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Fully commutative elements of the complex reflection groups

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We extend the usual notion of fully commutative elements from the finite Coxeter groups to the complex reflection groups. We decompose the sets of fully commutative elements into natural subsets according to their combinatorial properties, and investigate the structure of these decompositions. As a consequence, we enumerate and describe the form of these elements for the complex reflection groups.

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1. Introduction

An element w of a finite Coxeter group is said to be *fully commutative* if any reduced word for w can be obtained from any other by interchanges of adjacent commuting generators. In [11], Stembridge classified the finite Coxeter groups that have finitely many fully commutative elements. His results completed the work of Fan [3] and Graham [5], who had obtained such a classification for the simply-laced types and had shown that the fully commutative elements parameterize natural bases for the corresponding quotients of Hecke algebras. In type A_n, the quotients are isomorphic to the Temperley–Lieb algebras (see [6]). Fan and Stembridge also enumerated the set of fully commutative elements. In particular, they showed the following.

Proposition 1.1 *([3,12]). Let C_n be the nth Catalan number, i.e. $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then the numbers of fully commutative elements in the Coxeter groups of types A_n, B_n and D_n are given as follows:*

$$\begin{cases} C_{n+1} & \text{if the type is } A_n, \\ (n+2)\,C_n - 1 & \text{if the type is } B_n, \\ \frac{n+3}{2}\,C_n - 1 & \text{if the type is } D_n. \end{cases}$$

This paper focuses on the complex reflection groups $G(d, r, n)$, where $d, r, n \in \mathbb{Z}_{>0}$ such that $r|d$. These groups are generated by complex reflections and have the Coxeter groups of types A_{n-1}, B_n and D_n as special cases. As the complex reflection groups can be presented by analogues of simple reflections and braid relations, one can attempt to generalize the notion of full commutativity to these groups. However, a direct generalization using the usual set of braid relations does not work even for $G(d, 1, n)$ if $d \geq 3$. A breakdown comes from the fact that some reduced words may not be connected to others strictly using braid relations.

In this paper, we overcome the difficulty and define fully commutative elements for $G(d, 1, n)$, by proving that a slightly extended set of braid relations connects all the reduced words for an element of $G(d, 1, n)$ (see Example 3.2). The next task is to describe and enumerate all the fully commutative elements, and we take the approach of the paper [4] where the first and third named authors studied fully commutative elements of the Coxeter group of type D_n.

More precisely, we decompose the set of fully commutative elements into natural subsets, called *collections*, according to their canonical words, and group them together into *packets* $\mathcal{P}(n, k)$, $0 \leq k \leq n$, so that all the collections in a packet have the same cardinality. We show that the number of fully commutative elements in any collection belonging to the packet $\mathcal{P}(n, k)$ is equal to the Catalan triangle number $C(n, k)$. Then the total number of fully commutative elements in $G(d, 1, n)$ can be written as

$$\sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = d(d-1)\mathfrak{F}_{n,n-2}(d) + (2d-1)C_n - (d-1), \tag{1.1}$$

where $|\mathcal{P}(n,k)|$ is the number of collections in the (n,k)-packet and $\mathfrak{F}_{n,k}(x)$ is the Catalan triangle polynomial defined by

$$\mathfrak{F}_{n,k}(x) = \sum_{s=0}^{k} C(n,s)x^{k-s}. \tag{1.2}$$

When $d = 2$, the group $G(2,1,n)$ is isomorphic to the Coxeter group of type B_n, and our definition of fully commutative elements coincides with the usual definition for Coxeter groups, and we recover the known number $(n+2)\,C_n - 1$ from (1.1).

Our method exploits combinatorics of canonical words and establishes bijections among collections. In particular, we realize the Catalan triangle (Table 4.2) using collections of fully commutative elements.

For the group $G(d,r,n)$, $r > 1$, we fix an embedding into $G(d,1,n)$ and define $w \in G(d,r,n)$ to be fully commutative if its image under the embedding is fully commutative in $G(d,1,n)$. The main benefit of this definition is that the decomposition into collections and packets still works without any complications, and we obtain complete description and enumeration of fully commutative elements for all $G(d,r,n)$. On the other hand, a drawback of this definition is that some fully commutative elements in the Coxeter group of type D_n or $G(2,2,n)$ are not fully commutative in $G(2,1,n)$ after being embedded. That is, the usual definition of full commutativity for D_n is not compatible with the new definition.

Though it is not clear at the present, an intrinsic definition of full commutativity for $G(d,r,n)$, $r > 1$, which does not use an embedding, may be found. For such a definition, precise information about a complete set (or Gröbner–Shirshov basis) of relations would be very helpful. We leave it as a future direction.

The organization of this paper is as follows. In Section 2 we determine canonical words for the elements of the complex reflection groups. In the next section, we define fully commutative elements. Section 4 is devoted to a study of decomposition of the set of fully commutative elements into collections and packets for $G(d,1,n)$. In Section 5, we consider the packets of $G(d,r,n)$. The next section provides some examples, and the final section is an appendix with the list of reduced words for $G(3,3,3)$.

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2. Canonical forms

2.1. Complex reflection groups

For positive integers d and n, let $G(d, 1, n)$ be the finite complex reflection group generated by the elements $s_1, s_2, \cdots, s_n$ with defining relations:

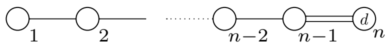
$$s_n^d = s_i^2 = 1 \qquad \text{for } 1 \le i \le n-1, \tag{2.1a}$$

$$s_i s_j = s_j s_i \qquad \text{for } j+1 < i \le n, \tag{2.1b}$$

$$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i \qquad \text{for } 1 \le i \le n-2, \tag{2.1c}$$

$$s_n s_{n-1} s_n s_{n-1} = s_{n-1} s_n s_{n-1} s_n. \tag{2.1d}$$

The group $G(d, 1, n)$ is isomorphic to the wreath product of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ and the symmetric group S_n. The corresponding diagram is given by



For each $d \ge 2$ and $n \ge 3$, let $G(d, d, n)$ be the complex reflection group generated by the elements $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_n$ with defining relations:

$$\tilde{s}_i^2 = 1 = (\tilde{s}_n \tilde{s}_{n-1})^d \qquad \text{for } 1 \le i \le n, \tag{2.2a}$$

$$\tilde{s}_i \tilde{s}_j = \tilde{s}_j \tilde{s}_i \qquad \text{for } j+1 < i \le n-1, \tag{2.2b}$$

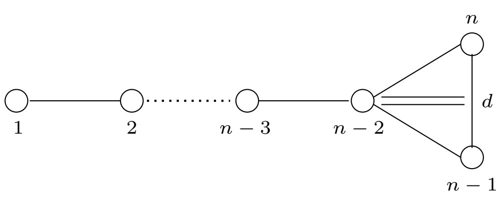
$$\tilde{s}_n \tilde{s}_j = \tilde{s}_j \tilde{s}_n \qquad \text{for } j \le n-3, \tag{2.2c}$$

$$\tilde{s}_{i+1} \tilde{s}_i \tilde{s}_{i+1} = \tilde{s}_i \tilde{s}_{i+1} \tilde{s}_i \qquad \text{for } 1 \le i \le n-2, \tag{2.2d}$$

$$\tilde{s}_n \tilde{s}_{n-2} \tilde{s}_n = \tilde{s}_{n-2} \tilde{s}_n \tilde{s}_{n-2}, \tag{2.2e}$$

$$(\tilde{s}_n \tilde{s}_{n-1} \tilde{s}_{n-2})^2 = (\tilde{s}_{n-2} \tilde{s}_n \tilde{s}_{n-1})^2. \tag{2.2f}$$

The corresponding diagram is the following.



Note that the complex reflection groups $G(2, 1, n)$ and $G(2, 2, n)$ are the finite Coxeter groups of types B_n and D_n, respectively.

For $r \mid d$ and $e = d/r$, let $G(d, r, n)$ be the complex reflection group generated by the elements $\tilde{s}_1, \tilde{s}_2 \cdots, \tilde{s}_n, \tilde{s}$ with defining relations:

$$\tilde{s}^e = \tilde{s}_i^2 = 1 \qquad \text{for } 1 \le i \le n, \tag{2.3a}$$
$$\tilde{s}_i\tilde{s}_j = \tilde{s}_j\tilde{s}_i \qquad \text{for } j+1 < i \le n-1, \tag{2.3b}$$
$$\tilde{s}\tilde{s}_j = \tilde{s}_j\tilde{s} \qquad \text{for } j \le n-2, \tag{2.3c}$$
$$\tilde{s}_n\tilde{s}_j = \tilde{s}_j\tilde{s}_n \qquad \text{for } j \le n-3, \tag{2.3d}$$
$$\tilde{s}_{i+1}\tilde{s}_i\tilde{s}_{i+1} = \tilde{s}_i\tilde{s}_{i+1}\tilde{s}_i \qquad \text{for } 1 \le i \le n-2, \tag{2.3e}$$
$$\tilde{s}\tilde{s}_n\tilde{s}_{n-1} = \tilde{s}_n\tilde{s}_{n-1}\tilde{s} \tag{2.3f}$$
$$\tilde{s}_n\tilde{s}_{n-2}\tilde{s}_n = \tilde{s}_{n-2}\tilde{s}_n\tilde{s}_{n-2}, \tag{2.3g}$$
$$(\tilde{s}_n\tilde{s}_{n-1}\tilde{s}_{n-2})^2 = (\tilde{s}_{n-2}\tilde{s}_n\tilde{s}_{n-1})^2 \tag{2.3h}$$
$$\tilde{s}\tilde{s}_n(\tilde{s}_{n-1}\tilde{s}_n)^{r-1} = \tilde{s}_{n-1}\tilde{s}. \tag{2.3i}$$

2.2. Canonical reduced words for $G(d,1,n)$

Every element $s_{i_1}\cdots s_{i_r} \in G(d,1,n)$ corresponds to the word $[i_1,\dots,i_r]$ in the alphabet $I := \{1,2,\dots,n\}$. For $1 \le i,j \le n$, we define the words $s_{i,j}$ by:

$$s_{i,j} = \begin{cases} [i,i-1,\dots,j] & \text{if } i > j, \\ [i] & \text{if } i = j, \\ [\,] & \text{if } i < j, \end{cases}$$

where $[\,]$ denotes the empty word that corresponds to the identity element of $G(d,1,n)$. We will often write $s_{i,i} = s_i$. We also define

$$s_{n,j}^{(k)} = s_n^k s_{n-1,j} \qquad \text{for } k \ge 1.$$

The following lemmas are useful to obtain a canonical form of the elements of $G(d,1,n)$.

