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Abstract

The weight multiplicities of finite dimensional simple Lie algebras can be computed individually using various methods. Still, it is hard to derive explicit closed formulas. Similarly, explicit closed formulas for the multiplicities of maximal weights of affine Kac–Moody algebras are not known in most cases. In this paper, we study weight multiplicities for both finite and affine cases of classical types for certain infinite families of highest weights modules. We introduce new classes of Young tableaux, called the (*spin*) *rigid tableaux*, and prove that they are equinumerous to the weight multiplicities of the highest weight modules under our consideration. These new classes of Young tableaux arise from crystal basis elements for dominant maximal weights of the integrable highest weight modules over affine Kac–Moody algebras. By applying combinatorics of tableaux such as the Robinson–Schensted algorithm and new insertion schemes, and using integrals over orthogonal groups, we reveal hidden structures in the sets of weight multiplicities and obtain explicit closed formulas for the weight multiplicities. In particular we show that some special families of weight multiplicities form the Pascal, Catalan, Motzkin, Riordan and Bessel triangles.

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Introduction

The irreducible representations $L(\omega)$ of finite dimensional complex simple Lie algebras are fundamental objects in mathematics. We understand their structures through the generating functions of weight multiplicities, i.e. the characters of the representations, which can be computed by the celebrated Weyl's character formula. Individual weight multiplicities can be computed using Kostant's formula or Freudenthal's recursive formula. One can also exploit the crystal basis theory, initiated by Kashiwara [**21**], and its realizations such as Kashiwara–Nakashima tableaux [**22**], Littelmann paths [**31**] and Mirkovic–Vilonen polytopes [**17**] to name a few.

Nonetheless there are only a few explicit closed formulas in the literature for weight multiplicities. Kostant's formula involves a summation over the Weyl group whose size becomes enormous as the rank increases, and Freudenthal's formula is recursive, and realizations of crystals convert computing weight multiplicities into challenging combinatorial problems.

The theory of finite-dimensional simple Lie algebras was generalized to that of Kac–Moody algebras in 1960's, and the first family of infinite dimensional Lie algebras is called *affine* Kac–Moody algebras. Representations of affine Kac–Moody algebras have been studied extensively as their applications have been found throughout mathematics and mathematical physics. In particular, weight multiplicities of an integrable highest weight module $V(\Lambda)$ over an affine Kac–Moody algebra are of great interests as they can be interpreted in several different ways such as generalized partition numbers [**30**], Fourier coefficients of certain modular forms [**16**], and numbers of isomorphism classes of irreducible modules over Hecke-type algebras [**1**, **28**]. However, our understanding of weight multiplicities is, in general, very limited though we can compute them individually through generalizations of classical formulas and constructions, e.g. [**23**].

The set of weights of $V(\Lambda)$ can be divided into δ-strings and the first weight of each string is called a *maximal weight*. Maximal weights and their multiplicities are fundamental in understanding the structure of $V(\Lambda)$. Since weight multiplicities are invariant under the Weyl group action, it is enough to consider *dominant* maximal weights, and it is well-known that the set of dominant maximal weights for each highest weight Λ is finite. Nevertheless, we do not have any explicit description of dominant maximal weights and their multiplicities in most cases. Except for trivial cases, only level 2 maximal weights of type $A_n^{(1)}$ and their multiplicities are completely known [**38**], and recently, some maximal weights of $V(k\Lambda_0 + \Lambda_s)$, $k \in \mathbb{Z}_{>0}$, $s = 0, 1, \ldots, n$, of type $A_n^{(1)}$ have been studied [**13**, **14**, **39**], where Λ_s are the fundamental weights. Other than type $A_n^{(1)}$, little is known about descriptions of dominant maximal weights and their multiplicities.

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In this paper, we study the multiplicities of dominant weights for finite types and those of dominant maximal weights for affine types *at the same time*. We introduce special classes of Young tableaux, called (spin) rigid Young tableaux, which are equinumerous to the weight multiplicities of the certain highest weight modules for finite and affine types simultaneously, and we derive explicit closed formulas for the weight multiplicities when they are of level $k \leqslant 6$ or $k \gg 0$. Our closed formulas are practically computable, and related to binomial coefficients, Catalan numbers, Motzkin numbers and their generalizations. We consider all classical finite types and affine types, but more focus will be made on finite types B_n and D_n and affine types $B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$.

We summarize the results of this paper in three main parts as follows.

First, we consider some families of highest weights Λ over affine Kac–Moody algebras of classical types, including all highest weights of levels 2 and 3, and determine dominant maximal weights. See, e.g., Theorems 5.9, 5.14 and 5.24. We observe that a majority of dominant maximal weights are *essentially finite* and can be associated with pairs of staircase partitions. We will denote the set of level k (essentially finite) maximal dominant weights, associated with pairs of staircase partitions, by $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ or $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$, depending on the corresponding finite types. Each $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ or $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ is given an *index* (m, s) recording the sizes of the associated staircase partitions.

A simple, yet crucial fact we prove is that two essentially finite dominant maximal weights of the same finite type with the same index (m, s) have the same weight multiplicity *even when their affine types are different.* This fact is related to a classification of the zero nodes of affine Dynkin diagrams (cf. [**29**]). Furthermore, for essentially finite weights, the weight multiplicities of affine Kac–Moody algebras are actually the same as those of the corresponding finite dimensional simple Lie algebras, and we may use the theory of finite dimensional simple Lie algebras. However, as indicated at the beginning of this introduction, explicit closed formulas are not available even for weight multiplicities of finite dimensional simple Lie algebras. Therefore, we utilize a realization of affine crystals to determine weight multiplicities.

Second, the realization of affine crystals we use is *Young walls* introduced by Kang [**18**] which are visualization of Kyoto paths. We first embed the crystals of $V(\Lambda)$ into a tensor product of Young walls of level 1 fundamental representations and investigate the sets of Young walls in the spaces of dominant maximal weights. A careful analysis of the patterns of the Young walls leads to new classes of skew standard Young tableaux that realize crystal basis elements of dominant maximal weights in the tensor product of crystals. Namely, we define the set ${}_s\mathfrak{B}_m^{(k)}$ of *rigid* Young tableaux and the set ${}_s\mathfrak{D}_m^{(k)}$ of *spin rigid* Young tableaux for any $k \geqslant 2$ and $0 \leqslant s \leqslant m$. Roughly speaking, a rigid Young tableau is a skew tableau for which a shift of the last row to the right by 1 makes the tableaux violate column-strictness. For example, the following are rigid tableaux:

$$\begin{array}{|c|c|c|c|}\hline \cdot & 7 & 5 & 4 \\ \hline \cdot & 3 & 1 \\ \cline{1-3} 6 & 2 \\ \cline{1-2}\end{array}\ , \qquad \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-6} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5}\end{array}\ .$$

Here we are using reverse standard Young tableaux and so the rows and columns are decreasing. Similarly, a spin rigid Young tableau is a skew tableau for which a shift of the last row to the right by 2 makes the tableaux violate column-strictness

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and whose shape satisfies certain conditions. For example, the following are spin rigid Young tableaux:

$$\begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 4 & 1 \\ \hline \cdot & \cdot & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}, \qquad \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}.$$

Using combinatorics of Young walls, we prove that the sets ${}_s\mathfrak{B}_m^{(k)}$ and ${}_s\mathfrak{D}_m^{(k)}$ are equinumerous to the weight multiplicities of highest weight modules of finite and affine types simultaneously (Theorems 6.8 and 6.14).

THEOREM 0.1. *Let $k \geqslant 2$ and $0 \leqslant s \leqslant m \leqslant n$.*

(1) *For $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ of index (m, s), we have*

$$\dim V(\Lambda)_\eta = |{}_s\mathfrak{B}_m^{(k)}| = \dim L((k-2)\omega_n + \tilde{\omega}_{n-s})_{(k-2)\omega_n + \tilde{\omega}_{n-m}},$$

where $L(\omega)$ is of type B_n, ω_t are the fundamental weights, and $\tilde{\omega}_t$ are defined by

$$\tilde{\omega}_t := \begin{cases} 2\omega_n & \text{if } t = n, \\ \omega_t & \text{otherwise.} \end{cases}$$

(2) *For $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ of index $(m, s-1)$, we have*

$$\dim V(\Lambda)_\eta = |{}_s\mathfrak{D}_m^{(k)}| = \dim L\big((k-2)\omega_n + \tilde{\omega}_{n-s}\big)_\mu,$$

where $L(\omega)$ is of type D_n, and $\tilde{\omega}_t$ are defined by

$$\tilde{\omega}_t = \begin{cases} \omega_t & \text{if } 1 \leqslant t < n-1, \\ \omega_{n-1} + \omega_n & \text{if } t = n-1, \\ 2\omega_n & \text{if } t = n, \end{cases}$$

and the weights μ are given by

$$\mu = \begin{cases} (k-2)\omega_n + \tilde{\omega}_{n-m-1} & \text{if } k = 2, \text{ or } k \geqslant 3 \text{ and } m \not\equiv_2 s, \\ (k-3)\omega_n + \omega_{n-1} + \tilde{\omega}_{n-m-1} & \text{if } k \geqslant 3 \text{ and } m \equiv_2 s. \end{cases}$$

Our methods unexpectedly reveal hidden structures of weight multiplicities. We consider highest weights in a family at the same time and form a triangular array consisting of $|{}_s\mathfrak{B}_m^{(k)}|$ or $|{}_s\mathfrak{D}_m^{(k)}|$ as highest weights vary in the family. Interestingly, the entries of the resulting triangular arrays count the number of certain lattice paths and we construct bijections between the sets of lattice paths and the corresponding sets of tableaux. These arrays are the Pascal, Catalan, Motzkin and Riordan triangles for various families of highest weights. See the triangular arrays in (4.3) and (4.6) for the Motzkin and Riordan triangles, respectively. See Example 8.23 for the case of generalized Motzkin paths. Moreover, the entries of the triangular arrays also represent some decomposition multiplicities of tensor products of $\mathfrak{sl}_2$-modules, invoking Schur–Weyl type dualities ([**2**, **7**]) into the structures of weight multiplicities.

Third, we use various combinatorial methods to find explicit formulas for the numbers $|{}_s\mathfrak{B}_m^{(k)}|$ and $|{}_s\mathfrak{D}_m^{(k)}|$ for $k = 2$ (Theorems 7.10, 7.16), for $k = 3$ (Theorems 8.1, 8.2), and for the number $|{}_0\mathfrak{D}_m^{(k)}|$ for $2 \leqslant k \leqslant 5$ (Theorem 10.2). In particular, we use the Robinson–Schensted algorithm and a new insertion scheme for the (spin) rigid tableaux, see Algorithm 8.18. We also use integrals over orthogonal groups to derive explicit formulas for $|{}_0\mathfrak{B}_m^{(k)}|$ (Theorem 10.9). The set ${}_0\mathfrak{B}_m^{(k)}$ is nothing but the set of (reverse) standard Young tableaux with m cells and at

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most k rows. In the literature an explicit formula for its cardinality is known only for $k \leqslant 5$ ([**10**, **34**]). We summarize our formulas as follows.

