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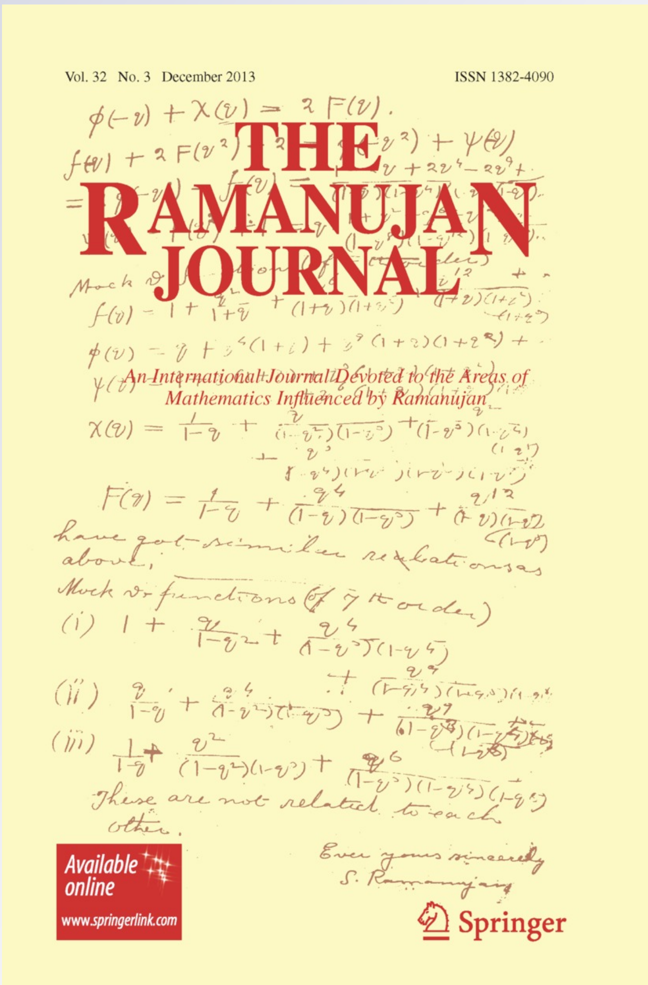

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Root multiplicities of hyperbolic Kac–Moody algebras and Fourier coefficients of modular forms

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Abstract In this paper we consider the hyperbolic Kac–Moody algebra $\mathcal{F}$ associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Its connection to Siegel modular forms of genus 2 was first studied by A. Feingold and I. Frenkel. The denominator function of $\mathcal{F}$ is not an automorphic form. However, Gritsenko and Nikulin extended $\mathcal{F}$ to a generalized Kac–Moody algebra whose denominator function is a Siegel modular form. Using the Borcherds denominator identity, the denominator function can be written as an infinite product. The exponents that appear in the product are given by Fourier coefficients of a weak Jacobi form. P. Niemann also constructed a generalized Kac–Moody algebra which contains $\mathcal{F}$ and whose denominator function is related to a product of Dedekind η-functions. In particular, root multiplicities of the generalized Kac–Moody algebra are determined by Fourier coefficients of a modular form. As the main results of this paper, we compute asymptotic formulas for these Fourier coefficients using the method of Hardy–Ramanujan–Rademacher, and obtain an asymptotic bound for root multiplicities of the algebra $\mathcal{F}$. Our method can be applied to other hyperbolic Kac–Moody algebras and to other modular forms as demonstrated in the later part of the paper.

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1 Introduction

After being introduced more than four decades ago, the Kac–Moody theory has been a great success. However, it makes one surprised to notice that little is known beyond the affine case. In particular, we do not have any single closed formula for the root multiplicities of a hyperbolic Kac–Moody algebra, even though hyperbolic Kac–Moody algebras constitute the simplest indefinite case. On the other hand, Borcherds' ingenious idea [1] was to consider generalized Kac–Moody algebras to extend some Kac–Moody algebras so that we may obtain automorphic forms from the denominator functions of the generalized Kac–Moody algebras. Pursuing Borcherds' idea, Gritsenko–Nikulin and Niemann constructed generalized Kac–Moody algebras to extend some hyperbolic Kac–Moody algebras [7, 8, 20].

Gritsenko–Nikulin's construction indeed produces Siegel modular forms as denominator functions of their generalized Kac–Moody algebras. For the hyperbolic Kac–Moody algebra $\mathcal{F}$ associated with the generalized Cartan matrix $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, they showed that there exists a generalized Kac–Moody algebra $\mathcal{G}$ which contains $\mathcal{F}$ and whose denominator function is the weight 35 Siegel cusp form $\Delta_{35}(Z)$, which is called the Igusa modular form. As a byproduct, they obtained the infinite product expression of $\Delta_{35}(Z)$:

$$\Delta_{35}(Z) = q^3 r s^2 \prod_{(n,l,m)\in\mathcal{D}} \left(1 - q^n r^l s^m\right)^{c_2(4nm-l^2)},$$

where the integers $c_2(N)$ are defined using Fourier coefficients of a weak Jacobi form of weight 0 and index 1. See (2.5) for the definition of $\mathcal{D}$. As a result, we have

$$\operatorname{mult}(\mathcal{F},\alpha) \le \operatorname{mult}(\mathcal{G},\alpha) = c_2\bigl(-2(\alpha,\alpha)\bigr),$$

where $\operatorname{mult}(\mathcal{F},\alpha)$ (resp. $\operatorname{mult}(\mathcal{G},\alpha)$) is the multiplicity of a root α in $\mathcal{F}$ (resp. in $\mathcal{G}$).

There has been a great deal of effort to compute $\operatorname{mult}(\mathcal{F},\alpha)$ (e.g. [5, 11, 12]), using Kac–Peterson type formulas or Berman–Moody type formulas. However, the complexity of these formulas grows fast as the length of a root becomes larger. In [6] I. Frenkel conjectured

$$\operatorname{mult}(\mathcal{F},\alpha) \le p\left(1 - \frac{1}{2}(\alpha,\alpha)\right), \tag{1.1}$$

where $p(n)$ is the usual partition function. This conjecture also appears in Exercise 13.37 of [10] as an open problem. (Note that the normalization of the standard form

$(\cdot|\cdot)$ in the exercise problem is different from ours.) For some other indefinite Kac–Moody algebras, Frenkel's conjecture turns out to be false (see e.g. [15]). However, it is still a tantalizing challenge for the algebra $\mathcal{F}$.

In his Ph.D. thesis [20], P. Niemann constructed a generalized Kac–Moody algebra $\mathcal{G}_{23}$ which contains $\mathcal{F}$. The denominator function of $\mathcal{G}_{23}$ is closely related to the eta product $\eta(z)\eta(23z)$, where η is the Dedekind η-function. If $q\eta^{-1}(z)\eta^{-1}(23z) = \sum_{n=0}^{\infty} p_\sigma(n)q^n$, $q = e^{2\pi i z}$, he showed that

$$\mathrm{mult}(\mathcal{F}, \alpha) \le \begin{cases} p_\sigma(1 - \frac{1}{2}(\alpha, \alpha)) & \text{if } \alpha \notin 23L^*, \\ p_\sigma(1 - \frac{1}{2}(\alpha, \alpha)) + p_\sigma(1 - \frac{1}{46}(\alpha, \alpha)) & \text{if } \alpha \in 23L^*, \end{cases}$$

where L is a certain lattice and L^* is its dual. This bound is quite close to Frenkel's conjecture.

Now that root multiplicities of $\mathcal{G}$ and $\mathcal{G}_{23}$ are given by Fourier coefficients of automorphic forms, we can apply analytic tools to get asymptotic formulas for these multiplicities; namely, we use the method of Hardy–Ramanujan–Rademacher to obtain asymptotic formulas for $c_2(N)$ and $p_\sigma(1+n)$. Since the space of weak Jacobi forms of weight 0 and index 1 is isomorphic to the Kohnen plus-space $\mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$, we will consider basis elements v_d of the space $\mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$ and the information on $c_2(N)$ will be obtained as a special case. We use the result of J. Lehner [17] on Fourier coefficients of modular forms, which adopts the method of Hardy–Ramanujan–Rademacher, and obtain Theorem 2.8 and Corollary 2.12.

As for Niemann's bound, we consider $f(z) = \eta(z)^{-1}\eta(23z)^{-1}$, which is a weakly holomorphic modular form of weight -1 with respect to $\Gamma_0(23)$. The method of Hardy–Ramanujan–Rademacher can be applied to this case too, and we obtain Theorem 3.5. Our result has an immediate implication on root multiplicities of the hyperbolic Kac–Moody algebra $\mathcal{F}$. For example, if $(\alpha, \alpha) = -56$ then our asymptotic formula gives 4578.99, while the actual value of the Fourier coefficient is 4576. The exact value of $\mathrm{mult}(\mathcal{F}, \alpha)$ is 4557. In this way, we can calculate a sharp upper bound for $\mathrm{mult}(\mathcal{F}, \alpha)$ even if $|(\alpha, \alpha)|$ is big. (See Example 3.6.)

Our method can be applied to other hyperbolic Kac–Moody algebras and to other modular forms. (See Sects. 5.3 and 5.4.)