Lemma 2.1. *The following relations hold in $G(d,1,n)$:*

$$s_{i,j}s_i = s_{i-1}s_{i,j} \qquad \textit{for } j < i \le n-1, \tag{2.4a}$$
$$s_{n,n-1}^{(k_1)}\, s_{n,n-1}^{(k_2)} = s_{n-1}s_{n,n-1}^{(k_2)}s_n^{k_1} \qquad \textit{for } k_1,k_2 \ge 1. \tag{2.4b}$$

Proof. We use downward induction and the commutative relation (2.1b) to establish (2.4a). When $j = i-1$, (2.4a) is nothing but (2.1c). By induction we have

$$s_{i,j-1}s_i = s_{i,j}s_{j-1}s_i = s_{i,j}s_is_{j-1} = s_{i-1}s_{i,j}s_{j-1} = s_{i-1}s_{i,j-1}.$$

To prove the relations (2.4b), we first establish

$$s_{n,n-1}\, s_{n,n-1}^{(k)} = s_{n-1}s_{n,n-1}^{(k)}s_n \qquad \text{for } k \ge 1. \tag{2.5}$$

When $k = 1$, (2.5) is just the defining relation (2.1d). For $k \geq 2$, we use induction on k and we obtain

$$\begin{aligned} s_{n,n-1}\, s^{(k)}_{n,n-1} &= s_{n,n-1}\, s^{(k-1)}_{n,n-1} s_{n-1} s_n s_{n-1} && (\text{using } s^2_{n-1} = 1) \\ &= s_{n-1} s^{(k-1)}_{n,n-1} s_n s_{n-1} s_n s_{n-1} && (\text{induction}) \\ &= s_{n-1} s^{(k-1)}_{n,n-1} s_{n-1} s_n s_{n-1} s_n && (\text{using (2.1d)}) \\ &= s_{n-1} s^{(k)}_{n,n-1} s_n && (\text{using } s^2_{n-1} = 1). \end{aligned}$$

We now prove (2.4b) by using induction to k_1. The case $k_1 = 1$ is obtained above, and when $k_1 \geq 2$, we see

$$\begin{aligned} s^{(k_1)}_{n,n-1}\, s^{(k_2)}_{n,n-1} &= s_n s^{(k_1-1)}_{n,n-1}\, s^{(k_2)}_{n,n-1} = s_{n,n-1} s^{(k_2)}_{n,n-1}\, s^{k_1-1}_{n} && (\text{induction}) \\ &= s_{n-1} s^{(k_2)}_{n,n-1} s^{k_1}_{n} && (\text{relation (2.5)}). \quad \square \end{aligned}$$

Lemma 2.2. *The following relations hold in $G(d, 1, n)$:*

$$s^{(k_1)}_{n,j}\, s^{(k_2)}_{n,j} = s_{n-1} s^{(k_2)}_{n,j}\, s^{(k_1)}_{n,j+1} \qquad \textit{for } j \leq n-1 \textit{ and } k_1, k_2 \geq 1. \tag{2.6}$$

Moreover, these relations are derived from (2.1b), (2.1c) *and* (2.4b).

Proof. We use downward induction on j. For $j = n - 1$ the result follows immediately from (2.4b). For $j < n-1$ we notice that $s^{(k)}_{n,j} = s^{(k)}_{n,j+1} s_j = s^{(k)}_{n,j+2} s_{j+1} s_j$, for every $k \geq 1$. Therefore,

$$\begin{aligned} s^{(k_1)}_{n,j}\, s^{(k_2)}_{n,j} &= s^{(k_1)}_{n,j+1} s_j s^{(k_2)}_{n,j+2} s_{j+1} s_j \\ &= s^{(k_1)}_{n,j+1} s^{(k_2)}_{n,j+2} s_j s_{j+1} s_j && (\text{relation (2.1b)}) \\ &= s^{(k_1)}_{n,j+1} s^{(k_2)}_{n,j+2} s_{j+1} s_j s_{j+1} && (\text{relation (2.1c)}) \\ &= s^{(k_1)}_{n,j+1} s^{(k_2)}_{n,j+1} s_j s_{j+1} = s_{n-1} s^{(k_2)}_{n,j+1} s^{(k_1)}_{n,j+2} s_j s_{j+1} && (\text{induction}) \\ &= s_{n-1} s^{(k_2)}_{n,j}\, s^{(k_1)}_{n,j+1} && (\text{relation (2.1b)}). \quad \square \end{aligned}$$

Let $\mathcal{R}$ be the following set of relations:

$$s^d_n = s^2_i = 1 \qquad \text{for } 1 \leq i \leq n-1, \tag{2.7a}$$

$$s_i s_j = s_j s_i \qquad \text{for } j+1 < i \leq n, \tag{2.7b}$$

$$s_{i,j} s_i = s_{i-1} s_{i,j} \qquad \text{for } j < i \leq n-1, \tag{2.7c}$$

$$s^{(k_1)}_{n,j}\, s^{(k_2)}_{n,j} = s_{n-1} s^{(k_2)}_{n,j}\, s^{(k_1)}_{n,j+1} \qquad \text{for } j \leq n-1 \text{ and } k_1, k_2 \geq 1. \tag{2.7d}$$

Proposition 2.3. *Using only the relations in $\mathcal{R}$, any element of the group $G(d,1,n)$ can be uniquely written in the following reduced form*

$$s_{1,i_1}s_{2,i_2}\cdots s_{n-1,i_{n-1}}s^{(k_1)}_{n,j_1}s^{(k_2)}_{n,j_2}\cdots s^{(k_\ell)}_{n,j_\ell} \tag{2.8}$$

where $1 \le i_p \le p+1$ for $1 \le p \le n-1$, and $1 \le j_1 < j_2 < \cdots < j_\ell \le n$ for $\ell \ge 0$, and $1 \le k_p \le d-1$ for $1 \le p \le \ell$.

Proof. Consider $w \in G(d,1,n)$ and an expression of w written in generators. Let ℓ be the number of occurrences of s_n^k in the expression of w for various k's, where k is maximal for each occurrence, i.e., if $w = \cdots s_i s_n^k s_j \cdots$, then $i \neq n$ and $j \neq n$. If $\ell = 0$ then w is an element of the subgroup of type A_{n-1} and it is well known that one can use only the relations (without s_n) in $\mathcal{R}$ to obtain the reduced form (2.8) (see, for example, [1]).

Assume that $\ell > 0$. Then we can write

$$w = w_1 s_n^k s_{1,p_1}s_{2,p_2}\cdots s_{n-1,p_{n-1}},$$

where s_n^k is the last occurrence of a power of s_n in the expression of w. By the commutativity relation (2.7b), we have

$$w = w_1 s_{1,p_1}s_{2,p_2}\cdots s_{n-2,p_{n-2}}s_n^k s_{n-1,p_{n-1}} = w_2\, s^{(k)}_{n,p_{n-1}},$$

where we set $w_2 = w_1 s_{1,p_1}s_{2,p_2}\cdots s_{n-2,p_{n-2}}$. By induction, the element w_2 can be written in the form (2.8), and we have

$$w = s_{1,i_1}s_{2,i_2}\cdots s_{n-1,i_{n-1}}s^{(k_1)}_{n,j_1}s^{(k_2)}_{n,j_2}\cdots s^{(k_{\ell-1})}_{n,j_{\ell-1}}s^{(k)}_{n,p_{n-1}}.$$

If $j_{\ell-1} < p_{n-1}$ or $j_{\ell-1} = n$ then we are done. If $j_{\ell-1} \ge p_{n-1}$ and $j_{\ell-1} < n$ then we use the relations (2.7b) and (2.7d) to obtain

$$\begin{aligned} s^{(k_{\ell-1})}_{n,j_{\ell-1}}s^{(k)}_{n,p_{n-1}} &= s^{(k_{\ell-1})}_{n,j_{\ell-1}}s^{(k)}_{n,j_{\ell-1}}s_{j_{\ell-1}-1,p_{n-1}} \\ &= s_{n-1}s^{(k)}_{n,j_{\ell-1}}s^{(k_{\ell-1})}_{n,j_{\ell-1}+1}s_{j_{\ell-1}-1,p_{n-1}} = s_{n-1}s^{(k)}_{n,p_{n-1}}s^{(k_{\ell-1})}_{n,j_{\ell-1}+1}. \end{aligned} \tag{2.9}$$

Using (2.9), we rewrite w and apply the induction hypothesis again. We repeat this process until we get the canonical form (2.8) in a finite number of steps.

