KAC–MOODY EISENSTEIN SERIES

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ABSTRACT. In this survey, we will present recent results on convergence and holomorphy of Kac–Moody Eisenstein series.

1. Motivation and Past Developments

After being developed by Langlands [La1, La2] in great generality, the theory of Eisenstein series has played fundamental roles in Langlands' Program. In particular, meromorphic continuation of Eisenstein series on reductive groups, established in the seminal work of Langlands [La2], has been a foundational basis for the study of L-functions by means of the Langlands–Shahidi method (e.g. [KimSh, Kim]), which exploits analytic properties of Eisenstein series together with the fact that L-functions appear in the Fourier coefficients of Eisenstein series.

Eisenstein series also appear in many other places throughout number theory and representation theory. The scope of applications extends to geometry and mathematical physics. For example, Eisenstein series on exceptional Lie groups have been shown to occur explicitly as coefficients of correction terms in certain maximally supersymmetric string theories [GRV, GMRV, GMV].

On the other hand, since we have seen many successful generalizations of finite dimensional constructions to infinite dimensional Kac–Moody algebras and groups [K, Ku], it is a natural question to ask whether one can generalize the theory of Eisenstein series to Kac–Moody groups. Such an attempt is not merely for the sake of generalization. Even though it is conjectural at the moment, a satisfactory theory of Eisenstein series on Kac–Moody groups would have huge impacts on some of the central problems in number theory related to Ramanujan Conjecture, Lindelöf Hypothesis and Langlands Functoriality [BFH, Sh].

It has also been discovered that discrete subgroups of Kac–Moody groups and their automorphic forms appear as symmetries in high-energy theoretical physics. In particular, the group $E_{10}(\mathbb{Z})$ is conjectured to be the discrete invariance group for certain functions that arise in 11-dimensional supersymmetric string theory [DKN, Ga]. Automorphic forms on E_{10} and E_{11} are conjectured to encode higher derivative corrections in string theory and M–theory [DN, DHH+, FGKP, W], and Eisenstein series on these Kac–Moody groups appear as central objects [FK, FKP].

All of these potential applications require establishing analytic properties of Kac-Moody Eisenstein series, including convergence and meromorphic continuation.

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In his pioneering work [G99, G04, G06], Garland extended the classical theory of Eisenstein series to arithmetic quotients $G_{\mathbb{Z}}\backslash G_{\mathbb{R}}/K$ of affine Kac–Moody groups G. In particular, he established absolute convergence for spectral parameters in a Godement range, and then proved a meromorphic continuation beyond it [GMS1, GMS2, GMS3, GMS4]. Absolute convergence has been generalized to affine Kac–Moody groups over number fields by Liu [Li]. Garland, Miller, and Patnaik [GMP] showed that affine Eisenstein series induced from cusp forms are entire functions of the spectral parameter. In the function field case, analogous works were established in [BK, Ka, LL, P].

Beyond the affine case, Carbone, Lee, and Liu [CLL1] studied Eisenstein series on the rank 2 hyperbolic Kac–Moody groups with symmetric generalized Cartan matrices, and established almost-everywhere convergence of the series. They also defined and calculated the degenerate Fourier coefficients for the Eisenstein series and showed that the cuspidal Eisenstein series are entire, extending the results on the affine Kac–Moody groups.

2. Recent Results

In this section, we will review recent results in [CGLLM] and [CLL2] for Eisenstein series on more general Kac–Moody groups.

2.1. Convergence.

In [CGLLM], Carbone, Garland, Lee, Liu and Miller studied the absolute convergence of Eisenstein series on general Kac–Moody groups. Almost-everywhere convergence is proven for an arbitrary symmetrizable Kac–Moody group, but everywhere convergence is established under an assumption on the root system (Property 2.1 below).

Let G be a representation-theoretic Kac–Moody group, and let $\mathfrak{g}$ be the corresponding real Kac–Moody algebra with a fixed Cartan subalgebra $\mathfrak{h}$. We assume that $\mathfrak{g}$ is infinite-dimensional and non-affine. Let $r = \dim(\mathfrak{h})$ denote the rank of G, I the index set $\{1, \ldots, r\}$, and Φ_+ (resp., Φ_-) the positive (resp., negative) roots of $\mathfrak{g}_{\mathbb{C}}$. Then $G_{\mathbb{R}}$ has the Iwasawa decomposition $G_{\mathbb{R}} = UA^+K$, where U is a maximal pro-unipotent subgroup, A^+ is the connected component of a maximal torus, and K is a subgroup of G playing the role of the maximal compact subgroup.

Define the Borel Kac–Moody Eisenstein series $E_\lambda(g)$ for $g \in G_{\mathbb{R}}$ and $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ by

$$E_\lambda(g) \;=\; \sum_{\gamma \in (\Gamma \cap B)\backslash \Gamma} a(\gamma g)^{\lambda+\rho}\,, \tag{2.1}$$

where $a(g)$ is the A^+–component of the Iwasawa decomposition of g, ρ is the Weyl vector, $\Gamma = G_{\mathbb{Z}}$ is the arithmetic subgroup, and $B \supset NA^+$ is a Borel subgroup. Let W be the Weyl group of $\mathfrak{g}$. We denote by $\ell(w)$ the length of $w \in W$ and define

$$\Phi_w \;:=\; \Phi_+ \cap w^{-1}\Phi_-,\; w \in W. \tag{2.2}$$

Consider the following property:

Property 2.1. *Every nontrivial $w \in W$ can be written as $w = vw_\beta$ for some $v \in W$, where $\ell(v) < \ell(w)$, w_β is the reflection associated to a positive simple root β, and $\alpha - \beta$ is never a real root for any $\alpha \in \Phi_v$.*

The main result in [CGLLM] is as follows:

Theorem 2.2 ([CGLLM])**.** *Assume that $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\langle \lambda, \alpha_i^\vee \rangle) > 1$ for each simple coroot $\alpha_i^\vee$, $i \in I$, and that Property 2.1 holds. Then the Kac–Moody Eisenstein series $E_\lambda(g)$ converges absolutely for $g \in \Gamma U A_{\mathfrak{C}} K$, where $A_{\mathfrak{C}} \subset A^+$ is the image of the Tits cone $\mathfrak{C}$ under the exponential map* $\exp : \mathfrak{h} \to A^+$.