2 Some automorphic forms

2.1 Jacobi forms and Siegel modular forms

Let $\mathbb{H}$ be the upper half-plane. A *Jacobi form ϕ of weight k and index m on $SL_2(\mathbb{Z})$* is a holomorphic function on $\mathbb{H} \times \mathbb{C}$ satisfying

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e\left(\frac{cmz^2}{c\tau + d}\right)\phi(\tau, z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

$$\phi(\tau, z + \lambda\tau + \mu) = e\left(-m\lambda^2\tau - 2m\lambda z\right)\phi(\tau, z), \quad (\lambda, \mu) \in \mathbb{Z}^2,$$

and having a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ r^2 \leq 4nm}} c(n,r)e(n\tau + rz), \tag{2.1}$$

where we write $e(z) = e^{2\pi i z}$. If ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r\in\mathbb{Z}} c(n,r)e(n\tau + rz), \tag{2.2}$$

instead of (2.1), the function ϕ is called a *weak Jacobi form*.

For $k \geq 4$ even, we define the *Jacobi–Eisenstein series of weight k and index m* by

$$E_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \sum_{\lambda\in\mathbb{Z}} (c\tau + d)^{-k} e\left(m\lambda^2 \frac{a\tau + b}{c\tau + d} + 2m\lambda \frac{z}{c\tau + d} - \frac{cmz^2}{c\tau + d}\right),$$

where a, b are chosen so that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Indeed, the series $E_{k,m}(\tau, z)$ is a Jacobi form of weight k and index m for $k \geq 4$, even. We also consider a Jacobi form of weight 12 and index 1:

$$\phi_{12,1}(\tau, z) = \frac{1}{144}\left(E_4^2(\tau)E_{4,1}(\tau, z) - E_6(\tau)E_{6,1}(\tau, z)\right),$$

where $E_k(\tau)$ are the usual Eisenstein series of weight k defined by

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} (c\tau + d)^{-k}.$$

Now we define a weak Jacobi form $\phi_{0,1}(\tau, z)$ of weight 0 and index 1 by

$$\phi_{0,1}(\tau, z) = \frac{\phi_{12,1}(\tau, z)}{\Delta_{12}(\tau)} = \sum_{n=0}^{\infty} \sum_{r\in\mathbb{Z}} c(n,r)e(n\tau + rz), \tag{2.3}$$

where $\Delta_{12}(\tau) = e(\tau)\prod_{n\geq 1}(1 - e(n\tau))^{24}$ and $c(n,r)$ are the Fourier coefficients. Since $\phi_{0,1}$ is of weight 0, we have $c(n,r) = c(n,-r)$ and $c(n,r)$ depends only on $4n - r^2$ (see [4, Theorem 2.2]). Therefore the following function is well-defined:

$$c(N) = \begin{cases} c(n,r) & \text{if } N = 4n - r^2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $c(0) = 10$, $c(-1) = 1$ and $c(n) = 0$ for $n < -1$. We use the function $c(N)$ to define

$$c_2(N) = 8c(4N) + 2\left(\left(\frac{-N}{2}\right) - 1\right)c(N) + c\left(\frac{N}{4}\right), \tag{2.4}$$

where we put

$$\left(\frac{D}{2}\right)=\begin{cases}1 & \text{for } D\equiv 1 \pmod 8,\\ -1 & \text{for } D\equiv 5 \pmod 8,\\ 0 & \text{for } D\equiv 0 \pmod 2).\end{cases}$$

Let $S_2(\mathbb{C})$ (resp. $S_2(\mathbb{Z})$) be the set of all symmetric 2×2 complex (resp. integer) matrices. The Siegel upper half-plane $\mathbb{H}_2$ of genus 2 is defined by

$$\mathbb{H}_2=\big\{Z=X+iY\in S_2(\mathbb{C})\mid Y \text{ is positive definite}\big\}.$$

We will use the coordinates z_1, z_2, z_3 for $\mathbb{H}_2$ so that $Z=\left(\begin{smallmatrix} z_1 & z_2\\ z_2 & z_3\end{smallmatrix}\right)\in\mathbb{H}_2$. A *Siegel modular form of weight k with respect to* $Sp_4(\mathbb{Z})$ is a holomorphic function $F(Z)$ on the Siegel upper half-plane $\mathbb{H}_2$ such that

$$F\big((AZ+B)(CZ+D)^{-1}\big)=\det(CZ+D)^k F(Z)$$

for each $M=\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\in Sp_4(\mathbb{Z})$. We write $q=e(z_1)$, $r=e(z_2)$ and $s=e(z_3)$. In [7, Theorem 1.5], Gritsenko and Nikulin proved that the following product represents a Siegel modular form $\Delta_{35}(Z)$ of weight 35:

$$\Delta_{35}(Z)=q^3rs^2\prod_{(n,l,m)\in\mathcal{D}}\big(1-q^nr^ls^m\big)^{c_2(4nm-l^2)}, \tag{2.5}$$

where the integers $c_2(N)$ are defined in (2.4) and we denote by $\mathcal{D}$ the set of integer triples $(n,l,m)\in\mathbb{Z}^3$ such that (1) $(n,l,m)=(-1,0,1)$ or (2) $n\ge 0, m\ge 0$ and either $n+m>0$ and l is arbitrary or $n=m=0$ and $l<0$.

Siegel modular forms and Jacobi forms are connected by the Fourier–Jacobi expansion:

Theorem 2.6 [4, *Theorem* 6.1] *Let F be a Siegel modular form of weight k with respect to* $Sp_4(\mathbb{Z})$ *and*

$$F(Z)=\sum_{m=0}^{\infty}\phi_m(z_1,z_2)s^m$$

be the Fourier–Jacobi expansion. Then $\phi_m(z_1,z_2)$ *is a Jacobi form of weight k and index m.*

2.2 Weakly holomorphic modular forms of weight $-\frac{1}{2}$

We use the notations from [21]. If d is an odd prime, let $(\frac{c}{d})$ be the usual Legendre symbol. For positive odd d, define $(\frac{c}{d})$ by multiplicativity. For negative odd d, let

$$\left(\frac{c}{d}\right)=\begin{cases}(\frac{c}{|d|}) & \text{if } d<0 \text{ and } c>0,\\ -(\frac{c}{|d|}) & \text{if } d<0 \text{ and } c<0.\end{cases}$$

We let $(\frac{0}{\pm 1}) = 1$. Define ϵ_d, for odd d, by $\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ i & \text{if } d \equiv 3 \bmod 4. \end{cases}$

Let $\mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$ be the *Kohnen plus-space* of weakly holomorphic modular forms with integer coefficients of weight $-\frac{1}{2}$ with respect to $\Gamma_0(4)$, namely, $f \in \mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$ if f is holomorphic on $\mathbb{H}$, and meromorphic at the cusps of $\Gamma_0(4)$, and

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{c}{d}\right)\epsilon_d(cz+d)^{-\frac{1}{2}} f(z),$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, and it has a Fourier expansion of the form

$$f(z) = \sum_{\substack{n \geq n_0 \\ n \equiv 0,3 \pmod 4}} a(n)q^n,$$

where we put $q = e(z)$ and let $\sqrt{z}$ be the branch of the square root having argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$. So for $z \in \mathbb{H}$, $(-z)^{\frac{1}{2}} = (-\frac{1}{z})^{-\frac{1}{2}} = -iz^{\frac{1}{2}}$. For each nonnegative integer $d \equiv 0, 1 \pmod 4$, there exists a unique modular form $v_d(z) \in \mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$ with a Fourier expansion ([23], page 19)

$$v_d(z) = q^{-d} + \sum_{\substack{n \geq 0 \\ n \equiv 0,3 \pmod 4}} a(n)q^n. \tag{2.7}$$

Here $v_d(z)$'s form a $\mathbb{Z}$-basis for $\mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$. Some examples are:

$$\begin{aligned}
v_1(z) &= q^{-1} + 10 - 64q^3 + 108q^4 - 513q^7 + \cdots, \\
v_4(z) &= q^{-4} + 70 + 32384q^3 + 131976q^4 + 4451328q^7 + \cdots, \\
v_5(z) &= q^{-5} + 48 - 131565q^3 + 656800q^4 - 35655680q^7 + \cdots, \\
v_8(z) &= q^{-8} + 120 + 4257024q^3 + 34867000q^4 + 6275241984q^7 + \cdots.
\end{aligned}$$

We will prove the following using the method of Hardy–Ramanujan–Rademacher.

Theorem 2.8 *In* (2.7), *suppose* $4|d$ *and* $d > 0$. *Then* $a(n)$ *is positive for all* n, *and*

$$a(n) = 2\left(d^{\frac{1}{2}} n^{-1}\cosh(\pi\sqrt{dn}) - \pi^{-1}n^{-\frac{3}{2}}\sinh(\pi\sqrt{dn})\right) + O\left(d^{\frac{3}{2}}\log(4\pi\sqrt{dn})e^{\frac{\pi\sqrt{dn}}{2}}\right).$$

If $d \equiv 1 \pmod 4$, $(-1)^n a(n)$ *is positive for all* n, *and*

$$\begin{aligned}
a(n) = {} & 2(-1)^n\left(d^{\frac{1}{2}} n^{-1}\cosh(\pi\sqrt{dn}) - \pi^{-1}n^{-\frac{3}{2}}\sinh(\pi\sqrt{dn})\right) \\
& + O\left(d^{\frac{3}{2}}\log(4\pi\sqrt{dn})e^{\frac{\pi\sqrt{dn}}{2}}\right).
\end{aligned}$$

In particular, if $d=1$,

$$a(n)=(-1)^n\frac{e^{\pi\sqrt{n}}}{n}\left(1-\frac{1}{\pi\sqrt{n}}\right)+O\left(\log(4\pi\sqrt{n})e^{\frac{\pi\sqrt{n}}{2}}\right). \tag{2.9}$$

Let $\tilde{J}_{0,1}$ be the space of weak Jacobi forms of weight 0 and index 1. The following theorem provides the connection of half-integral weakly holomorphic modular forms to weak Jacobi forms and then to root multiplicities of the hyperbolic Kac–Moody algebra $\mathcal{F}$.

