precisely when

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{g-1} \geq \lambda_g.$$

3 Identities for $g = 2$

In this section, we consider some disconnected subgroups H of $\mathrm{U}(2)$ and compute $\mathfrak{m}_{\lambda'}(H)$ for $\lambda \trianglelefteq (2^m)$ in (2.1). This computation produces identities involving Schur functions $S_\lambda^{\mathrm{U}(m)}$ for all $m \in \mathbb{Z}_{\geq 1}$.

We identify $\mathrm{U}(1)$ with the subgroup $\left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}, |t| = 1 \right\} \leq \mathrm{U}(2)$.

3.1 Subgroup $\langle \mathrm{U}(1), J \rangle$

Let us consider the subgroup H_2 of $\mathrm{U}(2)$ generated by $\mathrm{U}(1)$ and $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e.

$$H_2 = \langle \mathrm{U}(1), J \rangle \leq \mathrm{U}(2).$$

Then we have

$$H_2 = \mathrm{U}(1) \sqcup J\,\mathrm{U}(1). \tag{3.1}$$

Theorem 3.1 *For any partition* $(a, b) \trianglelefteq (m^2)$*, we have*

$$\mathfrak{m}_{(a,b)}(\mathrm{U}(1)) = \delta(a \equiv_2 b) \quad \text{and} \quad \mathfrak{m}_{(a,b)}(H_2) = \delta(a \equiv_4 b).$$

Furthermore, for any $m \in \mathbb{Z}_{\geq 1}$*, we have*

$$\prod_{i=1}^{m}(1+x_i^2) = \sum_{\substack{\lambda \trianglelefteq (2^m) \\ \lambda'=(b+2j,b)}} (-1)^j \; S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) = \sum_{b=0}^{m} \sum_{j=0}^{\lfloor \frac{m-b}{2} \rfloor} (-1)^j \; S_{(2^b,1^{2j})}^{\mathrm{U}(m)}(\mathbf{x}), \tag{3.2}$$

where we set $j := (a-b)/2$.

Proof For any $\gamma = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{U}(1)$, we have $\det(I + xJ\gamma) = 1 + x^2$. Let $du = du(\gamma)$ be the probability Haar measure on $\mathrm{U}(1) \leq \mathrm{U}(2)$. By Proposition 2.3 and (3.1), we have

$$\begin{aligned}
\sum_{\lambda \trianglelefteq (2^m)} \mathfrak{m}_{\lambda'}(H_2) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) &= \int_{H_2} \prod_{i=1}^{m} \det(I + x_i\gamma) d\gamma \\
&= \frac{1}{2}\int_{\mathrm{U}(1)} \prod_{i=1}^{m} \det(I + x_i\gamma) du + \frac{1}{2}\int_{\mathrm{U}(1)} \prod_{i=1}^{m} \det(I + x_i J\gamma) du \\
&= \frac{1}{2} \sum_{\lambda \trianglelefteq (2^m)} \mathfrak{m}_{\lambda'}(\mathrm{U}(1)) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) + \frac{1}{2}\int_{\mathrm{U}(1)} \prod_{i=1}^{m} (1 + x_i^2) du \\
&= \frac{1}{2} \sum_{\lambda \trianglelefteq (2^m)} \mathfrak{m}_{\lambda'}(\mathrm{U}(1)) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) + \frac{1}{2} \prod_{i=1}^{m} (1 + x_i^2).
\end{aligned} \tag{3.3}$$

Let $v_1 = (1, 0)$ and $v_2 = (0, 1)$ be the standard unit vectors of $V := \mathbb{C}^2$, and consider the standard representation of $\mathrm{U}(2)$ on V, and let **det** be the one-dimensional representation of $\mathrm{U}(2)$ defined by the determinant. For $\lambda' = (a, b) \trianglelefteq (m^2)$, we have $V(\lambda') \cong \mathbf{det}^b \otimes \mathrm{Sym}^{a-b}(V)$. Thus the trivial $\mathrm{U}(1)$-module is generated by $v_1^j v_2^j$ only when $a - b$ is even, where we set $j := (a - b)/2$. In other word, we have

$$\mathfrak{m}_{(a,b)}(\mathrm{U}(1)) = \delta(a \equiv_2 b).$$

Furthermore, since J sends $v_1 \mapsto -v_2$ and $v_2 \mapsto v_1$, we see that $v_1^j v_2^j$ is fixed by J when j is even. Therefore,

$$\mathfrak{m}_{(a,b)}(H_2) = \delta(j \equiv_2 0) = \delta(a \equiv_4 b).$$

Note that, when $\lambda' = (b + 2j, b)$, we have $\lambda = (2^b, 1^{2j})$. Now it follows from (3.3) that

$$\prod_{i=1}^{m}(1+x_i^2) = \sum_{b=0}^{m} \sum_{j=0}^{\lfloor \frac{m-b}{2} \rfloor} (2\delta(j \equiv_2 0) - 1) \; S_{(2^b,1^{2j})}^{\mathrm{U}(m)}(\mathbf{x}) = \sum_{b=0}^{m} \sum_{j=0}^{\lfloor \frac{m-b}{2} \rfloor} (-1)^j \; S_{(2^b,1^{2j})}^{\mathrm{U}(m)}(\mathbf{x}).$$

□

Remark 3.2 The left hand side of (3.2) can be written as a simple combination of the monomial symmetric functions. Namely, we have

$$\prod_{i=1}^{m}(1+x_i^2) = \sum_{k=0}^{m} \mathsf{m}_{(2^k)}^{(m)}(\mathbf{x}),$$

where $\mathsf{m}_{\lambda}^{(m)}$ denote the monomial symmetric functions in m-variables associated with partitions λ with $\ell(\lambda) \le m$.

Example 3.3 Let us see an example for the case $m = 10$ in Theorem 3.1. We have

$$\prod_{i=1}^{10}(1+x_i^2) = \sum_{b=0}^{10} \sum_{j=0}^{\lfloor \frac{10-b}{2} \rfloor} (-1)^j \; S^{\mathrm{U}(10)}_{(2^b,1^{2j})}(\mathbf{x}). \tag{3.4}$$

Note that $S^{\mathrm{U}(10)}_{(2^4,1^2)}(\mathbf{x})$ appears in the right hand side of (3.4) with the coefficient -1 since $j = 1$. As a polynomial itself, $S^{\mathrm{U}(10)}_{(2^4,1^2)}(\mathbf{x})$ contains 8701 monomial terms and $S^{\mathrm{U}(10)}_{(2^4,1^2)}(\mathbf{1}) = 29700$, where $\mathbf{1} = (1, 1, \ldots, 1)$. Actually, there are 15 Schur functions with negative coefficient -1 in the right hand side of (3.4) including $S^{\mathrm{U}(10)}_{(2^4,1^2)}(\mathbf{x})$. After amazing cancellations among Schur functions, we obtain a symmetric function in the left hand side of (3.4), which contains only 1024 monomial terms with coefficients all 1 in its expansion.

3.2 Subgroup $\langle \mathrm{U}(1), J, \boldsymbol{\zeta}_4 \rangle$

Set

$$\boldsymbol{\zeta}_4 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \in \mathrm{U}(2),$$

and denote by $H_{2,4}$ the subgroup of $\mathrm{U}(2)$ generated by $\mathrm{U}(1)$, J and $\boldsymbol{\zeta}_4$. That is, we define

$$H_{2,4} := \langle \mathrm{U}(1), J, \boldsymbol{\zeta}_4 \rangle \le \mathrm{U}(2).$$

Then we have

$$H_{2,4} = \mathrm{U}(1) \sqcup J\,\mathrm{U}(1) \sqcup \boldsymbol{\zeta}_4\,\mathrm{U}(1) \sqcup \boldsymbol{\zeta}_4 J\,\mathrm{U}(1).$$

Let $H'_{2,4}$ be the subgroup of $H_{2,4}$ generated by $U(1)$ and $\boldsymbol{\zeta}_4$.

Theorem 3.4 *For any partition $(a, b) \trianglelefteq (m^2)$, we have*

$$\mathfrak{m}_{(a,b)}(H'_{2,4}) = \delta(a+b \equiv_4 0) \quad \textit{and} \quad \mathfrak{m}_{(a,b)}(H_{2,4}) = \delta(a+b \equiv_4 0)\delta(a-b \equiv_4 0).$$

Moreover, for any $m \in \mathbb{Z}_{\ge 1}$, we have

$$\begin{aligned} \frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i^2) + \prod_{i=1}^{m}(1-x_i^2)\right) &= \sum_{\substack{(a,b)\trianglelefteq(m^2) \\ a+b\equiv_4 0}} (-1)^{\delta(a \not\equiv_4 b)} S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x}) \\ &= \sum_{\substack{(b+2j,b)\trianglelefteq(m^2) \\ b+j\equiv_2 0}} (-1)^j S^{\mathrm{U}(m)}_{(2^b,1^{2j})}(\mathbf{x}), \end{aligned} \tag{3.5}$$

where we set $j := (a-b)/2$ as before.

Proof Let $\lambda' = (a, b) \trianglelefteq (m^2)$. We keep the notations in the proof of Theorem 3.1 for $V(\lambda') \cong \mathbf{det}^b \otimes \mathrm{Sym}^{a-b}(V)$. A vector in $V(\lambda')$ is fixed by $\mathrm{U}(1)$ if it is of the form $v_1^j v_2^j$ up to scalar multiplication. Since $\det(\boldsymbol{\zeta}_4) = -1$, we get

$$\boldsymbol{\zeta}_4\, v_1^j v_2^j = (-1)^{b+j} v_1^j v_2^j.$$

We see that $b+j \equiv_2 0 \Leftrightarrow a+b \equiv_4 0$, and if $a+b \equiv_4 0$ then $a-b \equiv_2 0$. Thus we obtain

$$\mathfrak{m}_{(a,b)}(H'_{2,4}) = \delta(a+b \equiv_4 0).$$

As observed in the proof of Theorem 3.1, the vector $v_1^j v_2^j$ is fixed by J if and only if $a-b \equiv_4 0$. Therefore we have

$$\mathfrak{m}_{(a,b)}(H_{2,4}) = \delta(a+b \equiv_4 0)\delta(a-b \equiv_4 0).$$

