Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 313 (2007) 988–1004
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Linear algebraic approach to Gröbner–Shirshov basis theory

Seok-Jin Kang [a,1], Dong-il Lee [a,*,1], Kyu-Hwan Lee [b,2], Hyungju Park [c,3]

[a] *Department of Mathematical Sciences, Seoul National University, Seoul 151-747, South Korea*
[b] *Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA*
[c] *School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-722, South Korea*

Received 4 July 2006
Available online 9 February 2007
Communicated by Efim Zelmanov

Abstract

We construct a new efficient algorithm for finding Gröbner–Shirshov bases for noncommutative algebras and their representations. This algorithm uses the Macaulay matrix [F.S. Macaulay, On some formula in elimination, Proc. London Math. Soc. 33 (1) (1902) 3–27], and can be viewed as a representation theoretic analogue of the F_4 algorithm developed by J.C. Faugère. We work out some examples of universal enveloping algebras of Lie algebras and of their representations to illustrate the algorithm.

Keywords: Gröbner–Shirshov basis; Gröbner–Shirshov pair; Monomial basis; Macaulay matrix; Noncommutative algebra; Representation; Simple Lie algebra; Universal enveloping algebra

Introduction

The *Gröbner basis theory* introduced by Buchberger provides an effective algorithm for solving the reduction problem for commutative algebras [7]. In [1], Bergman generalized this to the

* Corresponding author.
E-mail addresses: sjkang@math.snu.ac.kr (S.-J. Kang), dilee@math.snu.ac.kr (D.-i. Lee), khlee@math.uconn.edu (K.-H. Lee), alanpark@kias.re.kr (H. Park).
[1] This research was supported by KRF Grant #2005-070-C00004.
[2] This research was supported by KOSEF Grant #R01-2003-000-10012-0.
[3] This research was supported by NSRI Grant #06006.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.02.001

Diamond Lemma for associative algebras. On the other hand, in [15], Shirshov independently established an equivalent theory for Lie algebras by proving the *Composition Lemma*, and in [2], Bokut applied this method to associative algebras. Since then, the *Gröbner–Shirshov basis theory* has been a useful tool for understanding the structure of associative algebras and their representations.

For finite-dimensional simple Lie algebras, the Gröbner–Shirshov bases were completely determined by Bokut and Klein [4–6]. For classical Lie superalgebras and their universal enveloping algebras, Bokut, Kang, Lee and Malcolmson developed the corresponding theory and gave an explicit construction of Gröbner–Shirshov bases [3].

The Gröbner–Shirshov basis theory for representations of associative algebras was developed in [9], where the notion of the *Gröbner–Shirshov pair* was introduced. More precisely, let $\mathcal{A}_X$ be the free associative algebra generated by a set X over a field. Suppose that (S, T) is a pair of subsets of $\mathcal{A}_X$, I is the two-sided ideal of $\mathcal{A}_X$ generated by S, and J is the left ideal of the algebra $A = \mathcal{A}_X/I$ generated by (the image of) T. Then the left A-module $M = A/J$ is said to be *defined by* the pair (S, T), and the pair (S, T) is called a *Gröbner–Shirshov pair* for M if the set of (S, T)-*standard monomials* forms a linear basis of M, or equivalently, if it is closed under composition. In [9], Kang and Lee proved a generalized version of Shirshov's Composition Lemma for a Gröbner–Shirshov pair (S, T). A Gröbner–Shirshov pair yields an explicit monomial basis of the left A-module. The monomial basis is useful in many respects; for example, one can easily compute the weight of each element in the basis. Moreover, one can give a colored oriented graph structure on the monomial basis, which is called the *Gröbner–Shirshov graph*.

The Gröbner–Shirshov basis theory for representations of associative algebras is quite general, and can be applied to the representation theory of various interesting algebras including finite-dimensional simple Lie (super)algebras, Kac–Moody (super)algebras and (affine) Hecke algebras. The Gröbner–Shirshov basis theory was successfully applied to the representation theory of Hecke algebras (of A-type) and Ariki–Koike algebras in [11] and [12], respectively.

A new efficient algorithm, called the F_4 *algorithm*, for computing Gröbner bases in commutative algebras was introduced by Faugère [8]. The core step in the F_4 algorithm is the reduction process by performing the elementary row operations on the *Macaulay matrix*. In this paper, we construct an analogue of the F_4 algorithm for noncommutative algebras and their representations. The main idea is essentially the same as in the commutative case. However, in our setting, we need to deal with a mixture of reduction procedures on two-sided ideals and on one-sided ideals. To illustrate the algorithm, we work out some examples of universal enveloping algebras and of their representations.

This paper consists of three sections. In the first section, we present a summary of the Gröbner–Shirshov basis theory for noncommutative algebras and their representations developed in [9,10]. In the next section, we describe an analogue of the F_4 algorithms for noncommutative algebras and for their representations, respectively, and we prove the validity of the algorithms. In the last section, we give illustrations for finding Gröbner–Shirshov bases in the cases of universal enveloping algebras of Lie algebras and their representations. The examples on the finite-dimensional simple Lie algebras of type A_2 and G_2 are given. Another explicit calculation of Gröbner–Shirshov pairs for irreducible B_2-modules has been done in [13].

1. Gröbner–Shirshov basis theory

Let X be a set and X^* be the free monoid of associative monomials on X. We denote the empty monomial by 1 and the *length* of a monomial u by $l(u)$. Thus we have $l(1) = 0$.

Definition 1.1. A well-ordering $\prec$ on X^* is called a *monomial order* if $x \prec y$ implies $axb \prec ayb$ for all $a, b \in X^*$.

Fix a monomial order $\prec$ on X^* and let $\mathcal{A}_X$ be the free associative algebra generated by X over a field $\mathbb{F}$. Given a nonzero element $p \in \mathcal{A}_X$, we denote by $\overline{p}$ the maximal monomial (called the *leading monomial*) appearing in p under the ordering $\prec$. Thus $p = \alpha\overline{p} + \sum \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \prec \overline{p}$. The α is called the *leading coefficient* of p and is denoted by $lc(p)$. If $lc(p) = 1$, p is said to be *monic*.

Let (S, T) be a pair of subsets of monic elements in $\mathcal{A}_X$, let I be the two-sided ideal of $\mathcal{A}_X$ generated by S, and let J be the left ideal of the algebra $A = \mathcal{A}_X/I$ generated by (the image of) T. Then we say that the algebra $A = \mathcal{A}_X/I$ is *defined by* S and that the left A-module $M = A/J$ is *defined by the pair* (S, T). The images of $p \in \mathcal{A}_X$ in A and in M under the canonical quotient map will also be denoted by p.

Definition 1.2.

(a) Given a subset S of monic elements in $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be S-*standard* if $u \neq a\overline{s}b$ for any $s \in S$ and $a, b \in X^*$. Otherwise, the monomial u is said to be S-*reducible*.
(b) Given a pair (S, T) of subsets of monic elements in $\mathcal{A}_X$, a monomial $u \in X^*$ is said to be (S, T)-*standard* if $u \neq a\overline{s}b$ and $u \neq c\overline{t}$ for any $s \in S$, $t \in T$ and $a, b, c \in X^*$. Otherwise, the monomial u is said to be (S, T)-*reducible*.

Lemma 1.3. *(See [9,10].)*

(a) *Given a subset S of monic elements in $\mathcal{A}_X$, every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum \alpha_i a_i s_i b_i + \sum \gamma_k u_k, \tag{1.1}$$

where $\alpha_i, \gamma_k \in \mathbb{F}$, $a_i, b_i, u_k \in X^*$, $s_i \in S$, $a_i\overline{s_i}b_i \preccurlyeq \overline{p}$, $u_k \preccurlyeq \overline{p}$ and u_k are S-standard.
(b) *Given a pair (S, T) of subsets of monic elements in $\mathcal{A}_X$, every $p \in \mathcal{A}_X$ can be expressed as*

$$p = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j + \sum \gamma_k u_k, \tag{1.2}$$

where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$, $a_i, b_i, c_j, u_k \in X^*$, $s_i \in S$, $t_j \in T$, $a_i\overline{s_i}b_i \preccurlyeq \overline{p}$, $c_j\overline{t_j} \preccurlyeq \overline{p}$, $u_k \preccurlyeq \overline{p}$ and u_k are (S, T)-standard.