Theorem 2.10 [4, Theorem 5.4] *The correspondence*

$$\sum_{\substack{N\geq n_0\\ N\equiv 0,3 \pmod 4}} c(N)q^N \mapsto \sum_{n,r\in\mathbb{Z}} c\left(4n-r^2\right)e(n\tau+rz)$$

gives an isomorphism between $\mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$ *and* $\tilde{J}_{0,1}$.

Remark 2.11 In [4], the above theorem is proved only for holomorphic forms. It is easy to extend the result to weakly holomorphic forms as stated above.

Under the above correspondence, the weak Jacobi form $\phi_{0,1}(\tau,z)$ in (2.3) corresponds to $v_1\in\mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$. Therefore, we have, by (2.4) and (2.9), the asymptotic formula for Fourier coefficients of $\phi_{0,1}(\tau,z)$:

Corollary 2.12

$$c(N)=(-1)^N\frac{e^{\pi\sqrt{N}}}{N}\left(1-\frac{1}{\pi\sqrt{N}}\right)+O\left(\log(4\pi\sqrt{N})e^{\frac{\pi\sqrt{N}}{2}}\right),$$

$$c_2(N)=\frac{2e^{2\pi\sqrt{N}}}{N}\left(1-\frac{1}{2\pi\sqrt{N}}\right)+O\left(\log(8\pi\sqrt{N})e^{\pi\sqrt{N}}\right).$$

3 Hyperbolic Kac–Moody algebra $\mathcal{F}$

Let $\mathcal{F}=\mathfrak{g}(A)$ be the hyperbolic Kac–Moody algebra associated to the generalized Cartan matrix

$$A=(a_{ij})=\begin{pmatrix}2&-2&0\\-2&2&-1\\0&-1&2\end{pmatrix}.$$

Let $\{\alpha_1,\alpha_2,\alpha_3\}$ be the set of simple roots. The Weyl group W of $\mathcal{F}$ is isomorphic to $PGL_2(\mathbb{Z})$ through the map given by

$$\sigma_1\mapsto\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad \sigma_2\mapsto\begin{pmatrix}-1&1\\0&1\end{pmatrix},\qquad \sigma_3\mapsto\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

where σ_i $(i=1,2,3)$ are the simple reflections corresponding to α_i.

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathcal{F}$. We define a standard bilinear form on $\mathfrak{h}^*$ by $(\alpha_i, \alpha_j) = a_{ij}$. We choose another basis $\{\gamma_1, \gamma_2, \gamma_3\}$ for $\mathfrak{h}^*$ to be

$$\gamma_1 = \alpha_1 + \alpha_2 + \alpha_3, \qquad \gamma_2 = \alpha_1, \qquad \gamma_3 = \alpha_1 + \alpha_2.$$

The dual basis of $\{\gamma_1, \gamma_2, \gamma_3\}$ with respect to $(\cdot,\cdot)$ is given by

$$\gamma_1^* = -\alpha_1 - \alpha_2, \qquad \gamma_2^* = \frac{1}{2}\alpha_1, \qquad \gamma_3^* = -\alpha_1 - \alpha_2 - \alpha_3.$$

The matrices $((\gamma_i, \gamma_j))$ and $((\gamma_i^*, \gamma_j^*))$ are

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

respectively.

We define a $\mathbb{C}$-linear map $\mu : \mathfrak{h}^* \to S_2(\mathbb{C})$ by

$$\mu\big(a\gamma_1^* + b\gamma_2^* + c\gamma_3^*\big) = \begin{pmatrix} -a & -b/2 \\ -b/2 & -c \end{pmatrix}.$$

Then we have

$$\mu(\alpha_1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \mu(\alpha_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mu(\alpha_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The group $PGL_2(\mathbb{Z})$ acts on $S_2(\mathbb{C})$ by $g(S) = gS({}^t g)$. Then the map μ is W-equivariant and we can identify $\mathfrak{h}^*$ with $S_2(\mathbb{C})$. We also obtain

$$(\alpha, \alpha) = -2 \det \mu(\alpha) \tag{3.1}$$

for $\alpha \in \mathfrak{h}^*$. (See Proposition 2.1 in [5].)

We write $z \in \mathfrak{h}^*$ as $z = z_1\gamma_1 + z_2\gamma_2 + z_3\gamma_3$. Then we define another map $\nu : \mathfrak{h}^* \to S_2(\mathbb{C})$ by

$$\nu(z) = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}.$$

For $\alpha = a\gamma_1^* + b\gamma_2^* + c\gamma_3^* \in \mathfrak{h}^*$, we have

$$(\alpha, z) = az_1 + bz_2 + cz_3 = -\operatorname{Tr}\big(\mu(\alpha)\nu(z)\big).$$

Let Δ_{re}^+ (resp. Δ_{im}^+) be the set of positive real (resp. imaginary) roots of $\mathcal{F}$ and we put $\Delta^+ = \Delta_{\mathrm{re}}^+ \cup \Delta_{\mathrm{im}}^+$.

Proposition 3.2 [5] *We have*

$$\mu\big(\Delta_{\mathrm{im}}^+\big) = \big\{N \in S_2(\mathbb{Z}) \mid N \geq 0 \ (\textit{i.e. } N \textit{ is semi positive definite})\big\} \quad \textit{and}$$

$$\mu(\Delta^+_{\mathrm{re}}) = \Bigg\{ N = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \in S_2(\mathbb{Z}) \,\Bigg|\, n_1 n_3 - n_2^2 = -1,\ n_2 \le n_1 + n_3,\ 0 \le n_1 + n_3,\ 0 \le n_3 \Bigg\}.$$

We denote by $\mathrm{mult}(\alpha)$ the multiplicity of $\alpha \in \Delta^+_{\mathrm{im}}$. In [6] I. Frenkel conjectured

$$\mathrm{mult}(\alpha) \le p\left(1 - \frac{1}{2}(\alpha, \alpha)\right) = p\big(1 + \det \mu(\alpha)\big) \quad \text{for } \alpha \in \Delta^+_{\mathrm{im}}, \tag{3.3}$$

where $p(n)$ is the usual partition function. This conjecture also appears in Exercise 13.37 of [10] as an open problem.

We define $p_\sigma(n)$ to be the coefficients of q^n in the expansion of $q\eta^{-1}(z)\eta^{-1}(23z)$, where η is the Dedekind η-function. Thus we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_\sigma(n) q^n &= \big(1 + q^{23} + 2q^{46} + \cdots\big) \prod_{n=1}^{\infty} \big(1 - q^n\big)^{-1} \\ &= \big(1 + q^{23} + 2q^{46} + \cdots\big) \sum_{n=0}^{\infty} p(n) q^n. \end{aligned} \tag{3.4}$$

In his Ph.D. thesis [20], P. Niemann studied root multiplicities of $\mathcal{F}$ and proved that

$$\mathrm{mult}(\alpha) \le \begin{cases} p_\sigma(1 - \frac{1}{2}(\alpha, \alpha)) & \text{if } \alpha \notin 23L^*, \\ p_\sigma(1 - \frac{1}{2}(\alpha, \alpha)) + p_\sigma(1 - \frac{1}{46}(\alpha, \alpha)) & \text{if } \alpha \in 23L^*, \end{cases}$$

where L^* is the dual of a certain lattice L.

We will prove the following asymptotics using the method of Hardy–Ramanujan–Rademacher:

Theorem 3.5

$$p_\sigma(n+1) = \frac{2\pi}{n\sqrt{23}} I_2\left(\frac{4\pi\sqrt{n}}{\sqrt{23}}\right) + O\left(n^{-\frac{1}{2}} I_2\left(\frac{2\pi\sqrt{n}}{\sqrt{23}}\right)\right),$$

where I_2 is the modified Bessel function of the first kind.

Using the fact $I_2(x) \sim \frac{e^x}{\sqrt{2\pi x}}$, we can see $p_\sigma(n+1) \sim \frac{e^{\frac{4\pi\sqrt{n}}{\sqrt{23}}}}{n^{\frac{5}{4}} 23^{\frac{1}{4}} \sqrt{2}}$. Compare it with $p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$. Using this result, we can calculate an upper bound for $\mathrm{mult}(\alpha)$ efficiently when $|(\alpha, \alpha)|$ is big.

Example 3.6 If $\alpha = 10\alpha_1 + 10\alpha_2 + 5\alpha_3 = (10, 10, 5)$ then $-\frac{1}{2}(\alpha, \alpha) = 25$ and we have $\mathrm{mult}(\alpha) = 2434$, and the main term of the asymptotics is 2437.16. We calculate more cases and make a table:

α	$-\frac{1}{2}(\alpha,\alpha)$	$\mathrm{mult}(\alpha)$	main term
(7, 7, 2)	10	56	56.65
(8, 10, 4)	20	792	793.19
(11, 11, 5)	30	6826	6867.52
(11, 14, 7)	40	44258	44975.14

(3.7)

(A table of $\mathrm{mult}(\alpha)$ can be found in [10, p. 205].)