Let $d\gamma' = d\gamma'(\gamma)$ be the probability Haar measure on $H'_{2,4}$. Since

$$\det(I + xJ\gamma) = 1 + x^2 \quad \text{and} \quad \det(I + x\boldsymbol{\zeta}_4 J\gamma) = 1 - x^2$$

for all $\gamma \in \mathrm{U}(1)$, we have

$$\begin{aligned}\int_{H_{2,4}} \Delta(\gamma)d\gamma &= \frac{1}{2}\int_{H'_{2,4}} \Delta(\gamma)d\gamma' + \frac{1}{4}\int_{J\,\mathrm{U}(1)} \Delta(\gamma)du + \frac{1}{4}\int_{\boldsymbol{\zeta}_4 J\,\mathrm{U}(1)} \Delta(\gamma)du \\ &= \frac{1}{2}\int_{H'_{2,4}} \Delta(\gamma)d\gamma' + \frac{1}{4}\prod_{i=1}^{m}(1+x_i^2) + \frac{1}{4}\prod_{i=1}^{m}(1-x_i^2),\end{aligned}$$

where we write $\Delta(\gamma) = \prod_{i=1}^m \det(I + x_i\gamma)$ for convenience. Applying Proposition 2.3 to the integrals, we obtain

$$\begin{aligned}\frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i^2) + \prod_{i=1}^{m}(1-x_i^2)\right) &= \sum_{(a,b)\trianglelefteq(m^2)} \left(2\mathfrak{m}_{(a,b)}(H_{2,4}) - \mathfrak{m}_{(a,b)}(H'_{2,4})\right) S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x}) \\ = \sum_{\substack{(a,b)\trianglelefteq(m^2)\\ a+b\equiv_4 0}} (-1)^{\delta(a\not\equiv_4 b)} S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x}) &= \sum_{\substack{(b+2j,b)\trianglelefteq(m^2)\\ b+j\equiv_2 0}} (-1)^j S^{\mathrm{U}(m)}_{(2^b,1^{2j})}(\mathbf{x}).\end{aligned}$$

□

Remark 3.5 (1) The identity (3.5) can be derived from (3.2). We will consider the alternate proof in Section 3.3.

(2) As in Remark (3.2), we observe that the left hand side of (3.5) is a simple combination of the monomial symmetric functions in m-variables:

$$\frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i^2) + \prod_{i=1}^{m}(1-x_i^2)\right) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \mathsf{m}^{(m)}_{(2^{2k})}(\mathbf{x}).$$

3.3 Pieri's rule

One can see that

$$\prod_{i=1}^{m}(1-x_i^2) = \sum_{\substack{\lambda\trianglelefteq(2^m)\\ \lambda'=(b+2j,b)}} (-1)^b\, S^{\mathrm{U}(m)}_{(2^b,1^{2j})}(\mathbf{x}) = \sum_{k=0}^{m}(-1)^k \mathsf{m}^{(m)}_{(2^k)}(\mathbf{x}). \tag{3.6}$$

Indeed, by replacing x_i with $\sqrt{-1}\,x_i$ in (3.2), we obtain

$$\prod_{i=1}^{m}(1-x_i^2) = \sum_{\substack{\lambda\trianglelefteq(2^m)\\ \lambda'=(b+2j,b)}} (-1)^j(-1)^{j+b}\, S^{\mathrm{U}(m)}_{(2^b,1^{2j})}(\mathbf{x}) = \sum_{\substack{\lambda\trianglelefteq(2^m)\\ \lambda'=(b+2j,b)}} (-1)^b\, S^{\mathrm{U}(m)}_{(2^b,1^{2j})}(\mathbf{x}),$$

and the second equality in (3.6) follows from the definition of the monomial symmetric function $\mathsf{m}^{(m)}_{(2^k)}$. Combining (3.2) with (3.6) yields the identity (3.5). If we combine (3.2) and (3.6) in a different way, we obtain

$$\frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i^2)-\prod_{i=1}^{m}(1-x_i^2)\right)=\sum_{\substack{(a,b)\trianglelefteq(m^2)\\ a+b\equiv_4 2}}(-1)^{\delta(a\not\equiv_4 b)}S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x})=\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\mathsf{m}^{(m)}_{(2^{2k+1})}(\mathbf{x}).$$

Moreover, (3.6) can also be proven using Pieri's rule. Since the idea can be used in other cases, let us see the proof. We first consider the trivial case $g=1$ to use the results for the case $g=2$. Let $\mathsf{m}^{(m)}_\lambda$ be the monomial symmetric function of m-variables associated to a partition λ with $\ell(\lambda)\le m$, as before. Then, in particular, we have

$$\mathsf{m}^{(m)}_{(1^k)}=S^{\mathrm{U}(m)}_{(1^k)}\quad\text{for } k\le m$$

and obtain

$$\prod_{i=1}^{m}(1+x_i)=\sum_{k=0}^{m}\mathsf{m}^{(m)}_{(1^k)}(\mathbf{x})=\sum_{\lambda\trianglelefteq(1^m)}S^{\mathrm{U}(m)}_\lambda(\mathbf{x}). \tag{3.7}$$

By replacing x_i with $-x_i$, we have

$$\prod_{i=1}^{m}(1-x_i)=\sum_{k=0}^{m}(-1)^k\mathsf{m}^{(m)}_{(1^k)}(\mathbf{x})=\sum_{\lambda\trianglelefteq(1^m)}(-1)^{|\lambda|}S^{\mathrm{U}(m)}_\lambda(\mathbf{x}). \tag{3.8}$$

Recall Pieri's rule from, e.g., Macdonald's book [27, (5.17)]:

$$S^{\mathrm{U}(m)}_\lambda(\mathbf{x})\times\prod_{i=1}^{m}(1+x_i)=S^{\mathrm{U}(m)}_\lambda(\mathbf{x})\times\left(\sum_{l=0}^{m}\mathsf{e}_l(\mathbf{x})\right)=\sum_{\lambda\trianglelefteq\mu\trianglelefteq\lambda+(1^m)}S^{\mathrm{U}(m)}_\mu(\mathbf{x}), \tag{3.9}$$

where $\mathsf{e}_l(\mathbf{x})$ denotes the elementary symmetric function of partition (l) of length 1 and $\lambda+(1^m)=(\lambda_1+1,\lambda_2+2,\dots,\lambda_m+1)$ for $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_m)$.

By (3.9) and (3.8), we have

$$\begin{aligned}\prod_{i=1}^{m}(1-x_i^2)&=\prod_{i=1}^{m}(1+x_i)\times\sum_{(1^k)\trianglelefteq(1^m)}(-1)^kS^{\mathrm{U}(m)}_{(1^k)}(\mathbf{x})\\&=\sum_{k=0}^{m}\sum_{(1^k)\trianglelefteq\mu\trianglelefteq(2^k,1^{m-k})}(-1^k)S^{\mathrm{U}(m)}_\mu(\mathbf{x})\end{aligned} \tag{3.10}$$

Thus for any partition $(2^b,1^{a-b})\trianglelefteq(2^m)$, the coefficient of $S^{\mathrm{U}(m-1)}_{(2^b,1^{a-b})}(\mathbf{x})$ in (3.10) is given by

$$\sum_{s=b}^{a}(-1)^s=\begin{cases}0 & \text{if } a-b\equiv_2 1,\\ 1 & \text{if } a-b\equiv_2 0 \text{ and } b\equiv_2 0,\\ -1 & \text{if } a-b\equiv_2 0 \text{ and } b\equiv_2 1.\end{cases}$$

When $a-b\equiv_2 0$, write $a-b=2j$. Then we have

$$\prod_{i=1}^{m}(1-x_i^2)=\sum_{\substack{\lambda\trianglelefteq(2^m)\\ \lambda'=(b+2j,b)}}(-1)^b\,S^{\mathrm{U}(m)}_{(2^b,1^{2j})}(\mathbf{x})$$

as desired.

Using a similar argument, we obtain two more identities.

Proposition 3.6 *For any $m \in \mathbb{Z}_{\geq 1}$, we have*

$$\prod_{i=1}^{m}(1+2x_i+x_i^2) = \sum_{(a,b)\trianglelefteq(m^2)} (a-b+1) S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x}), \tag{3.11}$$

$$\prod_{i=1}^{m}(1-2x_i+x_i^2) = \sum_{(a,b)\trianglelefteq(m^2)} (-1)^{a-b}(a-b+1) S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x}). \tag{3.12}$$

Proof By (3.9) and (3.7), we have

$$\begin{aligned}\prod_{i=1}^{m}(1+x_i)^2 &= \prod_{i=1}^{m}(1+x_i) \times \sum_{(1^k)\trianglelefteq(1^m)} S^{\mathrm{U}(m)}_{(1^k)}(\mathbf{x}) \\ &= \sum_{k=0}^{m} \sum_{(1^k)\trianglelefteq\mu\trianglelefteq(2^k,1^{m-k})} S^{\mathrm{U}(m)}_{\mu}(\mathbf{x})\end{aligned} \tag{3.13}$$

Thus for any partition $(2^b, 1^{a-b}) \trianglelefteq (2^m)$, the coefficient of $S^{\mathrm{U}(m)}_{(2^b,1^{a-b})}(\mathbf{x})$ in (3.13) is equal to

$$\sum_{s=b}^{a} 1 = a-b+1,$$

and the identity (3.11) follows. By replacing x_i with $-x_i$ in (3.11), we obtain (3.12).

□

Remark 3.7 The above use of Pieri's rule may not be applicable, in general, if one can try to obtain *closed-form formulas* for $g \geq 3$. For instance, based on Theorem 3.1 about $g = 2$, one can check the formula in Theorem 4.6 about $g = 3$ below using Pieri's rule, for *first several small values of m*. But, when $g \geq 3$, obtaining closed-form formula for the coefficient of $S^{\mathrm{U}(m)}_{\lambda}$, $m \geq 1$, seems not easy in this approach.

4 Identities for $g = 3$

In this section, we consider some disconnected subgroups H of U(3) and compute $\mathfrak{m}_{\lambda'}(H)$ for $\lambda \trianglelefteq (3^m)$ in (2.1). As with the case $g = 2$, our computation yields identities involving Schur functions $S^{\mathrm{U}(m)}_{\lambda}$ for all $m \in \mathbb{Z}_{\geq 1}$.

To begin with, we embed U(1) into U(3) via

$$U(1) \simeq \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{C}, |t| = 1 \right\},$$

and U(2) into U(3) via $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ for $A \in \mathrm{U}(2)$.

4.1 Subgroup $\langle \mathrm{U}(1), J\rangle$

Define

$$J := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}(3).$$

Note that

$$J^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}(1) \quad \text{and} \quad J^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Consider the subgroup H_3 of $\mathrm{U}(3)$ generated by J and $\mathrm{U}(1)$, i.e.

$$H_3 := \langle \mathrm{U}(1), J\rangle \le \mathrm{U}(3).$$

Then one can easily check that J normalizes $\mathrm{U}(1)$, and $H_3 = \mathrm{U}(1) \sqcup J\,\mathrm{U}(1)$. Note that

$$J \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t^{-1} & 0 \\ -t & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \det(I + xJ\gamma) = (1+x)(1+x^2) \tag{4.1}$$

for all $\gamma \in \mathrm{U}(1)$.