Now we claim that the number of the canonical words is exactly $n! \cdot d^n$. Indeed, since each i_p runs over the set $\{1, \cdots, p+1\}$, there are $2 \cdot 3 \cdots n = n!$ choices for the part $s_{1,i_1}s_{2,i_2}\cdots s_{n-1,i_{n-1}}$. Now for the part $s^{(k_1)}_{n,j_1}s^{(k_2)}_{n,j_2}\cdots s^{(k_\ell)}_{n,j_\ell}$ with the conditions $1 \le j_1 < j_2 < \cdots < j_\ell \le n$ ($\ell \ge 0$) and $1 \le k_p \le d-1$, we just need to consider the number of forms $s^{(k_1)}_{n,1}s^{(k_2)}_{n,2}\cdots s^{(k_\ell)}_{n,n}$ with each k_p running over the set $\{0, 1, \cdots, d-1\}$, setting $s^{(0)}_{n,j_p} = 1$ for convenience. In this way, this part has d^n elements. Thus altogether, we have $n! \cdot d^n$ canonical words as claimed.

We recall that $n! \cdot d^n$ is the order of $G(d, 1, n)$. Thus we have shown that every element of $G(d, 1, n)$ is uniquely written in the canonical form (2.8). □

Remark 2.4. Proposition 2.3 shows that $\mathcal{R}$ is a Gröbner–Shirshov basis for the group $G(d, 1, n)$. For details about Gröbner–Shirshov bases see [1]. For the group $G(2, 1, n)$, which is the Coxeter group of type B_n, the canonical form (2.8) is obtained by Bokut and Shiao [1, Lemma 5.2]. A different canonical form for the group $G(d, 1, n)$ can be found in [2,7,10].

Definition 2.5. The set of canonical words in (2.8) for $G(d, 1, n)$ will be denoted by $\mathcal{W}(d, 1, n)$. The left factor $s_{1,i_1} s_{2,i_2} \cdots s_{n-1,i_{n-1}}$ of a canonical word will be called the *prefix*, and similarly the right factor $s^{(k_1)}_{n,j_1} s^{(k_2)}_{n,j_2} \cdots s^{(k_\ell)}_{n,j_\ell}$ will be called the *suffix* of the reduced word. Given a reduced word $\boldsymbol{w}$ in the canonical form, we will denote by $\boldsymbol{w}_0$ and $\boldsymbol{w}'$ the prefix and the suffix of $\boldsymbol{w}$, respectively. We write $\boldsymbol{w} = \boldsymbol{w}_0 \boldsymbol{w}'$.

2.3. Canonical reduced words for $G(d, d, n)$

For $1 \le i \le n-1$, we define the words $\tilde{s}_{i,j}$ in the same way as with $s_{i,j}$ using the generators $\tilde{s}_i$. When $i = n$, we define

$$\tilde{s}_{n,j} = \begin{cases} [n, n-2, \ldots, j] & \text{if } j \le n-2, \\ [n] & \text{if } j \in \{n, n-1\}, \\ [\,] & \text{if } j > n. \end{cases} \tag{2.10}$$

The group $G(d, d, n)$ can be embedded into $G(d, 1, n)$ as a subgroup of index d. Indeed, we define $\iota : G(d, d, n) \to G(d, 1, n)$ by

$$\iota(\tilde{s}_n) = s_n^{d-1} s_{n-1} s_n \quad \text{and} \quad \iota(\tilde{s}_i) = s_i \text{ for } 1 \le i \le n-1.$$

Then one can check that ι is a well-defined group homomorphism. Furthermore, we have the following lemma.

Lemma 2.6. *The homomorphism ι is an embedding and its image consists of the elements whose canonical words are in the set*

$$\{s_{1,i_1} s_{2,i_2} \cdots s_{n-1,i_{n-1}} s^{(k_1)}_{n,j_1} s^{(k_2)}_{n,j_2} \cdots s^{(k_\ell)}_{n,j_\ell} \in \mathcal{W}(d, 1, n) : k_1 + k_2 + \cdots + k_\ell \equiv 0 \, (mod\ d)\}.$$

This set of reduced words will be denoted by $\mathcal{W}(d, d, n)$.

Proof. Since $\iota(\tilde{s}_n) = s^{(d-1)}_{n,n-1} s_n$, it is clear from (2.7d) that the elements in the image of ι have reduced forms in $\mathcal{W}(d, d, n)$. We have that $\mathcal{W}(d, d, n)$ has $n! \cdot d^{n-1}$ elements and, since the order of $G(d, d, n)$ is $n! \cdot d^{n-1}$, ι is an embedding. □

The preimage of an element with a reduced word in $\mathcal{W}(d,d,n)$ can be found in the following way. We note that

$$\iota(\tilde{s}_n\tilde{s}_{n-1})^k = s_{n-1}s^{(k)}_{n,n-1}s_n^{d-k}$$

and

$$\begin{aligned} s^{(k_1)}_{n,j_1}s^{(k_2)}_{n,j_2} &= s_{n-1}s_{n-1}s^{(k_1)}_{n,n-1}s_n^{d-k_1}s_n^{k_1}s_{n-2,j_1}s^{(k_2)}_{n,j_2} \\ &= \iota(\tilde{s}_{n-1})\,\iota(\tilde{s}_n\tilde{s}_{n-1})^{k_1}\,\iota(\tilde{s}_{n-2,j_1})\,s^{(k_1+k_2)}_{n,j_2} \\ &= \iota(\tilde{s}_{n-1}\tilde{s}_n)^{k_1}\,\iota(\tilde{s}_{n-1,j_1})\,s^{(k_1+k_2)}_{n,j_2}. \end{aligned} \tag{2.11}$$

Using (2.11) repeatedly, we can write

$$s_{1,i_1}s_{2,i_2}\cdots s_{n-1,i_{n-1}}s^{(k_1)}_{n,j_1}s^{(k_2)}_{n,j_2}\cdots s^{(k_\ell)}_{n,j_\ell} = \iota(\tilde{w})s^{(k_1+k_2+\cdots+k_\ell)}_{n,j_\ell} = \iota(\tilde{w}\tilde{s}_{n-1,j_\ell})$$

for some $\tilde{w} \in G(d,d,n)$ with the condition $k_1+k_2+\cdots+k_\ell \equiv 0 \pmod d$.

From now on, the group $G(d,d,n)$ will be identified with the image of ι and the set $\mathcal{W}(d,d,n)$ will be the set of *canonical words* for $G(d,d,n)$. As in the case of $G(d,1,n)$, we write $\boldsymbol{w} = \boldsymbol{w}_0\boldsymbol{w}' \in \mathcal{W}(d,d,n)$ as a product of the prefix $\boldsymbol{w}_0$ and the suffix $\boldsymbol{w}'$.

Remark 2.7. One can try to obtain canonical words for $G(d,d,n)$ without using an embedding into $G(d,1,n)$. Indeed, we obtain a set of *reduced* words for $G(3,3,3)$ in Appendix. However, we find that the set of reduced words is not compatible with the packet decomposition defined in Section 5.

2.4. Canonical reduced words for $G(d,r,n)$

The group $G(d,r,n)$ can be embedded into $G(d,1,n)$ as a subgroup of index r. Indeed, we define $\tau : G(d,r,n) \to G(d,1,n)$ by

$$\tau(\tilde{s}) = s_n^r, \quad \tau(\tilde{s}_n) = s_n^{d-1}s_{n-1}s_n \quad \text{and} \quad \tau(\tilde{s}_i) = s_i \text{ for } 1 \le i \le n-1.$$

Then one can check that τ is a well-defined group homomorphism.

Lemma 2.8. *The homomorphism τ is an embedding and its image consists of the elements whose canonical words are in the set*

$$\begin{aligned} \mathcal{W}(d,r,n) := \{ & s_{1,i_1}s_{2,i_2}\cdots s_{n-1,i_{n-1}}s^{(k_1)}_{n,j_1}s^{(k_2)}_{n,j_2}\cdots s^{(k_\ell)}_{n,j_\ell} \\ & \in \mathcal{W}(d,1,n) : k_1+k_2+\cdots+k_\ell \equiv 0\,(mod\ r)\}. \end{aligned}$$

Proof. Since $\tau(\tilde{s}_n) = s^{(d-1)}_{n,n-1}s_n$ and $\tau(\tilde{s}) = s_n^r$, it is clear that the image of τ is contained in $\mathcal{W}(d,r,n)$. One sees that the number of elements of $\mathcal{W}(d,r,n)$ is $n! \cdot d^n/r$ which is equal to the order of $G(d,r,n)$. Thus τ is an embedding. $\square$

As with the map ι for $G(d,d,n)$, we have

$$s_{n,j_1}^{(k_1)} s_{n,j_2}^{(k_2)} = \tau(\tilde{s}_{n-1}\tilde{s}_n)^{k_1}\, \tau(\tilde{s}_{n-1,j_1})\, s_{n,j_2}^{(k_1+k_2)}. \tag{2.12}$$

Using (2.12) repeatedly, we can write

$$s_{1,i_1}s_{2,i_2}\cdots s_{n-1,i_{n-1}} s_{n,j_1}^{(k_1)} s_{n,j_2}^{(k_2)} \cdots s_{n,j_\ell}^{(k_\ell)} = \tau(\tilde{w}) s_{n,j_\ell}^{(k_1+k_2+\cdots+k_\ell)} = \tau(\tilde{w}\,\tilde{s}^k\,\tilde{s}_{n-1,j_\ell})$$

for some $\tilde{w} \in G(d,r,n)$ and $k \in \mathbb{Z}_{\geq 0}$ with the condition $k_1 + k_2 + \cdots + k_\ell \equiv 0 \pmod r$.