Remark 3.8 In [5], Feingold and Frenkel proved that for the level one roots, $\mathrm{mult}(\alpha) = p(1-\frac{1}{2}(\alpha,\alpha))$, i.e., (3.3) holds with the equality. They also obtained the generating function of the multiplicities of the level 2 roots:

$$\begin{aligned}\sum_{n=0}^{\infty} M(n-1)q^n &= \frac{q^{-3}}{2}\left(\sum_{n=0}^{\infty} p(n)q^n\right)\prod_{j=1}^{\infty}\left(1-q^{4j-2}\right) \\ &\quad\times\left(\prod_{j=1}^{\infty}\left(1+q^{2j-1}\right)-\prod_{j=1}^{\infty}\left(1-q^{2j-1}\right)-2q\right) \\ &= \left(1-q^{20}+q^{22}-q^{24}+q^{26}-2q^{28}+\cdots\right)\sum_{n=0}^{\infty} p(n)q^n,\end{aligned}$$

where $M(2m) = \mathrm{mult}\,\mu^{-1}\left(\begin{smallmatrix} m & 0 \\ 0 & 2\end{smallmatrix}\right)$ and $M(2m-1) = \mathrm{mult}\,\mu^{-1}\left(\begin{smallmatrix} m & 1 \\ 1 & 2\end{smallmatrix}\right)$. Comparing with (3.4), we see more clearly the difference between actual multiplicities and bounds for these roots.

4 Automorphic correction for the Kac–Moody algebra $\mathcal{F}$

Let $M = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2 + \mathbb{Z}\gamma_3 \subset \mathfrak{h}^*$ and $M^* = \mathbb{Z}\gamma_1^* + \mathbb{Z}\gamma_2^* + \mathbb{Z}\gamma_3^* \subset \mathfrak{h}^*$. The lattice M has signature $(2, 1)$. We obtain the cone

$$V(M) = \left\{y \in M \otimes \mathbb{R} \subset \mathfrak{h}^* \mid (y|y) < 0\right\},$$

which is a union of two half cones. We choose one of these half cones as follows and denote it by $V^+(M)$:

$$V^+(M) = \left\{y_1\gamma_1 + y_2\gamma_2 + y_3\gamma_3 \in V(M) \mid y_1 > 0\right\}.$$

We consider the complexified cone $\Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M) \subset \mathfrak{h}^*$. Let $\mathbb{H}_2$ be the Siegel upper half-plane as before. By restricting the map $\nu : \mathfrak{h}^* \to S_2(\mathbb{C})$ to $\Omega(V^+(M))$, we have a biholomorphic map $\nu : \Omega(V^+(M)) \to \mathbb{H}_2$ and identify $z = z_1\gamma_1 + z_2\gamma_2 + z_3\gamma_3 \in \Omega(V^+(M))$ with the point $Z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_2 & z_3\end{smallmatrix}\right) \in \mathbb{H}_2$. With this identification, Siegel automorphic forms on $\mathbb{H}_2$ can be considered as automorphic forms on $\Omega(V^+(M))$. See [7, 8] for more details. We will write $\Phi(z) = \Phi(\nu(z)) = \Phi(Z)$ for a Siegel modular form $\Phi(Z)$.

A lattice Weyl vector is given by

$$\rho = 2\gamma_1 - \frac{1}{2}\gamma_2 + 3\gamma_3 = \frac{9}{2}\alpha_1 + 5\alpha_2 + 2\alpha_3 \in M \otimes \mathbb{Q}.$$

The vector ρ satisfies $(\rho, \alpha_i) = -1$ for $i = 1, 2, 3$. We define

$$\mathcal{M} = \big\{x \in V^+(M) \,|\, (x, \alpha_i) \le 0,\ i = 1, 2, 3\big\}.$$

Remark 4.1 In [10], the notation ρ denotes the vector whose paring with each simple (co)root is 1. Notice that our definition of ρ is the negative of that in [10].

In what follows, we will consider more than one generalized Kac–Moody algebra and we modify our notation to show which algebra it is attached to. Assume that $\mathcal{G}$ is a generalized Kac–Moody algebra. Let $\Delta(\mathcal{G})^+_{\mathrm{re}}$ (resp. $\Delta(\mathcal{G})^+_{\mathrm{im}}$) be the set of positive real (resp. imaginary) roots of $\mathcal{G}$ and we put $\Delta(\mathcal{G})^+ = \Delta(\mathcal{G})^+_{\mathrm{re}} \cup \Delta(\mathcal{G})^+_{\mathrm{im}}$. We denote by $\mathrm{mult}(\mathcal{G}, \alpha)$ the multiplicity of $\alpha \in \Delta(\mathcal{G})^+$ in the algebra $\mathcal{G}$.

The notion of an automorphic correction originated from an idea of Borcherds [1] and was further developed by Gritsenko and Nikulin [7, 8]. An *automorphic correction* for the Kac–Moody algebra $\mathcal{F}$ is defined to be an automorphic form $\Phi(z)$ on $\Omega(V^+(M))$ with Fourier expansion

$$\Phi(z) = \sum_{w \in W} \det(w) \Big(e\big(-\big(w(\rho), z\big)\big) - \sum_{a \in M^* \cap \mathcal{M}} m(a)\, e\big(-\big(w(\rho + a), z\big)\big) \Big),$$

where $m(a) \in \mathbb{Z}$ for all $a \in M^* \cap \mathcal{M}$ and $e(x) = e^{2\pi i x}$ as before. If $\Phi(z)$ is an automorphic correction for $\mathcal{F}$, we can construct a generalized Kac–Moody algebra $\mathcal{G}$ such that $\mathcal{F} \subset \mathcal{G}$ and the denominator function of $\mathcal{G}$ is $\Phi(z)$. Moreover, we obtain from the Weyl–Kac–Borcherds denominator identity

$$\Phi(z) = e\big(-(\rho, z)\big) \prod_{\alpha \in \Delta(\mathcal{G})^+} \big(1 - e\big(-(\alpha, z)\big)\big)^{\mathrm{mult}(\mathcal{G}, \alpha)},$$

where $\Delta(\mathcal{G})^+_{\mathrm{re}} = \Delta(\mathcal{F})^+_{\mathrm{re}}$ and $\Delta(\mathcal{G})^+_{\mathrm{im}} = \overline{V^+(M)} \cap M^* \supset \Delta(\mathcal{F})^+_{\mathrm{im}}$.

Theorem 4.2 [7, Theorem 3.1] *The Siegel modular form $\Delta_{35}(z)$ given in (2.5) is an automorphic correction for the hyperbolic Kac–Moody algebra $\mathcal{F}$.*

Let us look closely at the correspondence between $\Delta(\mathcal{G})^+$ and the set $\mathcal{D}$ in (2.5). If $\alpha \in \Delta(\mathcal{G})^+$ and $\mu(\alpha) = \left(\begin{smallmatrix} n & l/2 \\ l/2 & m \end{smallmatrix}\right)$, then we have

$$\mu(\alpha) \longleftrightarrow (n, l, m) \in \mathcal{D}.$$

Using Proposition 3.2, one can check that the set $\mathcal{D}$ indeed contains all the elements corresponding to the positive roots in $\Delta(\mathcal{F})^+$. Moreover, we obtain from (3.1)

$$4nm - l^2 = 4 \det \mu(\alpha) = -2(\alpha, \alpha). \tag{4.3}$$

Corollary 4.4 *For each $\alpha \in \Delta(\mathcal{F})_+ \subset \Delta(\mathcal{G})_+$, we have*

$$\mathrm{mult}(\mathcal{F},\alpha) \le \mathrm{mult}(\mathcal{G},\alpha) = c_2\big(-2(\alpha,\alpha)\big),$$

where the function $c_2(N)$ is defined in (2.4) *with the asymptotic formula in Corollary* 2.12.

Remark 4.5 Automorphic correction of certain rank 2 symmetric hyperbolic Kac–Moody algebras has been constructed in [14] using Hilbert modular forms. The first connection between rank 2 hyperbolic Kac–Moody algebras and the theory of Hilbert modular forms was made by Lepowsky and Moody in [19].

5 Method of Hardy–Ramanujan–Rademacher

We recall the result of J. Lehner [17] on Fourier coefficients of modular forms using the method of Hardy–Ramanujan–Rademacher. We refer to [17, 18] for unexplained notations. Let $f(z)$ be a weakly holomorphic modular form of weight $r<0$ with respect to a congruence subgroup Γ. Then we have the multiplier system υ such that $f(Mz)=\upsilon(M)(cz+d)^r f(z)$ for $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$. Let $p_0=\infty$, $p_1,\dots,p_{s-1}$ be the cusps of Γ, and

$$A_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad A_j=\begin{pmatrix}0&-1\\1&-p_j\end{pmatrix},\quad 1\le j\le s-1.$$

For $0\le j\le s-1$, we write $M^*=A_jM=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ for $M\in\Gamma$ and set

$$C_{j0}=\left\{c\,\middle|\,\begin{pmatrix}\cdot&\cdot\\c&\cdot\end{pmatrix}\in A_j\Gamma\right\},\qquad D_c=\left\{d\,\middle|\,\begin{pmatrix}\cdot&\cdot\\c&d\end{pmatrix}\in A_j\Gamma,\ 0<d\le c\right\}.$$

Choose the generator P_j $(1\le j\le s-1)$ of the cyclic subgroup of Γ which fixes p_j, such that

$$A_jP_jA_j^{-1}=\begin{pmatrix}1&\lambda_j\\0&1\end{pmatrix},\quad \lambda_j>0,\ 1\le j\le s-1.$$

This also defines λ_j. Let υ be a multiplier system belonging to Γ. Define κ_j $(1\le j\le s-1)$ by $e(\kappa_j)=\upsilon(P_j)$, $0\le\kappa_j<1$. It can be shown ([17], page 313) that given $c\in C_{j0}, d\in D_c$, there is a unique a such that $-c\lambda_j\le a<0$. For $k=1,\dots,s-1$, we have the expansion

$$(z-p_k)^r e\left(-\kappa_k\frac{A_kz}{\lambda_k}\right)f(z)=\sum_{n=-\mu_k}^{\infty}a(n)^{(k)}q_k^n,\quad q_k=e\left(\frac{A_kz}{\lambda_k}\right).$$

By replacing A_kz by z, this can be written as

$$f\left(p_k-\frac{1}{z}\right)=(-z)^r q^{\frac{\kappa_k}{\lambda_k}}\sum_{n=-\mu_k}^{\infty}a(n)^{(k)}q^{\frac{n}{\lambda_k}}. \tag{5.1}$$

We assume that $\lambda_0 = 1$, $\kappa_0 = 0$ for Γ. For $k = 0$, we have the usual Fourier expansion:

$$f(z) = \sum_{n=-\mu_0}^{\infty} a(n)q^n.$$

We have the following formula for $a(n)$ ([17], page 313).