THEOREM 0.2. *For* $0 \leqslant s \leqslant m$*, we have*

$$|{}_s\mathfrak{B}_m^{(2)}| = \binom{m}{\lfloor \frac{m-s}{2} \rfloor}, \qquad |{}_s\mathfrak{D}_{2u-1+s}^{(2)}| = \binom{2u+s-\delta_{s,0}}{u} \quad (u \geqslant 0),$$

$$|{}_s\mathfrak{B}_m^{(3)}| = \sum_{i=0}^{\lfloor s/2 \rfloor} \binom{m}{2i+m-s} \left(\binom{2i+m-s}{i} - \binom{2i+m-s}{i-1} \right),$$

$$|{}_s\mathfrak{D}_m^{(3)}| = \sum_{i=0}^{m+1-\delta_{s,0}-s} (-1)^i \left(|{}_s\mathfrak{B}^{(3)}_{m-\delta_{s,0}-i}| + |{}_{s-1}\mathfrak{B}^{(3)}_{m-\delta_{s,0}-i}| \right),$$

$$|{}_0\mathfrak{D}_{2m-1}^{(4)}| = \binom{\mathsf{C}_m+1}{2}, \qquad |{}_0\mathfrak{D}_{2m}^{(4)}| = \mathsf{C}_m\mathsf{C}_{m+1} - \mathsf{C}_m^2,$$

$$|{}_0\mathfrak{D}_{2m-1}^{(5)}| = \sum_{i=0}^{m} \binom{2m}{2i} \mathsf{C}_i\mathsf{C}_{i+1} - \sum_{i=0}^{m-1} \binom{2m}{2i+1} \mathsf{C}_{i+1}^2,$$

$$|{}_0\mathfrak{D}_{2m}^{(5)}| = \sum_{i=0}^{m} \frac{2i}{i+3} \binom{2m}{2i} \mathsf{C}_i\mathsf{C}_{i+1} - \sum_{i=0}^{m-1} \frac{2i}{i+3} \binom{2m}{2i+1} \mathsf{C}_{i+1}^2,$$

where $\mathsf{C}_i = \frac{1}{i+1}\binom{2i}{i}$ *is the* i*-th Catalan number.*

For integers $k \geqslant 1$ *and* $m \geqslant 0$*, we have*

$$|{}_0\mathfrak{B}_m^{(2k)}| = \sum_{t_1+\cdots+t_k=m} \binom{m}{t_1,\ldots,t_k} \det\left(\binom{t_i+2k-i-j}{\lfloor \frac{t_i+2k-i-j}{2} \rfloor} \right)_{i,j=1}^{k},$$

$$|{}_0\mathfrak{B}_m^{(2k+1)}| = \sum_{t_0+t_1+\cdots+t_k=m} \binom{m}{t_0,t_1,\ldots,t_k} \det\left(C\left(\frac{t_i+2k-i-j}{2} \right) \right)_{i,j=1}^{k},$$

where $C(x) = \mathsf{C}_x = \frac{1}{x+1}\binom{2x}{x}$ *if* x *is an integer and* $C(x) = 0$ *otherwise.*

When k increases, the numbers $|{}_s\mathfrak{B}_m^{(k)}|$ and $|{}_s\mathfrak{D}_m^{(k)}|$ (and thus the weight multiplicities) stabilize and we find their closed formulas (Corollary 9.8 and Theorem 9.9). In particular, from $\lim_{k\to\infty} |{}_s\mathfrak{D}_m^{(k)}|$, we obtain a triangular array of numbers, called *Bessel triangle*, consisting of the coefficients of Bessel polynomials, see (9.4).

The organization of this paper is as follows. In chapter 1, we fix notations and present basic definitions for affine Kac–Moody algebras and quantum affine algebras. Throughout this paper we mainly use the notations of affine types, even though we study finite types together. In chapter 2, after the theory of crystals is reviewed briefly, we describe constructions of Young walls and explain embeddings of highest weight crystals into tensor products of level 1 crystals. A connection between affine types and finite types is pointed out in section 2.2. In chapter 3, we explain a correspondence between Young walls and Young tableaux, and introduce some families of Young tableaux that will be used later. Chapter 4 is devoted to lattice paths and triangular arrays of numbers. The entries of the triangular arrays are the numbers of certain types of lattice paths and also the decomposition multiplicities of tensor products of $\mathfrak{sl}_2$-modules. All the entries of the triangular arrays are also to appear as weight multiplicities.

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In chapter 5, we determine dominant maximal weights for certain families of highest weight modules. These families include all highest weight modules of levels 2 and 3 except for types $A_n^{(1)}$ and $C_n^{(1)}$. We classify staircase dominant maximal weights according to their finite types. In chapter 6, we investigate the Young walls of dominant maximal weights and define (spin) rigid Young tableaux. Using combinatorics of Young walls, we prove that the sets of (spin) rigid tableaux count weight multiplicities.

Chapter 7 is concerned about the level 2 cases. We prove that the weight multiplicities form the Catalan triangle and the Pascal triangle. The main tool is an insertion scheme for tableaux. We also construct bijections between the set of lattice paths and the set of rigid Young tableaux in ${}_s\mathfrak{B}_m^{(2)}$. In chapter 8, we consider the level 3 cases and prove that the weight multiplicities form the Motzkin triangle for rigid Young tableaux and the Riordan triangle for spin rigid Young tableaux. We prove both cases using the Robinson–Schensted algorithm and provide a different proof for the Motzkin case using an insertion scheme which naturally realizes the Motzkin triangle through combinatorics of tableaux. An explicit bijection from the set of rigid tableaux in ${}_s\mathfrak{B}_m^{(3)}$ to the set of generalized Motzkin paths is also given.

In chapter 9, we investigate the limits of weight multiplicities of level k as k increases. We observe that the weight multiplicities given by the numbers of (spin) rigid Young tableaux stabilize as k increases, and compute the limits explicitly. The computation uses formulas for the numbers of involutions in the symmetric groups.

In the final chapter, we consider the set $\mathcal{S}_m^{(k,t)}$ of standard Young tableaux with m cells, at most k rows and exactly t rows of odd length. Both ${}_0\mathfrak{B}_m^{(k)}$ and ${}_0\mathfrak{D}_m^{(k)}$ can be considered as special cases of the set $\mathcal{S}_m^{(k,t)}$. Using the Robinson–Schensted algorithm, we find a relation between $|\mathcal{S}_m^{(k,0)}|$, $|\mathcal{S}_m^{(k,k)}|$ and $|{}_0\mathfrak{B}_m^{(k-1)}|$. Using this relation and some known results, we find an explicit formula for $|\mathcal{S}_m^{(k,t)}|$ for every $0 \leqslant t \leqslant k \leqslant 5$. We then express $|{}_0\mathfrak{B}_m^{(k)}|$ as an integral over the orthogonal group $\mathrm{O}(k)$. By evaluating this integral we obtain an explicit formula for $|{}_0\mathfrak{B}_m^{(k)}|$.

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CHAPTER 1

Affine Kac–Moody algebras

1.1. Preliminaries

Let $I = \{0, 1, ..., n\}$ be an index set. An *affine Cartan datum* $(\mathsf{A}, P^\vee, P, \Pi^\vee, \Pi)$ consists of

(a) a matrix $\mathsf{A} = (a_{ij})_{i,j\in I}$ of corank 1, called an *affine Cartan matrix*, satisfying, for all $i, j \in I$,

$$\text{(i) } a_{ii} = 2, \quad \text{(ii) } a_{ij} \in \mathbb{Z}_{\leqslant 0} \text{ for } i \neq j, \quad \text{(iii) } a_{ij} = 0 \text{ if } a_{ji} = 0,$$

(b) a free abelian group $P^\vee = \bigoplus_{i=0}^{n} \mathbb{Z}h_i \oplus \mathbb{Z}d$, the *dual weight lattice*, with $\mathfrak{h} := \mathfrak{C} \otimes_{\mathbb{Z}} P^\vee$,

(c) a free abelian group $P = \bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta \subset \mathfrak{h}^*$, the *weight lattice*,

(d) a linearly independent set $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, the set of *simple coroots*,

(e) a linearly independent set $\Pi = \{\alpha_i \mid i \in I\} \subset P$, the set of *simple roots*,

which satisfy, for all $i, j \in I$,

$$\begin{aligned} &\langle h_i, \alpha_j\rangle = a_{ij}, \quad \langle h_i, \Lambda_j\rangle = \delta_{i,j}, \quad \langle h_i, \delta\rangle = 0, \\ &\langle d, \alpha_i\rangle = \delta_{i,0}, \quad \langle d, \Lambda_i\rangle = 0, \qquad \langle d, \delta\rangle = 1. \end{aligned} \tag{1.1}$$

We call Λ_i the *i-th fundamental weight*, $\delta = \sum_{i\in I} a_i\alpha_i$ ($a_i \in \mathbb{Z}_{\geqslant 1}$) the *null root* and d the *degree derivation*.

Let $c = \sum_{i\in I} a_i^\vee h_i$ be the unique element such that $a_i^\vee \in \mathbb{Z}_{\geqslant 1}$ and

$$\mathbb{Z}c = \left\{ h \in \bigoplus_{i\in I} \mathbb{Z}h_i \;\middle|\; \langle h, \alpha_i\rangle = 0 \text{ for all } i \in I \right\}.$$

We say that a weight $\Lambda \in P$ is of *level* k if $\langle c, \Lambda\rangle = k$ (see [**15**, Chapters 5 and 6]). Note that, for $\Lambda = \sum_{i\in I} m_i\Lambda_i$, we have

$$\langle c, \Lambda\rangle = \sum_{i\in I} m_i a_i^\vee.$$

For reader's convenience, we list a_i's and $a_i^\vee$'s for classical affine types in Table 1.1.

Recall that A is *symmetrizable* in the sense that $D\mathsf{A}$ is symmetric where

$$D = \operatorname{diag}(\mathsf{d}_i := a_i^\vee a_i^{-1} \mid i \in I).$$

There exists a non-degenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ ([**15**, (6.2.2)]) which is defined in terms of the basis $\{\alpha_0, \alpha_1, \ldots, \alpha_n, \Lambda_0\}$ by

$$(\alpha_i|\alpha_j) = \mathsf{d}_i a_{ij}, \qquad (\alpha_i|\Lambda_0) = a_0^{-1}\delta_{i,0}, \qquad (\Lambda_0|\Lambda_0) = 0, \tag{1.2}$$

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Type	$(a_0, a_1, \dots, a_n)$	$(a_0^\vee, a_1^\vee, \dots, a_n^\vee)$
$A_n^{(1)}$	$(1,1,\dots,1)$	$(1,1,\dots,1)$
$B_n^{(1)}$	$(1,1,2,2,\dots,2)$	$(1,1,2,2,\dots,2,1)$
$C_n^{(1)}$	$(1,2,2,\dots,2,1)$	$(1,1,\dots,1)$
$D_n^{(1)}$	$(1,1,2,2,\dots,2,1,1)$	$(1,1,2,2,\dots,2,1,1)$
$A_{2n-1}^{(2)}$	$(1,1,2,2,\dots,2,1)$	$(1,1,2,2,\dots,2)$
$A_{2n}^{(2)}$	$(2,2,\dots,2,1)$	$(1,2,2,\dots,2)$
$D_{n+1}^{(2)}$	$(1,1,\dots,1)$	$(1,2,2,\dots,2,1)$

TABLE 1.1. a_i's and $a_i^\vee$'s for classical affine types

and which satisfies

$$\langle h_i, \alpha_j \rangle = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}$$

for all $i, j \in I$. In this paper, we choose $(\ |\ )$ such that

$$(\alpha|\alpha) = 2 \qquad \text{for a long root } \alpha.$$

We denote by $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geqslant 0},\ i \in I\}$ the set of *dominant integral weights*. The free abelian group $\mathsf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice* and we set $\mathsf{Q}^+ := \bigoplus_{i \in I} \mathbb{Z}_{\geqslant 0}\alpha_i$. For an element $\beta = \sum_{i \in I} k_i \alpha_i \in \mathsf{Q}^+$, we define the integer $\mathrm{ht}(\beta) := \sum_{i \in I} k_i$, called the *height* of β, and a subset $\mathrm{Supp}(\beta) = \{i \in I \mid k_i \neq 0\}$ of I, called the *support* of β.

DEFINITION 1.1. The *affine Kac-Moody algebra* $\mathfrak{g}$ associated with an affine Cartan datum $(\mathsf{A}, P^\vee, P, \Pi^\vee, \Pi)$ is the Lie algebra over $\mathfrak{C}$ generated by e_i, f_i $(i \in I)$ and $h \in P^\vee$ satisfying following relations:

(1) $[h, h'] = 0,\ [h, e_i] = \alpha_i(h) e_i,\quad [h, f_i] = -\alpha_i(h) f_i$ for all $h, h' \in P^\vee$ and $i \in I$,
(2) $[e_i, f_j] = \delta_{i,j} h_i$ for $i, j \in I$,
(3) $(\mathrm{ad}\ e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad}\ f_i)^{1-a_{ij}}(f_j) = 0$ if $i \neq j$, where $(\mathrm{ad}\ x)(y) := [x, y]$.

A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition*

$$V = \bigoplus_{\mu \in P} V_\mu$$

where $V_\mu = \{v \in V \mid h \cdot v = \langle h, \mu \rangle v \text{ for all } h \in P^\vee\}$. A weight module V over $\mathfrak{g}$ is *integrable* if all e_i and f_i $(i \in I)$ are locally nilpotent on V.

DEFINITION 1.2. The category $\mathscr{O}_{\mathrm{int}}$ consists of integrable $\mathfrak{g}$-modules V satisfying the following conditions:

(1) V admits a weight space decomposition $V = \bigoplus_{\mu \in P} V_\mu$ and $\dim_{\mathfrak{C}}(V_\mu) < \infty$ for each weight μ.
(2) There exist a finite number of elements $\lambda_1, \dots, \lambda_s \in P$ such that

$$\mathrm{wt}(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$

Here $\mathrm{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\}$ and $D(\lambda) := \{\lambda - \sum_{i \in I} k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$.