We prove a useful lemma.

Lemma 4.1 *For any $k \in \mathbb{Z}_{\ge 0}$, we have*

$$\mathfrak{m}_{(a+k,b+k,k)}(\mathrm{U}(1)) = \mathfrak{m}_{(a,b,0)}(\mathrm{U}(1)) \quad \textit{and} \quad \mathfrak{m}_{(a+k,b+k,k)}(H_3) = \mathfrak{m}_{(a,b,0)}(H_3).$$

Proof Let **det** be the one-dimensional representation of $\mathrm{U}(3)$ defined by the determinant. Then we have

$$V(a+k, b+k, k) = \mathbf{det}^k \otimes V(a, b, 0).$$

Since $\det(A) = 1$ for any $A \in \mathrm{U}(1)$ and $\det(J) = 1$, the assertion follows.

□

Thanks to Lemma 4.1, we need to consider the irreducible representations $V(\lambda')$ of $\mathrm{U}(3)$ only for $\lambda' = (a, b, 0)$. In what follows, we assume $\lambda' = (a, b, 0)$ and freely write $V(a, b) = V(\lambda')$. Note that we may also regard $V(a, b)$ as the irreducible representation of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight (a, b). More precisely, define $h_1, h_2 \in \mathfrak{sl}(3, \mathbb{C})$ by

$$h_1 = \mathrm{diag}(1, -1, 0) \quad \text{and} \quad h_2 = \mathrm{diag}(0, 1, -1),$$

and denote by $\mathfrak{h}$ the subspace of $\mathfrak{sl}(3, \mathbb{C})$ spanned by h_1 and h_2. We regard any partition $\mu = (\mu_1, \mu_2)$ as an element of $\mathfrak{h}^*$ by setting

$$\mu(h_1) = \mu_1 - \mu_2 \quad \text{and} \quad \mu(h_2) = \mu_2,$$

and μ is a weight of $\mathfrak{sl}(3, \mathbb{C})$.

Let V and $W = \bigwedge^2 V$ be the fundamental representations of $\mathrm{U}(3)$. Take a basis $\{v_1, v_2, v_3\}$ of V such that

$$Jv_1 = -v_2, \quad Jv_2 = v_1 \quad \text{and} \quad Jv_3 = v_3. \tag{4.2}$$

Write

$$w_{12} = v_1 \wedge v_2, \quad w_{13} = v_1 \wedge v_3, \quad w_{23} = v_2 \wedge v_3.$$

Then $\{w_{12}, w_{13}, w_{23}\}$ is a basis for W, and we have

$$Jw_{12} = w_{12}, \qquad Jw_{13} = -w_{23} \quad \text{and } Jw_{23} = w_{13}.$$

As $\mathfrak{sl}(3,\mathbb{C})$-representations, V and W are equivalent and can be described as follows:

$$V : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}, \qquad W : \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{2} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}. \tag{4.3}$$

The diagrams mean

$$f_1 v_1 = v_2, \quad f_2 v_2 = v_3, \quad f_2 w_{12} = w_{13}, \quad f_1 w_{13} = w_{23}$$

and $f_i v_j = 0$ and $f_i w_{jk} = 0$ for other choices of i, j, k, where we set $f_1 := \begin{pmatrix} 0&0&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$, $f_2 := \begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&1&0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C})$. Equivalently, if we set $e_1 := \begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&0&0 \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C})$, we have

$$e_2 v_3 = v_2, \quad e_1 v_2 = v_1, \quad e_1 w_{23} = w_{13}, \quad e_2 w_{13} = w_{12}$$

and $e_i v_j = 0$ and $e_i w_{jk} = 0$ for other choices of i, j, k. The embedding $\mathrm{U}(2) \hookrightarrow \mathrm{U}(3)$ corresponds to $\langle e_1, h_1, f_1 \rangle \cong \mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{sl}(3,\mathbb{C})$.

We realize the representation $V(a,b)$ for a partition (a,b) as the irreducible component of $\mathrm{Sym}^{a-b} V \otimes \mathrm{Sym}^b W$ generated by the highest weight vector $v_1^{a-b} \otimes w_{12}^b$. In particular, $V = V(1,0)$ and $W = V(1,1)$. We identify $V(a+1, b+1)$ with the image of the embedding

$$\iota_{a+1,b+1} : V(a+1, b+1) \hookrightarrow V(a,b) \otimes V(1,1) \tag{4.4}$$

given by

$$v_1^{a-b} \otimes w_{12}^{b+1} \mapsto (v_1^{a-b} \otimes w_{12}^b) \otimes w_{12}.$$

For a partition (a,b), set $z := a - b$ and define a set of partitions which interlace with (a,b):

$$\Phi(a,b) := \{(p,q) \mid a \geq p \geq b \geq q \geq 0\}.$$

Clearly, we have

$$|\Phi(a,b)| = (a-b+1)(b+1) = (z+1)(b+1).$$

It follows from Proposition 2.5 that

$$\text{the set } \Phi(a,b) \text{ is exactly the set of U(2)-highest weights in the restriction } V(a,b)|_{\mathrm{U}(2)}. \tag{4.5}$$

Example 4.2

(a) $\Phi(3,2) = \{(3,0), (2,0), (3,1), (2,1), (3,2), (2,2)\}$.
(b) $\Phi(4,3) = \{(4,0), (3,0), (4,1), (3,1), (4,2), (3,2), (4,3), (3,3)\}$.

For our purpose, we need to precisely describe U(2)-highest weight vectors in the restriction $V(a,b)|_{\mathrm{U}(2)}$. In what follows, we specify such vectors. We freely use the $\mathfrak{sl}(3,\mathbb{C})$-representation structure on $V(a,b)$ and apply actions of e_i, f_i $(i = 1,2)$ on vectors of $V(a,b)$.

For $V(a,0)$, define

$$v_{(k,0;a,0)} := v_1^k v_3^{a-k} \quad \text{for } (k,0) \in \Phi(a,0) \quad (\text{or equivalently, for } 0 \leq k \leq a).$$

By considering $\mathfrak{sl}(2,\mathbb{C})$ action from (4.3), one can see that $v_{(k,0;a,0)}$ are U(2)-highest weight vectors with highest weights $(k,0)$.

Next let us consider U(2)-highest weight vectors of $V(a+1, 1)$ via the embedding $\iota_{a+1,1}$ in (4.4). Obviously, the vectors

$$v_{(k+1,1;a+1,1)} := v_{(k,0;a,0)} \otimes w_{12}$$

are contained in $V(a+1, 1)$ from the construction of $V(a+1, 1)$, and each of them is a U(2)-highest weight vector of $V(a+1, 1)$, which generates a $(k+1)$-dimensional U(2)-module. Thus they correspond to the partitions $(k+1, 1)$ in $\Phi(a+1, 1)$. Similarly, the vectors

$$v_{(k+1,0;a+1,1)} := f_2\left(v_{(k,0;a,0)} \otimes w_{12}\right) = v_{(k,0;a,0)} \otimes w_{13}$$

are contained in $V(a+1, 1)$ and are U(2)-highest weight vectors of $V(a+1, 1)$. Each of them generates a $(k+2)$-dimensional U(2)-module. Hence they correspond to the partitions $(k+1, 0)$ in $\Phi(a+1, 1)$. Since

$$\Phi(a+1, 1) = \{(k+1, 1) \mid 0 \le k \le a\} \sqcup \{(k+1, 0) \mid 0 \le k \le a\},$$

we have obtained all the U(2)-highest weight vectors of the restriction $V(a+1, 1)|_{\mathrm{U}(2)}$.

Generally, for $(p, q) \in \Phi(a, b)$, define

$$v_{(p,q;a,b)} := v_1^{p-b} v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}, \qquad b \le p \le a.$$

Lemma 4.3 *The vectors $v_{(p,q;a,b)}$ are* U(2)*-highest weight vectors of $V(a, b)$ for $(p, q) \in \Phi(a, b)$.*

Proof It is straightforward to check that $v_{(p,q;a,b)}$ are U(2)-highest weight vectors. For induction, assume that $v_{(p,q;a-1,b-1)}$ are contained in $V(a-1, b-1)$. Then, by the construction of $V(a, b)$, we have

$$\begin{aligned} v_{(k+b-1,b-1;a-1,b-1)} \otimes w_{12} &= \left(v_1^k v_3^{a-b-k} \otimes w_{12}^{b-1}\right) \otimes w_{12} = v_1^k v_3^{a-b-k} \otimes w_{12}^b \\ &= v_{(k+b,b;a,b)} \in V(a, b) \end{aligned}$$

for $0 \le k \le a-b$. Since

$$f_2^l v_{(k+b,b;a,b)} = \tfrac{b!}{(b-l)!} v_1^k v_3^{a-b-k} \otimes w_{12}^{b-l} w_{13}^l = \tfrac{b!}{(b-l)!} v_{(k+b,b-l;a,b)} \quad \text{for } 0 \le l \le b,$$

we have $v_{(p,q;a,b)} \in V(a, b)$ for any $(p, q) \in \Phi(a, b)$.

□

The vectors in the above lemma are distinct, linearly independent and exhaust all the U(2)-highest weight vectors in $V(a, b)$. Note that

(J1) Since $J(v_{(p,q;a,b)}) = J(v_1^{p-b} v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}) = (-1)^{p-q} v_2^{p-b} v_3^{a-p} \otimes w_{12}^q w_{23}^{b-q}$, the vector $J(v_{(p,q;a,b)})$ is a U(2)-lowest vector in the U(2)-representation generated by $v_{(p,q;a,b)}$.

(J2) $Jf_1 = -e_1 J$.

Proposition 4.4 *For a partition $\lambda' = (a, b, 0)$, the multiplicity $\mathfrak{m}_{\lambda'}(\mathrm{U}(1))$ of the trivial representation in $V(\lambda')|_{\mathrm{U}(1)}$ is equal to the cardinality of the set*

$$\Phi^{(2)}(a, b) := \{(p, q) \in \Phi(a, b) \mid p \equiv_2 q\},$$

and the multiplicity $\mathfrak{m}_{\lambda'}(H_3)$ of the trivial representation in $V(\lambda')|_{H_3}$ is equal to the cardinality of the set

$$\Phi^{(4)}(a, b) := \{(p, q) \in \Phi(a, b) \mid p \equiv_4 q\}.$$

Proof From the embeddings of U(1) into U(3), we see that the multiplicity $\mathfrak{m}_{\lambda'}(\mathrm{U}(1))$ is equal to the number of linearly independent vectors in $V(\lambda')$ with weight μ such that

$$\mu(h_1) = 0.$$

Similarly, the multiplicity $\mathfrak{m}_{\lambda'}(H_3)$ is equal to the number of linearly independent vectors v in $V(\lambda')$ with weight μ such that

$$\mu(h_1) = 0 \quad \text{and} \quad Jv = v.$$

If we consider the restriction $V(\lambda')|_{\mathrm{U}(2)}$, then the condition $\mu(h_1) = 0$ means weight 0. A weight 0 vector occurs in $V_2(p,q)$ precisely when $p - q \equiv_2 0$ with multiplicity 1, where $V_2(p,q)$ is the irreducible representation of U(2) with highest weight (p,q). Thus the first assertion follows from (4.5).

