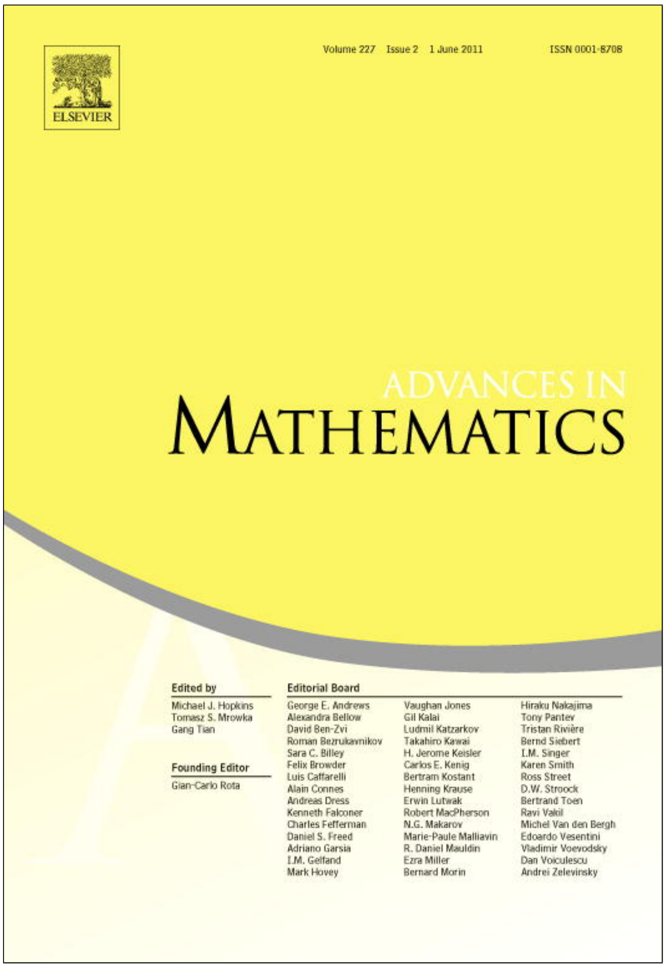
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Representation theory of p-adic groups and canonical bases

Henry H. Kim [a,1], Kyu-Hwan Lee [b,*]

[a] *Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada*
[b] *Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA*

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Abstract

In this paper, we interpret the Gindikin–Karpelevich formula and the Casselman–Shalika formula as sums over Kashiwara–Lusztig's canonical bases, generalizing the results of Bump and Nakasuji (2010) [7] to arbitrary split reductive groups. We also rewrite formulas for spherical vectors and zonal spherical functions in terms of canonical bases. In a subsequent paper Kim and Lee (preprint) [14], we will generalize these formulas to p-adic affine Kac–Moody groups.

Keywords: Canonical basis; Crystal basis; Gindikin–Karpelevich; Casselman–Shalika; Spherical function

0. Introduction

This paper was inspired by a paper by Bump and Nakasuji [7]. Their basic philosophy is that an integral over a maximal unipotent subgroup of p-adic group can sometimes be replaced by a sum over crystal bases defined by Kashiwara [13]. The same approach was made by McNamara in [19]. They demonstrated this for the Gindikin–Karpelevich formula and the Casselman–Shalika formula for GL_n. More precisely, let F be a p-adic field and N_- be the maximal unipotent subgroup of GL_n. Let χ be an unramified character of T, the maximal

* Corresponding author.
E-mail addresses: henrykim@math.toronto.edu (H.H. Kim), khlee@math.uconn.edu (K.-H. Lee).
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torus, and f^0 be the standard spherical vector corresponding to χ. Let $\mathbf{z}$ be the element of ${}^LT \subset GL_n(\mathbb{C})$, the L-group of GL_n, corresponding to χ by the Satake isomorphism. Then the Gindikin–Karpelevich formula for the longest Weyl group element can be written as

$$\int_{N_-(F)} f^0(n)\,dn = \prod_{\alpha\in\Phi^+} \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha} = \sum_{b\in\mathfrak{B}} G_\Omega(b) q^{\langle \mathrm{wt}(b),\rho\rangle} \mathbf{z}^{-\mathrm{wt}(b)}, \tag{0.1}$$

where $\mathfrak{B} = \mathfrak{B}(\infty)$ is the crystal basis for U^- (the negative part of the quantized enveloping algebra).

Let λ be a dominant integral weight and χ_λ be the irreducible character of $GL_n(\mathbb{C})$ with the highest weight λ. Then the Casselman–Shalika formula can be written as

$$\begin{aligned}\int_{N_-(F)} f^0(n)\psi_\lambda(n)\,dn &= \prod_{\alpha\in\Phi^+} \left(1-q^{-1}\mathbf{z}^\alpha\right)\chi_\lambda(\mathbf{z}) \\ &= \sum_{b\in\mathfrak{B}_{\lambda+\rho}\otimes\mathcal{T}_{-\lambda-\rho}} G_\Omega(b) q^{-\langle w_l(\mathrm{wt}(b)),\rho\rangle} \mathbf{z}^{w_l(\mathrm{wt}(b))},\end{aligned} \tag{0.2}$$

where w_l is the longest Weyl group element.

The definition of the coefficients $G_\Omega(b)$ in (0.1) and (0.2) is based on the "boxing rule" and "circling rule" in [4,5,7]. (See also [2,3,6].) We show that if we use canonical bases due to Lusztig [15] and tensor products of crystals, we obtain simple formulas for the coefficients in a uniform way. We can also generalize the above formulas. Namely, we prove, for any $w \in W$ and for any split reductive group,

$$\int_{N_w} f^0\left(w^{-1}n\right)dn = \prod_{\alpha\in\Phi(w)} \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha} = \sum_{b\in\mathbf{G}(w^{-1})} \left(1-q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)}, \tag{0.3}$$

$$\prod_{\alpha\in\Phi(w)} \left(1-q^{-1}\mathbf{z}^\alpha\right)^{-1} = \sum_{b\in\mathbf{G}(w^{-1})} q^{-\Sigma(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)},$$

$$\chi_\lambda(\mathbf{z}) \prod_{\alpha\in\Phi_+} \left(1-q^{-1}\mathbf{z}^{-\alpha}\right) = \mathbf{z}^{-\rho} \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b;q)\mathbf{z}^{\mathrm{wt}(b'\otimes b)}. \tag{0.4}$$

(See Propositions 1.4 and 2.1 for the notations.) Notice that in the Casselman–Shalika formula (0.4), we used crystal bases because they behave well with respect to the tensor product. Notice also that the left-hand side of (0.4) can be written as

$$(-t)^M \mathbf{z}^{-2\rho}\chi_\lambda(\mathbf{z}) \prod_{\alpha\in\Phi_+} \left(1-t^{-1}\mathbf{z}^\alpha\right),$$

where $M = |\Delta_+|$ and $t = q^{-1}$. Hence we obtain the product expansion of (0.2).

We first prove (0.3) by induction, and deduce (0.4) from (0.3) and the Weyl character formula. In the course of proof, we see that the Casselman–Shalika formula (0.4) can be considered as a q-deformation of the Weyl character formula.

K. Joshi and R. Raghunathan [12] constructed interesting infinite product identities for L-functions. As an application of our formulas, we write their identities in terms of canonical bases (see Remark 2.21).

In Section 3, we interpret some formulas in [9] such as for spherical vectors and zonal spherical functions due to Macdonald in terms of canonical bases. As a generalization of the Gindikin–Karpelevich formula, we write the action of intertwining operators on some Iwahori-fixed vectors in terms of canonical bases. In the subsequent paper [14], we will generalize all these formulas to p-adic affine Kac–Moody groups.