Theorem 5.2 *For $n > 0$,*

$$a(n) = 2\pi i^{-r} \sum_{j=0}^{s-1} \sum_{\nu=1}^{\mu_j} a(-\nu)^{(j)} \sum_{\substack{c\in C_{j0}\\ c>0}} c^{-1} A(c,n,\nu_j) L(c,n,\nu_j,-r), \tag{5.3}$$

where $\nu_j = \frac{\nu-\kappa_j}{\lambda_j}$, $M^* = A_j M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and

$$A(c,n,\nu_j) = \sum_{d\in D_c} v^{-1}(M) e\left(\frac{nd-\nu_j a}{c}\right),$$

$$L(c,n,\nu_j,r) = \left(\frac{\nu_j}{n}\right)^{\frac{r+1}{2}} I_{r+1}\left(\frac{4\pi\sqrt{n\nu_j}}{c}\right),$$

where I_{r+1} is the modified Bessel function of the first kind. It has the asymptotic expansion $I_{r+1}(x) = \frac{e^x}{\sqrt{2\pi x}}(1 + O(\frac{1}{x}))$.

5.1 Asymptotics of Fourier coefficients of modular forms of weight $-\frac{1}{2}$; proof of Theorem 2.8

Now we apply the theorem to $v_d \in \mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$. Recall that $\Gamma_0(4)$ has three cusps: $p_0 = \infty$, $p_1 = 0$, $p_2 = \frac{1}{2}$ ([16], page 108).

First, $p_0 = \infty$. In this case, $\lambda_0 = 1$, $\kappa_0 = 0$, and $A_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. If $c \in C_{00}$, then $4|c$, and the smallest $c \in C_{00}$ is 4, and we have $M = M^* = \left(\begin{smallmatrix} -3 & -1 \\ 4 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & -1 \\ 4 & 3 \end{smallmatrix}\right)$ for the set D_4. Because of (2.7), we only need to consider $\nu_0 = \mu_0$. In our case, $v(M) = (\frac{c'}{d'})\epsilon_{d'}$ for $M = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, and

$$A(4,n,\mu_0) = e\left(\frac{n+3\mu_0}{4}\right) - ie\left(\frac{3n+\mu_0}{4}\right).$$

So if $d = 4k$ then $\mu_0 = 4k$ and $A(4,n,\mu_0) = 1 - i$ for any $n \equiv 0, 3 \pmod 4$. If $d = 4k+1$ then $\mu_0 = 4k+1$ and $A(4,n,\mu_0) = (-1)^n(1-i)$ if $n \equiv 0, 3 \pmod 4$.

Second, $p_1 = 0$. In this case, $\lambda_1 = 4$, $\kappa_1 = 0$, and $A_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The smallest $c \in C_{10}$ is 1, and $M^* = \left(\begin{smallmatrix} -4 & -5 \\ 1 & 1 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 1 & 1 \\ 4 & 5 \end{smallmatrix}\right)$. Hence

$$A(1,n,\nu_1) = e(n+\nu) = 1, \quad \nu = 1, \ldots, \mu_1.$$

The Fourier expansion at 0 is

$$v_d\left(-\frac{1}{z}\right) = iz^{-\frac{1}{2}} \sum_{n=-\mu_1}^{\infty} a(n)^{(1)} q^{\frac{n}{4}}.$$

Third, $p_2 = \frac{1}{2}$. In this case, $\lambda_2 = 4$, $\kappa_2 = \frac{3}{4}$, and $A_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -\frac{1}{2} \end{smallmatrix}\right)$. The smallest $c \in C_{20}$ is 1, and $M^* = \left(\begin{smallmatrix} -4 & -3 \\ 1 & \frac{1}{2} \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 3 & 2 \\ 4 & 3 \end{smallmatrix}\right)$. Hence

$$A(1, n, \nu_2) = -i\, e\left(\frac{n}{2} + \nu - \frac{3}{4}\right) = e^{\pi i n} = (-1)^n, \quad \nu = 1, \ldots, \mu_2.$$

The Fourier expansion at $\frac{1}{2}$ is

$$v_d\left(-\frac{1}{z} + \frac{1}{2}\right) = iz^{-\frac{1}{2}} q^{\frac{3}{16}} \sum_{n=-\mu_2}^{\infty} a(n)^{(2)} q^{\frac{n}{4}}.$$

Combining all these calculations, we write (5.3) for $v_d(z)$ as follows:

$$\begin{aligned} 2\pi i^{\frac{1}{2}} \Bigg\{ & \frac{1}{4} A(4, n, d) L\left(4, n, d, \frac{1}{2}\right) + \sum_{\nu=1}^{\mu_1} a(-\nu)^{(1)} L\left(1, n, \nu_1, \frac{1}{2}\right) \\ & + \sum_{\nu=1}^{\mu_2} (-1)^n a(-\nu)^{(2)} L\left(1, n, \nu_2, \frac{1}{2}\right) + \sum_{\substack{c \in C_{00} \\ c > 4,\, 4|c}} c^{-1} A(c, n, d) L\left(c, n, d, \frac{1}{2}\right) \\ & + \sum_{j=1}^{2} \sum_{\substack{c \in C_{j0} \\ c > 1}} \sum_{\nu=1}^{\mu_j} a(-\nu)^{(j)} c^{-1} A(c, n, \nu_j) L\left(c, n, \nu_j, \frac{1}{2}\right) \Bigg\}. \end{aligned} \tag{5.4}$$

Let us consider $v_{4k}(z)$. Suppose $v_{4k}(z) = q^{-4k} + \sum_{n=1}^{\infty} a(n) q^n$. It follows from (5.1) that Fourier expansions at the other cusps are of the forms

$$v_{4k}\left(-\frac{1}{z}\right) = iz^{-\frac{1}{2}} \sum_{n=-\mu_1}^{\infty} a(n)^{(1)} q^{\frac{n}{4}},$$

$$v_{4k}\left(-\frac{1}{z} + \frac{1}{2}\right) = iz^{-\frac{1}{2}} q^{\frac{3}{16}} \sum_{n=-\mu_2}^{\infty} a(n)^{(2)} q^{\frac{n}{4}}.$$

Now we prove the following by imitating the proof of Lemma 14.2 in [2], and determine the principal parts of Fourier expansions at the other cusps.

Lemma 5.5

$$z^{\frac{1}{2}} v_{4k}\left(-\frac{1}{z}\right) = 2(1+i)\left(q^{-\frac{k}{4}} + \sum_{n=1}^{\infty} a(4n) q^{\frac{n}{4}}\right),$$

$$z^{\frac{1}{2}} v_{4k}\left(-\frac{1}{z}+\frac{1}{2}\right) = 2(1+i)q^{\frac{3}{16}} \sum_{n=0}^{\infty} a(4n+3)q^{\frac{n}{4}}.$$

Proof Let

$$h_0(z) = q^{-k} + \sum_{n=1}^{\infty} a(4n)q^n, \qquad h_1(z) = \sum_{n=0}^{\infty} a(4n+3)q^{n+\frac{3}{4}}.$$

Then $v_{4k}(z) = h_0(4z) + h_1(4z)$. Since $v_{4k} \in \mathcal{M}^+_{-\frac{1}{2}}(\Gamma_0(4))$,

$$v_{4k}\left(\frac{z}{4z+1}\right) = (4z+1)^{-\frac{1}{2}} v_{4k}(z).$$

By replacing $4z+1$ by z, and noting that $h_0(z \pm 1) = h_0(z)$, $h_1(z+1) = -ih_1(z)$, and $h_1(z-1) = ih_1(z)$, we have

$$h_0\left(-\frac{1}{z}\right) + ih_1\left(-\frac{1}{z}\right) = z^{-\frac{1}{2}}\big(h_0(z) - ih_1(z)\big).$$

Now let $z = iy$ and note that $h_0(iy)$ and $h_1(iy)$ are real. Then

$$h_0\left(\frac{i}{y}\right) = \frac{1}{\sqrt{2y}}\big(h_0(iy) + h_1(iy)\big), \qquad h_1\left(\frac{i}{y}\right) = \frac{1}{\sqrt{2y}}\big(h_0(iy) - h_1(iy)\big).$$

Since h_0 and h_1 are meromorphic functions, the above equalities are true by replacing iy by z with $\mathrm{Im}(z) > 0$. Hence

$$h_0\left(-\frac{1}{z}\right) = \frac{1+i}{2} z^{-\frac{1}{2}}\big(h_0(z) + h_1(z)\big), \qquad h_1\left(-\frac{1}{z}\right) = \frac{1+i}{2} z^{-\frac{1}{2}}\big(h_0(z) - h_1(z)\big).$$

Therefore,

$$v_{4k}\left(-\frac{1}{z}\right) = h_0\left(-\frac{4}{z}\right) + h_1\left(-\frac{4}{z}\right) = 2(1+i)z^{-\frac{1}{2}} h_0\left(\frac{z}{4}\right).$$