Write $p - q = 2k$. Then $v := f_1^k v_{(p,q;a,b)}$ is a weight 0 vector by Lemma 4.3. Using (J1) and (J2) in (4.6), we obtain

$$\begin{aligned} J(v) &= Jf_1^k(v_1^{p-b}v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}) = (-1)^k e_1^k J(v_1^{p-b}v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}) \\ &= (-1)^{3k} e_1^k (v_2^{p-b}v_3^{a-p} \otimes w_{12}^q w_{23}^{b-q}) = (-1)^{3k} f_1^k (v_1^{p-b}v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}) = (-1)^k v. \end{aligned}$$

Thus v is fixed only when k is even. Thus the second assertion follows. □

The cardinalities of the sets $\Phi^{(2)}$ and $\Phi^{(4)}$ are computed in the following proposition.

Proposition 4.5 *For a partition (a,b), write $z := a - b$. Then we have*

$$|\Phi^{(2)}(a,b)| = \lceil (z+1)(b+1)/2 \rceil ,$$

and

$$|\Phi^{(4)}(a,b)| = \frac{1}{2}\left(\lceil (z+1)(b+1)/2 \rceil + \tau(z,b)\right)$$

where $\tau(z,b) \in \{0, \pm 1\}$ is defined on the congruence classes of z and b modulo 4 as follows:

$\tau(z,b) =$

$z \backslash b$	*0*	*1*	*2*	*3*
0	1	1	0	0
1	1	0	−1	0
2	0	−1	−1	0
3	0	0	0	0

Proof The elements (p,q) in $\Phi(a,b)$ and the corresponding dimensions $p - q + 1$ can be each arranged into an array of size $(z+1) \times (b+1)$ as follows, where we put (p,q) in the left and its corresponding dimensions in the right:

$$\begin{array}{cccc|cccc} (a,0) & (a,1) & \cdots & (a,b) & a+1 & a & \cdots & a-b+1 \\ (a-1,0) & (a-1,1) & \cdots & (a-1,b) & a & a-1 & \cdots & a-b \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ (b,0) & (b,1) & \cdots & (b,b) & b+1 & b & \cdots & 1 \end{array} \tag{4.7}$$

By counting the number of odd integers in the right array, we obtain

$$|\Phi^{(2)}(a,b)| = \lceil (z+1)(b+1)/2 \rceil ,$$

and by counting the number of integers congruent to 1 modulo 4 in the right array, we get

$$|\Phi^{(4)}(a,b)| = \frac{1}{2}\left(\lceil (z+1)(b+1)/2 \rceil + \tau(z,b)\right).$$

□

Now we state and prove the main theorem of this subsection.

Theorem 4.6 *For a partition* $\lambda' = (a+k, b+k, k)$, $k \in \mathbb{Z}_{\geq 0}$, *we have*

$$\mathfrak{m}_{\lambda'}(\mathrm{U}(1)) = \lceil (z+1)(b+1)/2 \rceil \quad \textit{and} \quad \mathfrak{m}_{\lambda'}(H_3) = \frac{1}{2}\left(\lceil (z+1)(b+1)/2 \rceil + \tau(z,b)\right),$$

where we set $z := a - b$. *Furthermore, for any* $m \in \mathbb{Z}_{\geq 1}$, *we have*

$$\prod_{i=1}^{m}(1+x_i)(1+x_i^2) = \sum_{\substack{\lambda \trianglelefteq (3^m) \\ \lambda=(3^k,2^b,1^z)}} \tau(z,b)\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}), \tag{4.8}$$

$$\prod_{i=1}^{m}(1-x_i)(1+x_i^2) = \sum_{\substack{\lambda \trianglelefteq (3^m) \\ \lambda=(3^k,2^b,1^z)}} (-1)^{|\lambda|}\tau(z,b) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}). \tag{4.9}$$

Proof From Proposition 4.4, we obtain

$$\mathfrak{m}_{\lambda'}(\mathrm{U}(1)) = |\Phi^{(2)}(a,b)| \quad \text{and} \quad \mathfrak{m}_{\lambda'}(H_3) = |\Phi^{(4)}(a,b)|,$$

and the formulas for $\mathfrak{m}_{\lambda'}(\mathrm{U}(1))$ and $\mathfrak{m}_{\lambda'}(H_3)$ are from Proposition 4.5.

Let $d\gamma, d\gamma_1$ be the probability Haar measures on $H_3, \mathrm{U}(1) \leq \mathrm{U}(3)$, respectively. Since $H_3 = \mathrm{U}(1) \sqcup J\,\mathrm{U}(1)$, we use (4.1) and Proposition 2.3 to obtain

$$\begin{aligned}
\sum_{\lambda \trianglelefteq (3^m)} \mathfrak{m}_{\lambda'}(H_3) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) &= \int_{H_3} \prod_{i=1}^{m} \det(I + x_i\gamma) d\gamma \\
&= \frac{1}{2}\int_{\mathrm{U}(1)} \prod_{i=1}^{m} \det(1 + x_i\gamma) d\gamma_1 + \frac{1}{2}\int_{J\,\mathrm{U}(1)} \prod_{i=1}^{m}(1+x_i)(1+x_i^2) d\gamma_1 \\
&= \frac{1}{2}\sum_{\lambda \trianglelefteq (3^m)} \mathfrak{m}_{\lambda'}(\mathrm{U}(1)) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) + \frac{1}{2}\prod_{i=1}^{m}(1+x_i)(1+x_i^2).
\end{aligned}$$

Hence, using Lemma 4.1,

$$\prod_{i=1}^{m}(1+x_i)(1+x_i^2) = \sum_{\lambda \trianglelefteq (3^m)} \left(2\mathfrak{m}_{\lambda'}(H_3) - \mathfrak{m}_{\lambda'}(\mathrm{U}(1))\right) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) = \sum_{\substack{\lambda \trianglelefteq (3^m) \\ \lambda=(3^k,2^b,1^z)}} \tau(z,b) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}).$$

The identity (4.9) follows from (4.8) by replacing x_i with $-x_i$. □

Remark 4.7 The left hand side of (4.8) is a simple combination of the monomial symmetric functions $\mathsf{m}_\lambda^{(m)}$ associated with partitions λ in m-variables:

$$\prod_{i=1}^{m}(1+x_i)(1+x_i^2) = \sum_{\lambda \trianglelefteq (3^m)} \mathsf{m}_\lambda^{(m)}(\mathbf{x}).$$

Example 4.8 (1) Let us see an example for the case $m = 7$ in Theorem 4.6. We have

$$\prod_{i=1}^{7}(1+x_i)(1+x_i^2) = \sum_{\substack{\lambda \trianglelefteq (3^7) \\ \lambda=(3^k,2^b,1^z)}} \tau(z,b) S_\lambda^{\mathrm{U}(7)}(\mathbf{x}) \tag{4.10}$$

One can see that $S^{\mathrm{U}(7)}_{(3^2,2^2,1^1)}(\mathbf{x})$ appears in the right hand side of (4.10) with the coefficient -1 as $\tau(1,2) = -1$. As a polynomial, $S^{\mathrm{U}(7)}_{(3^2,2^2,1^1)}(\mathbf{x})$ contains 1778 monomial terms and $S^{\mathrm{U}(7)}_{(3^2,2^2,1^1)}(\mathbf{1}) = 7560$, where $\mathbf{1} = (1, 1, \dots, 1)$. There are 18 Schur functions with coefficient -1 in the right hand side of (4.10) including $S^{\mathrm{U}(7)}_{(3^2,2^2,1^1)}(\mathbf{x})$. On the other hand, we obtain a polynomial in the left hand side of (4.10), which contains 16384 monomial terms with coefficient all 1. One can check this, for example, using SAGEMATH.

(2) Let us consider the case $m = 20$ in Theorem 4.6. We have

$$\prod_{i=1}^{20}(1+x_i)(1+x_i^2) = \sum_{\substack{\lambda \trianglelefteq (3^{20}) \\ \lambda=(3^k,2^b,1^z)}} \tau(z,b) S^{\mathrm{U}(20)}_{\lambda}(\mathbf{x}) \tag{4.11}$$

Here $S^{\mathrm{U}(20)}_{(2^6,1^5)}(\mathbf{x})$ appears in the right hand side of (4.11) with the coefficient -1 as $\tau(5,6) = \tau(1,2) = -1$. Using WEYLCHARACTERRING in SAGEMATH, one can check that $S^{\mathrm{U}(20)}_{(2^6,1^5)}(\mathbf{1}) = 4557090720$. Including $S^{\mathrm{U}(20)}_{(2^6,1^5)}$, there are 315 Schur functions with coefficient -1 in the right hand side of (4.11) and the specialization of the left hand side of (4.11) at $\mathbf{1}$ is equal to $4^{20} = 1099511627776$. However, checking whether the right hand side of (4.11) coincides with the left hand side of (4.11) may well go beyond the capacity of a regular personal computer.