1. Gindikin–Karpelevich formula

Let $\mathfrak{g}$ be the finite-dimensional simple Lie algebra over $\mathbb{C}$. We have the set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, the set Φ of roots and the set Φ_+ of positive roots. Let W be the Weyl group of $\mathfrak{g}$ with generators $s_1, s_2, \ldots, s_r$, and let w_l be the longest element of W. We identify a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ for $w \in W$ with the k-tuple $\mathbf{i} = (i_1, i_2, \ldots, i_k)$. We will denote by $R(w)$ the set of all reduced expressions $\mathbf{i}$ for w.

Let U be the quantized enveloping algebra of $\mathfrak{g}$. Then U is a $\mathbb{Q}(v)$-algebra generated by the elements E_i, F_i, $K_i^{\pm 1}$, $i \in \{1, 2, \ldots, r\}$. Let U^+ be the subalgebra generated by the E_i's and let U^- be the subalgebra generated by the F_i's.

Let $T''_{i,-1}$ be the automorphism of U as in Section 37.1.3 of [17]. For $\mathbf{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{N}^k$ and $\mathbf{i} \in R(w)$, we let

$$F_{\mathbf{i}}^{\mathbf{c}} := F_{i_1}^{(c_1)} T''_{i_1,-1}\bigl(F_{i_2}^{(c_2)}\bigr) \cdots T''_{i_1,-1} T''_{i_2,-1} \cdots T''_{i_{k-1},-1}\bigl(F_{i_k}^{(c_k)}\bigr),$$

and define $G_{\mathbf{i}} = \{F_{\mathbf{i}}^{\mathbf{c}} \colon \mathbf{c} \in \mathbb{N}^k\}$, and denote by U_w^- the $\mathbb{Q}(v)$-span of $G_{\mathbf{i}}$. Let $^- : U \to U$ be the $\mathbb{Q}$-algebra automorphism of U taking E_i to E_i, F_i to F_i, K_i to K_i^{-1}, and v to v^{-1}.

Theorem 1.1. *(See [15,18].) Suppose that $\mathbf{i} \in R(w)$. The $\mathbb{Z}[v]$-span $\mathcal{L}_w$ of $G_{\mathbf{i}}$ is independent of $\mathbf{i}$. Let $\pi : \mathcal{L}_w \to \mathcal{L}_w / v\mathcal{L}_w$ be the natural projection. The image $\pi(G_{\mathbf{i}})$ is also independent of $\mathbf{i}$; we denote it by G_w. The restriction of π to $\mathcal{L}_w \cap \overline{\mathcal{L}_w}$ is an isomorphism of $\mathbb{Z}$-modules $\pi_1 : \mathcal{L}_w \cap \overline{\mathcal{L}_w} \to \mathcal{L}_w / v\mathcal{L}_w$, and $\mathbf{G}(w) = \pi_1^{-1}(G_w)$ is a $\mathbb{Q}(v)$-basis of U_w^-.*

When $w = w_l$, we obtain a $\mathbb{Q}(v)$-basis $\mathbf{G}_{w_l}$ of U^-, which is called the *canonical basis*. We will write $\mathbf{B} = \mathbf{G}_{w_l}$. We define a map $\phi_{\mathbf{i}} : \mathbf{G}(w) \to \mathbb{N}^k$ for $\mathbf{i} \in R(w)$ by setting $\phi_{\mathbf{i}}(b) = \mathbf{c}$, where $\mathbf{c} \in \mathbb{N}^k$ is given by

$$b \equiv F_{\mathbf{i}}^{\mathbf{c}} \mod v\mathcal{L}_w.$$

Then $\phi_{\mathbf{i}}$ is a bijection.

For $w \in W$, we set

$$\Phi(w) = \{\alpha \in \Phi_+ \mid w\alpha < 0\}.$$

If $\ell(ws_i) > \ell(w)$, we have

$$\Phi\bigl(s_i w^{-1}\bigr) = \Phi\bigl(w^{-1}\bigr) \cup \{w\alpha_i\}, \tag{1.2}$$

and if $\ell(s_i w) > \ell(w)$ then

$$\Phi\left(w^{-1}s_i\right) = s_i\left(\Phi\left(w^{-1}\right)\right) \cup \{\alpha_i\}. \tag{1.3}$$

For $\mathbf{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{N}^k$, we denote by $d(\mathbf{c})$ the number of nonzero c_i's.

Proposition 1.4. *For any* $\mathbf{i} \in R(w)$, $w \in W$, *we have*

$$\prod_{\alpha \in \Phi(w^{-1})} \frac{1 - q^{-1}\mathbf{z}^{\alpha}}{1 - \mathbf{z}^{\alpha}} = \sum_{b \in \mathbf{G}(w)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)}. \tag{1.5}$$

Proof. We will use induction on the length $\ell(w)$ of w. If $w = s_i$ for some i, then the identity (1.5) is easily verified. Assume that the identity (1.5) is true for $w = s_{i_1} \cdots s_{i_k} \in W$, and that $\ell(ws_i) = \ell(w) + 1$. We will write $\mathbf{i} = (i_1, \ldots, i_k)$ and $\mathbf{i}' = (i_1, \ldots, i_k, i)$. Using (1.2) and an induction argument, we obtain

$$\begin{aligned}
\prod_{\alpha \in \Phi(s_i w^{-1})} \frac{1 - q^{-1}\mathbf{z}^{\alpha}}{1 - \mathbf{z}^{\alpha}} &= \left(\prod_{\alpha \in \Phi(w^{-1})} \frac{1 - q^{-1}\mathbf{z}^{\alpha}}{1 - \mathbf{z}^{\alpha}} \right) \frac{1 - q^{-1}\mathbf{z}^{w\alpha_i}}{1 - \mathbf{z}^{w\alpha_i}} \\
&= \left(\sum_{b \in \mathbf{G}(w)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)} \right) \left(1 + \sum_{j \geqslant 1} \left(1 - q^{-1}\right) \mathbf{z}^{jw\alpha_i} \right) \\
&= \sum_{b \in \mathbf{G}(w)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)} \\
&\quad + \sum_{j \geqslant 1} \sum_{b \in \mathbf{G}(w)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))+1} \mathbf{z}^{-\mathrm{wt}(b)+jw\alpha_i}.
\end{aligned}$$

On the other hand, since $b' \in \mathbf{G}(ws_i)$ satisfies

$$b' \equiv bT''_{i_1,-1}T''_{i_2,-1} \cdots T''_{i_k,-1}\left(F_i^{(j)}\right) \quad \mathrm{mod}\ v\mathcal{L}_{ws_i}$$

for unique $b \in \mathbf{G}(w)$ and $j \geqslant 0$, we can write $\mathbf{G}(ws_i)$ as a disjoint union

$$\mathbf{G}(ws_i) = \bigcup_{j \geqslant 0} \left\{ b' \in \mathbf{G}(ws_i) \;\middle|\; \phi_{\mathbf{i}'}\left(b'\right) = (c_1, \ldots, c_k, j),\ c_i \in \mathbb{N} \right\}.$$

Now it is clear that

$$\begin{aligned}
&\sum_{b \in \mathbf{G}(ws_i)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}'}(b))} \mathbf{z}^{-\mathrm{wt}(b)} \\
&\quad = \sum_{b \in \mathbf{G}(w)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)} + \sum_{j \geqslant 1} \sum_{b \in \mathbf{G}(w)} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))+1} \mathbf{z}^{-\mathrm{wt}(b)+jw\alpha_i}.
\end{aligned}$$