From now on, the group $G(d,r,n)$ will be identified with the image of τ. An element of the set $\mathcal{W}(d,r,n)$ will be called a *canonical word*. As in the case of $G(d,1,n)$, a canonical word $\boldsymbol{w}$ will be written as $\boldsymbol{w} = \boldsymbol{w}_0\boldsymbol{w}'$, where $\boldsymbol{w}_0$ is the prefix and $\boldsymbol{w}'$ the suffix of $\boldsymbol{w}$.

3. Fully commutative elements

3.1. Case $G(d,1,n)$

In this subsection, let $W := G(d,1,n)$ be the complex reflection group defined in Section 2.1, with $S := \{s_1,\ldots,s_{n-1},s_n\}$ the set of generators and (2.1a), (2.1b), (2.1c) and (2.1d) the defining relations. We consider the free monoid $S^\star$ consisting of all finite length words $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ with $s_{i_j} \in S$. The multiplication in $S^\star$ is defined by the concatenation

$$(s_{i_1}\cdots s_{i_\ell})\cdot(s_{m_1}\cdots s_{m_t}) = s_{i_1}\cdots s_{i_\ell}s_{m_1}\cdots s_{m_t}.$$

We define a binary relation $\approx$ on $S^\star$ generated by the relations

$$s_is_j = s_js_i \qquad \text{for } j+1 < i \leq n, \tag{3.1a}$$

$$s_{i+1}s_is_{i+1} = s_is_{i+1}s_i \qquad \text{for } 1 \leq i \leq n-2, \tag{3.1b}$$

$$s_n^{k_1}s_{n-1}s_n^{k_2}s_{n-1} = s_{n-1}s_n^{k_2}s_{n-1}s_n^{k_1} \qquad \text{for } 1 \leq k_1,k_2 < d. \tag{3.1c}$$

We will call (3.1a), (3.1b) and (3.1c) the *(generalized) braid relations*. Define $R(w) \subset S^\star$ to be the set of reduced expressions for $w \in W$. Here, as usual, a reduced expression is a word of minimal length for w.

The following proposition is a generalization of Matsumoto's Theorem and is crucial to define fully commutative elements.

Proposition 3.1. *For any $w \in W$, the set $R(w)$ has exactly one equivalence class under $\approx$.*

Proof. Suppose that $\boldsymbol{w} \in R(w)$. We will show that $\boldsymbol{w}$ is related to the canonical word in $\mathcal{W}(d,1,n)$ under $\approx$. We follow the proof of Proposition 2.3. Let ℓ be the number of occurrences of s_n^k in $\boldsymbol{w}$ for various k's, where k is maximal for each occurrence. If $\ell = 0$ then $\boldsymbol{w}$ is a reduced word of an element of the subgroup of type A_{n-1} and one can use only relations (3.1a) and (3.1b) to obtain the canonical word by Matsumoto's Theorem.

Assume $\ell > 0$. Then the proof of Proposition 2.3 shows that the relations (2.7b) and (2.7d) are used to obtain the canonical word. As the relation (2.7b) is nothing but (3.1a), we have only to check if the relation (2.7d) is derived from (3.1a), (3.1b) and (3.1c), which follows from Lemma 2.2. $\square$

Example 3.2. Consider $W = G(3,1,2) =< s_1, s_2 \,|\, s_2^3 = s_1^2 = 1,\ s_2s_1s_2s_1 = s_1s_2s_1s_2 >$. Then the two reduced expressions $s_2s_1s_2^2s_1$ and $s_1s_2^2s_1s_2$ represent the same element in W. One cannot be transformed into the other, using only the defining braid relations (2.1b), (2.1c) and (2.1d). However, under $\approx$, the two expressions are related through (3.1c).

We define a weaker binary relation $\sim$ on $S^\star$ generated by the relations (3.1a) only. The equivalence classes under this relation are called *commutativity classes*. This gives the decomposition of $R(w)$ into commutativity classes:

$$R(w) = \mathcal{C}_1 \dot\cup\, \mathcal{C}_2 \cdots \dot\cup\, \mathcal{C}_\ell.$$

Definition 3.3. We say that $w \in W$ is *fully commutative* if $R(w)$ consists of a single commutativity class; i.e., any reduced word for w can be obtained from any other solely by using the commutation relations (3.1a) that correspond to commuting generators.

Throughout this paper, a subword always means a subword with all its letters in consecutive positions. We obtain the following lemma which is an analogue of Proposition 2.1 in [11].

Lemma 3.4. *An element $w \in W$ is fully commutative if and only if no member of $R(w)$ contains $s_{i+1}s_is_{i+1}$, $1 \le i \le n-2$, or $s_n^{k_1}s_{n-1}s_n^{k_2}s_{n-1}$, $1 \le k_1, k_2 < d$, as a subword.*

Proof. We will prove the contrapositive. If a word $\boldsymbol{w} \in R(w)$ has such a subword, the word $\boldsymbol{w}$ cannot be transformed into the canonical form only using commutative relations. Thus w cannot be fully commutative. Conversely, if w is not fully commutative, there must be a word $\boldsymbol{w} \in R(w)$ to which one of the relations (3.1b) and (3.1c) is applied. Then we see that there exists a word $\boldsymbol{w}_1$ obtained from $\boldsymbol{w}$, which has $s_{i+1}s_is_{i+1}$, $1 \le i \le n-2$, or $s_n^{k_1}s_{n-1}s_n^{k_2}s_{n-1}$, $1 \le k_1, k_2 < d$, as a subword. $\square$

The following proposition provides a practical criterion for full commutativity.

Proposition 3.5. *An element $w \in W$ is fully commutative if and only if there exists $\boldsymbol{w} \in R(w)$ whose commutativity class has no member that has as a subword any of the following*

$$s_{i+1}s_is_{i+1}, \qquad s_is_{i+1}s_i \qquad (1 \le i \le n-2), \tag{3.2a}$$
$$s_n^{k_1}s_{n-1}s_n^{k_2}s_{n-1}, \qquad s_{n-1}s_n^{k_1}s_{n-1}s_n^{k_2} \qquad (1 \le k_1, k_2 < d). \tag{3.2b}$$

Proof. Assume that w is fully commutative. Consider $\boldsymbol{w} \in R(w)$. Then by Lemma 3.4, the word $\boldsymbol{w}$ does not contain $s_{i+1}s_is_{i+1}$, $1 \le i \le n-2$, or $s_n^{k_1}s_{n-1}s_n^{k_2}s_{n-1}$, $1 \le k_1, k_2 < d$, as a subword. Moreover, $\boldsymbol{w}$ cannot contain $s_is_{i+1}s_i$, $1 \le i \le n-2$, or $s_{n-1}s_n^{k_1}s_{n-1}s_n^{k_2}$, $1 \le k_1, k_2 < d$, either. If it does, we obtain $\boldsymbol{w}_1 \in R(w)$ from $\boldsymbol{w}$ by applying (3.1b) or (3.1c), which contains $s_{i+1}s_is_{i+1}$, $1 \le i \le n-2$, or $s_n^{k_1}s_{n-1}s_n^{k_2}s_{n-1}$, $1 \le k_1, k_2 < d$. That is a contradiction. Thus any $\boldsymbol{w} \in R(w)$ does not contain any of the words in (3.2a) and (3.2b).

Conversely, assume that there exists $\boldsymbol{w} \in R(w)$ whose commutativity class $\mathcal{C}$ has no member that contains any of the words in (3.2a) and (3.2b). Then neither (3.1b) nor (3.1c) can be applied to any of the member of $\mathcal{C}$, and we must have $\mathcal{C} = R(w)$. Thus, by definition, w is fully commutative. □

As in Section 2, an expression $s_{i_1} \cdots s_{i_r} \in W$ will be identified with the word $[i_1, \dots, i_r]$. For $w \in W$, let $\boldsymbol{w} = [i_1, \dots, i_r] \in R(w)$. Define $\{i, i+1\}$-sequence of $\boldsymbol{w}$ to be the sequence of i's and $i+1$'s obtained by ignoring all entries of $\boldsymbol{w}$ different from i and $i+1$. For example, the $\{1, 2\}$-sequence of $\boldsymbol{w} = [1, 2, 1, 3, 4, 3, 2]$ is $[1, 2, 1, 2]$.

We have the following useful lemma due to Kleshchev and Ram.

Lemma 3.6 *([8]). The reduced words $\boldsymbol{w}$ and $\boldsymbol{v}$ are in the same commutativity class if and only if their $\{i, i+1\}$-sequences coincide for each $i = 1, 2, \dots, n-1$.*

Combining Proposition 3.5 and Lemma 3.6, we can easily check whether an element w is fully commutative or not.