Remark 1.4. The proof of the above lemma actually gives us an algorithm of writing an element p of $\mathcal{A}_X$ in the form (1.1) and (1.2). Thus it may be considered as a *division algorithm*.

The term $\sum \gamma_k u_k$ in the expressions (1.1) and (1.2) is called a *normal form* (or a *remainder*) of p with respect to the subset S and the pair (S, T), respectively. Note that it depends on the monomial order $\prec$. In general, a normal form is not unique.

As an immediate corollary of Lemma 1.3, we obtain:

Proposition 1.5. *(See [9,10].) Consider the algebra $A = \mathcal{A}_X/I$ defined by S and the left A-module $M = A/J$ defined by the pair (S, T), and view them as $\mathbb{F}$-vector spaces. Then,*

(a) *the set of* S*-standard monomials spans* A.
(b) *The set of* (S,T)*-standard monomials spans* $M = A/J$.

Definition 1.6.

(a) A subset S of monic elements in $\mathcal{A}_X$ is a *Gröbner–Shirshov basis* if the set of S-standard monomials forms a linear basis of the algebra $A = \mathcal{A}_X/I$ defined by the subset S. In this case, we say that S is a *Gröbner–Shirshov basis* for the algebra $A = \mathcal{A}_X/I$ defined by S.
(b) A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is a *Gröbner–Shirshov pair* if the set of (S,T)-standard monomials forms a linear basis of the left A-module $M = A/J$ defined by the pair (S,T). In this case, we say that (S,T) is a *Gröbner–Shirshov pair* for the module M defined by (S,T).

Let p and q be monic elements of $\mathcal{A}_X$ with leading monomials $\overline{p}$ and $\overline{q}$. We define the *composition* of p and q as follows.

Definition 1.7.

(a) If there exist a and b in X^* such that $\overline{p}a = b\overline{q} = w$ with $l(\overline{p}) > l(b)$, then the *composition of intersection* is defined to be $(p,q)_w = pa - bq$. Furthermore, if $a = 1$, the composition $(p,q)_w$ is called *right-justified*.
(b) If there exist a and b in X^* such that $a \neq 1$, $a\overline{p}b = \overline{q} = w$, then the *composition of inclusion* is defined to be $(p,q)_{a,b} = apb - q$.

Remark 1.8.

(1) In contrast to the commutative case, the role of a, b is important since there can be different choices of overlaps for given p and q.
(2) We do *not* consider a composition of the type $(p,q)_{a,b} = p - aqb$ with $b \neq 1$, $\overline{p} = a\overline{q}b$. This becomes critical when we consider the notion of *closedness* under composition for a pair (S,T).
(3) Given p, q and $w = a\overline{p}b = \overline{q}$, denoting the composition of inclusion by the notation $(p,q)_w$ is ambiguous. For example, if $p = x_2 + x_3$ and $q = x_1x_2^2x_3$, then $(p,q)_w$ may be $x_1px_2x_3 - q$ or $x_1x_2px_3 - q$. So the monomials a and b must be specified.

Let $p, q \in \mathcal{A}_X$ and $w \in X^*$. We define the *congruence relation* on $\mathcal{A}_X$ as follows: $p \equiv q \bmod (S,T;w)$ if and only if $p - q = \sum \alpha_i a_i s_i b_i + \sum \beta_j c_j t_j$, where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, c_j \in X^*$, $s_i \in S$, $t_j \in T$, $a_i\overline{s_i}b_i \prec w$, and $c_j\overline{t_j} \prec w$. When $T = \emptyset$, we simply write $p \equiv q \bmod (S;w)$.

Definition 1.9.

(a) A subset S of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if
    (i) $(p,q)_w \equiv 0 \bmod (S;w)$ and $(p,q)_{a,b} \equiv 0 \bmod (S;w)$ for all $p, q \in S$, $a, b \in X^*$ whenever the compositions $(p,q)_w$ and $(p,q)_{a,b}$ are defined.
(b) A pair (S,T) of subsets of monic elements in $\mathcal{A}_X$ is said to be *closed under composition* if the subset S satisfies the condition (i) and

(ii) $(p,q)_w \equiv 0 \bmod (S,T;w)$ for all $p,q \in T$, $w \in X^*$ whenever the right-justified composition $(p,q)_w$ is defined,

(iii) $(p,q)_w \equiv 0 \bmod (S,T;w)$ and $(p,q)_{a,b} \equiv 0 \bmod (S,T;w)$ for all $p \in S$, $q \in T$, $a,b \in X^*$ whenever the compositions $(p,q)_w$ and $(p,q)_{a,b}$ are defined.

The following theorem is the main result of [9,10], which is a generalization of Shirshov's Composition Lemma (for Lie algebras and associative algebras) to the representations of associative algebras.

Theorem 1.10. *(See [9].) Let (S,T) be a pair of subsets of monic elements in the free associative algebra $\mathcal{A}_X$ generated by X, let $A = \mathcal{A}_X/I$ be the associative algebra defined by S, and let $M = A/J$ be the left A-module defined by (S,T).*

(a) *If S is closed under composition and the image of $p \in \mathcal{A}_X$ is trivial in A, then the word $\overline{p}$ is S-reducible.*
(b) *If (S,T) is closed under composition and the image of $p \in \mathcal{A}_X$ is trivial in M, then the word $\overline{p}$ is (S,T)-reducible.*

As a corollary, we obtain:

Proposition 1.11. *(See [10].)*

(a) *Let S be a subset of monic elements in $\mathcal{A}_X$. Then the following conditions are equivalent*:
  (i) *S is a Gröbner–Shirshov basis.*
  (ii) *S is closed under composition.*
  (iii) *For each $p \in \mathcal{A}_X$, the normal form of p with respect to S is unique.*
(b) *Let (S,T) be a pair of subsets of monic elements in $\mathcal{A}_X$. Then the following conditions are equivalent*:
  (i) *(S,T) is a Gröbner–Shirshov pair.*
  (ii) *(S,T) is closed under composition.*
  (iii) *For each $p \in \mathcal{A}_X$, the normal form of p with respect to (S,T) is unique.*

The part (ii) of (a) and (b) in Proposition 1.11 gives an analogue of *Buchberger's algorithm* as we describe in the following. Suppose that (S,T) is a pair of subsets of monic elements in $\mathcal{A}_X$, $A = \mathcal{A}_X/I$ is the associative algebra defined by S, and $M = A/J$ is the left A-module defined by (S,T). We will show how one can enlarge the pair (S,T) to get a Gröbner–Shirshov pair for the A-module M. This can be considered as an analogue of *Buchberger's algorithm*. However, in general, there is no guarantee that this process will terminate in a finite number of steps. Nevertheless, such an algorithm works in many interesting cases.

For any subset R of $\mathcal{A}_X$, we define

$$\widehat{R} = \{p/\alpha \mid \alpha \in \mathbb{F} \text{ is the leading coefficient of } p \in R\}.$$

Let $S^{(0)} = S = \widehat{S}$. For $i \geqslant 0$, set

$$S_{(i)} = \{(f,g)_w \not\equiv 0,\ (f,g)_{a,b} \not\equiv 0 \bmod (S^{(i)};w) \mid f,g \in S^{(i)}\},$$
$$S^{(i+1)} = S^{(i)} \cup \widehat{S}_{(i)}.$$

Then, clearly, the set $\mathcal{S} = \bigcup_{i \geqslant 0} S^{(i)}$ is closed under composition. Note that the algebra A is defined also by $\mathcal{S}$.

Let $T^{(0)} = T = \widehat{T}$. For $i \geqslant 0$, set

$$T_{(i)} = \{(f, g)_w \not\equiv 0 \bmod (\mathcal{S}, T^{(i)}; w) \mid f, g \in T^{(i)},\ (f, g)_w \text{ is right-justified}\},$$
$$T^{(i+1)} = T^{(i)} \cup \widehat{T}_{(i)}.$$

Then the set $T^c = \bigcup_{i \geqslant 0} T^{(i)}$ is closed under the right-justified composition with respect to $\mathcal{S}$.