For $v_{4k}(-\frac{1}{z}+\frac{1}{2})$, note that $h_1(z+2) = -h_1(z)$. Then

$$v_{4k}\left(-\frac{1}{z}+\frac{1}{2}\right) = h_0\left(-\frac{4}{z}\right) - h_1\left(-\frac{4}{z}\right) = 2(1+i)z^{-\frac{1}{2}} h_1\left(\frac{z}{4}\right). \qquad \square$$

By Lemma 5.5, we obtain $A(4,n,d) = 1-i$ for $\mu_0 = d = 4k$ and the main term in (5.3) is

$$2\pi i^{\frac{1}{2}}\left(\frac{1-i}{4}\left(\frac{4k}{n}\right)^{\frac{3}{4}} I_{\frac{3}{2}}(\pi\sqrt{4nk}) + 2(1-i)\left(\frac{k}{4n}\right)^{\frac{3}{4}} I_{\frac{3}{2}}(2\pi\sqrt{nk})\right)$$
$$= 4\pi\left(\frac{k}{n}\right)^{\frac{3}{4}} I_{\frac{3}{2}}(2\pi\sqrt{nk}). \tag{5.6}$$

Since $I_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{z\cosh(z)-\sinh(z)}{z^{\frac{3}{2}}}$, the above term is equal to

$$4k^{\frac{1}{2}} n^{-1} \cosh(\pi\sqrt{4kn}) - \frac{2}{\pi} n^{-\frac{3}{2}} \sinh(\pi\sqrt{4kn}).$$

We consider the other terms:

$$\begin{aligned} &2\pi i^{\frac{1}{2}} \Bigg(\sum_{\substack{c\in C_{00} \\ c>4,\, 4|c}} c^{-1} A(c,n,4k) L\left(c,n,4k,\frac{1}{2}\right) \\ &\quad + \frac{1+i}{2} \sum_{\substack{c\in C_{10} \\ c>1}} c^{-1} A\left(c,n,\frac{k}{4}\right) L\left(c,n,\frac{k}{4},\frac{1}{2}\right)\Bigg) \\ &= 2\pi i^{\frac{1}{2}} \Bigg(\sum_{\substack{4c\in C_{00} \\ c>4,\, 4|c}} c^{-1} A(c,n,4k) \left(\frac{4k}{n}\right)^{\frac{3}{4}} I_{\frac{3}{2}}\left(\frac{4\pi\sqrt{4kn}}{c}\right) \\ &\quad + \frac{1+i}{2} \sum_{\substack{c\in C_{10} \\ c>1}} c^{-1} A\left(c,n,\frac{k}{4}\right) \left(\frac{k}{4n}\right)^{\frac{3}{4}} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{4kn}}{c}\right)\Bigg). \end{aligned}$$

We will prove that the above sum is smaller than the main term (5.6). We only deal with the first sum. The second sum is similar.

We divide the first sum into two regions: $4 < c \le 4\pi\sqrt{4kn}$ and $c > 4\pi\sqrt{4kn}$. By Weil's bound (cf. [22], page 26 and [9], page 403), $|A(c,n,4k)| \le (4k,n,c)^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c)$, where $\tau(c)$ is the number of positive divisors of c and $(4k,n,c)$ is the g.c.d. of $4k$, n and c. (Similarly, $(4k,n)$ will denote the g.c.d. of $4k$ and n.)

In the region $c \le 4\pi\sqrt{4kn}$, using the fact that $(4k,n,c) \le (4k,n) \le (4kn)^{\frac{1}{2}}$, the first sum is less than

$$\begin{aligned} &4\sqrt{2} k n^{-\frac{1}{2}} \sum_{4<c\le 4\pi\sqrt{4kn},\, 4|c} \tau(c) \cosh\left(\frac{4\pi\sqrt{4kn}}{c}\right) \\ &\quad \le 4\sqrt{2} k n^{-\frac{1}{2}} \cosh\left(\frac{\pi\sqrt{4kn}}{2}\right) \sum_{c\le 4\pi\sqrt{4kn}} \tau(c). \end{aligned}$$

By using the fact that $\sum_{c\le x} \tau(c) \le 2x\log x$, it is less than

$$32\sqrt{2}\pi k^{\frac{3}{2}} \log(4\pi\sqrt{4kn}) \cosh\left(\frac{\pi\sqrt{4kn}}{2}\right).$$

In the region $c > 4\pi\sqrt{4kn}$, we use the fact that for $0 < z < 1$, $I_{\frac{3}{2}}(z) \le \sqrt{\frac{2}{\pi}} z^{\frac{3}{2}}$. We use the trivial bound $|A(c, n, 4k)| \le c$. Then

$$\begin{aligned}
&\left| 2\pi i^{\frac{1}{2}} \left(\sum_{\substack{c \in C_{00} \\ c > 4\pi\sqrt{4kn}}} c^{-1} A(c, n, 4k) L\left(c, n, 4k, \frac{1}{2}\right)\right)\right| \\
&\quad \le 16\sqrt{2}\pi^2 k^{\frac{3}{2}} \sum_{c > 4\pi\sqrt{4kn}} c^{-\frac{3}{2}} \le 16\pi^{\frac{3}{2}} k^{\frac{5}{4}} n^{-\frac{1}{4}}.
\end{aligned}$$

Hence, the first sum is less than

$$16\pi^{\frac{3}{2}} k^{\frac{5}{4}} n^{-\frac{1}{4}} + 32\sqrt{2}\pi k^{\frac{3}{2}} \log(4\pi\sqrt{4kn}) \cosh\left(\frac{\pi\sqrt{4kn}}{2}\right).$$

Similarly, the second sum is less than

$$4\pi^{\frac{3}{2}} k^{\frac{5}{4}} n^{-\frac{1}{4}} + 32\pi^{\frac{3}{2}} k^{\frac{3}{2}} \log(4\pi\sqrt{4kn}) \cosh\left(\frac{\pi\sqrt{4kn}}{2}\right).$$

We can show easily that the sum of the above two terms is less than the main term (5.6) for $kn \ge 8$. Hence $a(n)$ is positive if $kn \ge 8$. When $kn < 8$, by looking at the tables, we see that $a(n)$ is positive. Therefore, we have shown that $a(n)$ is positive for all n. Also we have the following formula with error term:

$$\begin{aligned}
a(n) &= 2\left((4k)^{\frac{1}{2}} n^{-1} \cosh(\pi\sqrt{4kn}) - \pi^{-1} n^{-\frac{3}{2}} \sinh(\pi\sqrt{4kn})\right) \\
&\quad + O\left((4k)^{\frac{3}{2}} \log(4\pi\sqrt{4kn}) e^{\pi\sqrt{kn}}\right).
\end{aligned}$$

In the same way, we now consider v_{4k+1} and suppose $v_{4k+1}(z) = q^{-4k-1} + \sum_{n=0}^{\infty} a(n) q^n$. Then we can show as in Lemma 5.5:

Lemma 5.7

$$z^{\frac{1}{2}} v_{4k+1}\left(-\frac{1}{z}\right) = 2(1+i) \sum_{n=1}^{\infty} a(4n) q^{\frac{n}{4}},$$

$$z^{\frac{1}{2}} v_{4k+1}\left(-\frac{1}{z} + \frac{1}{2}\right) = 2(1+i) q^{\frac{3}{16}} \left(q^{-\frac{k+1}{4}} + \sum_{n=0}^{\infty} a(4n+3) q^{\frac{n}{4}}\right).$$

Proof Let

$$h_0(z) = \sum_{n=0}^{\infty} a(4n) q^n, \qquad h_1(z) = q^{-k-\frac{1}{4}} \sum_{n=0}^{\infty} a(4n+3) q^{n+\frac{3}{4}}.$$

Then $v_{4k+1}(z) = h_0(4z) + h_1(4z)$, and we follow the proof of Lemma 5.5. □

By Theorem 5.2 and Lemma 5.7, the main term of $a(n)$ is

$$2\pi i^{\frac{1}{2}}\left(\frac{1}{4}A(4,n,4k+1)L\left(4,n,4k+1,\frac{1}{2}\right)\right.$$
$$\left.+2(1-i)A\left(1,n,\frac{k+\frac{1}{4}}{4}\right)L\left(1,n,\frac{k+\frac{1}{4}}{4},\frac{1}{2}\right)\right).$$

Since $A(4,n,4k+1)=(-1)^n$, it is equal to

$$(-1)^n\pi\sqrt{2}\left(\frac{4k+1}{n}\right)^{\frac{3}{4}}I_{\frac{3}{2}}\left(\pi\sqrt{(4k+1)n}\right)$$
$$=2(-1)^n\left((4k+1)^{\frac{1}{2}}n^{-1}\cosh\left(\pi\sqrt{(4k+1)n}\right)-\pi^{-1}n^{-\frac{3}{2}}\sinh\left(\pi\sqrt{(4k+1)n}\right)\right).$$

For example, when $k=2, n=7$, $a(n)\sim -27774695413.6\ldots$. The actual value is -27774693612.