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It is well-known that the category $\mathcal{O}_{\rm int}$ is a semisimple tensor category with its irreducible objects being isomorphic to *the highest weight modules* $V(\Lambda)$ ($\Lambda \in P^+$), each of which is generated by a *highest weight vector* v_Λ. Recall, e.g. from [**15**, Chapter 10], that if $M,\ N \in \mathcal{O}_{\rm int}$, then

$$\text{(1.3)}\qquad \begin{array}{ll} \text{(a)} & M \simeq N \text{ if and only if } \operatorname{ch}(M) = \operatorname{ch}(N),\\ \text{(b)} & \operatorname{ch}(V(\Lambda)) = e^{-t\delta}\operatorname{ch}(V(\Lambda + t\delta)) \text{ for } \Lambda \in P^+ \text{ and } t \in \mathbb{Z}, \end{array}$$

where $\operatorname{ch}(M) := \sum_{\mu \in P} (\dim_{\mathbb{C}} M_\mu) e^\mu$ is the *character* of M.

For $\eta \in \operatorname{wt}(V(\Lambda))$, we define

$$\operatorname{Supp}_\Lambda(\eta) := \operatorname{Supp}(\Lambda - \eta).$$

The dimension of the μ-weight space $V(\Lambda)_\mu$ is called the *multiplicity* of μ in $V(\Lambda)$. A weight μ is *maximal* if $\mu + \delta \notin \operatorname{wt}(V(\Lambda))$. The set of all maximal weights of $V(\Lambda)$ of level k is denoted by $\max(\Lambda|k)$.

PROPOSITION 1.3. ([**15**, Chapter 12.6]) *For each $\Lambda \in P^+$ of level k, we have*

$$\operatorname{wt}(V(\Lambda)) = \bigsqcup_{\mu \in \max(\Lambda|k)} \{\mu - s\delta \mid s \in \mathbb{Z}_{\geqslant 0}\}.$$

We denote by $\max^+(\Lambda|k)$ the set of all dominant maximal weights of level k in $V(\Lambda)$, i.e.,

$$\max^+(\Lambda|k) := \max(\Lambda|k) \cap P^+.$$

It is well-known that

$$\max(\Lambda|k) = W \cdot \max^+(\Lambda|k), \qquad \text{where } W \text{ is the Weyl group of } \mathfrak{g}.$$

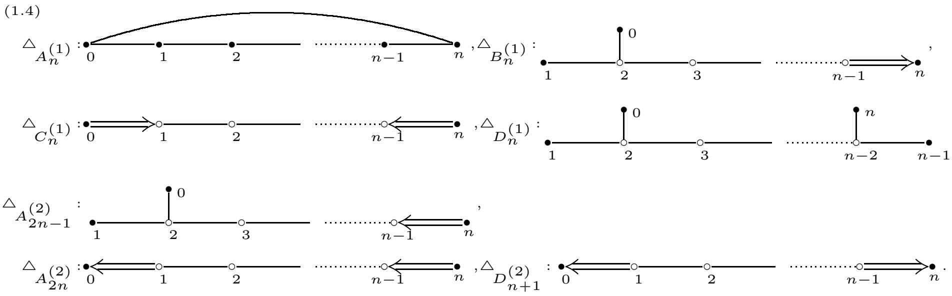
Let $\mathfrak{h}_0$ be the $\mathbb{C}$-vector space spanned by $\{h_i \mid i \in I_0\}$ for $I_0 := I\backslash\{0\}$. Define the *orthogonal projection* $^-\colon \mathfrak{h}^* \to \mathfrak{h}_0^*$ ([**15**, (6.2.7)]) by

$$\mu \longmapsto \overline{\mu} = \mu - \mu(c)\Lambda_0 - (\mu|\Lambda_0)\delta.$$

We denote by $\overline{\mathsf{Q}}$ the image of Q under the orthogonal projection $^-$.

For later use, we present the Dynkin diagrams of classical affine types.

(1.4)



For an affine Dynkin diagram $\triangle_{\mathfrak{g}}$ and a subset $J \subsetneq I$, we denote by $\triangle_{\mathfrak{g}}|_J$ the full-subdiagram of $\triangle_{\mathfrak{g}}$ whose vertices are in J. We call a vertex s in $\triangle_{\mathfrak{g}}$ *extremal* if $\triangle_{\mathfrak{g}}|_{I_s}$ for $I_s := I\backslash\{s\}$ is a connected Dynkin diagram of finite type. For example, every vertex in $\triangle_{A_n^{(1)}}$ is extremal, while 0, 1 and n are all the extremal vertices of $\triangle_{B_n^{(1)}}$. In (1.4), each solid dot $\bullet$ denotes an extremal vertex.

Let $\mathfrak{g}_s$ be the finite dimensional subalgebra of $\mathfrak{g}$ corresponding to $\triangle_{\mathfrak{g}}|_{I_s}$ for an extremal vertex s. Then each finite dimensional simple Lie algebra $\mathfrak{g}_{\rm fin}$ of classical type appears as the subalgebra $\mathfrak{g}_s$ of an affine Lie algebra $\mathfrak{g}$ as follows:

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$\mathfrak{g}_{\rm fin}$	A_n	B_n	C_n	D_n
$\mathfrak{g}$	$A_n^{(1)}$	$B_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$	$C_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$	$B_n^{(1)}$, $A_{2n-1}^{(2)}$, $D_n^{(1)}$

TABLE 1.2. Relationship between $\mathfrak{g}_{\rm fin}$ and $\mathfrak{g}$

By slightly abusing notations, we denote by $\Pi_0 = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots of $\mathfrak{g}_0$ and ω_i $(1 \leqslant i \leqslant n)$ the fundamental dominant weights of $\mathfrak{g}_0$. Then we have

$$\overline{\alpha}_i = \begin{cases} \alpha_i & \text{if } i \neq 0, \\ -a_0^{-1}\theta & \text{if } i = 0, \end{cases} \qquad \overline{\Lambda}_i = \begin{cases} \omega_i = \Lambda_i - a_i^\vee \Lambda_0 & \text{if } i \neq 0, \\ 0 & \text{if } i = 0, \end{cases} \tag{1.5}$$

where $\theta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ (see [**15**, §12]). Note that there exist bilinear forms $(\ |\ )$ and $\langle\ ,\ \rangle$ associated to $\mathfrak{g}_0$ which are induced by the projection $^-$.

Define

$$k\mathcal{C}_{\rm af} = \{\mu \in \mathfrak{h}_0^* \mid (\mu|\alpha_i) \geqslant 0 \text{ for } i \in I_0,\ (\mu|\theta) \leqslant k\} \quad \text{where } \theta := \delta - a_0\alpha_0.$$

PROPOSITION 1.4. ([**15**, Proposition 12.6]) *The map $\mu \longmapsto \overline{\mu}$ defines a bijection from* $\max^+(\Lambda|k)$ *onto* $k\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{\mathsf{Q}})$ *where Λ is of level k. In particular, the set* $\max^+(\Lambda|k)$ *is finite.*

CONVENTION 1.5. We denote an arbitrary fundamental weight of level 1 by boldfaced $\boldsymbol{\Lambda}$ to distinguish them from other (fundamental) weights.

1.2. Quantum affine algebras

We denote by γ the smallest positive integer such that $\gamma\frac{(\alpha_i|\alpha_i)}{2} \in \mathbb{Z}$ for all $i \in I$. Let q be an indeterminate and $m, n \in \mathbb{Z}_{\geqslant 0}$. For $i \in I$, let $q_i = q^{\mathsf{d}_i}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \qquad [n]_i! = \prod_{k=1}^{n} [k]_i, \qquad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

DEFINITION 1.6. The *quantum affine algebra* $U_q(\mathfrak{g})$ associated with an affine Cartan datum $(\mathsf{A}, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbb{Q}(q^{1/\gamma})$ with $\mathbf{1}$, generated by e_i, f_i $(i \in I)$ and q^h $(h \in \gamma^{-1}P^\vee)$ satisfying the following relations:

(1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$, $q^h e_i q^{-h} = q^{\langle h, \alpha_i\rangle} e_i$, $q^h f_i q^{-h} = q^{-\langle h, \alpha_i\rangle} f_i$ for $h, h' \in \gamma^{-1}P^\vee$,

(2) $e_i f_j - f_j e_i = \delta_{i,j} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q_i^{h_i}$,

(3) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ if $i \neq j$.

Here we set

$$e_i^{(n)} := e_i^n/[n]_i! \quad \text{and} \quad f_i^{(n)} := f_i^n/[n]_i!.$$

We define integrable $U_q(\mathfrak{g})$-modules, the category $\mathcal{O}^q_{\rm int}$, the character for $M \in \mathcal{O}^q_{\rm int}$ and highest weight modules $V^q(\Lambda)$ for $\Lambda \in P^+$ in the standard way ([**11**]). It is well-known that $\mathcal{O}^q_{\rm int}$ is a semisimple tensor category with its irreducible object being isomorphic to $V^q(\Lambda)$ for some $\Lambda \in P^+$ and

$$\operatorname{ch}\big(V(\Lambda)\big) = \operatorname{ch}\big(V^q(\Lambda)\big) \quad \text{and hence} \quad \dim_{\mathbb{Q}}(V(\Lambda)_\mu) = \dim_{\mathbb{Q}(q)}(V^q(\Lambda)_\mu) \tag{1.6}$$

for any $\mu \in P$.

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CHAPTER 2

Crystals and Young walls

In this chapter, we briefly review the theory of *crystals* developed by Kashiwara ([**20**, **21**]). Then we recall the combinatorial realization of affine crystals, called the *Young walls*, due to Kang ([**18**]).

2.1. Crystals

For an index $i \in I$ and $M = \bigoplus_{\mu \in P} M_\mu \in \mathcal{O}^q_{\mathrm{int}}$, every element $v \in M_\mu$ can be uniquely expressed as

$$v = \sum_{k \geqslant 0} f_i^{(k)} v_k,$$

where $\mu(h_i) + k \geqslant 0$ and $v_k \in \operatorname{Ker} e_i \cap M_{\mu + k\alpha_i}$. The *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\tilde{e}_i v = \sum_{k \geqslant 1} f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k \geqslant 0} f_i^{(k+1)} v_k. \tag{2.1}$$

Let $\mathbb{A}_0 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q],\ g(0) \neq 0\}$ and M a weight $U_q(\mathfrak{g})$-module.

Definition 2.1. A *crystal basis* of a $U_q(\mathfrak{g})$-module M consists of a pair (L, B) with the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ as follows:

(1) $L = \bigoplus_\mu L_\mu$ is a free $\mathbb{A}_0$-submodule of M such that

$$M \simeq \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L \quad \text{where} \quad L_\mu = L \cap M_\mu,$$

(2) $B = \bigsqcup_\mu B_\mu$ is a basis of the $\mathbb{Q}$-vector space L/qL, where $B_\mu = B \cap (L_\mu / qL_\mu)$,

(3) $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ are defined on L, i.e., $\tilde{e}_i L, \tilde{f}_i L \subset L$,

(4) the induced maps $\tilde{e}_i$ and $\tilde{f}_i$ on L/qL satisfy

$$\tilde{e}_i B, \tilde{f}_i B \subset B \sqcup \{0\}, \qquad \text{and} \qquad \tilde{f}_i b = b' \quad \text{if and only if} \quad b = \tilde{e}_i b' \quad \text{for } b, b' \in B.$$

The set B has a colored oriented graph structure as follows:

$$b \xrightarrow{\ i\ } b' \quad \text{if and only if} \quad \tilde{f}_i b = b'.$$

The graph structure encodes information on the structure of $M \in \mathcal{O}^q_{\mathrm{int}}$. For example,

- $|B_\mu| = \dim_{\mathbb{Q}(q)} M_\mu$ for all $\mu \in \mathrm{wt}(M)$,
- the graph of B is connected if and only if M is irreducible.