In Proposition 2.3, we prove that any element of W can be written as

$$s_{1,i_1}s_{2,i_2} \cdots s_{n-1,i_{n-1}}s_{n,j_1}^{(k_1)}s_{n,j_2}^{(k_2)} \cdots s_{n,j_\ell}^{(k_\ell)} \tag{3.3}$$

where $1 \le i_p \le p+1$ for $1 \le p \le n-1$, and $1 \le j_1 < j_2 < \cdots < j_\ell \le n$ for $\ell \ge 0$, and $1 \le k_p \le d-1$ for $1 \le p \le \ell$. Here we write

$$s_{n,j}^{(k)} = s_n^k s_{n-1,j} = s_n^k s_{n-1} \dots s_j \qquad \text{for } k \ge 1 \ \text{ and } \ j \le n.$$

Recall that the part $s_{n,j_1}^{(k_1)}s_{n,j_2}^{(k_2)} \cdots s_{n,j_\ell}^{(k_\ell)}$ in the canonical form (3.3) is called its suffix. Now we prove the following proposition.

Proposition 3.7. *Every suffix is a fully commutative element.*

Proof. By Proposition 3.5, we have to show that no member in the commutative class of a suffix contains as a subword any of the words in (3.2a) and (3.2b). By Lemma 3.6, we need to investigate relative positions of a letter p and their neighbors $p-1$ and $p+1$.

Assume that $1 \leq p \leq n-2$. Every consecutive occurrence of p in a suffix is of the form

$$[p, p-1, \ldots j_p, n^{k_p}, \ldots, p+1, p, \ldots, j_{p+1}] \quad \text{with } \; j_p < j_{p+1} < p+1. \tag{3.4}$$

Thus, neither the word $s_p s_{p+1} s_p$ nor the word $s_{p+1} s_p s_{p+1}$ can appear in any suffix or in any member of its commutative class. Similarly, one sees that neither the word $s_{n-1} s_n^{k_1} s_{n-1} s_n^{k_2}$ nor the word $s_n^{k_1} s_{n-1} s_n^{k_2} s_{n-1}$ can appear in any suffix or in its commutative class. Hence our assertion follows. □

3.2. Cases $G(d,d,n)$ and $G(d,r,n)$

Recall that we fixed embeddings of $G(d,d,n)$ and $G(d,r,n)$ into $G(d,1,n)$ and that these groups are identified with the images of the embeddings.

Definition 3.8. Let $W = G(d,d,n)$ or $G(d,r,n)$ be considered as a subgroup of $G(d,1,n)$ through the embedding ι or τ defined in Section 2, respectively. An element w of W is called *fully commutative* if w is fully commutative as an element of $G(d,1,n)$.

As mentioned in the introduction, this definition of fully commutative elements coincides with the usual definition for the Coxeter groups of type B_n when $d = 2$, $r = 1$. On the other hand, it is not compatible with the usual definition for the Coxeter groups of type D_n when $d = 2$, $r = 2$. This will be made more clear in the following sections.

4. Packets in $G(d,1,n)$

4.1. Collections

The words in $\mathcal{W}(d,1,n)$ which correspond to fully commutative elements will be called *fully commutative* and will be grouped based on their suffixes.

Definition 4.1. A *collection* $\mathbf{c}^n_{\boldsymbol{w}'} \subset \mathcal{W}(d,1,n)$ labeled by a suffix $\boldsymbol{w}'$ is defined to be the set of fully commutative words in $\mathcal{W}(d,1,n)$ whose suffix is $\boldsymbol{w}'$.

As in the case of type D studied in [4], some of the collections have the same number of elements as we will prove in the rest of this subsection. Since the proofs are essentially the same as in the case of type D, we will only sketch them here, referring the reader to [4] for more details.

Lemma 4.2. *For a fixed k, $0 \leq k \leq n-2$, any collection labeled by a suffix of the form*

$$s^{(t_1)}_{n,k+1} s^{(t_2)}_{n,j_2} s^{(t_3)}_{n,j_3} s^{(t_4)}_{n,j_4} \cdots s^{(t_\ell)}_{n,j_\ell} \quad (\ell \geq 2) \tag{4.1}$$

has the same set of prefixes. In particular, these collections have the same number of elements.

Proof. Let $\boldsymbol{w}'$ be a suffix of the form (4.1). Then $\boldsymbol{w}'$ has the suffix $\boldsymbol{w}_1 := s_{n,k+1}^{(t_1)} s_n$ as a subword, and any prefix appearing in the collection $\mathbf{c}^n_{\boldsymbol{w}'}$ also appears in $\mathbf{c}^n_{\boldsymbol{w}_1}$.

Conversely, assume that $\boldsymbol{w}_0$ is a prefix of a fully commutative word appearing in the collection $\mathbf{c}^n_{\boldsymbol{w}_1}$. Since the prefix and suffix of a fully commutative word are themselves fully commutative, we only assume that there is some letter $r \leq n-1$ which appears in both $\boldsymbol{w}_0$ and $\boldsymbol{w}'$. From the condition

$$1 \leq j_1 < j_2 < \cdots < j_\ell \leq n$$

on the suffix $\boldsymbol{w}'$, we see that the letter r also appears in $\boldsymbol{w}_1$. Then the full commutativity of $\boldsymbol{w}_0\boldsymbol{w}_1$ implies that $\boldsymbol{w}_0\boldsymbol{w}'$ is also fully commutative and that $\boldsymbol{w}_0$ is a prefix of $\boldsymbol{w}'$. □

Proposition 4.3. *For $1 \leq k \leq n-2$, the collection labeled by the suffix $s_{n,k}^{(t')}$, $1 \leq t' < d$, has the same number of elements as any of the collections labeled by the suffix of the form*

$$s_{n,k+1}^{(t_1)} s_{n,j_2}^{(t_2)} s_{n,j_3}^{(t_3)} s_{n,j_4}^{(t_4)} \cdots s_{n,j_\ell}^{(t_\ell)} \quad (\ell \geq 2).$$

Proof. Let $\boldsymbol{w}_1 = s_{n,k+1}^{(t_1)} s_n$ and $\boldsymbol{w}_2 = s_{n,k}^{(t')}$. By Lemma 4.2, it is enough to establish a bijection between the collections $\mathbf{c}^n_{\boldsymbol{w}_1}$ and $\mathbf{c}^n_{\boldsymbol{w}_2}$. We define a map $\sigma : \mathbf{c}^n_{\boldsymbol{w}_2} \to \mathbf{c}^n_{\boldsymbol{w}_1}$ as follows. Suppose that $\boldsymbol{w}_0$ is the prefix of the word $\boldsymbol{w} = \boldsymbol{w}_0\boldsymbol{w}_2 = \boldsymbol{w}_0[n^{t'}, n-1, \ldots, k] \in \mathbf{c}^n_{\boldsymbol{w}_2}$, and let r be the last letter of $\boldsymbol{w}_0$. Then by the condition of full commutativity, we must have $r < k$ or $r = n-1$.

If $r < k$ we simply define $\sigma(\boldsymbol{w}) = \boldsymbol{w}_0\boldsymbol{w}_1$. If $r = n-1$, we take $m \geq k$ to be the smallest letter such that the string $[m, m+1, \ldots, n-1]$ is a right factor of $\boldsymbol{w}_0$. Then we have $\boldsymbol{w} = s_{1,i_1} \cdots s_{m-1,i_{m-1}} s_m s_{m+1} \cdots s_{n-1} \boldsymbol{w}_2$, and we define

$$\sigma(\boldsymbol{w}) = s_{1,i_1} \cdots s_{m-1,i_{m-1}} s_{m,k} \boldsymbol{w}_1.$$

Then we have $\sigma(\mathbf{c}^n_{\boldsymbol{w}_2}) \subset \mathbf{c}^n_{\boldsymbol{w}_1}$.

Now we define a map $\eta : \mathbf{c}^n_{\boldsymbol{w}_1} \to \mathbf{c}^n_{\boldsymbol{w}_2}$. Suppose that $\boldsymbol{w} = \boldsymbol{w}_0\boldsymbol{w}_1 = \boldsymbol{w}_0[n^{t_1}, n-1, \ldots, k+1, n] \in \mathbf{c}^n_{\boldsymbol{w}_1}$, and let r be the last letter of $\boldsymbol{w}_0$. Then by the condition of full commutativity, we must have $1 \leq r \leq k$.

If $r < k$, then we define $\eta(\boldsymbol{w}_0\boldsymbol{w}_1) = \boldsymbol{w}_0\boldsymbol{w}_2$. If $r = k$, then the final non-empty segment of the prefix is $s_{m,k}$ for some m with $k \leq m \leq n-1$. We define

$$\eta(\boldsymbol{w}) = \eta(s_{1,i_1} \cdots s_{m-1,i_{m-1}} s_{m,k} \boldsymbol{w}_1) = s_{1,i_1} \cdots s_{m-1,i_{m-1}} s_m s_{m+1} \cdots s_{n-1} \boldsymbol{w}_2.$$

One sees that $\eta(\mathbf{c}^n_{\boldsymbol{w}_1}) \subset \mathbf{c}^n_{\boldsymbol{w}_2}$, and that η is both a left and a right inverse of σ. □

Lemma 4.4. *The collections labeled by the suffixes s_n^t and $s_{n,n-1}^{(t)}$ $(1 \le t < d)$ have the same set of prefixes.*

Proof. Assume that $\boldsymbol{w}_0$ is a prefix of s_n^t or $s_{n,n-1}^{(t)}$, i.e., $\boldsymbol{w}_0 s_n^t$ or $\boldsymbol{w}_0 s_{n,n-1}^{(t)}$ is fully commutative. Then replacing the suffix s_n^t with any of the suffixes $s_n^{t'}$ and $s_{n,n-1}^{(t')}$ $(1 \le t' < d)$ does not affect full commutativity. $\square$

4.2. Packets

The results in the previous subsection lead us to the following definition.