As in the case of v_{4k}, we can show that $(-1)^n a(n)$ is positive for all n, and we have the following formula with error term:

$$a(n)=2(-1)^n\left((4k+1)^{\frac{1}{2}}n^{-1}\cosh\left(\pi\sqrt{(4k+1)n}\right)-\pi^{-1}n^{-\frac{3}{2}}\sinh\left(\pi\sqrt{(4k+1)n}\right)\right)$$
$$+O\left((4k+1)^{\frac{3}{2}}\log\left(4\pi\sqrt{(4k+1)n}\right)e^{\frac{\pi\sqrt{(4k+1)n}}{2}}\right).$$

In particular, we have:

Corollary 5.8 *When $k=0$,*

$$a(4n)=\frac{e^{2\pi\sqrt{n}}}{4n}\left(1-\frac{1}{2\pi\sqrt{n}}\right)+O\left(\log(8\pi\sqrt{n})e^{\pi\sqrt{n}}\right).$$

5.2 Asymptotics of Niemann's bound; proof of Theorem 3.5

Recall that Niemann's bound for root multiplicities of $\mathcal{F}$ comes from $f(z):=\eta(z)^{-1}\eta(23z)^{-1}$ (see [20], pages 23–27). By [21], page 18, the function $f(z)$ is a weakly holomorphic modular form of weight -1 with respect to $\Gamma_0(23)$, namely,

$$f\left(\frac{az+b}{cz+d}\right)=\chi(d)(cz+d)^{-1}f(z),\quad \chi(d)=\left(\frac{-23}{d}\right),\quad \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(23).$$

We write

$$f(z)=q^{-1}+\sum_{n=0}^{\infty}a(n)q^n.$$

Here $\Gamma_0(23)$ has two cusps ([16], page 108): $p_0=\infty$ and $p_1=0$. We will use Theorem 5.2. In this case, $\lambda_0=1, \kappa_0=0$, and $\lambda_1=23$, $\kappa_1=0$. Since $\eta(-\frac{1}{z})=$

$(-iz)^{\frac{1}{2}}\eta(z)$, we can see that

$$f\left(-\frac{1}{z}\right) = i\sqrt{23}z^{-1}\left(q^{-\frac{1}{23}} + \sum_{n=0}^{\infty} a(n)q^{\frac{n}{23}}\right).$$

Then by Theorem 5.2, we obtain

$$a(n) = 2\pi i\left(\sum_{c=1}^{\infty}(23c)^{-1}A(23c, n, 1)L(23c, n, 1, 1) - i\sqrt{23}\sum_{c\in C_{10}} c^{-1}A\left(c, n, \frac{1}{23}\right)L\left(c, n, \frac{1}{23}, 1\right)\right),$$

where $L(23c, n, 1, 1) = n^{-1}I_2(\frac{4\pi\sqrt{n}}{23c})$, $L(c, n, \frac{1}{23}, 1) = (23n)^{-1}I_2(\frac{4\pi}{c}\sqrt{\frac{n}{23}})$, and

$$A\left(1, n, \frac{1}{23}\right) = 1, \qquad A(23, n, 1) = \sum_{d \ (\mathrm{mod}\ 23)} \left(\frac{-23}{d}\right)e\left(\frac{nd-a}{23}\right),$$

where $da \equiv 1 \pmod{23}$. Here the second sum is the Salié sum $T(n, -1, 23)$ ([9], page 323):

$$T(n, -1, 23) = -i\sqrt{23}\sum_{v^2\equiv -n \ (\mathrm{mod}\ 23)} e\left(\frac{2v}{23}\right).$$

In Niemann's notation [20], $a(n) = p_\sigma(1+n)$. Therefore, using the trivial bound for the Kloosterman sums, we have

$$p_\sigma(n+1) = \frac{2\pi}{n\sqrt{23}}I_2\left(\frac{4\pi\sqrt{n}}{\sqrt{23}}\right) + O\left(n^{-\frac{1}{2}}I_2\left(\frac{2\pi\sqrt{n}}{\sqrt{23}}\right)\right).$$

For example, when $n = 28$, we obtain $\frac{2\pi}{\sqrt{23}n}I_2(\frac{4\pi\sqrt{n}}{\sqrt{23}}) = 4578.99$. The actual value of $p_\sigma(29)$ is 4576 and the exact multiplicity of α in $\mathcal{F}$ is 4557 when $-\frac{1}{2}(\alpha, \alpha) = 28$.

5.3 Other hyperbolic Kac–Moody algebras

Niemann [20] obtained upper bounds for root multiplicities of other hyperbolic Kac–Moody algebras, and our method can be applied to produce asymptotic formulas for those bounds, too. We will consider two more hyperbolic Kac–Moody algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ whose Cartan matrices are

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}, \quad \text{respectively.}$$

In Niemann's notation [20], $\mathcal{F}_1$ is the case when $N = 11$ and $\mathcal{F}_2$ is when $N = 7$. We first consider $\mathcal{F}_1$, and set $f(z) = \eta(z)^{-2}\eta(11z)^{-2}$ and $\sum_{n=0}^{\infty} p_\sigma(n)q^n = q\eta(z)^{-2}\eta(11z)^{-2}$. By [21], page 18, the function $f(z)$ is a weakly holomorphic modular form of weight -2 with respect to $\Gamma_0(11)$, namely,

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{-2} f(z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(11).$$

We write

$$f(z) = q^{-1} + \sum_{n=0}^{\infty} a(n)q^n.$$

Note that $\Gamma_0(11)$ has two cusps ([16], page 108): $p_0 = \infty$ and $p_1 = 0$. In this case, $\lambda_0 = 1$, $\kappa_0 = 0$, and $\lambda_1 = 11$, $\kappa_1 = 0$. Then

$$f\left(-\frac{1}{z}\right) = -11z^{-2} f\left(\frac{z}{11}\right) = -11z^{-2}\left(q^{-\frac{1}{11}} + \sum_{n=0}^{\infty} a(n)q^{\frac{n}{11}}\right).$$

Then by Theorem 5.2, we obtain

$$\begin{aligned} a(n) = 2\pi i^2 \Bigg(& \sum_{c=1}^{\infty} (11c)^{-1} A(11c, n, 1) L(11c, n, 1, 1) \\ & - 11 \sum_{c \in C_{10}} c^{-1} A\left(c, n, \frac{1}{11}\right) L\left(c, n, \frac{1}{11}, 1\right)\Bigg), \end{aligned}$$

where $L(11c, n, 1, 1) = n^{-\frac{3}{2}} I_3(\frac{4\pi\sqrt{n}}{11c})$, $L(c, n, \frac{1}{11}, 1) = (11n)^{-\frac{3}{2}} I_3(\frac{4\pi}{c}\sqrt{\frac{n}{11}})$, and

$$A\left(1, n, \frac{1}{11}\right) = 1, \qquad A(11, n, 1) = \sum_{d \,(\mathrm{mod}\ 11)} e\left(\frac{nd - a}{11}\right),$$

where $da \equiv 1 \pmod{11}$. Here the second sum is the Kloosterman sum.

Therefore, when $N = 11$, $a(n) = p_\sigma(1+n)$, and

$$a(n) \sim \frac{2\pi}{\sqrt{11}n^{\frac{3}{2}}} I_3\left(4\pi\sqrt{\frac{n}{11}}\right).$$

When $n = 15$, $\frac{2\pi}{\sqrt{11}n^{\frac{3}{2}}} I_3(4\pi\sqrt{\frac{n}{11}}) = 5892.28$. The actual value of $p_\sigma(16)$ is 5894, and the corresponding root multiplicity of $\mathcal{F}_1$ is 5812.

Similarly, we consider $N = 7$, i.e., $f(z) = \eta(z)^{-3}\eta(7z)^{-3}$ in [20]. It satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^{-3} f(z), \quad \chi(d) = \left(\frac{-7}{d}\right), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(7).$$

We write

$$f(z)=q^{-1}+\sum_{n=0}^{\infty}a(n)q^n.$$

Then $f(-\frac{1}{z})=-i7^{\frac{3}{2}}z^{-3}f(\frac{z}{7})$, and we get

$$a(n)\sim\frac{2\pi}{\sqrt{7}n^2}I_4\left(4\pi\sqrt{\frac{n}{7}}\right).$$

Even more, we can consider $N=5,3,2$, and let $a_N=\frac{24}{N+1}$ and $f(z)=\eta(z)^{-a_N}\eta(Nz)^{-a_N}$ as in [20]. It satisfies

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^{-a_N}f(z),\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N).$$

If we write

$$f(z)=q^{-1}+\sum_{n=0}^{\infty}a(n)q^n,$$

then $f(-\frac{1}{z})=(-N)^{\frac{a_N}{2}}z^{-a_N}f(\frac{z}{N})$, and

$$a(n)\sim\frac{2\pi}{\sqrt{N}n^{\frac{a_N+1}{2}}}I_{a_N+1}\left(4\pi\sqrt{\frac{n}{N}}\right).$$

There are two hyperbolic Kac–Moody algebras when $N=5$ for which $a(n)$ gives a reasonable bound for root multiplicities. The cases $N=3,2$ do not seem to provide any good bounds for root multiplicities of a hyperbolic Kac–Moody algebra. See [20], Sect. 6.2, for more details.