We now consider the compositions between $\mathcal{S}$ and T^c. Let $X^{(0)} = T^c$. For $i \geqslant 0$, set

$$X_{(i)} = \{(f, g)_w \not\equiv 0,\ (f, g)_{a,b} \not\equiv 0 \bmod (\mathcal{S}, X^{(i)}; w) \mid f \in \mathcal{S},\ g \in X^{(i)}\},$$
$$X^{(i+1)} = (X^{(i)} \cup \widehat{X}_{(i)})^c.$$

Let $\mathcal{T} = \bigcup_{i \geqslant 0} X^{(i)}$. Then the A-module M is defined also by the new pair $(\mathcal{S}, \mathcal{T})$, and $(\mathcal{S}, \mathcal{T})$ is a Gröbner–Shirshov pair of M by construction.

2. Description of a new algorithm

In [8], J.C. Faugère introduced the F_4 *algorithm*, a new efficient algorithm for computing Gröbner bases in the commutative case. We construct an analogue of the F_4 algorithm for noncommutative algebras and their representations. The basic steps in the F_4 algorithm are the setting of reduction data, the reduction process by the elementary row operations and the adjunction of nontrivial compositions. We begin with some notations and definitions.

Let X be a finite set. We fix a monomial order $\prec$ on X^*. For a subset F of $\mathcal{A}_X$, we define $\overline{F} = \{\overline{f} \mid f \in F\}$ and $M(F) = \bigcup_{f \in F} M(f)$, where $M(f) \subseteq X^*$ is the set of all monomials appearing in f. We denote by $Id(F)$ the two-sided ideal of $\mathcal{A}_X$ generated by F. For given monic subsets S, T, T' of $\mathcal{A}_X$, the left ideal of $\mathcal{A}_X$ generated by T is denoted by $Ld(T)$, and we denote by $Ld(T) = Ld(T')$ if they are equal mod $Id(S)$, i.e., if they are equal in $A = \mathcal{A}_X / Id(S)$.

Definition 2.1. Consider noncommutative polynomials $f, p, q \in \mathcal{A}_X$ with $p, q \neq 0$ and a pair (P, Q) of subsets of $\mathcal{A}_X$.

(a) We say that f *reduces to* g *mod* p, and denote it by $f \underset{p}{\longrightarrow} g$, if there exist $m \in M(f)$ and $s, s' \in X^*$ such that $s\overline{p}s' = m$ and $g = f - \frac{c_m}{lc(p)} sps'$ where c_m is the coefficient of m in f. f *is reducible* mod p if there exists $g \in \mathcal{A}_X$ such that $f \underset{p}{\longrightarrow} g$, and in particular, f *is top-reducible* mod p if there exists $g \in \mathcal{A}_X$ such that $f \underset{p}{\longrightarrow} g$ and $\overline{g} \prec \overline{f}$. We say that f *reduces to* g mod P, denoted by $f \underset{P}{\longrightarrow} g$, if there exist sequences $p_1, \ldots, p_n \in P$ and $g_1, \ldots, g_n = g \in \mathcal{A}_X$ such that $f \underset{p_1}{\longrightarrow} g_1 \underset{p_2}{\longrightarrow} \cdots \underset{p_n}{\longrightarrow} g$. f *is reducible* mod P if there exist sequences $p_1, \ldots, p_n \in P$ and $g_1, \ldots, g_n \in \mathcal{A}_X$ such that $f \underset{p_1}{\longrightarrow} g_1 \underset{p_2}{\longrightarrow} \cdots \underset{p_n}{\longrightarrow} g_n$, and in particular, f *is top-reducible* mod P if there exist sequences $p_1, \ldots, p_n \in P$ and $g_1, \ldots, g_n \in \mathcal{A}_X$ such that $f \underset{p_1}{\longrightarrow} g_1 \underset{p_2}{\longrightarrow} \cdots \underset{p_n}{\longrightarrow} g_n$ and $\overline{g_n} \prec \overline{f}$.

(b) We say that f *reduces to* g mod (p,q), denoted by $f \underset{(p,q)}{\longrightarrow} g$, if there exist $m, m' \in M(f)$ and $s, s', s'' \in X^*$ such that $s\overline{p}s' = m$, $s''\overline{q} = m'$ and $g = f - \frac{c_m}{lc(p)} sps' - \frac{c_{m'}}{lc(q)} s''q$ where c_m and $c_{m'}$ are the coefficients of m and m' in f, respectively.

The following are similarly defined: f *is reducible* mod (p,q), f *is top-reducible* mod (p,q), f *reduces to* g mod (P,Q), f *is reducible* mod (P,Q), and f *is top-reducible* mod (P,Q).

Let $A = (a_{ij})$ be an $s \times n$ matrix over the field $\mathbb{F}$ and $M_A = (m_1, \dots, m_n)$ be an ordered set of distinct monomials. We define $\mathrm{Rows}(A, M_A)$ to be $\{\sum_{j=1}^{n} a_{ij} m_j \mid i = 1, \dots, s\} \setminus \{0\}$, the set of polynomials given by (A, M_A).

Conversely, for a set F of polynomials in $\mathcal{A}_X$, we make $M(F)$ an ordered set using the given monomial order. Then we obtain the $|F| \times |M(F)|$ matrix A_F, so-called the *Macaulay matrix* of F [14], whose (i,j)-entry is the coefficient of jth monomial in the ith polynomial. The matrix $\widetilde{A}$ denotes the unique reduced row echelon form of A, and we call the set $\widetilde{F} = \mathrm{Rows}(\widetilde{A_F}, M(F))$ the *reduced row echelon form of* F.

Let (S,T) be a pair of finite subsets of monic elements in $\mathcal{A}_X$.

Definition 2.2. We define the *composition pairs* of several types as follows.

(a) $\mathit{Comp}^1(p,q;w) := (pa, bq)$ for all $p, q \in S$, $w \in X^*$ whenever the composition of intersection $(p,q)_w = pa - bq$ is defined.
(b) $\mathit{Comp}^2(p,q;a,b) := (apb, q)$ for all $p, q \in S$, $a, b \in X^*$ whenever the composition of inclusion $(p,q)_{a,b} = apb - q$ is defined.

$\mathit{Comp}^i(p,q;-)$ $(i = 1,2)$ are called the *composition pairs from* S. The set of all composition pairs from S is denoted by $\mathscr{P}(S)$.

(c) $\mathit{Comp}^3(p,q;w) := (p, bq)$ for all $p, q \in T$, $w \in X^*$ whenever the right-justified composition $(p,q)_w = p - bq$ is defined.
(d) $\mathit{Comp}^4(p,q;w) := (pa, bq)$ for all $p \in S$, $q \in T$, $w \in X^*$ whenever the composition of intersection $(p,q)_w = pa - bq$ is defined.
(e) $\mathit{Comp}^5(p,q;a,b) := (apb, q)$ for all $p \in S$, $q \in T$, $a, b \in X^*$ whenever the composition of inclusion $(p,q)_{a,b} = apb - q$ is defined.

$\mathit{Comp}^i(p,q;-)$ $(i = 3,4,5)$ are called the *composition pairs from* (S,T). The set of all composition pairs from (S,T) is denoted by $\mathscr{P}(S,T)$.

Definition 2.3.

(a) The set of all *composition data from* S is defined by

$$\mathscr{D}(S) = \{ f \in \mathcal{A}_X \mid \text{there exists } g \in \mathcal{A}_X \text{ such that } (f,g) \in \mathscr{P}(S) \text{ or } (g,f) \in \mathscr{P}(S) \}.$$

(b) The set of all *composition data from* (S,T) is defined by

$$\mathscr{D}(S,T) = \{ f \in \mathcal{A}_X \mid \text{there exists } g \in \mathcal{A}_X \text{ such that } (f,g) \in \mathscr{P}(S,T) \text{ or } (g,f) \in \mathscr{P}(S,T) \}.$$

2.1. A linear algebraic algorithm for computing Gröbner–Shirshov bases for noncommutative algebras

Let us describe an analogue of the F_4 algorithm for Gröbner–Shirshov bases for noncommutative algebras.