Theorem 2.2 ([**21**]). *For $\Lambda \in P^+$, the module $V^q(\Lambda)$ has a crystal basis $(\mathbf{L}(\Lambda), \mathbf{B}(\Lambda))$ given as follows:*

(1) *$\mathbf{L}(\Lambda)$ is the $\mathbb{A}_0$-submodule generated by $\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\Lambda \mid r \geqslant 0, i_k \in I\}$,*

(2) *$\mathbf{B}(\Lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\Lambda + qL(\Lambda) \mid r \geqslant 0, i_k \in I\} \backslash \{0\}$.*

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By (1.3), (1.6) and the above theorem, we have that for $k \in \mathbb{Z}$, $|\mathbf{B}(\Lambda)_\mu| = |\mathbf{B}(\Lambda + k\delta)_{\mu+k\delta}|$ and

$$\operatorname{ch}(V(\Lambda)) = \sum_{\mu \in \operatorname{wt}(V(\Lambda))} |\mathbf{B}(\Lambda)_\mu| e^{\mu} = \sum_{\mu \in \operatorname{wt}(V(\Lambda))} |\mathbf{B}(\Lambda + k\delta)_\mu| e^{\mu - k\delta}. \tag{2.2}$$

DEFINITION 2.3. An (*affine*) *crystal* associated to an affine Cartan datum $(\mathsf{A}, P^\vee, P, \Pi^\vee, \Pi)$ is the set B together with maps

$$\operatorname{wt} : B \to P, \quad \varepsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \quad \text{and} \quad \tilde{e}_i, \tilde{f}_i : B \to B \sqcup \{0\} \quad (i \in I)$$

satisfying the following conditions:

(i) For $i \in I$, $b \in B$, we have

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \operatorname{wt}(b)\rangle, \operatorname{wt}(\tilde{e}_i b) = \operatorname{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \neq 0, \operatorname{wt}(\tilde{f}_i b) = \operatorname{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \neq 0,$$

(ii) if $\tilde{e}_i b \in B$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
(iii) if $\tilde{f}_i b \in B$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$,
(iv) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for all $i \in I$, $b, b' \in B$,
(v) if $\varepsilon_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

DEFINITION 2.4. The *tensor product* $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be the set $B_1 \times B_2$ whose crystal structure is given by

(i) $\operatorname{wt}(b_1 \otimes b_2) = \operatorname{wt}(b_1) + \operatorname{wt}(b_2)$,
(ii) $\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \operatorname{wt}(b_1)\rangle)$, $\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \operatorname{wt}(b_2)\rangle)$,
(iii) $\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$

$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2). \end{cases}$

THEOREM 2.5. [**20, 21**] *For M and $N \in \mathcal{O}^q_{\text{int}}$ with crystals B_M and B_N, the tensor product $B_M \otimes B_N$ is the crystal of $M \otimes N \in \mathcal{O}^q_{\text{int}}$.*

2.2. Connection to finite type

This section is important in understanding the results of this paper. We would like to make an emphasis on the fact that the formulas in this paper *simultaneously* cover the weight multiplicities of affine types and those of finite types. Furthermore, many of the formulas cover not merely one affine type but multiple affine types at the same time. Though these observations can easily be made from the graph structure (Theorem 2.6) of an affine crystal, they do not seem to be widely recognized in the literature. Therefore, we list the observations as two separate theorems (Theorems 2.7 and 2.9) below.

Recall that $\mathbf{B}(\Lambda)$ can be understood as a colored oriented graph. For an extremal vertex $s \in I$, we denoted by $\mathbf{B}(\Lambda)|_{I_s}$ the graph obtained by removing the arrows $\xrightarrow{s}$ of color s. Throughout this section, s denotes an extremal vertex in I.

THEOREM 2.6 ([**21**]). *As $\mathfrak{g}_s$-crystals, we have*

$$\mathbf{B}(\Lambda)|_{I_s} = \bigsqcup_{\omega'} \mathbf{B}(\omega'),$$

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where each $\mathbf{B}(\omega')$ *is a connected component of* $\mathbf{B}(\Lambda)|_{I_s}$*, which is a crystal of some irreducible module* $L(\omega')$ *over* $U_q(\mathfrak{g}_s)$*.*

For an extremal $s \in I$ and a highest weight $\Lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$, we denote by $\mathbf{B}^0(\Lambda)|_{I_s}$ the connected component of $\mathbf{B}(\Lambda)|_{I_s}$ originated from the highest weight element v_Λ in $\mathbf{B}(\Lambda)$. Then, by Theorem 2.6, $\mathbf{B}^0(\Lambda)|_{I_s} = \mathbf{B}(\omega)$ for $\omega = \sum_{i \in I\setminus\{s\}} m_i \omega_i$, where ω_i are the fundamental weights of $\mathfrak{g}_s$. Then the following theorem is obvious.

THEOREM 2.7. *Let* $\Lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$ *be a dominant integral weight for* $\mathfrak{g}$ *and* $\omega = \sum_{i \in I\setminus\{s\}} m_i \omega_i$ *for* $\mathfrak{g}_s$*. Then, for* $\mu = \Lambda - \sum_{i \in I} k_i \alpha_i \in \mathrm{wt}(V(\Lambda))$ *such that* $k_s = 0$ *and* $\eta = \omega - \sum_{i \in I\setminus\{s\}} k_i \alpha_i$*, we have*

$$\mathbf{B}(\Lambda)_\mu = \mathbf{B}(\omega)_\eta \quad \text{and} \quad \dim(V(\Lambda)_\mu) = \dim(L(\omega)_\eta). \tag{2.3}$$

Motivated by this theorem, we make a definition that will be useful later.

DEFINITION 2.8. Let $\Lambda = \sum_{i \in I} m_i \Lambda_i \in P^+$. A weight $\mu = \Lambda - \sum_{i \in I} k_i \alpha_i \in \mathrm{wt}(V(\Lambda))$ is called *essentially finite of type* X_n if there is an $s \in I$ such that $k_s = 0$ and $\mathfrak{g}_s$ is of finite type X_n with $X = A, B, C$ or D.

In chapter 5, we will see that most of the dominant maximal weights of affine Kac–Moody algebras are essentially finite.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two different affine Kac–Moody algebras, and consider their integrable highest weight modules $V(\Lambda)$ and $V(\Lambda')$, respectively. If $\eta \in \mathrm{wt}(V(\Lambda))$ and $\mu \in \mathrm{wt}(V(\Lambda'))$ are essentially finite of the same type X_n, then it is clear from Theorem 2.7 that the weight multiplicities of η and μ are determined by the finite type X_n without regard to the difference in their affine types. We make it precise in the theorem below whose proof is immediate from Theorem 2.7.

THEOREM 2.9. *We assume the following:*

(1) *For* $V(\Lambda)$ *over an affine* $\mathfrak{g}$ *and* $\eta \in \mathrm{wt}(V(\Lambda))$*, there exists an extremal* $s \notin \mathrm{Supp}_\Lambda(\eta)$ *such that* $\triangle_{\mathfrak{g}}|_{I_s}$ *is of finite type* X_n*.*
(2) *For* $V(\Lambda')$ *over another affine* $\mathfrak{g}'$ *and* $\mu \in \mathrm{wt}(V(\Lambda'))$*, there exists an extremal* $s' \notin \mathrm{Supp}_{\Lambda'}(\mu)$ *such that* $\triangle_{\mathfrak{g}'}|_{I_{s'}}$ *is of the same finite type* X_n*.*
(3) *There exists a bijection* $\sigma : I_s \to I_{s'}$ *which induces a diagram isomorphism* $\triangle_{\mathfrak{g}}|_{I_s} \simeq \triangle_{\mathfrak{g}'}|_{I_{s'}}$ *and a crystal isomorphism* $\mathbf{B}(\Lambda)^0|_{I_s} \simeq \mathbf{B}(\Lambda')^0|_{I_{s'}}$ *so that*

$$\eta = \Lambda - \sum_{i \in I_s} m_i \alpha_i \quad \text{and} \quad \mu = \Lambda' - \sum_{i \in I_s} m_{\sigma(i)} \alpha_{\sigma(i)}.$$

Then we have

$$\dim V(\Lambda)_\eta = \dim V(\Lambda')_\mu.$$

2.3. Young walls for level 1 representations

In [**18**], Kang constructed realizations of level 1 highest weight crystals $B(\mathbf{\Lambda})$ for all classical quantum affine algebras except $C_n^{(1)}$ in terms of *reduced Young walls*. For the rest of this section, we assume that $\mathfrak{g}$ is an affine Kac-Moody algebra of type $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$, $D_n^{(1)}$ or $D_{n+1}^{(2)}$.

Young walls are built from colored blocks. There are three types of blocks whose shapes are different and which appear depending on affine Cartan types as follows:

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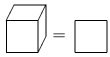
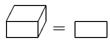
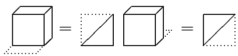
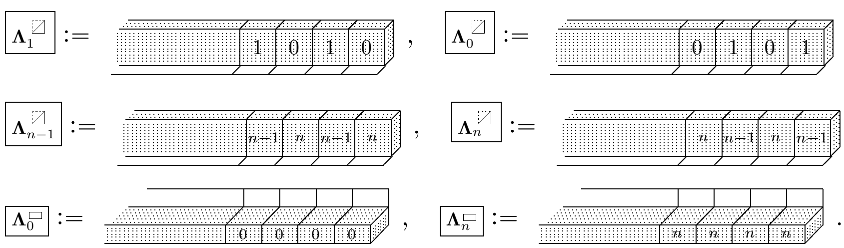
Shape	Width	Thickness	Height	Type
	1	1	1	all types
	1	1	1/2	$A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)}$
	1	1/2	1	$A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$

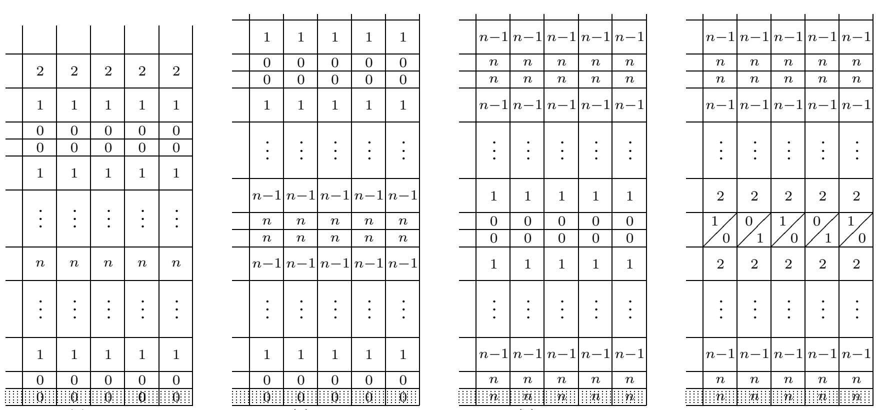
The walls are built on the ground-state wall $\boxed{\mathbf{\Lambda}}$, which is given below as the shaded part in (2.4), by the following rules:

(1) Blocks should be built in the pattern given below in (2.5), (2.6) or (2.7).
(2) No block can be placed on top of a column of half-unit thickness.
(3) There should be no free space to the right of any block except the right-most column.

Ground-state Young walls $\boxed{\mathbf{\Lambda}}$ corresponding to $\mathbf{\Lambda}$ are given as follows:

(2.4) 

Now we give the patterns mentioned above:

(2.5) 

$A_{2n}^{(2)}$, $\mathbf{\Lambda}_0$ $\qquad$ $D_{n+1}^{(2)}$, $\mathbf{\Lambda}_0$ $\qquad$ $D_{n+1}^{(2)}$, $\mathbf{\Lambda}_n$ $\qquad$ $B_n^{(1)}$, $\mathbf{\Lambda}_n$

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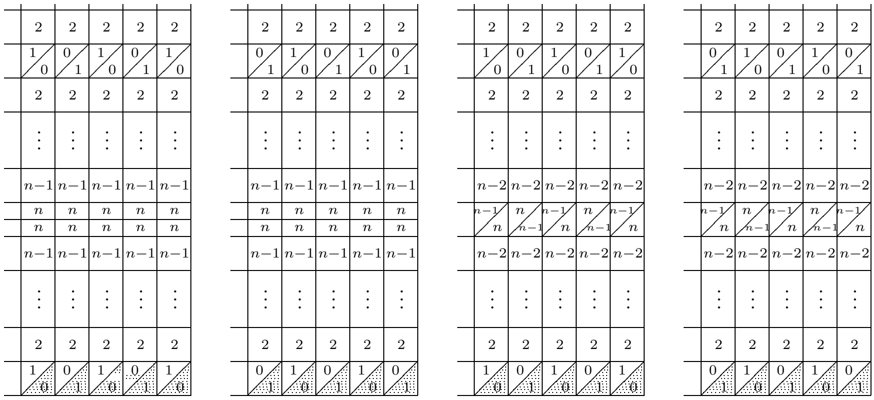
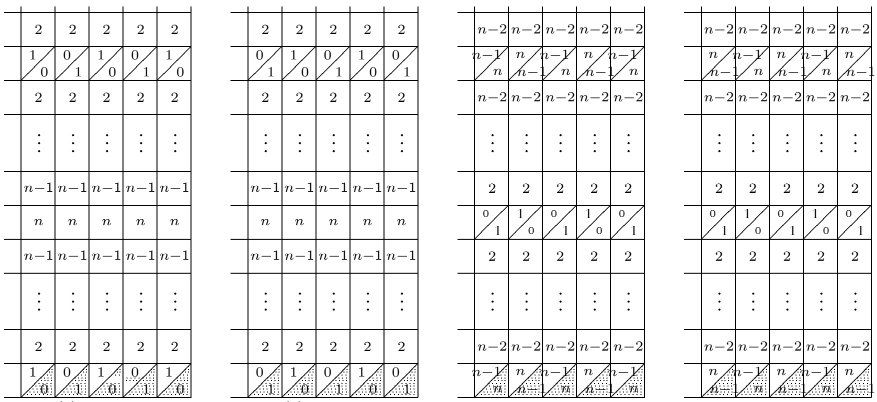
(2.6)



$B_n^{(1)}$, $\mathbf{\Lambda}_1$ $\quad$ $B_n^{(1)}$, $\mathbf{\Lambda}_0$ $\quad$ $D_n^{(1)}$, $\mathbf{\Lambda}_1$ $\quad$ $D_n^{(1)}$, $\mathbf{\Lambda}_0$

(2.7)



$A_{2n-1}^{(2)}$, $\mathbf{\Lambda}_1$ $\quad$ $A_{2n-1}^{(2)}$, $\mathbf{\Lambda}_0$ $\quad$ $D_n^{(1)}$, $\mathbf{\Lambda}_{n-1}$ $\quad$ $D_n^{(1)}$, $\mathbf{\Lambda}_n$

According to the ground-state Young walls in (2.4), we classify the fundamental weights $\mathbf{\Lambda}$ of level 1 into two types:

- Type $\mathfrak{B}$: those $\mathbf{\Lambda}$ whose ground-state Young wall consists of half-height blocks,
- Type $\mathfrak{D}$: those $\mathbf{\Lambda}$ whose ground-state Young wall consists of half-thickness blocks.