This completes the proof. $\square$

Let $\widetilde{E}_i$ and $\widetilde{F}_i$ be the Kashiwara operators on U^- as defined in [13]. Let $\mathcal{A} \subset \mathbb{Q}(v)$ be the subring of elements regular at $v = 0$, and let $\mathcal{L}'$ be the $\mathcal{A}$-lattice spanned by the set S given by

$$S = \{\widetilde{F_{j_1}}\widetilde{F_{j_2}}\cdots\widetilde{F_{j_m}} \cdot 1 \in U^- \mid m \geqslant 0,\ j_k = 1, 2, \ldots, r\}.$$

Theorem 1.6. *(See [13].)*

(1) *Let $\pi' : \mathcal{L}' \to \mathcal{L}'/v\mathcal{L}'$ be the natural projection, and let $B' = \pi'(S)$. Then B' is a $\mathbb{Q}$-basis of $\mathcal{L}'/v\mathcal{L}'$, called the* crystal basis.
(2) *The operators $\widetilde{E}_i$ and $\widetilde{F}_i$ act on $\mathcal{L}'/v\mathcal{L}'$ for each $i = 1, 2, \ldots, r$. They satisfy*

$$\widetilde{E}_i(B') \subseteq B' \cup \{0\} \quad \text{and} \quad \widetilde{F}_i(B') \subseteq B'.$$

For $b, b' \in B'$ we have $\widetilde{F}_i b = b'$ if and only if $\widetilde{E}_i b' = b$.
(3) *For each $b \in B'$, there is a unique element $\tilde{b} \in \mathcal{L}' \cap \overline{\mathcal{L}'}$ such that $\pi'(\tilde{b}) = b$. The set of elements $\{\tilde{b}\colon b \in B'\}$ forms a basis of U^-, called the* global basis *of U^-.*

It was shown by Lusztig [16] that Kashiwara's global basis coincides with his canonical basis. There is a parametrization of $\mathbf{B}$ arising from Kashiwara's construction, and the parametrization again depends on $\mathbf{i} \in R(w_l)$. Let $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in R(w_l)$ and $b \in \mathbf{B}$. Let a_1 be maximal such that $\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \bmod v\mathcal{L}'$; let a_2 be maximal such that $\widetilde{E}_{i_2}^{a_2}\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \bmod v\mathcal{L}'$, and so on. That is, a_j is maximal such that $\widetilde{E}_{i_j}^{a_j}\widetilde{E}_{i_{j-1}}^{a_{j-1}}\cdots\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \bmod v\mathcal{L}'$. We define a map $\psi_{\mathbf{i}} : \mathbf{B} \to \mathbb{N}^k$ by $\psi_{\mathbf{i}}(b) = \mathbf{a}$, where $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$ is determined as above. We obtain from Theorem 1.6(2) that $b \equiv \widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k} \cdot 1 \bmod v\mathcal{L}'$. The map $\psi_{\mathbf{i}}$ is injective, and we will write $C_{\mathbf{i}} = \psi_{\mathbf{i}}(\mathbf{B})$. It is known that $C_{\mathbf{i}}$ is a cone. For $\mathbf{i}, \mathbf{j} \in R(w_l)$, we define the *Berenstein–Zelevinsky function* $\sigma_{\mathbf{i}}^{\mathbf{j}} : C_{\mathbf{i}} \to \mathbb{N}^k$ by

$$\sigma_{\mathbf{i}}^{\mathbf{j}} = \phi_{\mathbf{j}}\psi_{\mathbf{i}}^{-1}.$$

Descriptions of the cone $C_{\mathbf{i}}$ and the function $\sigma_{\mathbf{i}}^{\mathbf{j}}$ can be obtained from Section 3 of [1].

For $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$ and $\mathbf{i} = (i_1, \ldots, i_k) \in R(w_l)$, we define

$$\mathbf{z}^{\mathbf{a}} = \mathbf{z}^{a_1\alpha_{i_1} + a_2\alpha_{i_2} + \cdots + a_k\alpha_{i_k}}.$$

Corollary 1.7. *For any $\mathbf{i}, \mathbf{j} \in R(w_l)$, we have*

$$\prod_{\alpha\in\Phi_+} \frac{1 - q^{-1}\mathbf{z}^{\alpha}}{1 - \mathbf{z}^{\alpha}} = \sum_{\mathbf{a}\in C_{\mathbf{i}}} (1 - q^{-1})^{d(\sigma_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}))}\mathbf{z}^{\mathbf{a}}.$$

Proof. Considering the case $w = w_l$ in Proposition 1.4, we obtain the identity of the corollary from the definitions. $\square$

Example 1.8. Let $\mathfrak{g} = \mathfrak{sl}_4$. We choose $\mathbf{i} = (1, 2, 1, 3, 2, 1) \in R(w_l)$. Then the cone $C_{\mathbf{i}}$ is determined by

$$\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \in C_{\mathbf{i}} \quad \Longleftrightarrow \quad \begin{cases} a_1 \geqslant 0, \ a_2 \geqslant a_3 \geqslant 0, \\ a_4 \geqslant a_5 \geqslant a_6 \geqslant 0, \end{cases}$$

and the Berenstein–Zelevinsky function $\sigma_{\mathbf{i}}^{\mathbf{i}} : C_{\mathbf{i}} \to \mathbb{N}^6$ is given by

$$\sigma_{\mathbf{i}}^{\mathbf{i}}(\mathbf{a}) = \sigma_{\mathbf{i}}^{\mathbf{i}}(a_1, a_2, a_3, a_4, a_5, a_6) = (a_1, a_3, a_2 - a_3, a_6, a_4 - a_5, a_5 - a_6).$$

One can easily see that the "circling rule" for a_i's in [5,7] is the same as having coordinates of $\sigma_{\mathbf{i}}^{\mathbf{i}}(\mathbf{a})$ become zero. Then we obtain from Corollary 1.7 the result of Theorem 2 in Bump and Nakasuji's paper [7].

2. Casselman–Shalika formula

For $\mathbf{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{N}^k$, we set $\Sigma(\mathbf{c}) = c_1 + c_2 + \cdots + c_k$.

Proposition 2.1. *For any $\mathbf{i} \in R(w)$ for $w \in W$, we have*

$$\prod_{\alpha \in \Phi(w^{-1})} \left(1 - q^{-1}\mathbf{z}^{\alpha}\right)^{-1} = \sum_{b \in \mathbf{G}(w)} q^{-\Sigma(\phi_{\mathbf{i}}(b))}\mathbf{z}^{-\mathrm{wt}(b)}. \tag{2.2}$$

Proof. One can prove the identity using an induction argument as in the proof of Proposition 1.4. We omit the details. □

When w is a particular Weyl group element (see w_0 below), the left-hand side of the identity in Proposition 2.1 can be considered as a local L-function, and we have written the local L-function as a sum over a canonical basis. We make it more precise. We refer the reader to [10, Part II, Chapters 4 and 6], for the notations and relevant material. In particular, see Example 4.2 on p. 120 and Lemma 6.1 on p. 137.