Let $S^{(0)} = S$. Assume, inductively, that the set $S^{(i)}$ of monic polynomials is given for $i \geqslant 0$. We set

$$D_i = \mathscr{D}\big(S^{(i)}\big), \quad \text{and} \quad P_i = D_i \setminus D_{i-1} \quad (D_{-1} = \emptyset).$$

Let $a_i = \min\{\deg f \mid f \in P_i\}$ and $b_i = \max\{\deg f \mid f \in P_i\}$, and let $P_i(d) \subseteq P_i$ be the subset consisting of elements of degree d, $a_i \leqslant d \leqslant b_i$. Define $F_i(d)$ to perform the reduction process, and one obtains $\widetilde{F_i(d)}^+$, the set of nontrivial compositions, for each d inductively from a_i to b_i, as described in the following:

Let $P_i(a_i)^{(0)} = P_i(a_i)$, and assume, inductively on k, that $P_i(a_i)^{(k)}$ is given for $k \geqslant 0$. Note that the monomials in $\overline{P_i}$ are reducible mod $S^{(i)}$. For each $m \in M(P_i(a_i)^{(k)}) \setminus \overline{P_i(a_i)^{(k)}}$ which is reducible mod $S^{(i)}$, choose a polynomial $f \in S^{(i)}$ and $m', m'' \in X^*$ such that $m = m'\overline{f}m''$. Set $P_i(a_i)^{(k+1)}$ to be the set of all such $m'fm''$, and define the set of all *reduction data* of degree a_i to be

$$F_i(a_i) = \bigcup_{k \geqslant 0} P_i(a_i)^{(k)}.$$

Then perform the elementary row operation, the core step in the F_4 algorithm, to obtain $\widetilde{F_i(a_i)}$. Define

$$\widetilde{F_i(a_i)}^+ = \big\{f \in \widetilde{F_i(a_i)} \bigm| \overline{f} \notin \overline{F_i(a_i)}\big\}.$$

The case for $a_i < d \leqslant b_i$ can be treated in a similar way. Let $P_i(d)^{(0)} = P_i(d)$, and assume, inductively on k, that $P_i(d)^{(k)}$ is given for $k \geqslant 0$. For each

$$m \in M\big(P_i(d)^{(k)}\big) \setminus \overline{P_i(d)^{(k)}} \quad \text{which is reducible mod } S^{(i)} \cup \bigcup_{j=a_i}^{d-1} \widetilde{F_i(j)}^+,$$

choose a polynomial $f \in S^{(i)} \cup \bigcup_{j=a_i}^{d-1} \widetilde{F_i(j)}^+$ and $m', m'' \in X^*$ such that $m = m'\overline{f}m''$. Set $P_i(d)^{(k+1)}$ to be the set of all such $m'fm''$, and define

$$F_i(d) = \bigcup_{k \geqslant 0} P_i(d)^{(k)}.$$

After calculating the reduced row echelon form $\widetilde{F_i(d)}$ of $F_i(d)$, define

$$\widetilde{F_i(d)}^+ = \big\{f \in \widetilde{F_i(d)} \bigm| \overline{f} \notin \overline{F_i(d)}\big\}.$$

Now write $F_i = \bigcup_{d=a_i}^{b_i} F_i(d)$, $\widetilde{F_i} = \bigcup_{d=a_i}^{b_i} \widetilde{F_i(d)}$ and $\widetilde{F_i}^+ = \bigcup_{d=a_i}^{b_i} \widetilde{F_i(d)}^+$. Finally, putting

$$S^{(i+1)} = S^{(i)} \cup \widetilde{F_i}^+$$

completes the ith step. After iterating this process, one obtains $\mathcal{S} = \bigcup_{i \geqslant 0} S^{(i)}$.

In the rest of this subsection, we will prove that the set $\mathcal{S}$ is a Gröbner–Shirshov basis for the noncommutative algebra A. We use the same notations as above.

Remark 2.4. For the purpose of illustrations in the next section, we use the degree-lexicographic order $\prec$. For each fixed degree d from a_i to b_i, note that $F_i(d)$ is a finite set because the number of monomials less than a fixed monomial is finite.

Proposition 2.5. *For any set S, we define V_S to be the vector space spanned by S. For any subset $F_i^- \subseteq F_i$ such that $|F_i^-| = |\overline{F_i}|$ and $\overline{F_i^-} = \overline{F_i}$, the set $G_i = F_i^- \cup \widetilde{F_i}^+$ is a triangular basis of V_{F_i}. That is, for any $f \in V_{F_i}$, there exist λ_k's in $\mathbb{F}$ and g_k's in G_i such that $f = \sum \lambda_k g_k$, $\overline{g_1} = \overline{f}$ and $\overline{g_k} \succ \overline{g_{k+1}}$.*

Remark 2.6. Note that, in order to have a subset F_i^-, we select only one polynomial from each set of polynomials in F_i with the same leading monomial.

Proof. Let $\widetilde{F_i}^- = \widetilde{F_i} \setminus \widetilde{F_i}^+$. We know $V_{F_i} = V_{\widetilde{F_i}}$ by the properties of elementary row operations, and clearly $V_{G_i} \subseteq V_{F_i} = V_{\widetilde{F_i}}$. Since G_i consists of polynomials with pairwise distinct leading monomials and $\overline{\widetilde{F_i}} = \overline{\widetilde{F_i}^-} \cup \overline{\widetilde{F_i}^+} = \overline{F_i^-} \cup \overline{\widetilde{F_i}^+} = \overline{G_i}$, we have $\dim V_{\widetilde{F_i}} = |\overline{\widetilde{F_i}}| = |\overline{G_i}| = \dim V_{G_i}$. Hence $V_{F_i} = V_{\widetilde{F_i}} = V_{G_i}$. □

Lemma 2.7. *For all $h \in \widetilde{F_i}^+$, we have $\bar{h} \notin Id(\overline{S^{(i)}})$. So $\widetilde{F_i}^+ = \{f \in \widetilde{F_i} \mid f$ is not top-reducible* mod $S^{(i)}\}$.

Proof. Suppose that there exists $h \in \widetilde{F_i}^+$ satisfying $\bar{h} \in Id(\overline{S^{(i)}})$. Then $\bar{h} = m\bar{g}m'$ for some $g \in S^{(i)}$ and $m, m' \in X^*$. Since $\bar{h}$ is in $M(\widetilde{F_i}^+) \subseteq M(\widetilde{F_i}) = M(F_i)$ and is reducible mod g, one gets $mgm' \in F_i$ (or another polynomial with the same leading monomial is in F_i) by the definition of F_i. But this is a contradiction to $\bar{h} \notin \overline{F_i}$. □

Lemma 2.8.

(a) *For each $i \geqslant 0$, $\widetilde{F_i}^+$ is a finite subset of $Id(S^{(i)})$.*
(b) *For all $f \in V_{P_i}$, f reduces to* 0 *mod $S^{(i)} \cup \widetilde{F_i}^+$.*

Proof. Since the algorithm starts with a finite set S, the sets P_i, F_i, $\widetilde{F_i}$ and $\widetilde{F_i}^+$ are also finite. Clearly, $P_i \subset Id(S^{(i)})$, $F_i \subset Id(S^{(i)})$ and $\widetilde{F_i} \subset Id(S^{(i)})$. This proves part (a). Also F_i^- satisfying the condition of Proposition 2.5 is a subset of $Id(S^{(i)})$. Since $G_i = F_i^- \cup \widetilde{F_i}^+$ is a basis of V_{F_i} and V_{P_i} is a subspace of V_{F_i}, we have proved (b). □

Using Lemma 2.8, we can prove that our algorithm yields a Gröbner–Shirshov basis $\mathcal{S}$ for the noncommutative algebra A.