REMARK 2.10. For classifying fundamental weights $\mathbf{\Lambda}_i$ of level 1, we use $\mathfrak{B}$ and $\mathfrak{D}$ by the following reason:

- When $\boxed{\mathbf{\Lambda}_i}$ consists of half-height blocks, the vertex i in the affine Dynkin diagram is an extremal vertex incident on a doubly-laced incoming arrow, which can be identified with the extremal vertex n in the Dynkin diagram $\triangle_{B_n}$.
- When $\boxed{\mathbf{\Lambda}_i}$ consists of half-thickness blocks, the vertex i in the affine Dynkin diagram is an extremal vertex incident on a simply-laced edge, which can be identified with an extremal vertex n or $n-1$ in the Dynkin diagram $\triangle_{D_n}$.

Later, we will see that this classification is closely related to finite simple Lie algebras of type B_n and D_n.

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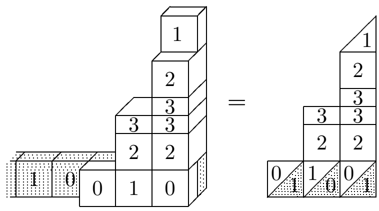
REMARK 2.11. For $\mathfrak{g} = B_n^{(1)}$ or $A_{2n-1}^{(2)}$, the patterns of Young walls based on $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_1$ are the same *up to one column*; that is, if we ignore the first column of the pattern for $\mathbf{\Lambda}_1$, then we get the pattern for $\mathbf{\Lambda}_0$ (see (2.6) and (2.7)).

We denote by $\mathsf{Y}_{\mathbf{\Lambda}}$ a Young wall stacked on $\boxed{\mathbf{\Lambda}}$ whose type will be clear from the context. For a Young wall $\mathsf{Y}_{\mathbf{\Lambda}}$, we write $\mathsf{Y}_{\mathbf{\Lambda}} = (y_k)_{k=1}^{\infty} = (\dots, y_2, y_1)$ as a sequence of its columns from the right. For $u \in \mathbb{Z}_{\geqslant 1}$, we define Young walls $(\mathsf{Y}_{\mathbf{\Lambda}})_{\geqslant u}$ and $(\mathsf{Y}_{\mathbf{\Lambda}})_{\leqslant u}$ as follows:

$$(\mathsf{Y}_{\mathbf{\Lambda}})_{\geqslant u} = (\dots, y_{u+2}, y_{u+1}, y_u), \qquad (\mathsf{Y}_{\mathbf{\Lambda}})_{\leqslant u} = (y_u, y_{u-1}, y_{u-2}, \dots, y_1).$$

EXAMPLE 2.12. For $\mathfrak{g} = B_3^{(1)}$ and $\mathbf{\Lambda}_0$, the following is an example of a Young wall $\mathsf{Y}_{\mathbf{\Lambda}_0}$:

(2.8)



DEFINITION 2.13.

(1) A column of a Young wall is called a *full column* if its height is a multiple of the unit length and its top is of unit thickness.
(2) A Young wall is said to be *proper* if none of the full columns have the same heights.
(3) An i-block of a proper Young wall $\mathsf{Y}_{\mathbf{\Lambda}}$ is called a *removable* i-block if $\mathsf{Y}_{\mathbf{\Lambda}}$ remains a proper Young wall after removing the block.
(4) A place in a proper Young wall $\mathsf{Y}_{\mathbf{\Lambda}}$ is called an *admissible* or *addable* i-slot if $\mathsf{Y}_{\mathbf{\Lambda}}$ remains a proper Young wall after adding an i-block at the place.

A *partition* λ of m is a weakly decreasing sequence of positive integers ($\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0$) such that $\|\lambda\| := \sum_{i=1}^{k} \lambda_i = m$, and we write $\lambda \vdash m$. Each integer λ_i is called a *part* of λ. For a given partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, we say that the integer $\ell(\lambda) := k$ is the *length* of λ. We denote by $\varnothing$ the empty partition. We say that a partition λ is *strict* if $\lambda_i > \lambda_{i+1}$ for $1 \leqslant i \leqslant \ell(\lambda) - 1$. We set $\lambda_i = 0$ when $i > \ell(\lambda)$.

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $1 \leqslant u \leqslant k$, we define partitions $\lambda_{\geqslant u}$ and $\lambda_{\leqslant u}$ as follows:

$$\lambda_{\geqslant u} = (\lambda_u, \lambda_{u+1}, \dots, \lambda_k), \qquad \lambda_{\leqslant u} = (\lambda_1, \lambda_2, \dots, \lambda_u).$$

DEFINITION 2.14.

(a) For a given proper Young wall $\mathsf{Y}_{\mathbf{\Lambda}} = (y_i)_{i=1}^{\infty}$, define $|\mathsf{Y}_{\mathbf{\Lambda}}| = (|y_1|, |y_2|, \dots)$ to be the sequence of nonnegative integers, where the $|y_i|$ is the number of blocks in the i-th column of $\mathsf{Y}_{\mathbf{\Lambda}}$ above the ground-state wall $\boxed{\mathbf{\Lambda}}$, and call $|\mathsf{Y}_{\mathbf{\Lambda}}|$ the *partition associated* to $\mathsf{Y}_{\mathbf{\Lambda}}$.
(b) For a strict partition λ and a fundamental weight $\mathbf{\Lambda}$ of level 1, we denote by $\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda}$ the Young wall with ground-state wall $\boxed{\mathbf{\Lambda}}$ if it is the *unique* Young wall whose associated partition $|\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda}|$ is equal to λ.

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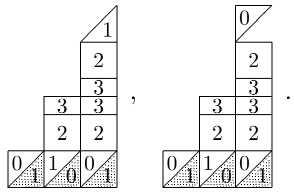
EXAMPLE 2.15. For the proper Young wall given in Example 2.12, the associated partition is $\lambda = (6, 3, 1)$. However, there are two proper Young walls corresponding to the partition $(6, 3, 1)$:



Thus $\mathsf{Y}_{\mathbf{\Lambda}_0}^{(6,3,1)}$ is not uniquely determined. On the other hand, for the partition $(5, 3, 1)$, one can easily see that $\mathsf{Y}_{\mathbf{\Lambda}_0}^{(5,3,1)}$ is well-defined (see [**33**] also).

For the rest of this paper, we will always deal with partitions λ such that the Young walls $\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda}$ are uniquely determined, unless otherwise stated.

We denote by $\mathcal{Z}(\mathbf{\Lambda})$ the set of all proper Young walls on $\boxed{\mathbf{\Lambda}}$, and define the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{Z}(\mathbf{\Lambda})$ as follows: Fix $i \in I$ and let $\mathsf{Y}_{\mathbf{\Lambda}} = (y_u)_{u=1}^{\infty}$ be a proper Young wall.

(a) To each column y_u of $\mathsf{Y}_{\mathbf{\Lambda}}$, assign

$$\begin{cases} -- & \text{if } y_u \text{ is twice } i\text{-removable,} \\ - & \text{if } y_u \text{ is once } i\text{-removable,} \\ -+ & \text{if } y_u \text{ is once } i\text{-removable and once } i\text{-addable,} \\ + & \text{if } y_u \text{ is once } i\text{-addable,} \\ ++ & \text{if } y_u \text{ is twice } i\text{-addable,} \\ \cdot & \text{otherwise.} \end{cases}$$

(b) From this sequence of $+$'s and $-$'s, we cancel out every $(+, -)$-pair to obtain a finite sequence of $-$'s followed by $+$'s, reading from left to right. This finite sequence $(- \cdots -, + \cdots +)$ is called the *i-signature* of $\mathsf{Y}_{\mathbf{\Lambda}}$ and is denoted by $\mathrm{sig}_i(\mathsf{Y}_{\mathbf{\Lambda}})$.

(c) We define $\tilde{e}_i\mathsf{Y}_{\mathbf{\Lambda}}$ to be the proper Young wall obtained from $\mathsf{Y}_{\mathbf{\Lambda}}$ by removing the i-block corresponding to the right-most $-$ in the i-signature of $\mathsf{Y}_{\mathbf{\Lambda}}$. We define $\tilde{e}_i\mathsf{Y}_{\mathbf{\Lambda}} = 0$ if there is no $-$ in the i-signature of $\mathsf{Y}_{\mathbf{\Lambda}}$.

(d) We define $\tilde{f}_i\mathsf{Y}_{\mathbf{\Lambda}}$ to be the proper Young wall obtained from $\mathsf{Y}_{\mathbf{\Lambda}}$ by adding an i-block to the column corresponding to the left-most $+$ in the i-signature of $\mathsf{Y}_{\mathbf{\Lambda}}$. We define $\tilde{f}_i\mathsf{Y}_{\mathbf{\Lambda}} = 0$ if there is no $+$ in the i-signature of $\mathsf{Y}_{\mathbf{\Lambda}}$.

For the $\mathsf{Y}_{\mathbf{\Lambda}_0}$ in Example 2.12, one can compute that

$$\mathrm{sig}_0(\mathsf{Y}_{\mathbf{\Lambda}_0}) = (-, \cdot, +), \mathrm{sig}_1(\mathsf{Y}_{\mathbf{\Lambda}_0}) = (\cdot, \cdot, -), \mathrm{sig}_2(\mathsf{Y}_{\mathbf{\Lambda}_0}) = (+, \cdot, \cdot), \mathrm{sig}_3(\mathsf{Y}_{\mathbf{\Lambda}_0}) = (\cdot, -+, \cdot).$$

We define

(a) $\mathrm{wt}(\mathsf{Y}_{\mathbf{\Lambda}}) = \mathbf{\Lambda} - \sum_{i \in I} m_i \alpha_i$,
(b) $\varepsilon_i(\mathsf{Y}_{\mathbf{\Lambda}})$(resp. $\varphi_i(\mathsf{Y}_{\mathbf{\Lambda}})$) = the number of $-$'s (resp $+$'s) in $\mathrm{sig}_i(\mathsf{Y}_{\mathbf{\Lambda}})$,

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where m_i is the number of i-blocks that have been added to the ground-state wall $\boxed{\mathbf{\Lambda}}$. We also define

$$\mathrm{cont}(\mathsf{Y}_{\mathbf{\Lambda}}) = \mathbf{\Lambda} - \mathrm{wt}(\mathsf{Y}_{\mathbf{\Lambda}}) = \sum_{i \in I} m_i \alpha_i$$

and call it the *content* of $\mathsf{Y}_{\mathbf{\Lambda}}$.

For the Young wall $\mathsf{Y}_{\mathbf{\Lambda}_0}$ in Example 2.12, we have

$\mathrm{wt}(\mathsf{Y}_{\mathbf{\Lambda}_0}) = \mathbf{\Lambda}_0 - (2\alpha_0 + 2\alpha_1 + 3\alpha_2 + 3\alpha_3)$ and $\mathrm{cont}(\mathsf{Y}_{\mathbf{\Lambda}_0}) = 2\alpha_0 + 2\alpha_1 + 3\alpha_2 + 3\alpha_3$.