Definition 4.5. For $0 \le k \le n$, we define the (n,k)-*packet* of collections:

- The $(n,0)$-packet is the set of collections labeled by suffixes of the form

$$s_{n,1}^{(t_1)} s_{n,j_2}^{(t_2)} s_{n,j_3}^{(t_3)} s_{n,j_4}^{(t_4)} \cdots s_{n,j_\ell}^{(t_\ell)} \quad (\ell \ge 2).$$

- The (n,k)-packet, $1 \le k \le n-2$, is the set of collections labeled by $s_{n,k}^{(t)}$ or by suffixes of the form $s_{n,k+1}^{(t_1)} s_{n,j_2}^{(t_2)} s_{n,j_3}^{(t_3)} s_{n,j_4}^{(t_4)} \cdots s_{n,j_\ell}^{(t_\ell)} \quad (\ell \ge 2)$.
- The $(n,n-1)$-packet contains the collections labeled by $s_n^{(t)} = [n^t]$ or $s_{n,n-1}^{(t)} = [n^t, n-1]$.
- The (n,n)-packet contains only the collection labeled by the empty suffix $[\ ]$.

We will denote the (n,k)-packet by $\mathcal{P}(n,k)$. As an example, Table 4.1 shows all packets for the case of $G(2,1,3)$ (or B_3).

We record the main property of a packet as a corollary.

Corollary 4.6. *The collections in a fixed packet $\mathcal{P}(n,k)$ have the same number of elements.*

Proof. The assertion follows from Lemma 4.2, Proposition 4.3 and Lemma 4.4. $\square$

We count the number of collections in a packet and obtain:

Proposition 4.7. *The size of the packet $\mathcal{P}(n,k)$ of $G(d,1,n)$ is*

$$|\mathcal{P}(n,k)| = \begin{cases} (d^{n-1}-1)(d-1) & \text{if } k=0, \\ d^{n-k-1}(d-1) & \text{if } 1 \le k \le n-2, \\ 2(d-1) & \text{if } k = n-1, \\ 1 & \text{if } k=n. \end{cases}$$

Hence we have $\sum_{k=0}^{n} |\mathcal{P}(n,k)| = d^n$.

Table 4.1
The packets of $G(2,1,3)$.

Packet	Collection	Elements
$\mathcal{P}(3,0)$	$\mathbf{c}^3_{[3,2,1,3]}$	$[3,2,1,3]$
	$\mathbf{c}^3_{[3,2,1,3,2]}$	$[3,2,1,3,2]$
	$\mathbf{c}^3_{[3,2,1,3,2,3]}$	$[3,2,1,3,2,3]$
$\mathcal{P}(3,1)$	$\mathbf{c}^3_{[3,2,1]}$	$[3,2,1]$, $[2,3,2,1]$, $[1,2,3,2,1]$
	$\mathbf{c}^3_{[3,2,3]}$	$[3,2,3]$, $[1,3,2,3]$, $[2,1,3,2,3]$
$\mathcal{P}(3,2)$	$\mathbf{c}^3_{[3,2]}$	$[3,2]$, $[2,3,2]$, $[1,3,2]$, $[2,1,3,2]$, $[1,2,3,2]$
	$\mathbf{c}^3_{[3]}$	$[3]$, $[2,3]$, $[1,3]$, $[2,1,3]$, $[1,2,3]$
$\mathcal{P}(3,3)$	$\mathbf{c}^3_{[\,]}$	$[\,]$, $[2]$, $[1]$, $[2,1]$, $[1,2]$

Proof. Assume that $k=0$. We consider the expression

$$s_{n,1}^{(k_1)} s_{n,2}^{(k_2)} s_{n,3}^{(k_3)} \cdots s_{n,n-1}^{(k_{n-1})} s_n^{k_n}.$$

The conditions for $(n,0)$-packet allows k_1 to vary from 1 to $d-1$ and k_i $(2 \leq i \leq n)$ from 0 to d, except from the case $k_2 = k_3 = \cdots = k_n = 0$. Thus the total number of collections in $\mathcal{P}(n,0)$ is $(d-1)(d^{n-1}-1)$.

Similar arguments can be applied to the other packets $\mathcal{P}(n,k)$ for $1 \leq k \leq n-1$, and it is clear that there is only one collection in $\mathcal{P}(n,n)$. The total sum can be checked straightforwardly. $\square$

Table 4.2
Catalan Triangle.

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

4.3. Catalan's Triangle

In this subsection, we will compute the size of a collection in a given packet, and thereby classify and enumerate all the fully commutative elements.

As in the case of type D studied in [4], the sizes of collections are given by *Catalan triangle numbers* $C(n, k)$ which are defined by

$$C(n, k) = \frac{(n + k)!(n - k + 1)}{k!(n + 1)!} \tag{4.2}$$

for $n \geq 0$ and $0 \leq k \leq n$. The numbers form the *Catalan Triangle* in Table 4.2 to satisfy the rule:

$$C(n, k) = C(n, k - 1) + C(n - 1, k), \tag{4.3}$$

where all entries outside of the range $0 \leq k \leq n$ are considered to be 0. One also sees that

$$C_n = C(n, n - 1) = C(n, n), \tag{4.4}$$

where C_n is the nth Catalan number.

Lemma 4.8. *For $n \geq 3$, the size of each collection in the packets $\mathcal{P}(n, n)$ and $\mathcal{P}(n, n - 1)$ is equal to the Catalan number C_n.*

Proof. The proof of Lemma 4.4 shows that prefixes in a collection belonging to one of the packets $\mathcal{P}(n, n)$ and $\mathcal{P}(n, n - 1)$ are exactly fully commutative words of type A_{n-1}, the total number of which is well known to be the Catalan number C_n. $\square$

The following theorem is an extension of Theorem 2.12 in [4] from the case of D_n to $G(d, 1, n)$. The proof is similar to that of type D_n, and we refer the reader to [4] for more details.

Theorem 4.9. *Assume that $n \geq 3$ and $0 \leq k \leq n$. Then any collection in the packet $\mathcal{P}(n,k)$ contains exactly $C(n,k)$ elements.*

Proof. We have already proved the cases when $k = n$ and $k = n-1$ in Lemma 4.8. Now consider the packet $\mathcal{P}(n,0)$, which consists of the collections labeled by the suffixes

$$s_{n,1}^{(t_1)} s_{n,j_2}^{(t_2)} s_{n,j_3}^{(t_3)} s_{n,j_4}^{(t_4)} \cdots s_{n,j_\ell}^{(t_\ell)} \quad (\ell \geq 2).$$

By Lemma 4.2, it is enough to consider the collection $\mathbf{c}$ labeled by $s_{n,1}^{(t_1)} s_n$. If a word $\boldsymbol{w} \in \mathbf{c}$ contains a non-empty prefix $\boldsymbol{w}_0$ ending with the letter r for $1 \leq r \leq n-1$, then the word $\boldsymbol{w}$ contains as a right factor $[r, n, n-1, \ldots, r+1, r, \ldots, 1, n]$ which contradicts full commutativity. Thus the collection $\mathbf{c}$, and hence every collection in $\mathcal{P}(n,0)$, contains only the suffix itself. Thus we have $C(n,0) = 1$.

For the other cases, we will combinatorially (or bijectively) obtain the identity (4.3). One can see that any collection in the packet $\mathcal{P}(3,k)$ contains exactly $C(3,k)$ elements for $0 \leq k \leq 3$. The case $G(2,1,3)$ is given in Table 4.1 and the case $G(3,1,3)$ in the second example of Section 6. Thus the assertion of the theorem is true for $n = 3$, and we will proceed by induction with the base cases $k = 0$ or $n = 3$. Further, by Corollary 4.6, it is enough to consider a single collection in each of the packets.

Assume that $n \geq 4$ and $1 \leq k \leq n-2$. We define

$$\boldsymbol{w}_1 = s_{n,k} s_n = [n, n-1, \ldots, k, n], \quad \boldsymbol{w}_2 = s_{n,k} = [n, n-1, \ldots, k],$$
$$\boldsymbol{w}_3 = [n-1, n-2, \ldots, k].$$

Then $\mathbf{c}^n_{\boldsymbol{w}_1} \in \mathcal{P}(n, k-1)$, $\mathbf{c}^n_{\boldsymbol{w}_2} \in \mathcal{P}(n,k)$ and $\mathbf{c}^{n-1}_{\boldsymbol{w}_3} \in \mathcal{P}(n-1,k)$. We will give an explicit bijection from $\mathbf{c}^n_{\boldsymbol{w}_1} \cup \mathbf{c}^{n-1}_{\boldsymbol{w}_3}$ to $\mathbf{c}^n_{\boldsymbol{w}_2}$.