Theorem 2.9. *The algorithm of this subsection computes a Gröbner–Shirshov basis $\mathcal{S}$ in $\mathcal{A}_X$ such that $S \subseteq \mathcal{S}$ and $Id(S) = Id(\mathcal{S})$.*

Proof. We have $\mathcal{S} = \bigcup_{i \geqslant 0} S^{(i)} = S \cup \bigcup_{i \geqslant 0} \widetilde{F}_i^+$. By Lemma 2.8(a), each $S^{(i)}$ is a finite subset of $\mathcal{A}_X$ such that $S \subseteq S^{(i)} \subseteq Id(S)$. For all $p, q \in S^{(i)}$ for which the compositions are defined, $(p,q)_w$ and $(p,q)_{a,b}$ are in $V_{D_i} = \bigoplus_{j=0}^{i} V_{P_j}$. Hence by Lemma 2.8(b), $(p,q)_w$ and $(p,q)_{a,b}$ reduce to 0 mod $S^{(i+1)}$. It follows that the set $\mathcal{S}$ is closed under composition, and by Proposition 1.11, $\mathcal{S}$ is a Gröbner–Shirshov basis for the algebra A. □

2.2. A linear algebraic algorithm for computing Gröbner–Shirshov pairs for representations of noncommutative algebras

Now let us describe an analogue of the F_4 algorithm for a Gröbner–Shirshov pair for the A-module M. The algorithm for the left-ideal part $T^{(i)}$ ($i \geqslant 0$) is essentially the same as the algebra part except for some necessary modifications.

Assume that $S^{(i)}$ has been obtained for each i by the algorithm described in the previous subsection. Let $T^{(0)} = T$. Inductively, assume that we are given the set $T^{(i)}$ of monic polynomials ($i \geqslant 0$). Set

$$D_i = \mathscr{D}\big(S^{(i+1)}, T^{(i)}\big), \quad \text{and} \quad P_i = D_i \setminus D_{i-1} \quad (D_{-1} = \emptyset).$$

Let $a_i = \min\{\deg f \mid f \in P_i\}$ and $b_i = \max\{\deg f \mid f \in P_i\}$. Let $P_i(d) \subseteq P_i$ be the subset consisting of elements of degree d, $a_i \leqslant d \leqslant b_i$. Define the sets $F_i(d)$ and $\widetilde{F_i(d)}^+$ for each d inductively from a_i to b_i as follows:

Let $P_i(d)^{(0)} = P_i(d)$, and assume, inductively on k, that $P_i(d)^{(k)}$ is given for $k \geqslant 0$. For each $m \in M(P_i(d)^{(k)}) \setminus \overline{P_i(d)^{(k)}}$ which is reducible mod $(S^{(i+1)}, T^{(i)} \cup \bigcup_{j=a_i}^{d-1} \widetilde{F_i(j)}^+)$ (mod $(S^{(i+1)}, T^{(i)})$ if $d = a_i$), choose $f \in S^{(i+1)}$ and $m', m'' \in X^*$ such that $m = m'\overline{f}m''$, or choose $f \in T^{(i)} \cup \bigcup_{j=a_i}^{d-1} \widetilde{F_i(j)}^+$ ($f \in T^{(i)}$ if $d = a_i$) and $m' \in X^*$ such that $m = m'\overline{f}$.

Set $P_i(d)^{(k+1)}$ to be the set of all such $m'fm''$ or $m'f$, and define

$$F_i(d) = \bigcup_{k \geqslant 0} P_i(d)^{(k)}.$$

Define

$$\widetilde{F_i(d)}^+ = \big\{ f \in \widetilde{F_i(d)} \bigm| \overline{f} \notin \overline{F_i(d)} \big\},$$

where $\widetilde{F_i(d)}$ is the reduced row echelon form of $F_i(d)$.

As in the algebra case, we write

$$F_i = \bigcup_{d=a_i}^{b_i} F_i(d), \qquad \widetilde{F}_i = \bigcup_{d=a_i}^{b_i} \widetilde{F_i(d)} \quad \text{and} \quad \widetilde{F}_i^+ = \bigcup_{d=a_i}^{b_i} \widetilde{F_i(d)}^+.$$

Finally, putting

$$T^{(i+1)} = T^{(i)} \cup \widetilde{F}_i^+.$$

completes the ith step. After repeating this process, we define $\mathcal{T} = \bigcup_{i \geqslant 0} T^{(i)}$.

In the rest of this subsection, we will prove that the pair $(\mathcal{S},\mathcal{T})$ is a Gröbner–Shirshov pair for the A-module M. We use the same notations as above.

Remark 2.10. Note that $F_i(d)$ is a finite set and Proposition 2.5 holds, as in the algebra case.

Lemma 2.11. *For all $h\in\widetilde{F_i}^+$, we have $\bar h\notin Id(\overline{S^{(i+1)}})\cup Ld(\overline{T^{(i)}})$. So $\widetilde{F_i}^+=\{f\in\widetilde{F_i}\mid f$ is not top-reducible* mod $(S^{(i+1)},T^{(i)})\}$.

Proof. By Lemma 2.7, $\bar h\notin Id(\overline{S^{(i+1)}})$. Suppose that there exists $h\in\widetilde{F_i}^+$ satisfying $\bar h\in Ld(\overline{T^{(i)}})$. Then $\bar h=m\bar g$ for some $g\in T^{(i)}$ and $m\in X^*$. Since $\bar h$ is in $M(\widetilde{F_i}^+)\subseteq M(\widetilde{F_i})=M(F_i)$ and is reducible mod g, one gets $mg\in F_i$ (or another polynomial with the same leading monomial is in F_i) by the definition of F_i. A contradiction to $\bar h\notin\overline{F_i}$. □

Lemma 2.12.

(a) *$\widetilde{F_i}^+$ is a finite subset of $Id(S^{(i+1)})\cup Ld(T^{(i)})$.*
(b) *For all $f\in V_{P_i}$, f reduces to* 0 mod $(S^{(i+1)},T^{(i)}\cup\widetilde{F_i}^+)$.

Proof. Since the algorithm starts with finite sets S and T, the sets P_i, F_i, $\widetilde{F_i}$ and $\widetilde{F_i}^+$ are also finite. Clearly, $P_i\subset Id(S^{(i+1)})\cup Ld(T^{(i)})$, $F_i\subset Id(S^{(i+1)})\cup Ld(T^{(i)})$ and $\widetilde{F_i}\subset Id(S^{(i+1)})\cup Ld(T^{(i)})$. Hence F_i^- satisfying the condition of Proposition 2.5 is a subset of $Id(S^{(i+1)})\cup Ld(T^{(i)})$. Since $G_i=F_i^-\cup\widetilde{F_i}^+$ is a basis of V_{F_i} and V_{P_i} is a subspace of V_{F_i}, part (b) follows. □

Using this Lemma 2.12, we can prove that our algorithm gives a Gröbner–Shirshov pair $(\mathcal{S},\mathcal{T})$ of the A-module M.

Theorem 2.13. *The algorithm of this subsection computes a Gröbner–Shirshov pair $(\mathcal{S},\mathcal{T})$ such that $S\subseteq\mathcal{S}$, $T\subseteq\mathcal{T}$, $Id(S)=Id(\mathcal{S})$ in $\mathcal{A}_X$ and $Ld(T)=Ld(\mathcal{T})$ in $A=\mathcal{A}_X/Id(S)$.*

Proof. We have $\mathcal{T}=\bigcup_{i\geqslant 0}T^{(i)}=T\cup\bigcup_{i\geqslant 0}\widetilde{F_i}^+$. By Lemma 2.12(a), each $T^{(i)}$ is a finite subset of $\mathcal{A}_X$ such that $T\subseteq T^{(i)}\subseteq Id(S)\cup Ld(T)$. For all $p\in S^{(i+1)}$, $q,q'\in T^{(i)}$ for which the compositions are defined, $(q,q')_w$, $(p,q)_w$ and $(p,q)_{a,b}$ are in $V_{D_i}=\bigoplus_{j=0}^{i}V_{P_j}$. Hence by Lemma 2.12(b), they reduce to 0 mod $(S^{(i+1)},T^{(i+1)})$. It follows that the pair $(\mathcal{S},\mathcal{T})$ is closed under composition. □

3. Examples

In this section, we give some examples to show how our algorithms work. The base field is fixed to be the complex field $\mathbb{C}$, and the degree-lexicographic order $\prec$ is used throughout this section. We keep all the notional conventions in the previous sections. Note that $\mathscr{D}(S)$ consists of all the components pa, bq, apb, q in each composition pair of types $Comp^1(p,q;w)$ and $Comp^2(p,q;a,b)$, and that $\mathscr{D}(S,T)$ consists of all the components p, bq, pa, bq, apb, q in each composition pair of types $Comp^3(p,q;w)$, $Comp^4(p,q;w)$ and $Comp^5(p,q;a,b)$.