Definition 2.16. Let $\mathsf{Y}_{\mathbf{\Lambda}} = (\ldots, y_2, y_1)$ and $\mathsf{Y}_{\mathbf{\Lambda}'} = (\ldots, y'_2, y'_1)$ be Young walls of the same affine type. For $t, u \in \mathbb{Z}_{\geqslant 1}$, we write

$$(\mathsf{Y}_{\mathbf{\Lambda}})_{\geqslant t} \supset (\mathsf{Y}_{\mathbf{\Lambda}'})_{\geqslant u}$$

if the following two conditions hold for each $s \in \mathbb{Z}_{\geqslant 0}$:

(a) the ground patterns for y_{t+s} and y'_{u+s} coincide with each other,
(b) $\mathrm{cont}(y_{t+s}) - \mathrm{cont}(y'_{u+s}) \in \mathsf{Q}^+$.

Here, for a column y of a Young wall $\mathsf{Y}_{\mathbf{\Lambda}}$, we define $\mathrm{cont}(y) = \sum_{i\in I} m_i\alpha_i$, where m_i is the number of i-blocks in the column y that have been added to the ground-state wall $\boxed{\mathbf{\Lambda}}$.

Recall we denote the null root by $\delta = a_0\alpha_0 + a_1\alpha_1 + \cdots + a_n\alpha_n$.

Definition 2.17. Set $d = 2$ if $\mathfrak{g} = D^{(2)}_{n+1}$ and $d = 1$, otherwise.

(i) A connected part of a column in a proper Young wall is called a δ-*column* if it contains da_0-many 0-blocks, da_1-many 1-blocks, ..., da_n-many n-blocks (see Table 1.1 for a_i's).
(ii) A δ-column in a proper Young wall $\mathsf{Y}_{\mathbf{\Lambda}}$ is *removable* if one can remove the δ-column from $\mathsf{Y}_{\mathbf{\Lambda}}$ and the result is still a proper Young wall.
(iii) A proper Young wall is said to be *reduced* if it has no removable δ-column.

We denote by $\mathcal{Y}(\mathbf{\Lambda})$ the set of all reduced proper Young walls on $\boxed{\mathbf{\Lambda}}$.

Theorem 2.18 ([**18**, **19**]).

(1) *The set $\mathcal{Z}(\mathbf{\Lambda})$ with $\tilde{e}_i, \tilde{f}_i, \mathrm{wt}, \varepsilon_i$ and φ_i is an affine crystal.*
(2) *The set $\mathcal{Y}(\mathbf{\Lambda})$ is an affine subcrystal which is isomorphic to $\mathbf{B}(\mathbf{\Lambda})$, where $\mathbf{B}(\mathbf{\Lambda})$ is the crystal of the highest weight module $V^q(\mathbf{\Lambda})$.*
(3) *As crystals,*

$$\mathcal{Z}(\mathbf{\Lambda}) = \begin{cases} \bigoplus_{m\geqslant 0} \mathbf{B}(\mathbf{\Lambda} - m\delta)^{\oplus p(m)} & \text{if } \mathfrak{g} \neq D^{(2)}_{n+1}, \\ \bigoplus_{m\geqslant 0} \mathbf{B}(\mathbf{\Lambda} - 2m\delta)^{\oplus p(m)} & \text{if } \mathfrak{g} = D^{(2)}_{n+1}, \end{cases}$$

where $p(m)$ denotes the number of partitions of m.

2.4. Higher level representations

In this section, we will realize the crystal $\mathbf{B}(\Lambda)$ for $\langle c, \Lambda\rangle \geqslant 2$ in terms of tensor products of Young walls. We will see that a crystal $\mathbf{B}(\Lambda)$ of level $k \in \mathbb{Z}_{\geqslant 1}$ (up to $\mathbb{Z}\delta$) appears in a connected component of $\otimes \mathbf{B}(\mathbf{\Lambda}^{(i)})^{\otimes k_i}$ $(\sum k_i = k)$ for some $\mathbf{\Lambda}^{(i)}$'s of level 1.

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To begin with, we consider the crystal $\mathbf{B}(k\boldsymbol{\Lambda})$ of level k and see that $\mathbf{B}(k\boldsymbol{\Lambda})$ is realized as

$$\text{(2.9)}\quad \begin{array}{l}\text{the subcrystal of } \mathcal{Y}(\boldsymbol{\Lambda})^{\otimes k} \text{ whose graph is the connected component}\\ \text{of the } k\text{-fold tensor of ground-state Young walls, denoted by } \boxed{k\boldsymbol{\Lambda}} := \\ \underbrace{\boxed{\boldsymbol{\Lambda}}\otimes\cdots\otimes\boxed{\boldsymbol{\Lambda}}}_{\text{k-times}}.\end{array}$$

Next we consider $\mathbf{B}(\Lambda_s)$ where Λ_s is a fundamental weight of level 2. In order to embed $\mathbf{B}(\Lambda_s)$ (up to $\mathbb{Z}\delta$, see (2.2)) into a tensor product $\mathcal{Y}(\boldsymbol{\Lambda}')\otimes\mathcal{Y}(\boldsymbol{\Lambda}'')$ of crystals $\mathcal{Y}(\boldsymbol{\Lambda}')$ and $\mathcal{Y}(\boldsymbol{\Lambda}'')$ (see (3) in Theorem 2.18) for some $\boldsymbol{\Lambda}'$ and $\boldsymbol{\Lambda}''$ of level 1, we first need equations of the form

$$\text{(2.10)}\qquad \Lambda_s - m\delta = \boldsymbol{\Lambda}' + \boldsymbol{\Lambda}'' - \sum_{i\in I} t_i\alpha_i \quad \text{for some } m \in \mathbb{Z} \text{ and } t_i \in \mathbb{Z}_{\geqslant 0},\ i \in I.$$

For each $\mathfrak{g}$ and a fundamental weight Λ_s of level 2, an equation of the form (2.10) is explicitly given in what follows according to whether $\boldsymbol{\Lambda}'$ and $\boldsymbol{\Lambda}''$ are of type $\mathfrak{D}$ or $\mathfrak{B}$. Using (1.2) with the basis $\{\alpha_0, \alpha_1, \ldots, \alpha_n, \Lambda_0\}$, one can check that equations in the below hold:
(2.11)

$$\begin{aligned}
\text{Type } \mathfrak{D}:\qquad & \Lambda_{2u} - u\delta = 2\boldsymbol{\Lambda}_0 - \left(u\alpha_0 + (u-1)\alpha_1 + \sum_{i=2}^{2u-1}(2u-i)\alpha_i\right),\\
& \Lambda_{2u+1} - u\delta = \boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_0 - \left(u\alpha_0 + u\alpha_1 + \sum_{i=2}^{2u}(2u+1-i)\alpha_i\right),\\
& \Lambda_{n-2u} = 2\boldsymbol{\Lambda}_n - \left(u\alpha_n + (u-1)\alpha_{n-1} + \sum_{i=n-2}^{n-2u+1}(i-(n-2u))\alpha_i\right),\\
& \Lambda_{n-2u-1} = \boldsymbol{\Lambda}_{n-1} + \boldsymbol{\Lambda}_n - \left(u\alpha_n + u\alpha_{n-1} + \sum_{i=n-2}^{n-2u}(i-(n-2u-1))\alpha_i\right).
\end{aligned}$$

(2.12)

$$\text{Type } \mathfrak{B}:\qquad \Lambda_u = 2\boldsymbol{\Lambda}_n - \sum_{i=u+1}^{n}(i-u)\alpha_i, \qquad \Lambda_u - u\delta = 2\boldsymbol{\Lambda}_0 - \sum_{i=0}^{u-1}(u-i)\alpha_i.$$

Here $\Lambda_\bullet$ denotes a fundamental weight of level 2 and hence the range of u is determined by the affine type of given Young walls (see Table 1.1). We observe that what is subtracted in the right-hand side of each of the formulas in (2.11) and (2.12) corresponds to a specific type of partitions. To be precise, we need the following definition.

DEFINITION 2.19. For a positive integer m, we denote by $\lambda(m)$ the strict partition given by

$$\lambda(m) = (m, m-1, \ldots, 2, 1),$$

and call $\lambda(m)$ the *m-th staircase partition*. We also set $\lambda(m) = \varnothing$ for any non-positive integer m.

Now, for each Λ_s of level 2, the crystal $\mathbf{B}(\Lambda_s)$ is realized up to a weight shift by an element of $\mathbb{Z}\delta$ as the subcrystal of $\mathcal{Y}(\boldsymbol{\Lambda}')\otimes\mathcal{Y}(\boldsymbol{\Lambda}'')$ generated by a highest weight crystal $\boxed{\boldsymbol{\Lambda}'}\otimes \mathsf{Y}^{\lambda(s)}_{\boldsymbol{\Lambda}''}$ for some staircase partition $\lambda(s)$. Concretely, we associate a

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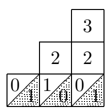
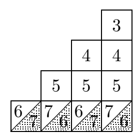
tensor product of Young walls to a fundamental weight Λ_s of level 2 using (2.11) and (2.12).

(i) For type $\mathfrak{D}$,
$$\Lambda_{2u} - u\delta = 2\mathbf{\Lambda}_0 - \left(u\alpha_0 + (u-1)\alpha_1 + \sum_{i=2}^{2u-1}(2u-i)\alpha_i\right) \longleftrightarrow \boxed{\Lambda^{0,0}_{2u}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(2u-1)}_{\mathbf{\Lambda}_0},$$
$$\Lambda_{2u+1} - u\delta = \mathbf{\Lambda}_1 + \mathbf{\Lambda}_0 - \left(u\alpha_0 + u\alpha_1 + \sum_{i=2}^{2u}(2u+1-i)\alpha_i\right) \longleftrightarrow \boxed{\Lambda^{1,0}_{2u+1}} := \boxed{\mathbf{\Lambda}_1} \otimes \mathsf{Y}^{\lambda(2u)}_{\mathbf{\Lambda}_0},$$
$$\text{or } \boxed{\Lambda^{0,1}_{2u+1}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(2u)}_{\mathbf{\Lambda}_1},$$
$$\Lambda_{n-2u} = 2\mathbf{\Lambda}_n - \left(u\alpha_n + (u-1)\alpha_{n-1} + \sum_{i=n-2}^{n-2u+1}(i-(n-2u))\alpha_i\right) \longleftrightarrow \boxed{\Lambda^{n,n}_{n-2u}} := \boxed{\mathbf{\Lambda}_n} \otimes \mathsf{Y}^{\lambda(2u-1)}_{\mathbf{\Lambda}_n},$$
$$\Lambda_{n-2u-1} = \mathbf{\Lambda}_{n-1} + \mathbf{\Lambda}_n - \left(u\alpha_n + u\alpha_{n-1} + \sum_{i=n-2}^{n-2u}(i-(n-2u-1))\alpha_i\right) \longleftrightarrow \boxed{\Lambda^{n,n-1}_{n-2u-1}} := \boxed{\mathbf{\Lambda}_n} \otimes \mathsf{Y}^{\lambda(2u)}_{\mathbf{\Lambda}_{n-1}},$$
$$\text{or } \boxed{\Lambda^{n-1,n}_{n-2u-1}} := \boxed{\mathbf{\Lambda}_{n-1}} \otimes \mathsf{Y}^{\lambda(2u)}_{\mathbf{\Lambda}_n}.$$

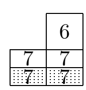
EXAMPLE 2.20. For $\mathfrak{g} = D_7^{(1)}$, we describe $\boxed{\Lambda_4^{0,0}}$ and $\boxed{\Lambda_2^{6,7}}$:

$$\boxed{\Lambda_4^{0,0}} = \boxed{\mathbf{\Lambda}_0} \otimes$$  , $$\boxed{\Lambda_2^{6,7}} = \boxed{\mathbf{\Lambda}_6} \otimes$$  .

(ii) For type $\mathfrak{B}$,
$$\Lambda_u = 2\mathbf{\Lambda}_n - \left(\sum_{i=u+1}^{n}(i-u)\alpha_i\right) \longleftrightarrow \boxed{\Lambda_u^{n,n}} := \boxed{\mathbf{\Lambda}_n} \otimes \mathsf{Y}^{\lambda(n-u)}_{\mathbf{\Lambda}_n},$$
$$\Lambda_u - u\delta = 2\mathbf{\Lambda}_0 - \left(\sum_{i=0}^{u-1}(u-i)\alpha_i\right) \longleftrightarrow \boxed{\Lambda_u^{0,0}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(u-1)}_{\mathbf{\Lambda}_0}.$$

Note that all $\boxed{\Lambda}$ of level 2 are contained in $\mathcal{Y}(\mathbf{\Lambda}) \otimes \mathcal{Y}(\mathbf{\Lambda}')$ for some $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$.