Let α be a simple root and set $\theta = \Delta - \{\alpha\}$. Then θ determines a maximal parabolic subgroup $P = MN$, where M is the Levi subgroup. Let w_0 be the unique Weyl group element such that $w_0(\alpha) < 0$ and $w_0(\theta) \subset \Delta$. Let π_p be a spherical representation of $M(\mathbb{Q}_p)$ such that $\pi_p \hookrightarrow I(\chi_p)$, where χ_p is a quasi-character of the maximal split torus. Then for $s \in \mathbb{C}$, we have $I(s, \pi_p) \hookrightarrow I(\chi_p \otimes \exp(s\lambda, H_B(\cdot)))$, where λ is the fundamental weight corresponding to α. By Theorem 6.7 of [10], the identity (1.5) becomes

$$\prod_{i=1}^{m} \frac{L(is, \pi_p, r_i)}{L(1 + is, \pi_p, r_i)} = \sum_{b \in \mathbf{G}(w_0^{-1})} \left(1 - p^{-1}\right)^{d(\phi_{\mathbf{i}}(b))}\mathbf{z}^{-\mathrm{wt}(b)},$$

where $\mathbf{z}$ is the Satake parameter in ${}^L A \subset {}^L G$ corresponding to the character $\eta = \chi_p \otimes \exp(s\lambda, H_B(\cdot))$ of A such that $\eta \circ \beta^{\vee} = \beta^{\vee}(\mathbf{z})$ for any root β. Here we are considering $\beta^{\vee}$ on the right side as a root of ${}^L G$.

Suppose that π_p is generic, i.e., it has a Whittaker model. Then by Theorem 8.11 of [10], the identity (2.2) becomes

$$\prod_{i=1}^{m} L(1 + is, \pi_p, r_i) = \sum_{b \in \mathbf{G}(w_0^{-1})} p^{-\Sigma(\phi_{\mathbf{i}}(b))}\mathbf{z}^{-\mathrm{wt}(b)}.$$

A special case is when $G = GL_{n+1}$ and $M = GL_n \times GL_1$. Consider the cuspidal representation $\pi \otimes \xi^{-1}$ of $M/\mathbb{Q}$, where $\pi = \bigotimes \pi_p$ is a cuspidal representation of $GL_n/\mathbb{Q}$ and $\xi = \bigotimes \xi_p$ is a Dirichlet character modulo N. Suppose $\pi_p \hookrightarrow I(\mu_1, \dots, \mu_n)$. Then

$$I\left(s, \pi_p \otimes \xi_p^{-1}\right) \hookrightarrow I\left(\mu_1 \left| \, \right|^{\frac{s}{n+1}} \otimes \cdots \otimes \mu_n \left| \, \right|^{\frac{s}{n+1}} \otimes \xi_p^{-1} \left| \, \right|^{-\frac{ns}{n+1}}\right).$$

Hence $\mathbf{z} = (\mu_1(p) p^{-\frac{s}{n+1}}, \dots, \mu_n(p) p^{-\frac{s}{n+1}}, \xi_p^{-1}(p) p^{\frac{ns}{n+1}})$. Therefore we have

$$\frac{L(s, \pi_p \times \xi_p)}{L(1+s, \pi_p \times \xi_p)} = \sum_{b \in \mathbf{G}(w_0^{-1})} \left(1 - p^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)}$$

and

$$L(1+s, \pi_p \times \xi_p) = \sum_{b \in \mathbf{G}(w_0^{-1})} p^{-\Sigma(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)}.$$

Let P^+ be the set of dominant integral weights and $Q^+ = \sum_{i=1}^r (\mathbb{Z}_{\geqslant 0}) \alpha_i$ be the $\mathbb{Z}_{\geqslant 0}$-span of Δ. For $\lambda \in P^+$, let L_λ be the irreducible highest weight module of $\mathfrak{g}$ with the highest weight λ.

Definition 2.3. Let $\lambda \in P^+$ and $\mathbf{i} \in R(w_l)$. We define $H_\lambda(\cdot; q) : Q^+ \to \mathbb{Z}[q^{-1}]$ using the generating series

$$\sum_{\mu \in Q^+} H_\lambda(\mu; q) \mathbf{z}^{\lambda - \mu} = \sum_{w \in W} (-1)^{\ell(w)} \sum_{b \in \mathbf{B}} \left(1 - q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{w\lambda + \mathrm{wt}(b)},$$

and we write

$$\chi_q(L_\lambda) = \sum_{\mu \in Q^+} H_\lambda(\mu; q) \mathbf{z}^{\lambda - \mu}.$$

In what follows, we will see that $H_\lambda(\cdot; q)$ does not depend on the choice of $\mathbf{i}$.

We denote by $\chi(L_\lambda)$ the usual character of L_λ. (It was denoted by χ_λ in the Introduction.) By the Weyl character formula,

$$\frac{\sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Phi_+} (1 - \mathbf{z}^{-\alpha})} = \chi(L_\lambda).$$

In particular, if $\lambda = 0$, then

$$\sum_{w \in W} (-1)^{\ell(w)} \mathbf{z}^{w\rho} = \mathbf{z}^\rho \prod_{\alpha \in \Phi_+} \left(1 - \mathbf{z}^{-\alpha}\right). \tag{2.4}$$

By Proposition 1.4,

$$\sum_{b\in\mathbf{B}}\left(1-q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))}\mathbf{z}^{\mathrm{wt}(b)}=\prod_{\alpha\in\Phi_+}\left(\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right).$$

Thus we obtain

$$\begin{aligned}\chi_q(L_\rho)&=\left(\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w\rho}\right)\left(\sum_{b\in\mathbf{B}}\left(1-q^{-1}\right)^{d(\phi_{\mathbf{i}}(b))}\mathbf{z}^{\mathrm{wt}(b)}\right)\\&=\mathbf{z}^\rho\prod_{\alpha\in\Phi_+}\left(1-\mathbf{z}^{-\alpha}\right)\prod_{\alpha\in\Phi_+}\left(\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right)\\&=\mathbf{z}^\rho\prod_{\alpha\in\Phi_+}\left(1-q^{-1}\mathbf{z}^{-\alpha}\right).\end{aligned}$$

Therefore we have proved the following.

Proposition 2.5.

$$\chi_q(L_\rho)=\mathbf{z}^\rho\prod_{\alpha\in\Phi_+}\left(1-q^{-1}\mathbf{z}^{-\alpha}\right).\tag{2.6}$$

When $q=-1$ in (2.6), we have the following identity.

Lemma 2.7.

$$\chi_{-1}(L_\rho)=\mathbf{z}^\rho\prod_{\alpha\in\Phi_+}\left(1+\mathbf{z}^{-\alpha}\right)=\chi(L_\rho).$$

Proof. In (2.4), by replacing $\mathbf{z}^{\alpha_i}$ by $\mathbf{z}^{2\alpha_i}$ for each $i=1,\dots,r$, we have

$$\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{2w\rho}=\mathbf{z}^{2\rho}\prod_{\alpha\in\Phi_+}\left(1-\mathbf{z}^{-2\alpha}\right).$$

Since

$$\chi(L_\rho)=\frac{\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{2w\rho}}{\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w\rho}},$$

we obtain the result. $\square$

Remark 2.8. By Definition 2.3,

$$\chi_{-1}(L_\rho)=\sum_{\mu\in Q^+}H_\rho(\mu;-1)\mathbf{z}^{\rho-\mu}=\mathbf{z}^\rho\prod_{\alpha\in\Phi_+}\left(1+\mathbf{z}^{-\alpha}\right).$$

Therefore, if $H_\rho(\mu; -1) \neq 0$, $\rho - \mu$ must be a weight of L_ρ, and $H_\rho(\mu; -1)$ is the multiplicity of $\rho - \mu$ in L_ρ.

Now we have the following corollary which is a generalization of the second equality of the Casselman–Shalika formula (0.2).