Define a map $\varphi_1 : \mathbf{c}^n_{\boldsymbol{w}_1} \to \mathbf{c}^n_{\boldsymbol{w}_2}$ by

$$\varphi_1(\boldsymbol{w}_0 \boldsymbol{w}_1) = \varphi(\boldsymbol{w}_0[n, n-1, \ldots, k, n]) = \boldsymbol{w}_0 \boldsymbol{w}_2 = \boldsymbol{w}_0[n, n-1, \ldots, k],$$

and another map $\varphi_2 : \mathbf{c}^{n-1}_{\boldsymbol{w}_3} \to \mathbf{c}^n_{\boldsymbol{w}_2}$ by

$$\varphi_2(\boldsymbol{w}_0 \boldsymbol{w}_3) = \boldsymbol{w}_0 s_{n-1} \boldsymbol{w}_2 = \boldsymbol{w}_0 s_{n-1}[n, n-1, n-2, \ldots, k].$$

Then it can be checked that the maps φ_1 and φ_2 are well defined ant that the images are disjoint. Finally, combining φ_1 and φ_2, we define a map $\varphi : \mathbf{c}^n_{\boldsymbol{w}_1} \cup \mathbf{c}^{n-1}_{\boldsymbol{w}_3} \to \mathbf{c}^n_{\boldsymbol{w}_2}$, i.e. the restriction of φ to $\mathbf{c}^n_{\boldsymbol{w}_1}$ is φ_1 and the restriction of φ to $\mathbf{c}^{n-1}_{\boldsymbol{w}_3}$ is φ_2.

Conversely, define the map $\rho : \mathbf{c}^n_{\boldsymbol{w}_2} \to \mathbf{c}^n_{\boldsymbol{w}_1} \cup \mathbf{c}^{n-1}_{\boldsymbol{w}_3}$ to be given by the rule:

$$\rho(\boldsymbol{w}_0 \boldsymbol{w}_2) = \rho(\boldsymbol{w}_0[n, n-1, \ldots, k]) = \begin{cases} \boldsymbol{w}_0[n-2, \ldots, k] \in \mathbf{c}^{n-1}_{\boldsymbol{w}_3}, & \text{if } \boldsymbol{w}_0 \text{ ends with } n-1, \\ \boldsymbol{w}_0 \boldsymbol{w}_1 \in \mathbf{c}^n_{\boldsymbol{w}_1}, & \text{otherwise.} \end{cases}$$

One can check that the map ρ is well defined.

Now one can see that ρ is the two-sided inverse of φ. In particular, if we restrict ρ to the words whose prefixes end with $n-1$, then we obtain the inverse for φ_2, while if we restrict to the prefixes not ending in $n-1$, we have the inverse for φ_1.

This establishes, for each k,

$$|\mathbf{c}^n_{\boldsymbol{w}_2}| = |\mathbf{c}^n_{\boldsymbol{w}_1}| + |\mathbf{c}^{n-1}_{\boldsymbol{w}_3}|,$$

which is the same identity as (4.3) inductively. This proves that $|\mathbf{c}^n_{\boldsymbol{w}_2}| = C(n,k)$ as desired. □

Let us recall the Catalan triangle polynomial introduced in [9, Definition 2.11]:

Definition 4.10. For $0 \le k \le n$, we define the *Catalan triangle polynomial* $\mathfrak{F}_{n,k}(x)$ by

$$\mathfrak{F}_{n,k}(x) = \sum_{s=0}^{k} C(n,s)x^{k-s}. \tag{4.5}$$

We need some special values of the polynomial $\mathfrak{F}_{n,k}(x)$.

Lemma 4.11. *[9, Corollary 2.9] For $0 \le k < n$, we have*

$$\mathfrak{F}_{n,k}(2) = \binom{n+1+k}{k}. \tag{4.6}$$

In light of the above lemma, the numbers $\mathfrak{F}_{n,k}(d)$, $d > 2$, can be considered as a certain generalization of binomial coefficients. Interestingly, we need $\mathfrak{F}_{n,n-2}(d)$ to write a formula for the number of fully commutative elements in $G(d,1,n)$ in the following corollary. This is also the case for $G(d,r,n)$. See Corollaries 5.3 and 5.6.

Corollary 4.12. *For $n \ge 3$, the number of fully commutative elements of $G(d,1,n)$ is equal to*

$$\sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = d(d-1)\mathfrak{F}_{n,n-2}(d) + (2d-1)C_n - (d-1). \tag{4.7}$$

In particular, when $d = 2$, we recover the result of [11] on B_n-type using (4.6):

$$\sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = 2\binom{2n-1}{n-2} + 3C_n - 1 = (n+2)C_n - 1.$$

Proof. The assertion follows from Proposition 4.7 and the definitions. □

5. Packets in $G(d, r, n)$

In this section, we assume that $1 \le r \le d$ and $r|d$. Thus the family of groups $G(d, r, n)$ includes the case $G(d, d, n)$. The results will be presented for $G(d, d, n)$ first for simplicity, and then will be generalized to the case $G(d, r, n)$.

Recall that we consider $G(d, r, n)$ as subgroups of $G(d, 1, n)$ through the embedding τ, and that Lemma 2.8 describes the elements of $G(d, r, n)$. We define the *packets of* $G(d, r, n)$ to be those of $G(d, 1, n)$ which are contained in $G(d, r, n)$.

Proposition 5.1. *The size of the packet $\mathcal{P}(n, k)$ of $G(d, d, n)$ is*

$$|\mathcal{P}(n,k)| = \begin{cases} d^{n-k-2}(d-1) & \text{if } 0 \le k \le n-2, \\ 1 & \text{if } k = n. \end{cases}$$

Hence we have $\sum_{k=0}^{n} |\mathcal{P}(n, k)| = d^{n-1}$.

Proof. We use Lemma 2.6 to determine which packets of $G(d, 1, n)$ in Definition 4.5 are contained in $G(d, d, n)$. Clearly, the $(n, n-1)$-packet cannot occur in $G(d, d, n)$, and there is still only one collection in the (n, n)-packet. For $0 \le k \le n-2$, the suffixes $s_{n,k}^{(t)}$ cannot appear in $G(d, d, n)$ and the number of suffixes of the form $s_{n,k+1}^{(t_1)} s_{n,j_2}^{(t_2)} s_{n,j_3}^{(t_3)} s_{n,j_4}^{(t_4)} \cdots s_{n,j_\ell}^{(t_\ell)}$ $(\ell \ge 2)$ that appear in $G(d, d, n)$ is $d^{n-k-2}(d-1)$. □

More generally, we have the following.

Proposition 5.2. *The size of the packet $\mathcal{P}(n, k)$ of $G(d, r, n)$ is*

$$|\mathcal{P}(n,k)| = \begin{cases} \dfrac{d^{n-1}}{r}(d-1) - \left(\dfrac{d}{r} - 1\right) & \text{if } k = 0, \\ \dfrac{d^{n-k-1}}{r}(d-1) & \text{if } 1 \le k \le n-2, \\ 2\left(\dfrac{d}{r} - 1\right) & \text{if } k = n-1, \\ 1 & \text{if } k = n. \end{cases}$$

Hence we have $\sum_{k=0}^{n} |\mathcal{P}(n, k)| = d^n/r$.

Proof. We use Lemma 2.8 to determine which packets of $G(d, 1, n)$ in Definition 4.5 are contained in $G(d, r, n)$. Clearly, there is still only one collection in the (n, n)-packet. For the $(n, n-1)$-packet, each of $s_n^{(t)}$ and $s_{n,n-1}^{(t)}$ has $\frac{d}{r} - 1$ possibilities to satisfy the conditions $t \equiv 0 \pmod{r}$ and $1 \le t < d$.

As for the $(n, 0)$-packet, we consider the expression

$$s_{n,1}^{(k_1)} s_{n,2}^{(k_2)} s_{n,3}^{(k_3)} \cdots s_{n,n-1}^{(k_{n-1})} s_n^{k_n}.$$

Then k_1 varies from 1 to $d-1$ and k_i $(2 \le i \le n-1)$ from 0 to d and then k_n has d/r choices, except the cases that $k_1 \equiv 0 \pmod r$ and $k_2 = k_3 = \cdots = k_n = 0$. Thus the total number of collections in $\mathcal{P}(n,0)$ is $\dfrac{d^{n-1}}{r}(d-1) - \left(\dfrac{d}{r} - 1\right)$.

The sizes of (n,k)-packets for $1 \le k \le n-2$ can be checked similarly. □

Corollary 5.3. *The number of fully commutative elements in the group $G(d,d,n)$ is equal to*

$$\sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = (d-1)\mathfrak{F}_{n,n-2}(d) + C_n. \tag{5.1}$$

In particular, when $d=2$, we obtain from (4.6)

$$\sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = \binom{2n-1}{n-2} + C_n = \frac{n-1}{2}C_n + C_n = \frac{n+1}{2}C_n. \tag{5.2}$$

Remark 5.4. The number $\frac{n+1}{2}C_n$ in (5.2) is different from the number $\frac{n+3}{2}C_n - 1$ of fully commutative elements of type D_n considered in [12,4] without embedding ι. Thus our definition of fully commutative elements of $G(2,2,n)$ is not equivalent to that of D_n in [12,4]. See the first example in Section 6 for more details.

Remark 5.5. Let $c(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ be the generating function of the Catalan numbers C_n. The generating function of the numbers of fully commutative elements in $G(d,d,n)$ is given by

$$\frac{1-(d-1)\,x\,c(x)}{1-d\,x\,c(x)}.$$

Corollary 5.6. *The number of fully commutative elements in the group $G(d,r,n)$ is equal to*

$$\sum_{k=0}^{n} C(n,k)\,|\mathcal{P}(n,k)| = \frac{d(d-1)}{r}\mathfrak{F}_{n,n-2}(d) + \left(\frac{2d}{r} - 1\right)C_n - \left(\frac{d}{r} - 1\right). \tag{5.3}$$

6. Examples

(1) The group $G(2,2,n)$ is isomorphic to the Coxeter group of type D_n which has its own definition of fully commutative elements without invoking the embedding ι into $G(2,1,n)$. For example, the element $\tilde{s}_3\tilde{s}_4\tilde{s}_2\tilde{s}_1 \in G(2,2,4)$ is fully commutative before being embedded into $G(2,1,4)$, but we have

$$\iota(\tilde{s}_3\tilde{s}_4\tilde{s}_2\tilde{s}_1) = s_3s_4s_3s_4s_2s_1,$$

which is not fully commutative in $G(2,1,4)$. The number of fully commutative elements of D_4 (without embedding) is 48, whereas the number of fully commutative elements of $G(2,2,4)$ (after being embedded) is 35.