EXAMPLE 2.21. For $\mathfrak{g} = B_7^{(1)}$, we have

$$\boxed{\Lambda_5^{7,7}} = \boxed{\mathbf{\Lambda}_7} \otimes$$  .

The tensor products of Young walls given above will be denoted by $\boxed{\Lambda_s}$ without superscripts if there is no possible confusion. One can see that the crystal $\mathbf{B}(\Lambda_s)$ is realized as the subcrystal of $\mathcal{Y}(\mathbf{\Lambda}') \otimes \mathcal{Y}(\mathbf{\Lambda}'')$ generated by $\boxed{\Lambda_s}$ for each fundamental weight Λ_s of level 2.

(2.13) Next, the crystal $\mathbf{B}((k-2)\mathbf{\Lambda} + \Lambda_s)$ of level k is realized as the subcrystal of $\mathcal{Y}(\mathbf{\Lambda})^{\otimes k-2} \otimes \mathcal{Y}(\mathbf{\Lambda}') \otimes \mathcal{Y}(\mathbf{\Lambda}'')$ generated by the highest weight crystal $\boxed{(k-2)\mathbf{\Lambda}} \otimes \boxed{\Lambda_s}$ whose weight is $(k-2)\mathbf{\Lambda} + \Lambda_s$ up to $\mathbb{Z}\delta$ (see (2.2)). Here, $\boxed{k\mathbf{\Lambda}} = \boxed{\mathbf{\Lambda}}^{\otimes k}$ as defined in (2.9).

REMARK 2.22.

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(1) There are several other possible realizations of $\mathbf{B}((k-2)\boldsymbol{\Lambda}+\Lambda_s)$ depending on the choice of highest weight crystals. For example, the connected component originated from

$$\boxed{a\boldsymbol{\Lambda}} \otimes \boxed{\Lambda_s} \otimes \boxed{b\boldsymbol{\Lambda}} \subset \mathcal{Y}(\boldsymbol{\Lambda})^{\otimes a} \otimes \mathcal{Y}(\boldsymbol{\Lambda}') \otimes \mathcal{Y}(\boldsymbol{\Lambda}'') \otimes \mathcal{Y}(\boldsymbol{\Lambda})^{\otimes b} \quad (a+b=k-2)$$

is also a realization of $\mathbf{B}((k-2)\boldsymbol{\Lambda}+\Lambda_s)$, and we can also choose different highest weight crystals for Λ_s.

(2) For each $\Lambda \in P^+$ of level k with $\Lambda = \sum_{i=0}^n m_i\Lambda_i$, the crystal $\mathbf{B}(\Lambda)$ can be realized as the subcrystal of $\mathcal{Y}(\boldsymbol{\Lambda}_{i_1}) \otimes \mathcal{Y}(\boldsymbol{\Lambda}_{i_2}) \otimes \cdots \otimes \mathcal{Y}(\boldsymbol{\Lambda}_{i_k})$ for some $(i_1, i_2, \ldots, i_k)$, which is generated by $\otimes_{i=0}^n \boxed{\Lambda_i}^{\otimes m_i}$. (Here we abuse notations a little bit and write $\boxed{\Lambda_i} = \boxed{\boldsymbol{\Lambda}_i}$ even when $\boldsymbol{\Lambda}_i$ is of level 1.)

Throughout this paper we will use the following notational convention.

Convention 2.23. For a statement P, the number $\delta(P)$ is equal to 1 if P is true and 0 if P is false. Sometimes, we will write δ_P for $\delta(P)$.

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CHAPTER 3

Young tableaux and almost even tableaux

In this chapter we make connections between tensor products of Young walls and Young tableaux.

3.1. Young tableaux

For partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, we define the partition $\lambda^{(1)} * \lambda^{(2)}$ by rearranging the parts of $\lambda^{(1)}$ and $\lambda^{(2)}$ in a weakly decreasing way. As an obvious generalization, for partitions $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k-1)}, \lambda^{(k)}$, we set

$$\mathop{*}_{t=1}^{k} \lambda^{(t)} := \lambda^{(1)} * \lambda^{(2)} * \cdots * \lambda^{(k-1)} * \lambda^{(k)}. \tag{3.1}$$

Example 3.1. For partitions $\lambda^{(1)} = (7,3,1)$, $\lambda^{(2)} = (8,6,6,3)$ and $\lambda^{(3)} = (7,5,4,1)$, we have

$$\mathop{*}_{t=1}^{3} \lambda^{(t)} = (8,7,7,6,6,5,4,3,3,1,1).$$

The *Young diagram* Y^λ associated to a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a finite collection of cells arranged in left-justified rows, with the i-th row length given by λ_i.

We also define a partial order $\subset$ on the set of all partitions, called the *inclusion order*, in the following way:

$$\mu \subset \lambda \quad \text{if and only if} \quad Y^\mu \subset Y^\lambda.$$

A *skew partition*, denoted by λ/μ, is a pair of two partitions λ and μ satisfying $\mu \subset \lambda$. For a skew partition λ/μ, the *skew Young diagram* $Y^{\lambda/\mu}$ is the diagram obtained by removing cells corresponding to Y^μ from Y^λ. The notation $\lambda/\mu \vdash m$ means that the number of cells in $Y^{\lambda/\mu}$ is m.

We will identify a usual partition λ with the skew partition $\lambda/\varnothing$. In this identification, every definition on the skew partitions in this section induces a definition on the usual partitions.

Definition 3.2.

(1) A *tableau* T is a filling of the cells in the skew Young diagram $Y^{\lambda/\mu}$ with integers $1, 2, \dots, m$ for some skew partition $\lambda/\mu \vdash m$ such that every integer $1 \leqslant i \leqslant m$ appears exactly once. In this case we say that the *shape* $\mathsf{Sh}(T)$ of the tableau T is λ/μ.
(2) A *standard Young tableau* is a tableau in which the entries in each row and each column are increasing. We denote by $\mathcal{S}^{\lambda/\mu}$ the set of standard Young tableaux of shape λ/μ.
(3) A *reverse standard Young tableau* is a tableau in which the entries in each row and each column are decreasing. We denote by $\mathcal{R}^{\lambda/\mu}$ the set of reverse standard Young tableaux of shape λ/μ.

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EXAMPLE 3.3. The following tableaux are a reverse standard Young tableau of shape $(4,3,1)$ and a standard Young tableau of shape $(4,3,1)$:

$$T = \begin{array}{|c|c|c|c|}\hline 8&6&4&3\\\hline 7&2&1\\\cline{1-3} 5\\\cline{1-1}\end{array} \in \mathcal{R}^{(4,3,1)}, \qquad T' = \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&7&8\\\cline{1-3} 4\\\cline{1-1}\end{array} \in \mathcal{S}^{(4,3,1)}.$$

Note that there is an obvious bijection between $\mathcal{R}^{\lambda/\mu}$ and $\mathcal{S}^{\lambda/\mu}$ that replaces each integer i by $m+1-i$, where $\lambda/\mu \vdash m$. The two tableaux in Example 3.3 correspond to each other under this bijection. Thus we have $|\mathcal{R}^{\lambda/\mu}| = |\mathcal{S}^{\lambda/\mu}|$. We will sometimes identify reverse standard Young tableaux and standard Young tableaux using this bijection. We denote by $f^\lambda = |\mathcal{R}^\lambda| = |\mathcal{S}^\lambda|$. Recall that there is a well known formula for f^λ called the hook-length formula.

In this paper, we only consider reverse standard Young tableaux except the last 3 sections. Hence, for simplicity, we call a reverse standard Young tableau just a Young tableau.

For later use, we define another notation related to a tableau.

DEFINITION 3.4. For a Young tableau T with m cells, we denote by $T_{>s}$ for $1 \leqslant s \leqslant m$ the tableau which is obtained by removing all cells filled with t such that $t \leqslant s$ and replacing $u > s$ with $u - s$ for all $u > s$.

For T in Example 3.3,

$$T_{>1} = \begin{array}{|c|c|c|c|}\hline 7&5&3&2\\\hline 6&1\\\cline{1-2} 4\\\cline{1-1}\end{array}.$$

Let $\mathfrak{B}_m^{(k)}$ denote the set of Young tableaux with m cells and at most k rows. It is well known that the cardinality of $\mathfrak{B}_m^{(k)}$ is equinumerous to the number of $(k+1,k,\dots,1)$-avoiding involutions in the symmetric group $\mathfrak{S}_m$.

In the literature an explicit formula for $|\mathfrak{B}_m^{(k)}|$ is known only for $k \leqslant 5$ as follows.

THEOREM 3.5. [**10, 34**] *We have*

$$|\mathfrak{B}_m^{(2)}| = \binom{m}{\lfloor \frac{m}{2} \rfloor}, \ |\mathfrak{B}_m^{(3)}| = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \mathsf{C}_i \binom{m}{2i},$$

$$|\mathfrak{B}_m^{(4)}| = \mathsf{C}_{\lceil \frac{m+1}{2} \rceil} \mathsf{C}_{\lfloor \frac{m+1}{2} \rfloor}, |\mathfrak{B}_m^{(5)}| = 6 \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2i} \frac{(2i+2)!\mathsf{C}_i}{(i+2)!(i+3)!},$$

where $\mathsf{C}_m = \frac{1}{m+1}\binom{2m}{m}$ *is the m-th Catalan number.*

Note that each element T in $\mathfrak{B}_m^{(k)}$ can be identified with a sequence $\underline{\lambda}_T = (\lambda^{(1)}, \dots, \lambda^{(k)})$ of strict partitions, where $\lambda^{(i)}$ is the partition obtained by reading the ith row of T. Using this identification we have

$$\mathfrak{B}_m^{(k)} = \left\{ \underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \,|\, \ell \leqslant k, \lambda^{(i)} \supset \lambda^{(i+1)} (1 \leqslant i < \ell) \text{ and } \lambda^{(1)} * \cdots * \lambda^{(\ell)} = \lambda(m) \right\}.$$

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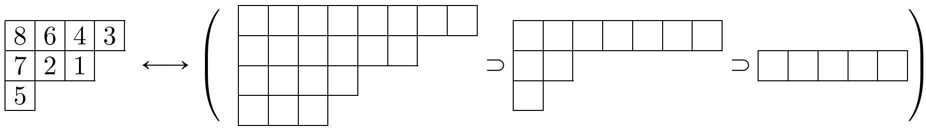
In Example 3.3, the tableau T can be identified with $((8,6,4,3) \supset (7,2,1) \supset (5)) \in \mathfrak{B}_8^{(3)}$:



3.2. Tensor products of Young walls

As we have seen in Definition 2.14, we can construct a Young wall when we have a partition λ and a fundamental weight $\mathbf{\Lambda}$ of level 1. As before, for a given skew shape ρ/μ with k rows, we will identify a (skew) Young tableau T of shape ρ/μ with the sequence $\underline{\lambda}_T = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ of strict partitions, where $\lambda^{(i)}$ is the partition obtained by reading the ith row of T. Then we can make a correspondence between a (skew) Young tableau T of shape ρ/μ with k rows and a k-fold tensor product of Young walls,

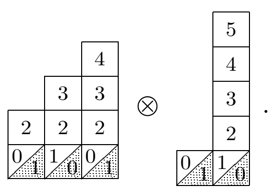
$$\mathbb{Y}^{T}_{\underline{\mathbf{\Lambda}}} \text{ or } \mathbb{Y}^{\underline{\lambda}}_{\underline{\mathbf{\Lambda}}} := \mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}_{i_1}} \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}_{i_2}} \otimes \cdots \otimes \mathsf{Y}^{\lambda^{(k)}}_{\mathbf{\Lambda}_{i_k}} \quad \text{with } \underline{\lambda} = \underline{\lambda}_T,$$

for a fixed sequence $\underline{\mathbf{\Lambda}} = (\mathbf{\Lambda}_{i_1}, \mathbf{\Lambda}_{i_2}, \ldots, \mathbf{\Lambda}_{i_k})$ of fundamental weights of level 1.

EXAMPLE 3.6. For $\mathfrak{g} = D_7^{(1)}$, let $\underline{\mathbf{\Lambda}} = (\mathbf{\Lambda}_0, \mathbf{\Lambda}_0)$ and consider the Young wall $\boxed{\Lambda_4^{0,0}}$ in Example 2.20. Then we have the correspondence

$$T = \begin{array}{|c|c|c|}\hline \cdot & \cdot & \cdot \\ \hline 3 & 2 & 1 \\ \hline \end{array} \quad \longleftrightarrow \quad \mathbb{Y}^{(\varnothing, \lambda(3))}_{(\mathbf{\Lambda}_0, \mathbf{\Lambda}_0)} = \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(3)}_{\mathbf{\Lambda}_0} = \boxed{\Lambda_4^{0,0}}.$$

For $\mathfrak{g} = B_7^{(1)}$, consider $T = \begin{array}{|c|c|c|c|}\hline \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2} \end{array}$ of shape $(4,2)/(1)$ and $\underline{\mathbf{\Lambda}} = (\mathbf{\Lambda}_0, \mathbf{\Lambda}_1)$. Then the corresponding Young wall $\mathbb{Y}^{T}_{\underline{\mathbf{\Lambda}}}$ is given by



3.3. Some families of Young tableaux

In this section, we shall introduce special families of Young tableaux and study the cardinalities of them.