Corollary 2.9.

$$\chi_q(L_{\lambda+\rho}) = \chi(L_\lambda)\chi_q(L_\rho). \tag{2.10}$$

Proof. By Definition 2.3 and Proposition 1.4,

$$\chi_q(L_{\lambda+\rho}) = \left(\sum_{w\in W}(-1)^{\ell(w)}\mathbf{z}^{w(\lambda+\rho)}\right)\prod_{\alpha\in\Phi_+}\left(\frac{1-q^{-1}\mathbf{z}^{-\alpha}}{1-\mathbf{z}^{-\alpha}}\right).$$

By the Weyl character formula and Proposition 2.5, the right-hand side is $\chi(L_\lambda)\chi_q(L_\rho)$. $\square$

Remark 2.11. When $q = 1$, we see that $\chi_1(L_{\lambda+\rho})\mathbf{z}^{-\rho}$ is the numerator of the Weyl character formula. Hence we can think of (2.10) as a q-deformation of the Weyl character formula. Since $\chi_\infty(L_\rho) = \mathbf{z}^\rho$, by setting $q = \infty$, we have

$$\chi_\infty(L_{\lambda+\rho}) = \mathbf{z}^\rho\chi(L_\lambda).$$

Hence by Definition 2.3,

$$\sum_{\mu\in Q_+} H_{\lambda+\rho}(\mu;\infty)\mathbf{z}^{\lambda-\mu} = \chi(L_\lambda).$$

Therefore, $H_{\lambda+\rho}(\mu;\infty)$ is the multiplicity of the weight $\lambda - \mu$ in L_λ.

By putting $q = -1$ in (2.10), and by Lemma 2.7,

Lemma 2.12.

$$\begin{aligned}\chi_{-1}(L_{\lambda+\rho}) &= \sum_{\mu\in Q^+} H_{\lambda+\rho}(\mu;-1)\mathbf{z}^{\lambda+\rho-\mu}\\ &= \chi(L_\lambda)\chi(L_\rho) = \chi(L_\lambda\otimes L_\rho).\end{aligned}$$

Hence, $H_{\lambda+\rho}(\mu;-1)$ is the multiplicity of the weight $\lambda+\rho-\mu$ in the tensor product $L_\lambda\otimes L_\rho$.

Before we further investigate the implication of the Casselman–Shalika formula (2.10), we need the following lemma.

Lemma 2.13. *Assume that $\lambda_1, \lambda_2 \in P_+$. Then the set of weights of $L_{\lambda_1}\otimes L_{\lambda_2}$ is the same as that of $L_{\lambda_1+\lambda_2}$.*

Proof. By Proposition 21.3, p. 114 in [11] μ is a weight of L_λ if and only if μ and all its Weyl conjugates are less than λ.

Now let η_1, η_2 be the weights of $L_{\lambda_1}, L_{\lambda_2}$, respectively. Then all weights of $L_{\lambda_1} \otimes L_{\lambda_2}$ are of the form $\eta_1 + \eta_2$ [11, p. 117, Exercise 7]. Hence it is enough to show that $\eta_1 + \eta_2$ is a weight of $L_{\lambda_1+\lambda_2}$. Then we are done, since it is clear that $\eta_1 + \eta_2$ and all its Weyl conjugates are less than $\lambda_1 + \lambda_2$. $\square$

Now we use crystal bases, namely, bases at $v = 0$, since they behave nicely under tensor products. Let $\mathfrak{B}_\lambda$ be the crystal basis associated to a dominant integral weight $\lambda \in P_+$. We choose $G_\rho(\cdot; q) : \mathfrak{B}_\rho \to \mathbb{Z}[q^{-1}]$ by assigning any element of $\mathbb{Z}[q^{-1}]$ to each $b \in \mathfrak{B}_\rho$ so that

$$H_\rho(\mu; q) = \sum_{\substack{b\in\mathfrak{B}_\rho \\ \mathrm{wt}(b)=\rho-\mu}} G_\rho(b; q). \tag{2.14}$$

By Remark 2.8, it is enough to consider $\mu \in Q_+$ such that $\rho - \mu$ is a weight of $b \in \mathfrak{B}_\rho$.

Using the function $G_\rho(\cdot; q)$, we can rewrite the Casselman–Shalika formula in Corollary 2.9 in a familiar form:

Corollary 2.15.

$$\begin{aligned}\sum_{\mu\in Q^+} H_{\lambda+\rho}(\mu; q)\mathbf{z}^{\lambda+\rho-\mu} &= \chi(L_\lambda)\mathbf{z}^\rho \prod_{\alpha\in\Phi_+}\left(1 - q^{-1}\mathbf{z}^{-\alpha}\right) \\ &= \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b; q)\mathbf{z}^{\mathrm{wt}(b'\otimes b)}. \end{aligned}\tag{2.16}$$

Proof. The first equality is obvious from Proposition 2.5 and Corollary 2.9. For the second equality, we obtain

$$\begin{aligned}\chi(L_\lambda)\mathbf{z}^\rho \prod_{\alpha\in\Phi_+}\left(1 - q^{-1}\mathbf{z}^{-\alpha}\right) &= \chi(L_\lambda)\chi_q(L_\rho) \\ &= \left(\sum_{b'\in\mathfrak{B}_\lambda} \mathbf{z}^{\mathrm{wt}(b')}\right)\left(\sum_{\mu\in Q^+} H_\rho(\mu; q)\mathbf{z}^{\rho-\mu}\right) \\ &= \left(\sum_{b'\in\mathfrak{B}_\lambda} \mathbf{z}^{\mathrm{wt}(b')}\right)\left(\sum_{b\in\mathfrak{B}_\rho} G_\rho(b; q)\mathbf{z}^{\mathrm{wt}(b)}\right) \\ &= \sum_{b'\otimes b\in\mathfrak{B}_\lambda\otimes\mathfrak{B}_\rho} G_\rho(b; q)\mathbf{z}^{\mathrm{wt}(b'\otimes b)}. \qquad \square\end{aligned}$$

The following proposition provides useful information on $H_{\lambda+\rho}(\mu; q) \in \mathbb{Z}[q^{-1}]$.

Proposition 2.17. *Assume that $\lambda \in P_+$. Then we have $H_{\lambda+\rho}(\mu; q)$ is a nonzero polynomial if and only if $\lambda + \rho - \mu$ is a weight of $L_{\lambda+\rho}$.*