3.1. Lie algebra $\mathfrak{sl}_3$

Recall that the special linear Lie algebra $\mathfrak{sl}_3$ is the Kac–Moody algebra associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. Hence the algebra U_- is the associative algebra defined by the set $S = \{[[f_2 f_1] f_1], [f_2[f_2 f_1]]\}$ of the Serre relations in $\mathcal{A}_X$, where $X = \{f_1, f_2\}$ and $[xy] = xy - yx$. We define $f_1 \prec f_2$.

Let $S^{(0)} = S = \{p := f_2^2 f_1 - 2 f_2 f_1 f_2 + f_1 f_2^2, q := f_2 f_1^2 - 2 f_1 f_2 f_1 + f_1^2 f_2\}$. The possible composition pair from $S^{(0)}$ is $Comp^1(p, q; f_2^2 f_1^2) = (p f_1, f_2 q)$, which yields

$$P_0(4) = \left\{ f_2^2 f_1^2 - 2 f_2 f_1 f_2 f_1 + f_1 f_2^2 f_1, f_2^2 f_1^2 - 2 f_2 f_1 f_2 f_1 + f_2 f_1^2 f_2 \right\}.$$

Note that each polynomial is homogeneous of degree 4.

Since $f_1 f_2^2 f_1 = f_1 \overline{p}$ and $f_2 f_1^2 f_2 = \overline{q} f_2$, we have

$$P_0(4)^{(1)} = \{f_1 p, q f_2\} = \left\{ f_1 f_2^2 f_1 - 2 f_1 f_2 f_1 f_2 + f_1^2 f_2^2, f_2 f_1^2 f_2 - 2 f_1 f_2 f_1 f_2 + f_1^2 f_2^2 \right\}.$$

None of the monomials in $M(P_0(4)^{(1)}) \setminus \overline{P_0(4)^{(1)}}$ are reducible mod $S^{(0)}$. From $F_0(4) = P_0(4) \cup P_0(4)^{(1)}$, we get

$$A_{F_0(4)} = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{A_{F_0(4)}} = \begin{pmatrix} 1 & -2 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since there is no polynomial with new leading monomial, we have $\widetilde{F_0(4)}^+ = \emptyset$ and $S^{(1)} = S^{(0)}$. Therefore the algorithm terminates at the first loop; that is, S is already closed under compositions and is a Gröbner–Shirshov basis.

3.2. Irreducible $\mathfrak{sl}_3$-module $V(\Lambda_1 + \Lambda_2)$

Let $V(\lambda)$ be the irreducible highest weight module over $\mathfrak{sl}_3$ with highest weight $\lambda = \Lambda_1 + \Lambda_2$. Then, as a left U_--module, $V(\lambda)$ is defined by the pair (S, T_λ), where

$$S = S^{(0)} = \left\{ p := f_2^2 f_1 - 2 f_2 f_1 f_2 + f_1 f_2^2,\ q := f_2 f_1^2 - 2 f_1 f_2 f_1 + f_1^2 f_2 \right\},$$
$$T_\lambda = T^{(0)} = \left\{ f_1^2, f_2^2 \right\}.$$

As computed in the last subsection, $S^{(1)} = S^{(0)}$. The possible composition pairs from $(S^{(1)}, T^{(0)})$ are

$$Comp^4\left(p, f_1^2; f_2^2 f_1^2\right) = \left(p f_1, f_2^2 f_1^2\right),$$
$$Comp^4\left(q, f_1^2; f_2 f_1^3\right) = \left(q f_1, f_2 f_1^3\right),$$
$$Comp^4\left(q, f_1^2; f_2 f_1^2\right) = \left(q, f_2 f_1^2\right).$$

Note that each component of composition pairs is homogeneous of degree 3 or 4.

First, observe that $P_0(3) = \{f_2 f_1^2 - 2 f_1 f_2 f_1 + f_1^2 f_2,\ f_2 f_1^2\}$. Since none of the monomials in $M(P_0(3)) \setminus \overline{P_0(3)}$ are reducible mod $(S^{(1)}, T^{(0)})$, we have $F_0(3) = P_0(3)$ and $M(F_0(3)) = \{f_2 f_1^2,\ f_1 f_2 f_1,\ f_1^2 f_2\}$. Thus

$$A_{F_0(3)} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{A_{F_0(3)}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}.$$

Since the polynomial $f_1 f_2 f_1 - \frac{1}{2} f_1^2 f_2$ has a new leading monomial,

$$\widetilde{F_0(3)}^+ = \left\{ t_1 := f_1 f_2 f_1 - \frac{1}{2} f_1^2 f_2 \right\}.$$

Next, note that $P_0(4) = \{f_2^2 f_1^2 - 2 f_2 f_1 f_2 f_1 + f_1 f_2^2 f_1,\ f_2^2 f_1^2,\ f_2 f_1^3 - 2 f_1 f_2 f_1^2 + f_1^2 f_2 f_1,\ f_2 f_1^3\}$. An explicit computation shows that

$$A_{F_0(4)} = \begin{pmatrix}
1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\
0 & 0 & 1 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}, \qquad
\widetilde{A_{F_0(4)}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Since the polynomial $f_1^3 f_2$ has a new leading monomial, $\widetilde{F_0(4)}^+ = \{t_2 := f_1^3 f_2\}$. Thus

$$T^{(1)} = \left\{ f_1^2,\ f_2^2,\ t_1 := f_1 f_2 f_1 - \frac{1}{2} f_1^2 f_2,\ t_2 := f_1^3 f_2 \right\}.$$

From $S^{(2)} = S$ and $T^{(1)}$, we compute the set P_1. The possible composition pairs are

$$Comp^4\big(p, t_1;\ f_2^2 f_1 f_2 f_1\big) = \big(p f_2 f_1,\ f_2^2 t_1\big), \qquad Comp^4\big(q, t_1;\ f_2 f_1^2 f_2 f_1\big) = (q f_2 f_1,\ f_2 f_1 t_1),$$
$$Comp^4\big(q, t_2;\ f_2 f_1^3 f_2\big) = (q f_1 f_2,\ f_2 t_2), \qquad Comp^4\big(p, t_2;\ f_2^2 f_1^3 f_2\big) = \big(p f_1^2 f_2,\ f_2^2 t_2\big),$$
$$Comp^4\big(q, t_2;\ f_2 f_1^4 f_2\big) = \big(q f_1^2 f_2,\ f_2 f_1 t_2\big),$$

which consist of monomials of homogeneous degree 5 or 6. We compute $F_1(5)$ and $\widetilde{F_1(5)}$ to obtain

$$\widetilde{F_1(5)}^+ = \big\{ f_2 f_1 f_2 f_1 f_2,\ f_1^2 f_2 f_1 f_2 \big\}.$$

For the degree 6, we have $\widetilde{F_1(6)}^+ = \emptyset$. Thus

$$T^{(2)} = \left\{ f_1^2,\ f_2^2,\ t_1 := f_1 f_2 f_1 - \frac{1}{2} f_1^2 f_2,\ t_2 := f_1^3 f_2,\ t_3 := f_2 f_1 f_2 f_1 f_2,\ t_4 := f_1^2 f_2 f_1 f_2 \right\}.$$

From $S^{(3)} = S$ and $T^{(2)}$, a computation shows $\widetilde{F}_2^+ = \emptyset$, that is, the algorithm terminates. Finally, we obtain

$$\mathcal{S} = S,$$
$$\mathcal{T} = T^{(2)} = T_\lambda \cup \widetilde{F_0}^+ \cup \widetilde{F_1}^+ = \left\{ f_1^2, f_2^2, f_1(f_2f_1) - \frac{1}{2} f_1^2 f_2, f_1^3 f_2, (f_2f_1)^2 f_2, f_1^2(f_2f_1)f_2 \right\}.$$

Note that the $\mathcal{S}$-standard monomials are of the form $f_1^a(f_2f_1)^b f_2^c$ $(a, b, c \geqslant 0)$ and that the set of $(\mathcal{S}, \mathcal{T})$-standard monomials is given by

$$\{1, f_1, f_2, (f_2f_1), f_1f_2, f_1^2f_2, (f_2f_1)f_2, f_1(f_2f_1)f_2\},$$

whose cardinality is 8, equal to the dimension of $V(\Lambda_1 + \Lambda_2)$. This result coincides with that of [9], and our linear algebraic algorithm works well in this case.