The condition on λ is precisely the Godement range, and appears in the classical theory [La2]. Property 2.1 holds when G is of rank 2 or when the Cartan matrix is symmetric and has sufficiently large entries. However, Property 2.1 is not true for all Kac–Moody root systems – it even fails in finite-dimensional groups.

In order to prove Theorem 2.2 we consider the constant term $E_\lambda^\sharp(g)$ of the series $E_\lambda(g)$, which is computed by the Gindikin–Karpelevich formula, and establish its absolute convergence. Unlike Theorem 2.2, which assumes Property 2.1, convergence of the constant term $E_\lambda^\sharp(g)$ holds for *all* symmetrizable G. As a consequence, almost-everywhere convergence of $E_\lambda(g)$ is also true for all symmetrizable G. When we compare $E_\lambda^\sharp(g)$ with $E_\lambda(g)$ to prove everywhere convergence, we need Property 2.1. Though Theorem 2.2 is stated only for Borel Eisenstein series, its conclusions hold for *cuspidally* induced Eisenstein series as well, at least for parabolics with finite-dimensional Levi components. This fact was used in [CLL2].

2.2. **Entirety.** In the process of these developments, a striking difference between the finite-dimensional case and the affine case was observed in [BK, GMP], where the affine Eisenstein series induced from cusp forms on finite-dimensional Levi subgroups were shown to be *entire* and not just meromorphic as they are in the finite-dimensional case. This phenomenon is not restricted to the affine case, and it was shown in [CLL1] that cuspidal Eisenstein series on rank 2 symmetric hyperbolic Kac–Moody groups are also entire.

To understand the analytic properties of cuspidal Eisenstein series on Kac–Moody groups, one may naturally ask which parabolic subgroups of a Kac–Moody group make cuspidal Eisenstein series entire. To answer this question, a natural condition on parabolic subgroups (Property RD) is introduced in [CLL2], and it was shown that cuspidal Eisenstein series attached to a parabolic subgroup satisfying Property RD is holomorphic on the full complex plane.

More precisely, let P be a maximal parabolic subgroup of $G_{\mathbb{R}}$ with fixed Levi decomposition and associated finite-dimensional Levi subgroup M, which is associated with a subset $\theta \subseteq I$. We denote by α_P the simple root associated to the one element index in $I \backslash \theta$. Let L be the derived subgroup of M. For a cusp form f on $(L \cap \Gamma)\backslash L$, we recall that f is unramified if f is right invariant under the action of $L \cap K$.

For such an f, we define the Eisenstein series $E_f(s, g)$, $s \in \mathbb{C}$, $g \in G_{\mathbb{R}}$, in analogy with the classical case. We use the reduction mechanism of [Bo, MW] to obtain absolute convergence of the cuspidal Eisenstein series $E_f(s, g)$ from that of Borel Eisenstein series established in [CGLLM].

Denote by ϖ_P the fundamental weight associated to α_P, and by ρ_M the Weyl vector of M. Let

$$W^\theta = \{ w \in W : w^{-1}\alpha_i > 0, i \in \theta \}.$$

Property 2.3. *A parabolic group P is said to satisfy* Property RD *if there exists a constant $D > 0$, such that for every nontrivial element $w \in W^\theta$ we have*

$$\langle D\varpi_P + \rho_M, \alpha^\vee \rangle \leq 0$$

for any positive root α such that $w^{-1}\alpha < 0$.

Property RD states that the coefficient of the simple root α_P grows faster than the coefficients of the simple roots in the subset θ. This property allows us to make use of the rapid decay of cusp forms on parabolic subgroups.

The main theorem in [CLL2] can be stated as follows:

Theorem 2.4 ([CLL2])**.** *Let f be an unramified cusp form on $(L \cap \Gamma)\backslash L$. If the maximal parabolic subgroup P satisfies Property RD, then for any compact subset $\mathfrak{S}$ of $A_{\mathcal{C}}$, there exists a measure zero subset S_0 of $(\Gamma \cap U)\backslash U\mathfrak{S}$ such that $E_f(s, g)$ is an entire function of $s \in \mathbb{C}$ for $g \in (\Gamma \cap U)\backslash U\mathfrak{S}K - S_0K$.*

Here a measure zero set appears because absolute convergence is only established almost everywhere for the Eisenstein series in general. In the setting of everywhere convergence established in [CGLLM], the measure zero set is not needed.

The main idea of the proof is to exploit rapid decay of a cusp form on the maximal parabolic subgroup, guaranteed by Property RD. It turns out that a large class of Kac–Moody groups have parabolic subgroups satisfying Property RD, including the Kac–Moody group G associated with the Feingold–Frenkel rank 3 hyperbolic Kac–Moody algebra [FF]. It would be an interesting question for future investigation to characterize the full class of Kac–Moody groups that admit parabolic subgroups satisfying Property RD.

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