4.2 Subgroup $\langle \mathrm{U}(1), J, \boldsymbol{\zeta}_4 \rangle$

Let us set

$$\boldsymbol{\zeta}_4 := \begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & \sqrt{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in U(3) \setminus U(1).$$

Then we have

$$\det(I + x\boldsymbol{\zeta}_4 J\gamma) = (1+x)(1-x^2) \quad \text{and} \quad \det(I + xJ\gamma) = (1+x)(1+x^2). \tag{4.12}$$

Define $H_{3,4}$ to be the subgroup generated by $\mathrm{U}(1)$, J, $\boldsymbol{\zeta}_4$, i.e.

$$H_{3,4} := \langle \mathrm{U}(1), J, \boldsymbol{\zeta}_4 \rangle \leq \mathrm{U}(3).$$

Then we have the coset decomposition

$$H_{3,4} = \mathrm{U}(1) \sqcup J\,\mathrm{U}(1) \sqcup \boldsymbol{\zeta}_4\,\mathrm{U}(1) \sqcup \boldsymbol{\zeta}_4 J\,\mathrm{U}(1). \tag{4.13}$$

Let $H'_{3,4}$ be the subgroup of $H_{3,4}$ generated by $\mathrm{U}(1)$, $\boldsymbol{\zeta}_4$. Note that

$$H_{3,4} = H'_{3,4} \sqcup J H'_{3,4}. \tag{4.14}$$

Lemma 4.9 *For any $c \in \mathbb{Z}_{\geq 2}$, we have*

$$\mathfrak{m}_{(a,b,c)}(H_{3,4}) = \mathfrak{m}_{(a-2,b-2,c-2)}(H_{3,4}) \quad \textit{and} \quad \mathfrak{m}_{(a,b,c)}(H'_{3,4}) = \mathfrak{m}_{(a-2,b-2,c-2)}(H'_{3,4}).$$

Proof As before, let $\mathbf{det}$ be the one-dimensional representation of $\mathrm{U}(3)$ defined by the determinant. Then we have

$$V(a,b,c) = \mathbf{det}^2 \otimes V(a-2, b-2, c-2).$$

Since $\det(A) = 1$ for any $A \in \mathrm{U}(1)$, $\det(J) = 1$ and $\det(\boldsymbol{\zeta}_4) = -1$, the assertion follows. □

By Lemma 4.9, it suffices to consider partitions of the form

$$(a, b, \epsilon), \qquad \epsilon \in \{0, 1\}.$$

For a partition (a, b, ϵ) ($\epsilon \in \{0, 1\}$), define

$$\Phi(a, b, \epsilon) := \{(p, q) \in \mathbb{Z}^2 \mid a \geq p \geq b \geq q \geq \epsilon\}.$$

For example, $\Phi(3, 2, 1) = \{(3, 2), (2, 2), (3, 1), (2, 1)\}$. From Proposition 2.5, we see that the set $\Phi(a, b, \epsilon)$ is exactly the set of U(2)-highest weights in the restriction $V(a, b, \epsilon)|_{\mathrm{U}(2)}$. (4.15)

Proposition 4.10 *For a partition $\lambda' = (a, b, \epsilon)$ ($\epsilon \in \{0, 1\}$), the multiplicity $\mathfrak{m}_{\lambda'}(H'_{3,4})$ is the same as the cardinality of the set*

$$\Phi^{(2,4)}(\lambda') = \{(p, q) \in \Phi(\lambda') \mid p + q \equiv_4 0\},$$

and the multiplicity $\mathfrak{m}_{\lambda'}(H_{3,4})$ is the same as the cardinality of the set

$$\Phi^{(4,4)}(\lambda') = \{(p, q) \in \Phi(\lambda') \mid p - q \equiv_4 0,\ \ p + q \equiv_4 0\}.$$

Proof Let v be a weight vector fixed by $H'_{3,4}$. Then, in particular, v is fixed by U(1), and we may write $v = f_1^k v_{(p,q;a,b)}$ for $p - q = 2k$ as in the proof of Proposition 4.4. Since

$$v = f_1^k (v_1^{p-b} v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}) = C v_1^{p-b-k} v_2^k v_3^{a-p} \otimes w_{12}^q w_{13}^{b-q}$$
$$+(\text{other terms with the same weight}),$$

for some constant C, the number of v_1 factors and the number of v_2 factors are the same, which is equal to $k + q = (p + q)/2$, and we have

$$\boldsymbol{\zeta}_4 v = (\sqrt{-1})^{(p+q)} v = v.$$

Thus we have $p + q \equiv_4 0$, which implies $p - q \equiv_2 0$.

If v is also fixed by J, the we obtain an additional condition $p - q \equiv_4 0$ as in (the proof of) Proposition 4.4. □

The cardinalities of the sets $\Phi^{(2,4)}(\lambda')$ and $\Phi^{(4,4)}(\lambda')$ will be computed in the next section, and we present the resulting formulas in the following two propositions.

Proposition 4.11 *For a partition $\lambda' = (a, b, \epsilon)$ ($\epsilon \in \{0, 1\}$), write $z := a - b$ and $b' := b - \epsilon$. Then we have*

$$|\Phi^{(2,4)}(\lambda')| = \frac{(z + 1)(b' + 1) + \kappa_\epsilon(z, b')}{4}, \tag{4.16}$$

where $\kappa_\epsilon(z, b')$ are defined on the congruence classes of z and b' modulo 4 by

$\kappa_0(z, b) =$

$z \backslash b$	0	1	2	3
0	3	−2	1	0
1	2	−4	2	0
2	1	−2	3	0
3	0	0	0	0

and $\kappa_1(z, b') =$

$z \backslash b'$	0	1	2	3
0	−1	2	1	0
1	−2	4	−2	0
2	1	2	−1	0
3	0	0	0	0

.

Proposition 4.12 *For a partition $\lambda' = (a, b, \epsilon)$ ($\epsilon \in \{0, 1\}$), write $z := a - b$ and $b' := b - \epsilon$. Then we have*

$$|\Phi^{(4,4)}(\lambda')| = \frac{b'z + \eta_\epsilon(z, b') \cdot (b', z) + \xi_\epsilon(z, b')}{8}, \tag{4.17}$$

where $(x, y) \cdot (b', z) = xb' + yz$, *and* $\eta_\epsilon(z, b')$ *and* $\xi_\epsilon(z, b')$ *are defined on the congruence classes of* z *and* b' *by*

$\eta_0(z, b) =$

$z\backslash b$	0	1
0	(2, 2)	(0, 1)
1	(1, 2)	(1, 1)

, $\eta_1(z, b') =$

$z\backslash b'$	0	1
0	(0, 0)	(2, 1)
1	(1, 0)	(1, 1)

,

$\xi_0(z, b) =$

$z\backslash b$	0	1	2	3
0	8	0	4	0
1	6	−3	2	1
2	4	−4	4	0
3	2	1	2	1

and $\xi_1(z, b') =$

$z\backslash b'$	0	1	2	3
0	0	6	0	2
1	0	5	−4	1
2	0	2	−4	2
3	0	1	0	1

.

Corollary 4.13 *For each partition* $\lambda' = (a, b, \epsilon)$ $(\epsilon \in \{0, 1\})$, *write* $z := a - b$ *and* $b' := b - \epsilon$. *Then we have*

$$\omega_\epsilon(z, b') := 2|\Phi^{(4,4)}(\lambda')| - |\Phi^{(2,4)}(\lambda')| = \frac{\alpha_\epsilon(z, b') \cdot (b', z) + \beta_\epsilon(z, b')}{4},$$

where $(x, y) \cdot (b', z) = xb' + yz$, *and* $\alpha_\epsilon(z, b')$ *and* $\beta_\epsilon(z, b')$ *are defined on the congruence classes of* z *and* b' *by*

$\alpha_0(z, b) =$

$z\backslash b$	0	1
0	(1, 1)	(−1, 0)
1	(0, 1)	(0, 0)

, $\alpha_1(z, b') =$

$z\backslash b'$	0	1
0	(−1, −1)	(1, 0)
1	(0, −1)	(0, 0)

,

$\beta_0(z, b) =$

$z\backslash b$	0	1	2	3
0	4	1	2	−1
1	3	0	−1	0
2	2	−3	0	−1
3	1	0	1	0

and $\beta_1(z, b') =$

$z\backslash b'$	0	1	2	3
0	0	3	−2	1
1	1	0	−3	0
2	−2	−1	−4	1
3	−1	0	−1	0

.

Now we state and prove the main result of this subsection.

Theorem 4.14 *For a partition* $\lambda' = (a + 2k, b + 2k, \epsilon + 2k)$ $(k \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\})$, *write* $z := a - b$ *and* $b' := b - \epsilon$. *Then we have*

$$\mathfrak{m}_{\lambda'}(H'_{3,4}) = \frac{(z+1)(b'+1)+\kappa_\epsilon(z, b')}{4} \quad \textit{and} \quad \mathfrak{m}_{\lambda'}(H_{3,4}) = \frac{b'z+\eta_\epsilon(z, b') \cdot (b', z)+\xi_\epsilon(z, b')}{8}.$$

Furthermore, for any $m \in \mathbb{Z}_{\geq 1}$, *we have*

$$\frac{1}{2}\left(\prod_{i=1}^{m}(1 + x_i)(1 + x_i^2) + \prod_{i=1}^{m}(1 + x_i)(1 - x_i^2)\right) = \sum_{\lambda \trianglelefteq (3^m)} \omega_\epsilon(z, b')\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}), \tag{4.18}$$

where z, b', ϵ *are determined by the transpose* λ' *of* $\lambda \trianglelefteq (3^m)$.

Proof The first assertion follows from Propositions 4.10, 4.11, 4.12 and Lemma 4.9. Let $d\gamma, d\gamma_1, d\gamma_2$ be the probability Haar measures on $H_{3,4}$, $H'_{3,4}$, $\mathrm{U}(1) \leq \mathrm{U}(3)$, respectively. We use the coset decompositions (4.13) and (4.14) and the computation (4.12) and Proposition 2.3 to obtain

$$\sum_{\lambda \trianglelefteq (3^m)} \mathfrak{m}_{\lambda'}(H_{3,4}) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) = \int_{H_{3,4}} \prod_{i=1}^{m} \det(I + x_i\gamma)d\gamma$$

$$= \frac{1}{2}\int_{H'_{3,4}} \prod_{i=1}^{m} \det(I+x_i\gamma)d\gamma_1 + \frac{1}{4}\int_{J\,\mathrm{U}(1)} \prod_{i=1}^{m} \det(I+x_i\gamma)d\gamma_2 + \frac{1}{4}\int_{\zeta_4 J\,\mathrm{U}(1)} \prod_{i=1}^{m} \det(I+x_i\gamma)d\gamma_2$$

$$= \frac{1}{2} \sum_{\lambda \trianglelefteq (3^m)} \mathfrak{m}_{\lambda'}(H'_{3,4}) S_\lambda^{\mathrm{U}(m)}(\mathbf{x}) + \frac{1}{4} \prod_{i=1}^{m} (1+x_i)(1+x_i^2) + \frac{1}{4} \prod_{i=1}^{m} (1+x_i)(1-x_i^2).$$

Hence, using Corollary 4.13,

$$\frac{1}{2} \left(\prod_{i=1}^{m} (1+x_i)(1+x_i^2) + \prod_{i=1}^{m} (1+x_i)(1-x_i^2) \right) = \sum_{\lambda \trianglelefteq (3^m)} \left(2\mathfrak{m}_{\lambda'}(H_{3,4}) - \mathfrak{m}_{\lambda'}(H'_{3,4}) \right) S_\lambda^{\mathrm{U}(m)}(\mathbf{x})$$
$$= \sum_{\lambda \trianglelefteq (3^m)} \omega_\epsilon(z, b') S_\lambda^{\mathrm{U}(m)}(\mathbf{x}).$$

□

Remark 4.15 (1) The left hand side of (4.18) can be written as a simple combination of the monomial symmetric functions in m-variables:

$$\frac{1}{2} \left(\prod_{i=1}^{m} (1+x_i)(1+x_i^2) + \prod_{i=1}^{m} (1+x_i)(1-x_i^2) \right) = \sum_{\substack{\lambda' \trianglelefteq (m^3) \\ \lambda'_2 \equiv_2 0}} \mathsf{m}_\lambda^{(m)}(\mathbf{x}).$$

where $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3)$.