(2) The group $G(3,1,3)$ has 59 fully commutative elements. We list them below in packets and collections.

Packets	Collections	$C(3,k)$
$\mathcal{P}(3,0)$	$\mathbf{c}_{[3,2,1,3]}$, $\mathbf{c}_{[3,2,1,3^2]}$, $\mathbf{c}_{[3,2,1,3,2]}$, $\mathbf{c}_{[3,2,1,3,2,3]}$, $\mathbf{c}_{[3,2,1,3,2,3^2]}$, $\mathbf{c}_{[3,2,1,3^2,2]}$, $\mathbf{c}_{[3,2,1,3^2,2,3]}$, $\mathbf{c}_{[3,2,1,3^2,2,3^2]}$, $\mathbf{c}_{[3^2,2,1,3]}$, $\mathbf{c}_{[3^2,2,1,3^2]}$, $\mathbf{c}_{[3^2,2,1,3,2]}$, $\mathbf{c}_{[3^2,2,1,3,2,3]}$, $\mathbf{c}_{[3^2,2,1,3,2,3^2]}$, $\mathbf{c}_{[3^2,2,1,3^2,2]}$, $\mathbf{c}_{[3,2,1,3^2,2,3]}$, $\mathbf{c}_{[3,2,1,3^2,2,3^2]}$	1
$\mathcal{P}(3,1)$	$\mathbf{c}_{[3,2,3]} = \{s_3s_2s_3, s_1s_3s_2s_3, s_2s_1s_3s_2s_3\}$ $\mathbf{c}_{[3,2,3^2]} = \{s_3s_2s_3^2, s_1s_3s_2s_3^2, s_2s_1s_3s_2s_3^2\}$ $\mathbf{c}_{[3^2,2,3]} = \{s_3^2s_2s_3, s_1s_3^2s_2s_3, s_2s_1s_3^2s_2s_3\}$ $\mathbf{c}_{[3^2,2,3^2]} = \{s_3^2s_2s_3^2, s_1s_3^2s_2s_3^2, s_2s_1s_3^2s_2s_3^2\}$ $\mathbf{c}_{[3,2,1]} = \{s_3s_2s_1, s_2s_3s_2s_1, s_1s_2s_3s_2s_1\}$ $\mathbf{c}_{[3^2,2,1]} = \{s_3^2s_2s_1, s_2s_3^2s_2s_1, s_1s_2s_3^2s_2s_1\}$	3
$\mathcal{P}(3,2)$	$\mathbf{c}_{[3]} = \{s_3, s_1s_3, s_2s_3, s_1s_2s_3, s_2s_1s_3\}$ $\mathbf{c}_{[3^2]} = \{s_3^2, s_1s_3^2, s_2s_3^2, s_1s_2s_3^2, s_2s_1s_3^2\}$ $\mathbf{c}_{[3,2]} = \{s_3s_2, s_1s_3s_2, s_2s_3s_2, s_1s_2s_3s_2, s_2s_1s_3s_2\}$ $\mathbf{c}_{[3^2,2]} = \{s_3^2s_2, s_1s_3^2s_2, s_2s_3^2s_2, s_1s_2s_3^2s_2, s_2s_1s_3^2s_2\}$	5
$\mathcal{P}(3,3)$	$\mathbf{c}_{[]} = \{[\ ], s_1, s_2, s_1s_2, s_2s_1\}$	5

(3) The set of reduced words for the group $G(3,3,3)$ is given in Appendix. To the canonical words, one applies the embedding $\iota : G(3,3,3) \hookrightarrow G(3,1,3)$ and sees that the group has 17 fully commutative elements. We list them all below, where we write $[i_1i_2\dots i_k]$ for $\tilde{s}_{i_1}\tilde{s}_{i_2}\cdots\tilde{s}_{i_k} \in G(3,3,3)$ and $\langle i_1i_2\dots i_k\rangle$ for $s_{i_1}s_{i_2}\cdots s_{i_k} \in G(3,1,3)$.

$$
\begin{array}{rlrlrl}
[] &\longmapsto \langle\rangle, & [1] &\longmapsto \langle 1\rangle, & [2] &\longmapsto \langle 2\rangle,\\
[3] &\longmapsto \langle 3^223\rangle, & [12] &\longmapsto \langle 12\rangle, & [13] &\longmapsto \langle 13^223\rangle,\\
[21] &\longmapsto \langle 21\rangle, & [31] &\longmapsto \langle 3^2231\rangle = \langle 3^2213\rangle, & [213] &\longmapsto \langle 213^223\rangle,\\
[232] &\longmapsto \langle 23^2232\rangle = \langle 323^2\rangle, & [312] &\longmapsto \langle 3^22132\rangle, & [1232] &\longmapsto \langle 123^2232\rangle = \langle 1323^2\rangle,\\
[2321] &\longmapsto \langle 3213^2\rangle, & [12132] &\longmapsto \langle 21323^2\rangle, & [13123] &\longmapsto \langle 3^2213^223^2\rangle,\\
[23121] &\longmapsto \langle 3213^22\rangle, & [23213] &\longmapsto \langle 321323\rangle. & &
\end{array}
$$

The $(3,0)$-packet has 6 collections, each of which has only one element:

$$\mathcal{P}(3,0) = \big\{\{\langle 3^2213\rangle\}, \{\langle 3^22132\rangle\}, \{\langle 3213^2\rangle\}, \{\langle 3^2213^223^2\rangle\}, \{\langle 3213^22\rangle\}, \{\langle 321323\rangle\}\big\}.$$

There are 2 collections in the $(3,1)$-packet, each of which has 3 elements:

$$\mathcal{P}(3,1) = \big\{\{\langle 3^223\rangle, \langle 13^223\rangle, \langle 213^223\rangle\}, \{\langle 323^2\rangle, \langle 1323^2\rangle, \langle 21323^2\rangle\}\big\}.$$

Recall that there is no $(3,2)$-packet. There is only one collection in the $(3,3)$-packet and it has 5 elements:

$$\mathcal{P}(3,3) = \{\{\langle\rangle, \langle 1\rangle, \langle 2\rangle, \langle 12\rangle, \langle 21\rangle\}\}.$$

All together we have

$$1 \times 6 + 3 \times 2 + 5 \times 1 = 17.$$

Remark 6.1. Before taking the embedding $G(3,3,3) \hookrightarrow G(3,1,3)$, we may want to say that the element $\tilde{s}_2\tilde{s}_3 = [23]$ is fully commutative. After the embedding, we have

$$[23] \longmapsto \langle 23^2 23\rangle,$$

and the element is not fully commutative.

7. Appendix: reduced words for $G(3,3,3)$ without an embedding

In this appendix, we write $[i_1 i_2 \dots i_k]$ for $\tilde{s}_{i_1}\tilde{s}_{i_2}\cdots\tilde{s}_{i_k}$.

Lemma 7.1. *The following relations hold in $G(3,3,3)$:*

$$\begin{aligned}
&[31232] = [13123], && [32131] = [23213], && [213121] = [131213], && [213123] = [131231],\\
&[213213] = [132132], && [231213] = [123121], && [231231] = [123123], && [232132] = [132131],\\
&[312131] = [121312], && [312132] = [121321], && [312312] = [123123], && [321321] = [132132].
\end{aligned}$$

Proof. All the relations are derived from the defining relations. For example, we have the defining relation $[313] = [131]$. Multiplying both sides by $\tilde{s}_2\tilde{s}_3$ from the right, we obtain

$$[31323] = [31232] = [13123],$$

where we use another defining relation $[323] = [232]$. Thus we obtain $[31232] = [13123]$. $\square$

Proposition 7.2. *A set of reduced words for $G(3,3,3)$ is given by*

$$\begin{array}{lllllllll}
[\ ], \\
[1], & [2], & [3], \\
[12], & [13], & [21], & [23], & [31], & [32] \\
[121], & [123], & [131], & [132], & [213], & [231], & [232], & [312], & [321], \\
[1213], & [1231], & [1232], & [1312], & [1321], & [2131], \\
[2132], & [2312], & [2321], & [3121], & [3123], & [3213], \\
[12131], & [12132], & [12312], & [12321], & [13121], & [13123], & [13213], & [21312], \\
[21321], & [23121], & [23123], & [23213], & [31213], & [31231], & [32132], \\
[121312], & [121321], & [123121], & [123123], & [123213], & [131213], & [131231], & [132132].
\end{array}$$

Proof. We set an ordering $1 < 2 < 3$ on the alphabet $I = \{1, 2, 3\}$ and use the degree-lexicographic ordering on the set of words on I. Then one can see that the words in the list above do not contain as a subword any of the leading words of the defining relations for $G(3, 3, 3)$ and of the relations of Lemma 7.1. Further it can be checked that the list has all the words with this property. The number of words in the list is 54, which is exactly the order of $G(3, 3, 3)$. Thus it follows from the Gröbner–Shirshov basis theory that the list is a set of reduced words for $G(3, 3, 3)$. $\square$

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