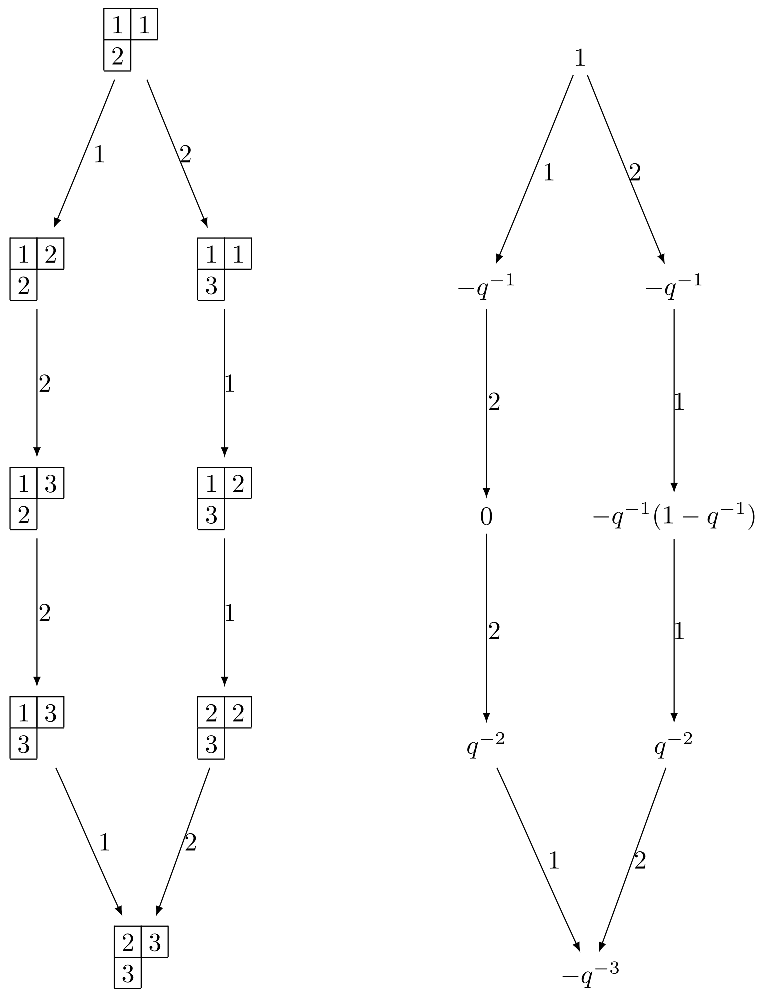
Proof. We obtain from (2.16) that if $H_{\lambda+\rho}(\mu; q) \neq 0$ then $\lambda + \rho - \mu$ is a weight of $L_\lambda \otimes L_\rho$. Then $\lambda + \rho - \mu$ is a weight of $L_{\lambda+\rho}$ by Lemma 2.13. Conversely, assume that $\lambda + \rho - \mu$ is a weight of $L_{\lambda+\rho}$, so a weight of $L_\lambda \otimes L_\rho$. By Lemma 2.12,

$$\sum_{\mu' \in Q^+} H_{\lambda+\rho}\big(\mu'; -1\big)\mathbf{z}^{\lambda+\rho-\mu'} = \chi(L_\lambda \otimes L_\rho).$$

Since $\lambda + \rho - \mu$ is a weight of $L_\lambda \otimes L_\rho$, the coefficient $H_{\lambda+\rho}(\mu; -1) \neq 0$. Then $H_{\lambda+\rho}(\mu; q)$ is a nonzero polynomial. $\square$

Remark 2.18. We have proved that $H_{\lambda+\rho}(\mu; q) = a_0 + a_1 q^{-1} + \cdots + a_k q^{-k}$, $a_i \in \mathbb{Z}$, and $H_{\lambda+\rho}(\mu; \infty) = a_0$ is the multiplicity of the weight $\lambda - \mu$ in L_λ, and $H_{\lambda+\rho}(\mu; -1)$ is the multiplicity of the weight $\lambda + \rho - \mu$ in the tensor product $L_\lambda \otimes L_\rho$. It would be interesting to study how $H_{\lambda+\rho}(\mu; q)$ is related to the Kazhdan–Lusztig polynomials. We will pursue this in the subsequent paper [14].

Example 2.19. We consider the case $\mathfrak{g} = \mathfrak{sl}_3$ and fix $\mathbf{i} = (1, 2, 1) \in R(w_l)$. Using the circling and boxing rules in [5,7], we define $G_\rho(b; q)$ for each $b \in \mathfrak{B}_\rho$. Comparing Corollary 2.9 and Theorem 1 in Bump and Nakasuji's paper [7], we see that the condition in (2.14) is satisfied.



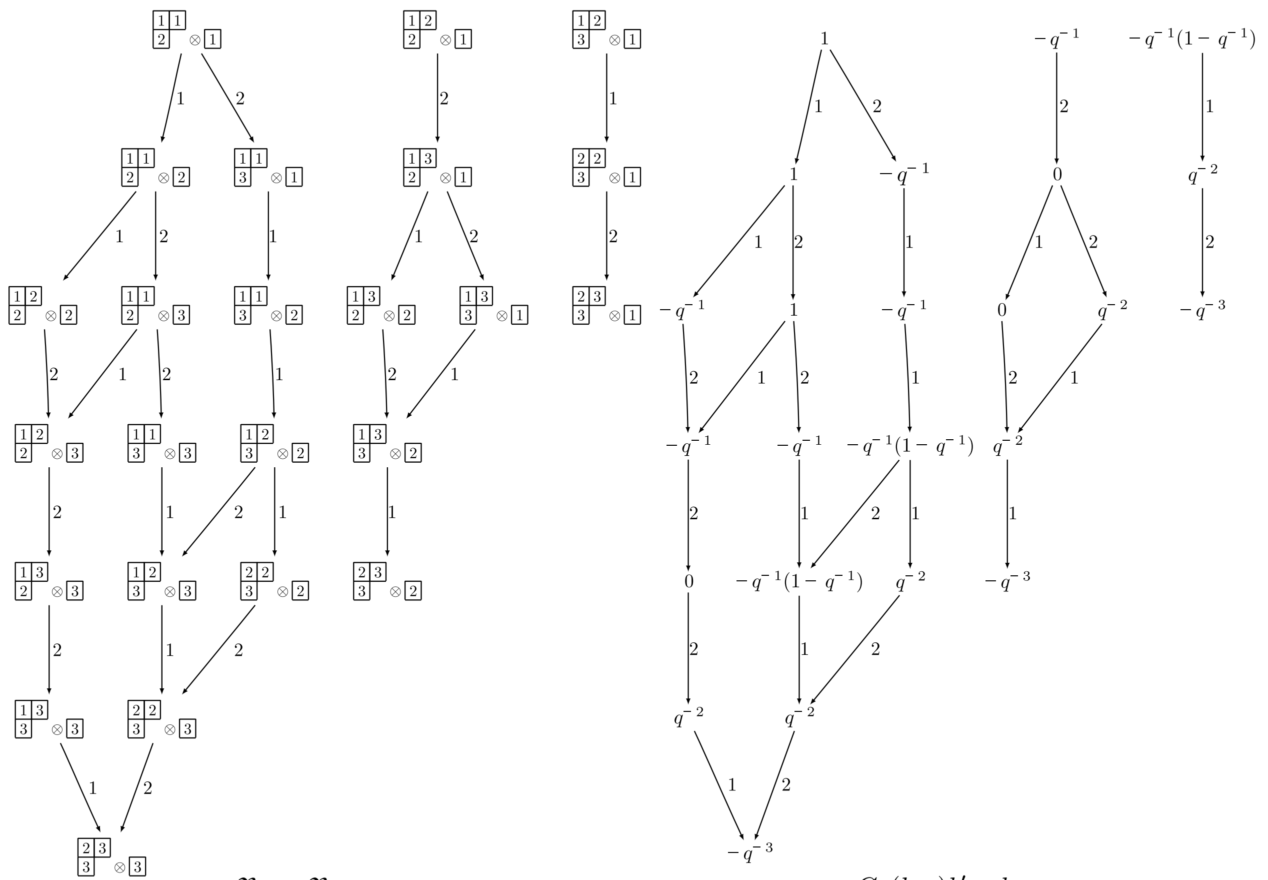
$\mathfrak{B}_\rho$ $\qquad$ G_ρ

We let $\lambda = \Lambda_1$ and consider $\mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho$. We present a crystal graph of $\mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho$ in the following figure. The tensor product should be read in the far-eastern reading in the figure. We put $G_\rho(b;q)$ for each $b' \otimes b \in \mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho$. We can calculate $H_{\lambda+\rho}(\mu)$ by taking the sum of $G_\rho(b;q)$ over the crystals $b' \otimes b$ with $\mathrm{wt}(b' \otimes b) = \lambda + \rho - \mu$, i.e.

$$H_{\lambda+\rho}(\mu;q) = \sum_{\substack{b' \otimes b \in \mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho \\ \mathrm{wt}(b' \otimes b) = \lambda+\rho-\mu}} G_\rho(b;q).$$

If we define $G_{\lambda+\rho}(b;q)$ for $b \in \mathfrak{B}_{\lambda+\rho}$ using the circling and boxing rules as in [5,7], it follows from Corollary 2.9 and Theorem 1 in [7] that

$$H_{\lambda+\rho}(\mu;q) = \sum_{\substack{b \in \mathfrak{B}_{\lambda+\rho} \\ \mathrm{wt}(b) = \lambda+\rho-\mu}} G_{\lambda+\rho}(b;q).$$



$\mathfrak{B}_\lambda \otimes \mathfrak{B}_\rho$ $\qquad$ $G_\rho(b;q)b' \otimes b$

Remark 2.20. Since we can define G_ρ in an arbitrary way under the condition (2.14) and still obtain $H_{\lambda+\rho}$, the circling and boxing rules in [5,7] seem to be very special.