(2) By replacing x_i by $-x_i$ in (4.18), we obtain

$$\frac{1}{2} \left(\prod_{i=1}^{m} (1-x_i)(1+x_i^2) + \prod_{i=1}^{m} (1-x_i)(1-x_i^2) \right) = \sum_{\lambda \trianglelefteq (3^m)} (-1)^{|\lambda|} \omega_\epsilon(z, b') S_\lambda^{\mathrm{U}(m)}(\mathbf{x}).$$

Example 4.16 (1) Let us see the case $m = 7$ in Theorem 4.14. Then we have

$$\frac{1}{2} \left(\prod_{i=1}^{7} (1+x_i)(1+x_i^2) + \prod_{i=1}^{7} (1+x_i)(1-x_i^2) \right) = \sum_{\lambda \trianglelefteq (3^7)} \omega_\epsilon(z, b')\, S_\lambda^{\mathrm{U}(7)}(\mathbf{x}) \qquad (4.19)$$

The function $S^{\mathrm{U}(7)}_{(3,2^2,1)}(\mathbf{x})$ appears in the right hand side of (4.19) with the coefficient -2 as $\lambda' = (4, 3, 1)$ and hence $z = 1$ and $b' = 2$. As a polynomial itself, $S^{\mathrm{U}(7)}_{(3^2,2^2,1^1)}(\mathbf{x})$ contains 1239 monomial terms and $S^{\mathrm{U}(7)}_{(3,2^2,1^1)}(\mathbf{1}) = 3402$. There are 36 Schur functions with negative coefficients in the right hand side of (4.19) including $S^{\mathrm{U}(7)}_{(3,2^2,1)}(\mathbf{x})$. However, we obtain a polynomial of 8192 monomial terms with all coefficient 1 in the left hand side of (4.19).

(2) Let us see the case $m = 20$ in Theorem 4.14:

$$\frac{1}{2} \left(\prod_{i=1}^{20} (1+x_i)(1+x_i^2) + \prod_{i=1}^{20} (1+x_i)(1-x_i^2) \right) = \sum_{\lambda \trianglelefteq (3^m)} \omega_\epsilon(z, b')\, S_\lambda^{\mathrm{U}(20)}(\mathbf{x}) \qquad (4.20)$$

Here $S^{\mathrm{U}(20)}_{(2^9,1^2)}(\mathbf{x})$ appears in the right hand side of (4.20) with the coefficient -3 as $\lambda' = (11, 9)$ and hence $z = 2$ and $b' \equiv_4 1$. One can check that $S^{\mathrm{U}(20)}_{(2^9,1^2)}(\mathbf{1}) = 12342120700$. There are 590 Schur functions with negative coefficients in the right hand side of (4.20), and the specialization of the right hand side of (4.20) at $\mathbf{1}$ is equal to 2^{39}. This shows some systematic, even miraculous, cancelations of monomial terms in Schur functions involved in this example.

Combining Theorems 4.6 and 4.14, we obtain the following identity.

Corollary 4.17 *For each partition* $\lambda' = (a+2k, b+2k, \epsilon+2k)$ $(k \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\})$, *write* $z := a - b$ *and* $b' := b - \epsilon$. *Then we have*

$$\prod_{i=1}^{m}(1+x_i)(1-x_i^2) = \sum_{(\lambda_1',\lambda_2',\lambda_3') \trianglelefteq (m^3)} (-1)^{\delta(\lambda_2' \equiv_2 1)} \mathsf{m}_\lambda^{(m)}(\mathbf{x}) = \sum_{\lambda \trianglelefteq (3^m)} \widetilde{\omega}_\epsilon(z, b')\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}),$$

where

$$\widetilde{\omega}_\epsilon(z, b') := \frac{\alpha_\epsilon(z, b') \cdot (b', z) + \widetilde{\beta}_\epsilon(z, b')}{2}.$$

Here $\widetilde{\beta}_\epsilon(z, b')$ *are defined on the congruence classes of* z *and* b' *by*

$\widetilde{\beta}_0(z, b) =$

$z\backslash b$	*0*	*1*
0	2	−1
1	1	0

and $\widetilde{\beta}_1(z, b') =$

$z\backslash b'$	*0*	*1*
0	−2	1
1	−1	0

.

Similarly, combining Theorem 4.6 and Corollary 4.17, we obtain another identity below.

Corollary 4.18 *For each partition* $\lambda' = (a+2k, b+2k, \epsilon+2k)$ $(k \in \mathbb{Z}_{\geq 0}, \epsilon \in \{0, 1\})$, *write* $z := a - b$ *and* $b' := b - \epsilon$. *Then we have*

$$\frac{1}{2}\left(\prod_{i=1}^{m}(1+x_i)(1+x_i^2) - \prod_{i=1}^{m}(1+x_i)(1-x_i^2)\right) = \sum_{\substack{(\lambda_1',\lambda_2',\lambda_3') \trianglelefteq (m^3) \\ \lambda_2' \not\equiv_2 1}} \mathsf{m}_\lambda^{(m)}(\mathbf{x})$$
$$= \sum_{\lambda \trianglelefteq (3^m)} \widehat{\omega}_\epsilon(z, b')\, S_\lambda^{\mathrm{U}(m)}(\mathbf{x}),$$

where

$$\widehat{\omega}_\epsilon(z, b') := = \frac{-\alpha_\epsilon(z, b') \cdot (b', z) + \widehat{\beta}_\epsilon(z, b')}{4}.$$

Here $\widehat{\beta}_\epsilon(z, b')$ *are defined on the congruence classes of* z *and* b' *by*

$\widehat{\beta}_0(z, b) =$

$z\backslash b$	*0*	*1*	*2*	*3*
0	0	3	−2	1
1	1	0	−3	0
2	−2	−1	−4	1
3	−1	0	−1	0

and $\widehat{\beta}_1(z, b') =$

$z\backslash b'$	*0*	*1*	*2*	*3*
0	4	1	2	−1
1	3	0	−1	0
2	2	−3	0	−1
3	1	0	1	0

.

Remark 4.19 Replacing x_i with $-x_i$, we obtain identities for

$$\prod_{i=1}^{m}(1-x_i)(1+x_i^2) \qquad \text{and} \qquad \prod_{i=1}^{m}(1-x_i)(1-x_i^2)$$

and hence identities for

$$\frac{1}{2}\left(\prod_{i=1}^{m}(1-x_i)(1+x_i^2) \pm \prod_{i=1}^{m}(1-x_i)(1-x_i^2)\right).$$

5 Proofs for the Cardinalities of $\Phi^{(2,4)}$ and $\Phi^{(4,4)}$

In this section, we will prove the explicit closed-form formulas of $\Phi^{(2,4)}$ and $\Phi^{(4,4)}$, which are presented in Propositions 4.10 and 4.11, respectively. Throughout this section, we will explain the reason why the functions $\kappa_\epsilon(z,b)$, $\eta_\epsilon(z,b)$ and $\xi_\epsilon(z,b)$ are defined with respect to congruence classes modulo 2 or 4. We start this section with an example.

Let us consider the partition $\nu' = (a,b,0) = (a,b) = (7,2)$ of length 2. Note that

- $z = a - b = 5$ and hence $z+1 = 6$, $b+1 = 3$,
- among the partitions (a,b) with

$$a - b = 5 \quad \text{and} \quad b \equiv_4 2, \tag{5.1}$$

the partition $(7,2)$ is the smallest one.

Thus the set $\Phi(7,2)$ can be displayed by the following (6×3)-array as in (4.7):

$$\boxed{\begin{array}{ccc} (7,2) & (7,1) & (7,0) \\ (6,2) & (6,1) & (6,0) \\ (5,2) & (5,1) & (5,0) \\ (4,2) & (4,1) & (4,0) \\ (3,2) & (3,1) & (3,0) \\ (2,2) & (2,1) & (2,0) \end{array}}_{\,2}^{\,z} = \boxed{\begin{array}{ccc} (z+b,b) & (z+b,b-1) & (z+b,0) \\ (z+b-1,b) & (z+b-1,b-1) & (z+b-1,0) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (b+1,b) & (b+2,1) & (b+1,0) \\ (b,b) & (b+1,1) & (b,0) \end{array}}_{\,b}^{\,z} \tag{5.2}$$

Motivated by (5.2), for $(k,l) \in \mathbb{Z}^2$ and $z, b \in \mathbb{Z}_{\ge 0}$, we define the set $\boxed{(k,l)}_{\,b}^{\,z}$ of pairs of integers:

$$\boxed{(k,l)}_{\,b}^{\,z} := \{(p,q) \in \mathbb{Z}^2 \mid k+b \ge p \ge k,\ z+l \ge q \ge l\}, \tag{5.3}$$

whose cardinality is $(z+1)\times(b+1)$.

In (5.2), one can check that, for $\nu' = (7,2)$, we have

- $\Phi^{(2,4)}(\nu') = \{(4,0),\ (3,1),\ (7,1),\ (6,2),\ (2,2)\}$,
- $\Phi^{(4,4)}(\nu') = \{(4,0),\ (6,2),\ (2,2)\}$,

and hence $\Phi^{(2,4)}(\nu') \setminus \Phi^{(4,4)}(\nu') = \{(3,1),\ (7,1)\}$. Here one can notice that $(p,q) \in \Phi^{(2,4)}(\nu') \setminus \Phi^{(4,4)}(\nu')$ are located in the second column of (5.2), since

$$q \equiv_4 1 \text{ and hence } p \equiv_4 3, \text{ yielding } p - q \not\equiv_4 0. \tag{5.4}$$

Let us consider a partition $\mu' = (11,6)$ which is the second smallest case in the sense of (5.1). Then its corresponding $\Phi(11,6)$ can be represented by the following $(6\times(3+4))$-array as follows:

$$\boxed{\begin{array}{ccc} (11,6) & (11,5) & (11,4) \\ (10,6) & (10,5) & (10,4) \\ (9,6) & (9,5) & (9,4) \\ (8,6) & (8,5) & (8,4) \\ (7,6) & (7,5) & (3,4) \\ (6,6) & (6,5) & (6,4) \end{array}}_{\,2}^{\,z} \boxed{\begin{array}{cccc} (11,3) & (11,2) & (11,1) & (11,0) \\ (10,3) & (10,2) & (10,1) & (10,0) \\ (9,3) & (9,2) & (9,1) & (9,0) \\ (8,3) & (8,2) & (8,1) & (8,0) \\ (7,3) & (7,2) & (7,1) & (7,0) \\ (6,3) & (6,2) & (6,1) & (6,0) \end{array}}_{\,3}^{\,z} \tag{5.5}$$

Here the left (resp. right) part of (5.5) coincides with $\boxed{(b+4,4)}_{\,2}^{\,z}$ $\left(\text{resp. } \boxed{(b+4,0)}_{\,3}^{\,z}\right)$.