3.3. *Lie algebra* G_2

Let us consider the exceptional Lie algebra G_2. Recall that the simple Lie algebra G_2 is a Kac–Moody algebra associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -3 & 2 \end{smallmatrix}\right)$. Hence the algebra U_- is the associative algebra defined by the set $S = \{[[f_2f_1]f_1], [f_2[f_2[f_2[f_2f_1]]]]\}$ of the Serre relations in $\mathcal{A}_X$, where $X = \{f_1, f_2\}$ and $[xy] = xy - yx$. We define $f_1 \prec f_2$.

Let $S^{(0)} = S = \{q := f_2f_1^2 - 2f_1f_2f_1 + f_1^2f_2,\ p := f_2^4f_1 - 4f_2^3f_1f_2 + 6f_2^2f_1f_2^2 - 4f_2f_1f_2^3 + f_1f_2^4\}$. The possible composition pair from $S^{(0)}$ is $Comp^1(p, q; f_2^4f_1^2) = (pf_1, f_2^3q)$, which yields $P_0(6) = \{f_2^4f_1^2 - 4f_2^3f_1f_2f_1 + 6f_2^2f_1f_2^2f_1 - 4f_2f_1f_2^3f_1 + f_1f_2^4f_1,\ f_2^4f_1^2 - 2f_2^3f_1f_2f_1 + f_2^3f_1^2f_2\}$. Note that each polynomial is homogeneous of degree 6.

An explicit computation shows

$$A_{F_0(6)} = \begin{pmatrix} 1 & -4 & 0 & 6 & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 \end{pmatrix},$$

$$\widetilde{A_{F_0(6)}} = \begin{pmatrix} 1 & 0 & 0 & -6 & 4 & 0 & 4 & -4 & 0 & 0 & -4 & 6 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 & 2 & -1 & 0 & 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 6 & -4 & 1 \end{pmatrix}.$$

By applying the algorithm in a straightforward manner, we can compute $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$, from which we obtain $\widetilde{F_3}^+ = \emptyset$ and $S^{(4)} = S^{(3)}$. That is, the algorithm terminates. Finally,

$$\begin{aligned} \mathcal{S} = S^{(3)} &= S \cup \widetilde{F_0}^+ \cup \widetilde{F_1}^+ \cup \widetilde{F_2}^+ \\ &= \{(f_2f_1)f_1 - 2f_1(f_2f_1) + f_1^2f_2, \\ &\quad f_2(f_2^3f_1) - 4(f_2^3f_1)f_2 + 6(f_2^2f_1)f_2^2 - 4(f_2f_1)f_2^3 + f_1f_2^4, \\ &\quad (f_2^3f_1)(f_2f_1) - 3(f_2^2f_1)^2 + (f_2^2f_1f_2f_1)f_2 + 2(f_2f_1)(f_2^3f_1) - (f_2f_1)^2f_2^2 \\ &\quad - 2f_1(f_2^3f_1)f_2 + 3f_1(f_2^2f_1)f_2^2 - f_1(f_2f_1)f_2^3, \\ &\quad (f_2^3f_1)(f_2^2f_1) - 4(f_2^2f_1)(f_2^3f_1) + 3(f_2^2f_1)^2f_2 - 2(f_2^2f_1f_2f_1)f_2^2 \end{aligned}$$

$$+12(f_2f_1)(f_2^3f_1)f_2 - 21(f_2f_1)(f_2^2f_1)f_2^2 + 16(f_2f_1)^2f_2^3 - 8f_1(f_2^3f_1)f_2^2$$
$$+17f_1(f_2^2f_1)f_2^3 - 22f_1(f_2f_1)f_2^4 + 8f_1^2f_2^5,$$
$$(f_2^2f_1f_2f_1)(f_2f_1) - 4(f_2f_1)(f_2^2f_1f_2f_1) + 5(f_2f_1)^2(f_2^2f_1) - 2(f_2f_1)^3f_2 + 3f_1(f_2^2f_1)^2$$
$$-2f_1(f_2^2f_1f_2f_1)f_2 - 7f_1(f_2f_1)(f_2^3f_1) + 6f_1(f_2f_1)(f_2^2f_1)f_2 + 9f_1^2(f_2^3f_1)f_2$$
$$-17f_1^2(f_2^2f_1)f_2^2 + 10f_1^2(f_2f_1)f_2^3 - 2f_1^3f_2^4,\ (f_2^2f_1)(f_2^2f_1f_2f_1) - 4(f_2^2f_1f_2f_1)(f_2^2f_1)$$
$$+3(f_2f_1)(f_2^2f_1)^2 + 8(f_2f_1)(f_2^2f_1f_2f_1)f_2 + 2(f_2f_1)^2(f_2^3f_1) - 16(f_2f_1)^2(f_2^2f_1)f_2$$
$$+6(f_2f_1)^3f_2^2 - 5f_1(f_2^2f_1)(f_2^3f_1) + 24f_1(f_2f_1)(f_2^3f_1)f_2 - 26f_1(f_2f_1)(f_2^2f_1)f_2^2$$
$$+12f_1(f_2f_1)^2f_2^3 - 25f_1^2(f_2^3f_1)f_2^2 + 46f_1^2(f_2^2f_1)f_2^3 - 36f_1^2(f_2f_1)f_2^4 + 10f_1^3f_2^5.$$

Note that the $\mathcal{S}$-standard monomials are of the form $f_1^a(f_2f_1)^b(f_2^2f_1f_2f_1)^c(f_2^2f_1)^d \times (f_2^3f_1)^e f_2^f$ where $a, b, c, d, e, f \geqslant 0$. This result matches with that of [5].

3.4. Irreducible G_2-module $V(\Lambda_2)$

Let $V(\lambda)$ be the irreducible highest weight module over G_2 with highest weight $\lambda = \Lambda_2$. Then, as a left U_--module, $V(\lambda)$ is defined by the pair (S, T_λ), where

$$S = S^{(0)}$$
$$= \{q := f_2f_1^2 - 2f_1f_2f_1 + f_1^2f_2,\ p := f_2^4f_1 - 4f_2^3f_1f_2 + 6f_2^2f_1f_2^2 - 4f_2f_1f_2^3 + f_1f_2^4\},$$

$$T = T^{(0)} = T_\lambda = \{f_1, f_2^2\}.$$

As computed in the last subsection, we have

$$S^{(1)} = \{q := f_2f_1^2 - 2f_1(f_2f_1) + f_1^2f_2,$$
$$p := f_2^4f_1 - 4(f_2^3f_1)f_2 + 6(f_2^2f_1)f_2^2 - 4(f_2f_1)f_2^3 + f_1f_2^4,$$
$$s_1 := (f_2^3f_1)(f_2f_1) - 3(f_2^2f_1)^2 + (f_2^2f_1f_2f_1)f_2 + 2(f_2f_1)(f_2^3f_1) - (f_2f_1)^2f_2^2$$
$$-2f_1(f_2^3f_1)f_2 + 3f_1(f_2^2f_1)f_2^2 - f_1(f_2f_1)f_2^3\}.$$

The possible composition pairs from $(S^{(1)}, T^{(0)})$ are

$$Comp^4(q, f_1; f_2f_1^2) = (q, f_2f_1^2), \qquad Comp^4(p, f_1; f_2^4f_1) = (p, f_2^4f_1),$$
$$Comp^4(s_1, f_1; f_2^3f_1f_2f_1) = (s_1, f_2^3f_1f_2f_1).$$

Note that each composition data is homogeneous of degree 3, 5 or 6.