5.4 Other modular forms

We can apply the same technique to other modular forms. We first consider the function $j_m(z)$ in [21], page 23. It is defined as $j_0(z)=1$, $j_1(z)=j(z)-744$, and for $m\geq 2$,

$$j_m(z)=j_1(z)|T_0(m)=\sum_{\substack{d|m\\ ad=m}}\sum_{b=0}^{d-1}j_1\left(\frac{az+b}{d}\right).$$

It has the q-expansion

$$j_m(z)=q^{-m}+\sum_{n=1}^{\infty}c_m(n)q^n.$$

From the definition, it is clear that $c_m(n)$ are all positive integers. We have the following series expression for $c_m(n)$ ([17], page 314):

$$c_m(n) = 2\pi \sum_{k=1}^{\infty} \frac{A(k,n,m)}{k} \left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{mn}}{k}\right) + O_m(1),$$

where

$$A(k,n,m) = \sum_{\substack{h \pmod{k},\ (h,k)=1,\\ hh' \equiv -1 \pmod{k}}} e\left(-\frac{nh+mh'}{k}\right).$$

Hence we have

$$c_m(n) \sim \frac{m^{\frac{1}{4}} e^{4\pi\sqrt{mn}}}{n^{\frac{3}{4}}\sqrt{2}}.$$

Next, we consider elements of the Kohnen plus-space $M^+_{\frac{3}{2}}(\Gamma_0(4))$. For each positive integer $D \equiv 0, 1 \pmod 4$, let $g_D(z) \in \mathcal{M}^+_{\frac{3}{2}}(\Gamma_0(4))$ be the unique modular form with a Fourier expansion of the form ([21], page 72)

$$g_D(z) = q^{-D} + \sum_{\substack{d \geq 0\\ d \equiv 0,3 \pmod 4}} B(D,d)\, q^d.$$

We can get an asymptotic expression for $B(D,d)$. In this case, we have the following formula with an error term. For $g(z) \in \mathcal{M}^+_{\frac{3}{2}}(\Gamma_0(4))$, we write $g(z) = \sum_{n=-\mu_0}^{\infty} a(n) q^n$ and keep the notations in the beginning of Sect. 4.

Theorem 5.9 [18] *For $n > 0$,*

$$a(n) = 2\pi i^{-\frac{3}{2}} \sum_{j=0}^{s-1} \sum_{\nu=1}^{\mu_j} a(-\nu)^{(j)} \sum_{\substack{c \in C_{j0}\\ 0<c<\sqrt{n}}} c^{-1} A(c,n,\nu_j) M\left(c,n,\nu_j,\frac{3}{2}\right) + E(n), \tag{5.10}$$

where $\nu_j = \frac{\nu - \kappa_j}{\lambda_j}$, and

$$A(c,n,\nu_j) = \sum_{d \in D_c} v^{-1}(M) e\left(\frac{nd - \nu_j a}{c}\right), \quad M = A_j^{-1} M^*$$

$$M\left(c,n,\nu_j,\frac{3}{2}\right) = \left(\frac{n}{\nu_j}\right)^{\frac{1}{4}} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{n\nu_j}}{c}\right).$$

Here $E(n) = O(n^{\frac{3}{4}})$, where the implied constant is independent of n.

We apply the above theorem to $g_{4k}(z) = q^{-4k} + \sum_{n=0}^{\infty} a(n)q^n$. In this case, $\kappa_1 = 0, \kappa_2 = \frac{3}{4}$. So it has Fourier expansions at the other cusps of the following forms:

$$g_{4k}\left(-\frac{1}{z}\right) = iz^{\frac{3}{2}} \sum_{n=-\mu_1}^{\infty} a(n)^{(1)} q^{\frac{n}{4}}, \qquad g_{4k}\left(-\frac{1}{z} + \frac{1}{2}\right) = iz^{\frac{3}{2}} q^{\frac{3}{16}} \sum_{n=-\mu_2}^{\infty} a(n)^{(2)} q^{\frac{n}{4}}.$$

Let

$$h_0(z) = q^{-k} + \sum_{n=0}^{\infty} a(4n)q^n, \qquad h_1(z) = \sum_{n=0}^{\infty} a(4n+3)q^{n+\frac{3}{4}}.$$

Then $g_{4k}(z) = h_0(4z) + h_1(4z)$. In the same way as in Lemma 5.5 for v_{4k}, we can show

$$g_{4k}\left(-\frac{1}{z}\right) = \frac{1+i}{8} z^{\frac{3}{2}} h_0\left(\frac{z}{4}\right), \qquad g_{4k}\left(-\frac{1}{z} + \frac{1}{2}\right) = \frac{1+i}{8} z^{\frac{3}{2}} h_1\left(\frac{z}{4}\right).$$

We have $\mu_0 = 4k$ and $A(4, n, \mu_0) = 1 - i$ in (5.10). Hence we obtain

$$\begin{aligned} a(n) &\sim 2\pi i^{-\frac{3}{2}} \left(\frac{1-i}{4\sqrt{2}} \left(\frac{n}{k}\right)^{\frac{1}{4}} I_{\frac{1}{2}}(2\pi\sqrt{nk}) + \frac{\sqrt{2}(1-i)}{8} \left(\frac{n}{k}\right)^{\frac{1}{4}} I_{\frac{1}{2}}(2\pi\sqrt{nk}) \right) \\ &= -\pi \left(\frac{n}{k}\right)^{\frac{1}{4}} I_{\frac{1}{2}}(2\pi\sqrt{nk}). \end{aligned}$$

Since $I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{\sinh(z)}{\sqrt{z}}$, we have

$$a(n) \sim -\frac{2\sinh(\pi\sqrt{4kn})}{\sqrt{4k}}.$$

We can also show that $a(n)$ is negative for all sufficiently large n. However, due to the error term $E(n)$, we cannot show it for all n. In order to show that $a(n)$ is negative for all n, a different approach needs to be taken. See Remark 5.11 below.

Now we consider $g_{4k+1}(z) = q^{-4k-1} + \sum_{n=0}^{\infty} a(n)q^n$. In the same way as we did for g_{4k}, we can show that

$$a(n) \sim (-1)^{n-1} \frac{2\sinh(\pi\sqrt{(4k+1)n})}{\sqrt{4k+1}}.$$

For example, when $k = 2, n = 7$, $a(n) \sim 22505067826.5\ldots$. The actual value is 22505066244.

Remark 5.11 Zagier [23] proved that $B(D, d) = -A(D, d)$, where

$$v_d(z) = q^{-d} + \sum_{D>0} A(D, d)q^D \in M^{+}_{\frac{1}{2}}\big(\Gamma_0(4)\big).$$

In [13], we showed that if $4|d$, $A(D,d)$ is a positive integer for all $D>0$, using the explicit formula (without error term) in [3]. Hence if $4|D$, the coefficient $B(D,d)$ is a negative integer for all d. We can also prove it directly, using the explicit formula in [3].

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References

1. Borcherds, R.E.: Monstrous moonshine and monstrous Lie superalgebras. Invent. Math. **109**(2), 405–444 (1992)
2. Borcherds, R.E.: Automorphic forms on $O_{s+2,2}(R)$ and infinite products. Invent. Math. **120**(1), 161–213 (1995)
3. Bringmann, K., Ono, K.: Arithmetic properties of coefficients of half-integral weight Maass–Poincare series. Math. Ann. **337**, 591–612 (2007)
4. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Progress in Mathematics, vol. 55. Birkhäuser Boston, Inc., Boston (1985)
5. Feingold, A.J., Frenkel, I.B.: A hyperbolic Kac–Moody algebra and the theory of Siegel modular forms of genus 2. Math. Ann. **263**(1), 87–144 (1983)
6. Frenkel, I.B.: Representations of Kac–Moody algebras and dual resonance models. In: Applications of Group Theory in Physics and Mathematical Physics, Chicago, 1982. Lectures in Appl. Math., vol. 21, pp. 325–353. Amer. Math. Soc., Providence (1985)
7. Gritsenko, V.A., Nikulin, V.V.: Igusa modular forms and "the simplest" Lorentzian Kac–Moody algebras. Mat. Sb. **187**(11), 27–66 (1996); translation in Sb. Math. **187**(11), 1601–1641 (1996)
8. Gritsenko, V.A., Nikulin, V.V.: Siegel automorphic form corrections of some Lorentzian Kac–Moody Lie algebras. Am. J. Math. **119**(1), 181–224 (1997)
9. Iwaniec, H., Kowalski, E.: Analytic Number Theory. American Mathematical Society Colloquium Publications, vol. 53 (2004)
10. Kac, V.G.: Infinite-Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
11. Kang, S.-J.: Root multiplicities of the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$. J. Algebra **160**(2), 492–523 (1993)
12. Kang, S.-J.: On the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$. Trans. Am. Math. Soc. **341**(2), 623–638 (1994)
13. Kim, H.H., Lee, K.-H.: A family of generalized Kac–Moody algebras and deformation of modular forms. Int. J. Number Theory **8**(5), 1107–1131 (2012)
14. Kim, H.H., Lee, K.-H.: Rank 2 symmetric hyperbolic Kac–Moody algebras and Hilbert modular forms. Preprint, arXiv:1209.1860
15. Klima, V.W., Misra, K.C.: Root multiplicities of the indefinite Kac–Moody algebras of symplectic type. Commun. Algebra **36**(2), 764–782 (2008)
16. Koblitz, N.: Introduction to Elliptic Curves and Modular Forms. Springer, Berlin (1984)
17. Lehner, J.: Discontinuous Groups and Automorphic Functions. Mathematical Surveys, No. VIII. American Mathematical Society, Providence (1964)
18. Lehner, J.: On automorphic forms of negative dimension. Ill. J. Math. **8**, 395–407 (1964)
19. Lepowsky, J., Moody, R.V.: Hyperbolic Lie algebras and quasiregular cusps on Hilbert modular surfaces. Math. Ann. **245**(1), 63–88 (1979)
20. Niemann, P.: Some generalized Kac–Moody algebras with known root multiplicities. Mem. Amer. Math. Soc. **157**(746) (2002)
21. Ono, K.: The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series. CBMS Regional Conference Series in Mathematics, vol. 102. American Mathematical Society, Providence (2004)
22. Sarnak, P.: Some Applications of Modular Forms. Cambridge University Press, Cambridge (1990)
23. Zagier, D.: Traces of singular moduli. In: Bogomolov, F., Katzarkov, L. (eds.) Motives, Polylogarithms, and Hodge Theory, pp. 211–244. Intl. Press, Somerville (2003)