Then one can see that (i) the left part of (5.5) can be obtained from (5.2) by adding (4, 4) for each entry and (ii) the right part of (5.5) consists of four columns. Since the number of (p,q)'s satisfying $p+q \equiv_4 0$ in $\boxed{(b+4,4)}_{\,2}^{\,z}$ is the same as $|\Phi^{(2,4)}(\nu')|$ by (i), and each row in $\boxed{(b+4,0)}_{\,3}^{\,z}$ contains (p,q) with $p+q \equiv_4 0$ exactly once by (ii), we have

$$|\Phi^{(2,4)}(\mu')| = |\Phi^{(2,4)}(\nu')| + (z+1) = 5 + 6.$$

For the same reason as in (5.4), the (p, q)'s with $p + q \equiv_4 0$ with $q \equiv_4 1, 3$ cannot satisfy the condition $p - q \equiv_4 0$. Thus we obtain

$$|\Phi^{(4,4)}(\mu')| = |\Phi^{(4,4)}(\nu')| + \frac{(z+1)}{2} = 3 + 3.$$

Since

$$\Phi^{(4,4)}(c+4, d+4) = \left(\Phi^{(4,4)}(c, d) + (4, 4)\right) \sqcup \boxed{(d+4, 0)}_3^z$$

and the number of (p, q)'s satisfying $p + q \equiv_4 0$ (resp. $p + q \equiv_4 0$ and $p - q \equiv_4 0$) in $\boxed{(d+4, 0)}_3^z$ does not change for any (c, d) with $c - d = 5$ and $d \equiv_4 2$, the closed-form formulas are written in (4.16) and (4.17), which are determined by $|\Phi^{(k,4)}(\nu')|, |\Phi^{(k,4)}(\mu')| - |\Phi^{(k,4)}(\nu')|$ $(k = 2, 4)$ depending on z and on the congruent classes of z and b modulo 4.

Now we generalize the argument above. We begin with a definition.

Definition 5.1 Fix $\epsilon \in \{0, 1\}$, $\mathsf{b}' \in \{0, 1, 2, 3\}$ and $z \in \mathbb{Z}_{\geq 0}$. We call the partition

$$\nu'(z, \mathsf{b}', \epsilon) := (z + \mathsf{b}' + \epsilon, \mathsf{b}' + \epsilon, \epsilon)$$

the *base* partition for the triple $(z, \mathsf{b}', \epsilon)$

For any finite set X of pairs of integers, we denote by $\phi^{(2,4)}(X)$ (resp. $\phi^{(4,4)}(X)$) the number of (p, q)'s in X satisfying $p + q \equiv_4 0$ (resp. $p + q \equiv_4 0$ and $p - q \equiv_4 0$). In particular, we write

$$\phi^{(u,4)}(\nu') := \phi^{(u,4)}(\Phi(\nu')) = |\Phi^{(u,4)}(\nu')| \qquad (u = 2, 4)$$

for a partition ν' of the form $(z + b' + \epsilon, b' + \epsilon, \epsilon)$ $(z, b' \in \mathbb{Z}_{\geq 0},\ \epsilon \in \{0, 1\})$. Then the following lemma is obvious:

Lemma 5.2 *For any* $\boxed{(k, l)}_b^z$ *and* $(m, n) \in \mathbb{Z}^2$*, we have*

$$\phi^{(u,4)}\left(\boxed{(k, l)}_b^z\right) = \phi^{(u,4)}\left(\boxed{(k, l)}_b^z + (4m, 4n)\right) \qquad (u = 2, 4).$$

Lemma 5.3 *For any partition* (a, b, ϵ) *with* $a - b = z \in \mathbb{Z}_{\geq 0}$ *and* $\epsilon \in \{0, 1\}$*, we have*

$$\phi^{(2,4)}(a+4, b+4, \epsilon) - \phi^{(2,4)}(a, b, \epsilon) = z + 1.$$

Proof Let $z = a - b$ and $b' = b - \epsilon$. Then we have

$$\Phi(a+4, b+4, \epsilon) = \boxed{(b+4, \epsilon)}_{b'+4}^z, \quad \Phi(a, b, \epsilon) = \boxed{(b, \epsilon)}_{b'}^z$$

and hence

$$\Phi(a+4, b+4, \epsilon) = (\Phi(a, b, \epsilon) + (4, 4)) \bigsqcup \boxed{(b, \epsilon)}_3^z. \tag{5.6}$$

By Lemma 5.2, $\phi^{(u,4)}(a, b, \epsilon) = \phi^{(u,4)}\left(\boxed{(b, \epsilon)}_{b'}^z + (4, 4)\right)$ $(u = 2, 4)$ and hence

$$\phi^{(u,4)}(a+4, b+4, \epsilon) - \phi^{(2,4)}(a, b, \epsilon) = \phi^{(u,4)}\left(\boxed{(b, \epsilon)}_3^z\right) \qquad (u = 2, 4).$$

Recall that $\boxed{(b, \epsilon)}_3^z$ consists of four columns with height $z + 1$. Thus we have

$$\phi^{(2,4)}\left(\boxed{(b, \epsilon)}_3^z\right) = z + 1,$$

because each row in $\boxed{(b, \epsilon)}_3^z$ contains (p, q) with $p + q \equiv_4 0$ exactly once. □

Lemma 5.4 *For any base partition $\nu' = \nu'(z, \mathsf{b}', \epsilon)$, we have*

$$\phi^{(2,4)}(\nu') = \frac{(\mathsf{b}'+1)(z+1) + \kappa_\epsilon(z, \mathsf{b}')}{4}.$$

Proof (1) Note that, when $\mathsf{b}' = 3$ (resp. $z \equiv_3 4$), the integer $\kappa_\epsilon(z, \mathsf{b}') = 0$. The assertion for this case comes from the fact that

$$\Phi(\nu') = \boxed{(\mathsf{b}'+\epsilon, \epsilon)}^{\,z}_{\,\mathsf{b}'}$$

consists of 4-columns with height $z+1$ (resp. $(\mathsf{b}'+1)$-columns with height $z+1 = 4k$ for some $k \in \mathbb{Z}_{\geq 1}$), implying that each row in the set contains (p, q) with $p+q \equiv_4 0$ exactly once (resp. each column in the set contains (p, q)'s with $p+q \equiv_4 0$ exactly k-times).

(2) Now let us consider the case when $\mathsf{b}' = 2$, $z \equiv_4 1$ and $\epsilon = 1$. Write $z = 4k+1$ for some $k \in \mathbb{Z}_{\geq 0}$. Then $\boxed{(\mathsf{b}'+\epsilon, \epsilon)}^{\,z}_{\,\mathsf{b}'} = \boxed{(\mathsf{b}'+\epsilon, \epsilon)}^{\,z}_{\,2}$ can be described as follows:

$$\boxed{\begin{matrix} (\mathsf{b}'+z+1, 3) & (\mathsf{b}'+z+1, 2) & (\mathsf{b}'+z+1, 1) \\ (\mathsf{b}'+z, 3) & (\mathsf{b}'+z, 2) & \underline{(\mathsf{b}'+z, 1)} \\ (\mathsf{b}'+z-1, 3) & \underline{(\mathsf{b}'+z-1, 2)} & (\mathsf{b}'+z-1, 1) \\ \vdots & \vdots & \vdots \\ \underline{(b'+3, 3)} & (\mathsf{b}'+3, 2) & (\mathsf{b}'+3, 1) \\ (\mathsf{b}'+2, 3) & (\mathsf{b}'+2, 2) & (\mathsf{b}'+2, 1) \\ (\mathsf{b}'+1, 3) & (\mathsf{b}'+1, 2) & \underline{(\mathsf{b}'+1, 1)} \end{matrix}}^{\,z}_{\,2} \tag{5.7}$$

Here the underlined pairs $\underline{(p, q)}$ of integers satisfy the condition that $p+q \equiv_4 0$. One sees that the number t_i of underlined (p, q)'s in the column with $q = i$ $(i = 0, 1, 2)$ is given by

$$t_0 = k+1, \quad t_1 = k \quad \text{and} \quad t_2 = k. \tag{5.8}$$

Thus we have

$$\phi^{(2,4)}(\nu') = 3k+1 = \frac{3 \times (4k+2) - 2}{4}$$

which implies $\kappa_1(1, 2) = -2$.

(3) For the remaining cases, one can prove the formula by a similar argument. □

Proof *(Proof of Proposition 4.11)* By Lemma 5.3 and Lemma 5.4, for a partition (a, b, ϵ) of the form

$$(z+\mathsf{b}'+4k+\epsilon, \mathsf{b}'+4k+\epsilon, \epsilon) \qquad (z = a-b,\ k \in \mathbb{Z}_{\geq 0} \text{ and } \mathsf{b}' \in \{0, 1, 2, 3\}),$$

we have

$$\begin{aligned} \phi^{(2,4)}(a, b, \epsilon) &= \frac{(\mathsf{b}'+1)(z+1) + \kappa_\epsilon(z, \mathsf{b}')}{4} + k \times (z+1) \\ &= \frac{(b'+1)(z+1) + \kappa_\epsilon(z, b')}{4} \qquad (\because b' = b - \epsilon), \end{aligned}$$

as we desired. □

Lemma 5.5 *For any partition (a, b, ϵ) with $a - b = z \in \mathbb{Z}_{\geq 0}$, we have*

$$\phi^{(4,4)}(a+4, b+4, \epsilon) - \phi^{(4,4)}(a, b, \epsilon) = \frac{z + \eta_\epsilon(z, b)_1}{2},$$

where $\eta_\epsilon(z, b)_1$ denotes the first component of $\eta_\epsilon(z, b)$.