We continue this calculation to obtain $S^{(4)} = S^{(3)}$ and $\widetilde{F_3}^+ = \emptyset$. That is, the algorithm terminates,

$$
\begin{aligned}
\mathcal{S} = \big\{ & (f_2 f_1) f_1 - 2 f_1 (f_2 f_1) + f_1^2 f_2, \\
& f_2\big(f_2^3 f_1\big) - 4\big(f_2^3 f_1\big) f_2 + 6\big(f_2^2 f_1\big) f_2^2 - 4(f_2 f_1) f_2^3 + f_1 f_2^4, \\
& \big(f_2^3 f_1\big)(f_2 f_1) - 3\big(f_2^2 f_1\big)^2 + \big(f_2^2 f_1 f_2 f_1\big) f_2 + 2(f_2 f_1)\big(f_2^3 f_1\big) - (f_2 f_1)^2 f_2^2 \\
& - 2 f_1\big(f_2^3 f_1\big) f_2 + 3 f_1\big(f_2^2 f_1\big) f_2^2 - f_1 (f_2 f_1) f_2^3, \\
& \big(f_2^3 f_1\big)\big(f_2^2 f_1\big) - 4\big(f_2^2 f_1\big)\big(f_2^3 f_1\big) + 3\big(f_2^2 f_1\big)^2 f_2 - 2\big(f_2^2 f_1 f_2 f_1\big) f_2^2 \\
& + 12 (f_2 f_1)\big(f_2^3 f_1\big) f_2 - 21 (f_2 f_1)\big(f_2^2 f_1\big) f_2^2 + 16 (f_2 f_1)^2 f_2^3 - 8 f_1\big(f_2^3 f_1\big) f_2^2 \\
& + 17 f_1\big(f_2^2 f_1\big) f_2^3 - 22 f_1 (f_2 f_1) f_2^4 + 8 f_1^2 f_2^5, \big(f_2^2 f_1 f_2 f_1\big)(f_2 f_1) - 4 (f_2 f_1)\big(f_2^2 f_1 f_2 f_1\big) \\
& + 5 (f_2 f_1)^2\big(f_2^2 f_1\big) - 2 (f_2 f_1)^3 f_2 + 3 f_1\big(f_2^2 f_1\big)^2 - 2 f_1\big(f_2^2 f_1 f_2 f_1\big) f_2 - 7 f_1 (f_2 f_1)\big(f_2^3 f_1\big) \\
& + 6 f_1 (f_2 f_1)\big(f_2^2 f_1\big) f_2 + 9 f_1^2\big(f_2^3 f_1\big) f_2 - 17 f_1^2\big(f_2^2 f_1\big) f_2^2 + 10 f_1^2 (f_2 f_1) f_2^3 - 2 f_1^3 f_2^4, \\
& \big(f_2^2 f_1\big)\big(f_2^2 f_1 f_2 f_1\big) - 4\big(f_2^2 f_1 f_2 f_1\big)\big(f_2^2 f_1\big) + 3 (f_2 f_1)\big(f_2^2 f_1\big)^2 + 8 (f_2 f_1)\big(f_2^2 f_1 f_2 f_1\big) f_2 \\
& + 2 (f_2 f_1)^2\big(f_2^3 f_1\big) - 16 (f_2 f_1)^2\big(f_2^2 f_1\big) f_2 + 6 (f_2 f_1)^3 f_2^2 - 5 f_1\big(f_2^2 f_1\big)\big(f_2^3 f_1\big) \\
& + 24 f_1 (f_2 f_1)\big(f_2^3 f_1\big) f_2 - 26 f_1 (f_2 f_1)\big(f_2^2 f_1\big) f_2^2 + 12 f_1 (f_2 f_1)^2 f_2^3 - 25 f_1^2\big(f_2^3 f_1\big) f_2^2 \\
& + 46 f_1^2\big(f_2^2 f_1\big) f_2^3 - 36 f_1^2 (f_2 f_1) f_2^4 + 10 f_1^3 f_2^5, \\
\mathcal{T} = T^{(3)} & = T_\lambda \cup \widetilde{F}_0^{+} \cup \widetilde{F}_1^{+} \cup \widetilde{F}_2^{+} \\
& = \big\{ f_1, f_2^2, f_1^2 f_2, \big(f_2^3 f_1\big) f_2, \big(f_2^2 f_1 f_2 f_1\big) f_2, f_1 (f_2 f_1) f_2, \big(f_2^2 f_1\big)^2 f_2, f_1 (f_2 f_1)\big(f_2^2 f_1\big) f_2, \\
& \quad f_1^2\big(f_2^2 f_1\big) f_2 \big\}.
\end{aligned}
$$

Note that the $\mathcal{S}$-standard monomials are of the form $f_1^a (f_2 f_1)^b (f_2^2 f_1 f_2 f_1)^c (f_2^2 f_1)^d \times (f_2^3 f_1)^e f_2^f$, $a, b, c, d, e, f \geqslant 0$, and that the set of $(\mathcal{S}, \mathcal{T})$-standard monomials is given by

$$\big\{1, f_2, f_1 f_2, (f_2 f_1) f_2, \big(f_2^2 f_1\big) f_2, f_1\big(f_2^2 f_1\big) f_2,\ (f_2 f_1)\big(f_2^2 f_1\big) f_2\big\},$$

whose cardinality is 7, equal to the dimension of $V(\Lambda_2)$. Hence we conclude that $(\mathcal{S}, \mathcal{T})$ is a Gröbner–Shirshov pair for the U_--module $V(\Lambda_2)$.

References

[1] G.M. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (1978) 178–218.

[2] L.A. Bokut, Imbedding into simple associative algebras, Algebra Logic 15 (1976) 117–142.

[3] L.A. Bokut, S.-J. Kang, K.-H. Lee, P. Malcolmson, Gröbner–Shirshov bases for Lie superalgebras and their universal enveloping algebras, J. Algebra 217 (1999) 461–495.

[4] L.A. Bokut, A.A. Klein, Serre relations and Gröbner–Shirshov bases for simple Lie algebras I, Internat. J. Algebra Comput. 6 (1996) 389–400;
L.A. Bokut, A.A. Klein, Serre relations and Gröbner–Shirshov bases for simple Lie algebras II, Internat. J. Algebra Comput. 6 (1996) 401–412.

[5] L.A. Bokut, A.A. Klein, Gröbner–Shirshov bases for exceptional Lie algebras I, J. Pure Appl. Algebra 133 (1998) 51–57.

[6] L.A. Bokut, A.A. Klein, Gröbner–Shirshov bases for the exceptional Lie algebras E_6, E_7, E_8, in: Algebra and Combinatorics, Hong Kong, 1997, Springer-Verlag, Singapore, 1999, pp. 37–46.

[7] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional ideal, PhD thesis, University of Innsbruck, 1965.
[8] J.C. Faugère, A new efficient algorithm for computing Gröbner bases (F_4), J. Pure Appl. Algebra 139 (1999) 61–88.
[9] S.-J. Kang, K.-H. Lee, Gröbner–Shirshov bases for representation theory, J. Korean Math. Soc. 37 (2000) 55–72.
[10] S.-J. Kang, K.-H. Lee, Gröbner–Shirshov bases for irreducible sl_{n+1}-modules, J. Algebra 232 (2000) 1–20.
[11] S.-J. Kang, I.-S. Lee, K.-H. Lee, H. Oh, Hecke algebras, Specht modules and Gröbner–Shirshov bases, J. Algebra 252 (2002) 258–292.
[12] S.-J. Kang, I.-S. Lee, K.-H. Lee, H. Oh, Representations of Ariki–Koike algebras and Gröbner–Shirshov bases, Proc. London Math. Soc. (3) 89 (2004) 54–70.
[13] D.-I. Lee, Gröbner–Shirshov bases for irreducible $\mathfrak{sp}_4$-modules, J. Korean Math. Soc., in press.
[14] F.S. Macaulay, On some formula in elimination, Proc. London Math. Soc. 33 (1) (1902) 3–27.
[15] A.I. Shirshov, Some algorithmic problems for Lie algebras, Siberian Math. J. 3 (1962) 292–296.