Remark 2.21. K. Joshi and R. Raghunathan [12] have interesting infinite product identities for the L-functions. We can interpret their identities in terms of canonical bases. Let $\pi = \bigotimes \pi_p$ be a cuspidal representation of $GL_n/\mathbb{Q}$. Given $N \in \mathbb{N}$, let X_N be the set of all Dirichlet characters modulo N. Let p be a fixed prime and ξ_p be the local character at p associated to $\xi \in X_N$ for some N. Since $\xi(p) = 0$ if $p|N$, Proposition 2.5 of [12] states

$$\frac{L(s,\pi_p)}{L(s+1,\pi_p)} = \prod_{N=1}^{\infty} \prod_{\xi \in X_N} L(s+1, \pi_p \times \xi_p).$$

Then by using Propositions 1.3 and 2.1, we can write the above identity in terms of canonical bases and obtain

$$\sum_{b \in \mathbf{G}(w_0^{-1})} \left(1-p^{-1}\right)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)} = \prod_{N=1}^{\infty} \prod_{\xi \in X_N} \left(\sum_{b \in \mathbf{G}(w_0^{-1})} p^{-\Sigma(\phi_{\mathbf{i}}(b))} \mathbf{z}_{\xi}{}^{-\mathrm{wt}(b)} \right),$$

where $\pi_p \hookrightarrow I(\mu_1, \dots, \mu_n)$, and

$$\mathbf{z} = \left(\mu_1(p) p^{-\frac{s}{n+1}}, \dots, \mu_n(p) p^{-\frac{s}{n+1}}, p^{\frac{ns}{n+1}}\right),$$
$$\mathbf{z}_{\xi} = \left(\mu_1(p) p^{-\frac{s}{n+1}}, \dots, \mu_n(p) p^{-\frac{s}{n+1}}, \xi_p(p)^{-1} p^{\frac{ns}{n+1}}\right).$$

3. Spherical functions and generalization of the Gindikin–Karpelevich formula

Let us recall some notations and results from [9]. Let G be a split reductive group. Let $\mathcal{O}$ be the valuation ring of a p-adic field F with its maximal ideal $\mathcal{P}$ and $k = \mathcal{O}/\mathcal{P}$ be the residue field. We abuse notation and write $G = G(F)$. Let $K = G(\mathcal{O})$ and I be the Iwahori subgroup of K defined as the inverse image under the projection $G(\mathcal{O}) \to G(k)$.

We define the G-projection $\mathscr{P}_\chi$ from C_c^∞ onto $I(\chi)$ by

$$\mathscr{P}_\chi(f)(g) = \int_B \chi^{-1} \delta_B^{\frac{1}{2}}(b) f(bg)\, db.$$

For each $w \in W$, let $\phi_{w,\chi} = \mathscr{P}_\chi(1_{IwI})$, and let $\phi_{K,\chi} = \mathscr{P}_\chi(1_K)$. Here given a subset S of G, the notation 1_S denotes the characteristic function of S. We will sometimes omit the reference to χ. Then the functions ϕ_w, $w \in W$, form a basis of $I(\chi)^I$, and the function ϕ_K is a basis of the 1-dimensional space $I(\chi)^K$.

The intertwining operator $T_w : I(\chi) \to I(w\chi)$ satisfies

$$T_w(\phi_{K,\chi}) = c_w(\chi)\phi_{K,w\chi},$$

where we set

$$c_w(\chi) = \prod_{\alpha \in \Phi(w)} c_\alpha(\chi) \quad \text{and} \quad c_\alpha(\chi) = \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}.$$

We have another basis $\{f_w\}$ of $I(\chi)^I$ such that

$$T_w(f_x)(1) = \int\limits_{N_w} f_x\big(w^{-1}n\big)\,dn = \begin{cases} 1 & \text{if } x = w, \\ 0 & \text{if } x \neq w, \end{cases}$$

where we set $N_w = (wNw^{-1} \cap N)\backslash N$. Then we have, for the longest Weyl group element w_l,

$$f_{w_l} = \phi_{w_l},$$

and

$$\phi_K = \sum_{w \in W} \phi_w = \sum_{w \in W} c_w(\chi) f_w.$$

Using Proposition 1.4, we rewrite the above formula as follows.

Proposition 3.1.

$$\begin{aligned} \phi_K &= \sum_{w \in W} f_w \Bigg(\sum_{b \in \mathbf{G}(w^{-1})} \big(1 - q^{-1}\big)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}^{-\mathrm{wt}(b)} \Bigg) \\ &= \sum_{b \in \mathbf{B}} \mathbf{z}^{-\mathrm{wt}(b)} \Bigg(\sum_{\substack{w \in W \\ b \in \mathbf{G}(w^{-1})}} \big(1 - q^{-1}\big)^{d(\phi_{\mathbf{i}}(b))} f_w \Bigg). \end{aligned}$$

Now we study the effect of intertwining operators on Iwahori-fixed vectors. Casselman [9] proved that if $\ell(s_i w) > \ell(w)$ for some $i = 1, \ldots, r$, we have

$$T_{s_i}(\phi_{w,\chi}) = \big(c_{s_i}(\chi) - 1\big)\phi_{w,s_i\chi} + q^{-1}\phi_{s_i w, s_i\chi}, \tag{3.2}$$

$$T_{s_i}(\phi_{s_i w,\chi}) = \phi_{w,s_i\chi} + \big(c_{s_i}(\chi) - q^{-1}\big)\phi_{s_i w, s_i\chi}. \tag{3.3}$$

One can find the following results in Reeder's paper [20, p. 323].

Lemma 3.4.

(1) *If $T_{x^{-1}}(\phi_w)(1) \neq 0$, then $w \leqslant x$.*
(2) $T_{w^{-1}}(\phi_w)(1) = 1$.
(3) *If $w \leqslant s_i w$, then $T_{w^{-1}s_i}(\phi_w)(1) = c_{s_i}(\chi) - 1$.*

In the following proposition, we write the action of intertwining operators on some Iwahori-fixed vectors in terms of canonical bases. This can be regarded as a generalization of the Gindikin–Karpelevich formula. A relevant result and interesting conjectures can be found in [8].