Proof By (5.6), it suffices to show that

$$\phi^{(4,4)}\left(\boxed{(b,\epsilon)}_{3}^{z}\right) = \frac{z+\eta_\epsilon(z,b)_1}{2},$$

where $\phi^{(2,4)}\left(\boxed{(b,\epsilon)}_{3}^{z}\right) = z+1$ by Lemma 5.3 and the set $\boxed{(b,\epsilon)}_{3}^{z}$ can be described as follows:

$$\boxed{\begin{array}{cccc} (b+z,\epsilon+3) & (b+z,\epsilon+2) & (b+z,\epsilon+1) & (b+z,\epsilon) \\ (b+z-1,\epsilon+3) & (b+z-1,\epsilon+2) & (b+z-1,\epsilon+1) & (b+z-1,\epsilon) \\ (b+z-2,\epsilon+3) & (b+z-2,\epsilon+2) & (b+z-2,\epsilon+1) & (b+z-2,\epsilon) \\ \vdots & \vdots & \vdots & \vdots \\ (b+2,\epsilon+3) & (b+2,\epsilon+2) & (b+2,\epsilon+1) & (b+2,\epsilon) \\ (b+1,\epsilon+3) & (b+1,\epsilon+2) & (b+1,\epsilon+1) & (b+1,\epsilon) \\ (b,\epsilon+3) & (b,\epsilon+2) & (b,\epsilon+1) & (b,\epsilon) \end{array}}_{3}^{z} \tag{5.9}$$

Then, for each adjacent two rows, there are exactly two pairs of integers (p_i, q_i) $(i = 1, 2)$ satisfying $p_i + q_i \equiv_4 0$, where exactly one of them does not satisfy the condition $p - q \equiv_4 0$ and the another satisfies the condition, as in (5.4).

(1) For $z \equiv_2 1$, we have always $\eta_\epsilon(z, b) = 1$ and hence $\dfrac{z+\eta_\epsilon(z,b)_1}{2} = \dfrac{z+1}{2}$. Thus the assertion for this case follows from the fact that $z + 1 \equiv_2 0$.

(2) Let us consider the case when $z \equiv_2 0$, $b \equiv_2 1$ and $\epsilon = 0$. Then it suffices to consider the bottom row in (5.9):

$$\boxed{(b,3)\ \ (b,2)\ \ (b,1)\ \ (b,0)}$$

Then exactly one of $(b, 1)$ and $(b, 3)$ satisfies the condition that $p + q \equiv_4 0$ but it does not satisfies the condition that $p - q \equiv_4 0$, since $q \equiv_4 1, 3$. Hence

$$\phi^{(4,4)}\left(\boxed{(b,0)}_{3}^{z}\right) = \frac{z}{2}.$$

(3) The remaining 3-cases can be proved by a similar argument. □

Lemma 5.6 *For any base partition* $\nu' = \nu'(z, \mathsf{b}', \epsilon)$, *we have*

$$\phi^{(4,4)}(\nu') = \frac{b'z + \eta_\epsilon(z,\mathsf{b}')\cdot(\mathsf{b}',z) + \xi_\epsilon(z,\mathsf{b}')}{8}.$$

Proof (1) As in Lemma 5.4, let us consider the case when $\mathsf{b}' = 2$, $z \equiv_4 1$ and $\epsilon = 1$. Write $z = 4k + 1$ for some $k \in \mathbb{Z}_{\geq 0}$. Then the underlined pairs (p, q) of integers in columns with $q = 1$ or 3 in (5.7) cannot satisfy the condition $p - q \equiv_4 0$, while the ones in the column with $q = 2$ in (5.7) satisfy the condition as $p \equiv_4 2$. Thus $\phi^{(2,4)}(\nu') = k$ by (5.8). Since

$$k = \frac{2(4k+1) + (1,0)\cdot(2,4k+1) - 4}{8},$$

our assertion holds in this case.

(2) For the remaining cases, one can prove the formula by a similar argument. □

Proof *(Proof of Proposition 4.12)* By Lemma 5.5 and Lemma 5.6, for a partition (a, b, ϵ) of the form

$$(z + \mathsf{b}' + 4k + \epsilon, \mathsf{b}' + 4k + \epsilon, \epsilon) \qquad (z = a - b,\ k \in \mathbb{Z}_{\geq 0} \text{ and } \mathsf{b}' \in \{0, 1, 2, 3\}),$$

we have

$$\phi^{(2,4)}(a,b,\epsilon) = \frac{b'z + \eta_\epsilon(z,\mathsf{b}')\cdot(\mathsf{b}',z) + \xi_\epsilon(z,\mathsf{b}')}{8} + k \times \frac{z+\eta_\epsilon(z,b)_1}{2}$$
$$= \frac{b'z + \eta_\epsilon(z,b')\cdot(b',z) + \xi_\epsilon(z,b')}{8} \qquad (\because b' = b - \epsilon),$$

as we desired. □

Remark 5.7 After the first version of this paper was posted on the arXiv, Ronald C. King informed us that identities (1.5)-(1.5) in **Application** can be extracted from the work of Yang–Wybourne, Lascoux–Pragacz and King–Wybourne–Yang [24, 25, 31] that appeared in the mathematical physics literature in the 1980's. Their methods are purely combinatorial and do not have direct connections to auto-correlation functions.

Author Contributions KHL and SjO have made equal contributions.

Funding KHL was partially supported by a grant from the Simons Foundation (#712100). SjO was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2022R1A2C1004045).

Availability of data and materials Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Ethical Approval Ethical approval is not applicable to this article.

Competing of interest The authors have no conflicts of interests or competing interests to declare.

References

1. Andreev, A.V., Simons, B.D.: Correlators of spectral determinants in quantum chaos. Phys. Rev. Lett. **75**, 2304–2307 (1995)
2. Baik, J., Deift, P., Strahov, E.: Products and ratios of characteristic polynomials of random Hermitian matrices. Integrability, Topological Solitons and Beyond **44**, 3657–3670 (2003)
3. Brézin, E., Hikami, S.: Characteristic polynomials of random matrices. Comm. Math. Phys. **214**(1), 111–135 (2000)
4. Brézin, E., Hikami, S.: Characteristic polynomials of real symmetric random matrices. Comm. Math. Phys. **223**(2), 363–382 (2001)
5. Borodin, A., Strahov, E.: Averages of characteristic polynomials in random matrix theory. Comm. Pure Appl. Math. **59**(2), 161–253 (2006)
6. Bump, D. (2013) Lie Groups, vol. 225 of Graduate Texts in Mathematics. Springer, New York, 2nd edn
7. Bump, D., Gamburd, A.: On the averages of characteristic polynomials from classical groups. Comm. Math. Phys. **265**(1), 227–274 (2006)
8. Conrey, J.B., Farmer, D.W., Keating, J.P., Rubinstein, M.O., Snaith, N.C.: Autocorrelation of random matrix polynomials. Comm. Math. Phys. **237**(3), 365–395 (2003)
9. Conrey, J.B., Farmer, D.W., Keating, J.P., Rubinstein, M.O. and Snaith, N.C. (2005) Integral moments of L-functions. Proc. London Math. Soc. (3), 91(1):33–104
10. Conrey, J.B., Farmer, D.W. and Zirnbauer, M.R. (2005) Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups $U(N)$. arXiv:math-ph/0511024
11. Conrey, J.B., Farmer, D.W., Zirnbauer, M.R.: Autocorrelation of ratios of L-functions. Commun. Number Theory Phys. **2**(3), 593–636 (2008)

12. Clozel, L. Harris, M. and Taylor, R. (2008) Automorphy for some l-adic lifts of automorphic mod l Galois representations. Publ. Math. Inst. Hautes Études Sci., (108):1–181 With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras
13. Fité, F., Kedlaya, K.S., Rotger, V., Sutherland, A.V.: Sato-Tate distributions and Galois endomorphism modules in genus 2. Compos. Math. **148**(5), 1390–1442 (2012)
14. Fité, F., Kedlaya, K.S., Rotger, V. and Sutherland, A.V. (2019) Sato–Tate groups of abelian threefolds. preprint, arXiv:2106.13759
15. Fyodorov, Y.V., Strahov, E.: Characteristic polynomials of random Hermitian matrices and Duistermaat-Heckman localisation on non-compact Kähler manifolds. Nuclear Phys. B **630**(3), 453–491 (2002)
16. Fyodorov, Y.V., Strahov, E.: On correlation functions of characteristic polynomials for chiral Gaussian unitary ensemble. Nuclear Phys. B **647**(3), 581–597 (2002)
17. Harris, M., Shepherd-Barron, N. and Taylor, R. (2010) A family of Calabi–Yau varieties and potential automorphy. Ann. of Math. (2), 171(2):779–813
18. Howe, R. (1995) Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In: The Schur lectures (1992) (Tel Aviv), vol. 8 of Israel Math. Conf. Proc., pp. 1–182. Bar-Ilan Univ., Ramat Gan
19. Jimbo, M. and Miwa, T. (1985) On a duality of branching rules for affine Lie algebras. In: Algebraic groups and related topics (Kyoto/Nagoya, 1983), vol. 6 of Adv. Stud. Pure Math., pp. 17–65. North-Holland, Amsterdam
20. Katz, N.M. and Sarnak, P. (1999) Random matrices, Frobenius eigenvalues, and monodromy, vol. 45 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI
21. Keating, J.P., Snaith, N.C.: Random matrix theory and L-functions at $s = 1/2$. Comm. Math. Phys. **214**(1), 91–110 (2000)
22. Keating, J.P., Snaith, N.C.: Random matrix theory and $\zeta(1/2 + it)$. Comm. Math. Phys. **214**(1), 57–89 (2000)
23. Kedlaya, K.S., Sutherland, A.V.: Hyperelliptic curves, L-polynomials, and random matrices. Contemp. Math. **487**, 119–162 (2009)
24. King, R.C., Wybourne, B.G., Yang, M.: Slinkies and the S-function content of certain generating functions. J. Phys. A **22**, 4519–4535 (1989)
25. Lascoux, A., Pragacz, P.: S-function series. J. Phys. A **21**, 4105–4114 (1988)
26. Lee, K.-H., Oh, S.-J.: Auto-correlation functions of Sato-Tate distributions and identities of symplectic characters. Adv. Math. **401**(4), 108309 (2022)
27. Macdonald, I.G.: Symmetric functions and Hall polynomials. Oxford University Press (1998)
28. Mehta, M.L., Normand, J.-M.: Moments of the characteristic polynomial in the three ensembles of random matrices. J. Phys. A **34**(22), 4627–4639 (2001)
29. Serre, J.-P.: Propriétés conjecturales des groupes de Galois motiviques et des représentations l-adiques, in Motives (Seattle, WA, 1991). Proc. Sympos. Pure Math. **55**, 377–400 (1994)
30. Taylor, R.: Automorphy for some l-adic lifts of automorphic mod l Galois representations. II. Publ. Math. Inst. Hautes Études Sci. **108**, 183–239 (2008)
31. Yang, M., Wybourne, B.G.: New S function series and non-compact Lie groups. J. Phys. A **19**(17), 3513–3515 (1986)