Proposition 3.5. *Assume that $\ell(w'w) = \ell(w') + \ell(w)$ for $w, w' \in W$. We write $w' = s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$ in a reduced expression and suppose that*

$$w \leqslant s_{i_j} w, \quad \textit{for each } j = 1, \ldots, k, \tag{3.6}$$

$$s_{i_{j+1}} w \nleqslant s_{i_j} \cdots s_{i_1} w, \quad \textit{for each } j = 1, \ldots, k-1. \tag{3.7}$$

Then

$$T_{w^{-1}w'^{-1}}(\phi_w)(1) = \prod_{\alpha\in\Phi(w'^{-1})}\left(\frac{1-q^{-1}\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}}-1\right) = \left(1-q^{-1}\right)^{\ell(w')}\mathbf{z}^{\rho-w'^{-1}\rho}\sum_{b\in\mathbf{G}(w')}\mathbf{z}^{-\mathrm{wt}(b)}.$$

Proof. Using induction on the length of w', we first prove

$$T_{w^{-1}w'^{-1}}(\phi_w)(1) = \prod_{\alpha\in\Phi(w'^{-1})}\left(\frac{1-q^{-1}\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}}-1\right). \tag{3.8}$$

If $\ell(w')=1$, we obtain (3.8) from Lemma 3.4(3).

Now we consider the case $\ell(w')=k$. Applying Casselman's formulas (3.2), Lemma 3.4 and the assumptions (3.6), (3.7), we obtain

$$\begin{aligned}
T_{w^{-1}s_{i_1}\cdots s_{i_k}}(\phi_{w,\chi})(1) &= T_{w^{-1}s_{i_1}\cdots s_{i_{k-1}}}T_{s_{i_k}}(\phi_{w,\chi})(1)\\
&= T_{w^{-1}s_{i_1}\cdots s_{i_{k-1}}}\left(\left(c_{s_{i_k}}(\chi)-1\right)\phi_{w,s_{i_k}\chi}+q^{-1}\phi_{s_{i_k}w,s_{i_k}\chi}\right)(1)\\
&= \left(c_{s_{i_k}}(\chi)-1\right)T_{w^{-1}s_{i_1}\cdots s_{i_{k-1}}}\phi_{w,s_{i_k}\chi}(1)\\
&= \left(c_{s_{i_k}}(\chi)-1\right)\prod_{\alpha\in\Phi(s_{i_1}\cdots s_{i_{k-1}})}\left(\frac{1-q^{-1}\mathbf{z}^{s_{i_k}\alpha}}{1-\mathbf{z}^{s_{i_k}\alpha}}-1\right)\\
&= \prod_{\alpha\in\Phi(w'^{-1})}\left(\frac{1-q^{-1}\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}}-1\right).
\end{aligned}$$

The last two equalities come from induction and (1.3), respectively.

We see that

$$\prod_{\alpha\in\Phi(w'^{-1})}\left(\frac{1-q^{-1}\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}}-1\right) = \left(1-q^{-1}\right)^{\ell(w')}\prod_{\alpha\in\Phi(w'^{-1})}\frac{\mathbf{z}^{\alpha}}{1-\mathbf{z}^{\alpha}}. \tag{3.9}$$

Recall that we have

$$\sum_{\alpha\in\Phi(w'^{-1})}\alpha = \rho - w'^{-1}\rho.$$

Applying Proposition 2.1 to the right-hand side of (3.9) with $q=1$, we obtain the second equality of the proposition. $\square$

We give examples where the assumption (3.6) or (3.7) is not satisfied. We consider the root system of type C_2. Let $\sigma = s_{\alpha_1}$, $\tau = s_{\alpha_2}$ be the simple reflections with respect to the short and long simple roots, respectively. Then the Weyl group is given by

$$W = \{1, \sigma, \tau, \sigma\tau, \tau\sigma, \sigma\tau\sigma, \tau\sigma\tau, \sigma\tau\sigma\tau = \tau\sigma\tau\sigma\}.$$

We first consider $T_{\sigma\tau\sigma}(\phi_{\sigma,\chi})(1)$. In this case, $\sigma \nleqslant \sigma^2$. So (3.6) is not satisfied, and we obtain

$$\begin{aligned} T_{\sigma\tau\sigma}(\phi_{\sigma,\chi})(1) &= T_{\sigma\tau}\big(\phi_{1,\sigma\chi} + \big(c_\sigma(\chi) - q^{-1}\big)\phi_{\sigma,\sigma\chi}\big)(1) \\ &= \big(c_{\sigma\alpha_2}(\chi) - 1\big)\big(c_{\sigma\tau\alpha_1}(\chi) - 1\big) + \big(c_\sigma(\chi) - q^{-1}\big)\big(c_{\sigma\alpha_2}(\chi) - 1\big). \end{aligned}$$

Next we consider $T_{\sigma\tau\sigma}(\phi_1)(1)$. Then $\sigma \leqslant \tau\sigma$. So (3.7) is not satisfied, and we get

$$\begin{aligned} T_{\sigma\tau\sigma}(\phi_{1,\chi})(1) &= T_{\sigma\tau}\big(\big(c_\sigma(\chi) - 1\big)\phi_{1,\sigma\chi} + q^{-1}\phi_{\sigma,\sigma\chi}\big)(1) \\ &= \prod_{\alpha\in\Phi(\sigma\tau\sigma)} \big(c_\alpha(\chi) - 1\big) + q^{-1}\big(c_{\sigma\alpha_2}(\chi) - 1\big). \end{aligned}$$

In general, if the assumption (3.6) is satisfied and (3.7) is not, then

$$T_{w^{-1}w'^{-1}}(\phi_w)(1) = \prod_{\alpha\in\Phi(w'^{-1})} \big(c_\alpha(\chi) - 1\big) + \text{some lower terms}.$$

Let Γ_χ be the zonal spherical function corresponding to χ. It satisfies

$$\Gamma_\chi(1) = 1, \qquad \Gamma_\chi(k_1 g k_2) = \Gamma_\chi(g) \quad \text{for all } k_1, k_2 \in K \text{ and } g \in G.$$

We have the Cartan decomposition $G = KT^-K$, where T^- corresponds to the set of dominant integral coweights. Then Macdonald's identity shows that for $t \in T^-$, we have

$$\Gamma_\chi(t) = Q^{-1}\delta_B^{\frac{1}{2}}(t) \sum_{w\in W} c_{w_l}\big((w\chi)^{-1}\big)(w\chi)(t),$$

where $Q = \sum_{w\in W} q^{-l(w)}$. Let $\mathbf{z}_{(w\chi)^{-1}}$ be the Satake parameter corresponding to $(w\chi)^{-1}$. Then we see that ${\mathbf{z}_{(w\chi)^{-1}}}^{-\mathrm{wt}(b)} = {\mathbf{z}_{w\chi}}^{\mathrm{wt}(b)}$. So by Proposition 1.4, we have

$$c_{w_l}\big((w\chi)^{-1}\big) = \sum_{b\in\mathbf{B}} \big(1 - q^{-1}\big)^{d(\phi_{\mathbf{i}}(b))} \mathbf{z}_{w\chi}^{\mathrm{wt}(b)}.$$

Hence we obtain the following proposition.

Proposition 3.10.

$$\begin{aligned} \Gamma_\chi(t) &= Q^{-1}\delta_B^{\frac{1}{2}}(t) \sum_{w\in W} (w\chi)(t)\left(\sum_{b\in\mathbf{B}} \big(1 - q^{-1}\big)^{d(\phi_{\mathbf{i}}(b))} {\mathbf{z}_{w\chi}}^{\mathrm{wt}(b)}\right) \\ &= Q^{-1}\delta_B^{\frac{1}{2}}(t) \sum_{b\in\mathbf{B}} \big(1 - q^{-1}\big)^{d(\phi_{\mathbf{i}}(b))} \left(\sum_{w\in W} (w\chi)(t)\mathbf{z}_{w\chi}^{\mathrm{wt}(b)}\right). \end{aligned}$$

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