A *composition* λ of m is a sequence $(\lambda_1, \ldots, \lambda_k)$ of nonnegative integers such that $\sum_{i=1}^{k} \lambda_i = m$.

DEFINITION 3.7. We say that a composition λ of m is *almost even* if there are exactly one or two odd parts. We write $\lambda \Vdash_0 m$ to denote an almost even composition λ of m.

We denote by $\mathfrak{D}_m^{(k)}$ $(k \leqslant m)$ the subset of $\mathfrak{B}_m^{(k)}$ consisting of the tableaux T such that $\mathsf{Sh}(T) \Vdash_0 m$ and call $T \in \mathfrak{D}_m^{(k)}$ an *almost even tableau* of m with at most k rows.

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EXAMPLE 3.8. For $m = 5, 6$ and $k = 2$, we have

$$\ytableaushort{532,41} \in \mathfrak{D}_5^{(2)}, \quad \ytableaushort{5432,1} \in \mathfrak{D}_5^{(2)} \quad \text{and} \quad \ytableaushort{6532,41} \notin \mathfrak{D}_6^{(2)}, \quad \ytableaushort{65432,1} \in \mathfrak{D}_6^{(2)}.$$

For $\epsilon \in \{0, 1\}$ and $k \leqslant m$, we denote by ${}^{\epsilon}\mathfrak{P}_m^{(k)}$ the subset of $\mathfrak{B}_m^{(k)}$ consisting of the tableaux T satisfying

$$\lambda := \mathsf{Sh}(T) \vdash m \quad \text{and} \quad \lambda_i \equiv \epsilon \pmod 2 \quad \text{for all } 1 \leqslant i \leqslant k.$$

We say that $T \in {}^{\epsilon}\mathfrak{P}_m^{(k)}$ is an *ϵ-parity tableau* for $\epsilon \in \{0, 1\}$. We set $\mathfrak{P}_m^{(k)} = {}^{0}\mathfrak{P}_m^{(k)} \sqcup {}^{1}\mathfrak{P}_m^{(k)}$ and call it the set of *parity tableaux* of m cells with at most k rows.

EXAMPLE 3.9. The following are examples of parity tableaux:

$$\ytableaushort{532,4,1} \in {}^{1}\mathfrak{P}_5^{(3)}, \qquad \ytableaushort{6521,43} \in {}^{0}\mathfrak{P}_6^{(2)}.$$

On the other hand, we have

$$\ytableaushort{432,1} \notin \mathfrak{P}_4^{(3)} = {}^{0}\mathfrak{P}_4^{(3)} \bigsqcup {}^{1}\mathfrak{P}_4^{(3)} \text{ (note that } \lambda_3 = 0 \not\equiv 1 \bmod 2), \qquad \ytableaushort{532,41} \notin \mathfrak{P}_5^{(2)}.$$

REMARK 3.10. Note that $\mathfrak{D}_{2m-1}^{(2)} = \mathfrak{B}_{2m-1}^{(2)}$ $(m \geqslant 1)$, and by Theorem 3.5, we have

$$|\mathfrak{D}_{2m-1}^{(2)}| = |\mathfrak{B}_{2m-1}^{(2)}| = \binom{2m-1}{m}.$$

Furthermore, one can observe that, for each $m \geqslant 1$,

- $\mathfrak{B}_{2m}^{(2)} = \mathfrak{D}_{2m}^{(2)} \sqcup {}^{0}\mathfrak{P}_{2m}^{(2)}$ and $\mathfrak{D}_{2m}^{(2)} = {}^{1}\mathfrak{P}_{2m}^{(2)}$,
- there exists a bijection $\psi : \mathfrak{D}_{2m}^{(2)} \to {}^{0}\mathfrak{P}_{2m}^{(2)}$ such that $\psi(T)$ is the tableau which is obtained by moving the cell filled with 1 from its row in T to the other row.

Since $|\mathfrak{B}_{2m}^{(2)}| = \binom{2m}{m}$ by Theorem 3.5, we have

$$|\mathfrak{D}_{2m}^{(2)}| = |{}^{0}\mathfrak{P}_{2m}^{(2)}| = |{}^{1}\mathfrak{P}_{2m}^{(2)}| = \frac{1}{2}\binom{2m}{m} = \binom{2m-1}{m} = |\mathfrak{B}_{2m-1}^{(2)}|. \tag{3.2}$$

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CHAPTER 4

Lattice paths and triangular arrays

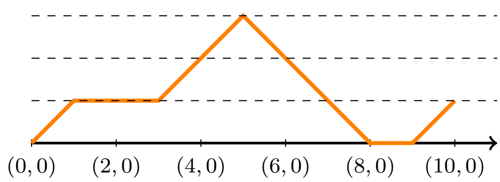
In this chapter, we find an interesting relationship among Young tableaux with at most $k = 2$ or 3 rows, triangular arrays related to lattice paths and composition multiplicities of m-fold tensor products of irreducible $\mathfrak{sl}_2$-modules.

4.1. Motzkin triangle

DEFINITION 4.1. A *Motzkin path* is a path on the lattice $\mathbb{Z}^2$ starting from $(0,0)$, having three kinds of steps called an *up step* $U = (1,1)$, a *horizontal step* $H = (1,0)$, and a *down step* $D = (1,-1)$, and not going below the x-axis.

EXAMPLE 4.2. The following path is a Motzkin path from $(0,0)$ to $(10,1)$:

(4.1)



We also express the above path as a sequence of steps by $UHHUUDDDHU$.

DEFINITION 4.3. A *generalized Motzkin number* $\mathsf{M}_{(m,s)}$ for $m \geqslant s \geqslant 0$ is the number of all Motzkin paths ending at the lattice point (m,s). In particular, we write $\mathsf{M}_m = \mathsf{M}_{(m,0)}$ and call it the *m-th Motzkin number*.

Interestingly, the Motzkin number M_m is also equal to the number of all Young tableaux with m cells and at most 3 rows, see [**4**]. That is, we have

$$\mathsf{M}_m = |\mathfrak{B}_m^{(3)}| = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \mathsf{C}_i \binom{m}{2i}.$$

A recursive formula and a closed formula for $\mathsf{M}_{(m,s)}$ are known and easy to derive:

$$\begin{aligned} \mathsf{M}_{(m,s)} &= \mathsf{M}_{(m-1,s)} + \mathsf{M}_{(m-1,s-1)} + \mathsf{M}_{(m-1,s+1)} \\ &= \sum_{i=0}^{\lfloor (m-s)/2 \rfloor} \binom{m}{2i+s} \left(\binom{2i+s}{i} - \binom{2i+s}{i-1} \right). \end{aligned} \tag{4.2}$$

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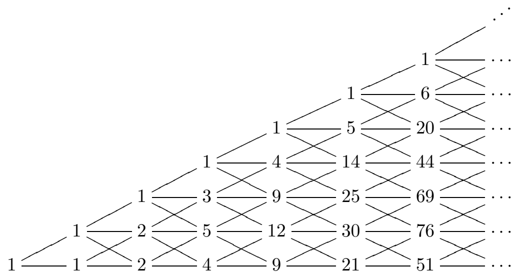
Consider the following triangular array consisting of $\mathsf{M}_{(m,s)}$ and reflecting the recursive relation (4.2).

(4.3)



Here a solid line represents the contribution of a number to the number connected by the line in the next column. For example, we obtain 76 as $25 + 30 + 21$ from the previous column. We call this triangular array the *Motzkin triangle*.

REMARK 4.4. For $m \in \mathbb{Z}_{\geqslant 0}$, let V_m be the $(m+1)$-dimensional irreducible module over $\mathfrak{sl}_2$. In particular, the standard module $\mathbf{V}$ is (isomorphic to) V_1 and the adjoint module $\mathbb{V}$ is V_2. The Clebsch–Gordan formula yields

$$V_m \otimes \mathbb{V} \simeq V_{m-2} \oplus V_m \oplus V_{m+2} \qquad \text{for } m \geqslant 2.$$

Using (4.2), one can show that $\mathsf{M}_{(m,s)}$ is equal to the multiplicity of V_{2s+1} in $\mathbf{V} \otimes \mathbb{V}^{\otimes m}$. The same observation holds for $\mathsf{V} := V_1 \oplus V_0$ (see [**2**]); that is, one can check that $\mathsf{M}_{(m,s)}$ is equal to the multiplicity of V_s in $\mathsf{V}^{\otimes m}$. Thus we have

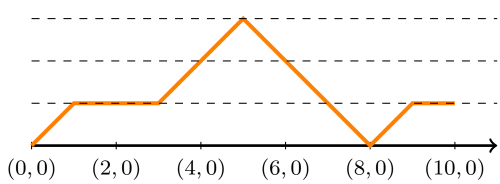
$$3^m = \sum_{s=0}^{m} (s+1)\mathsf{M}_{(m,s)}.$$

4.2. Riordan triangle

DEFINITION 4.5. A *Riordan path* is a Motzkin path without horizontal steps on the x-axis.

EXAMPLE 4.6. The following path is a Riordan path:

(4.4)



Note that the path in (4.1) is *not* a Riordan path.

DEFINITION 4.7. A *generalized Riordan number* $\mathsf{R}_{(m,s)}$ for $m \geqslant s \geqslant 0$ is the number of all Riordan paths ending at the lattice point (m, s). In particular, we write $\mathsf{R}_m = \mathsf{R}_{(m,0)}$ and call it the *m-th Riordan number*.

The Riordan number R_m has a closed formula: $\mathsf{R}_0 = 1$, $\mathsf{R}_1 = 0$ and

$$\mathsf{R}_m = \frac{1}{m+1} \sum_{i=1}^{\lfloor m/2 \rfloor} \binom{m+1}{i} \binom{m-i-1}{i-1} \qquad \text{for } m \geqslant 2.$$

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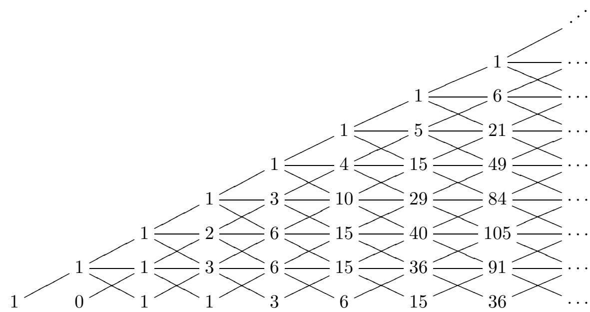
We see that $\mathsf{R}_{(m,s)}$ has a recursive formula

$$\mathsf{R}_{(m,s)} = \begin{cases} \mathsf{R}_{(m-1,s)} + \mathsf{R}_{(m-1,s-1)} + \mathsf{R}_{(m-1,s+1)} & \text{if } s \geqslant 1, \\ \mathsf{R}_{(m-1,1)} & \text{if } s = 0. \end{cases} \tag{4.5}$$

Consider the following triangular array consisting of $\mathsf{R}_{(m,s)}$ and reflecting the recursive formula (4.5).

(4.6)



We call this triangular array the *Riordan triangle.*

REMARK 4.8. Let $\mathbb{V}$ be the adjoint representation of $\mathfrak{sl}_2$ as before. By the same argument as in Remark 4.4, the number $\mathsf{R}_{(m,s)}$ is equal to the multiplicity of V_{2s} in the decomposition of $\mathbb{V}^{\otimes m}$. Then we have the identity

$$3^m = \sum_{s=0}^{m} (2s+1)\mathsf{R}_{(m,s)}.$$

Let $\overline{\mathsf{R}}_{(m,s)} = \mathsf{M}_{(m,s)} - \mathsf{R}_{(m,s)}$. In other words, $\overline{\mathsf{R}}_{(m,s)}$ is the number of Motzkin paths ending at (m,s) which have at least one horizontal step on the x-axis.

LEMMA 4.9. *For $m \geqslant s \geqslant 1$, we have*

$$\mathsf{R}_{(m,s)} = \overline{\mathsf{R}}_{(m,s-1)}.$$

PROOF. We prove this by constructing a bijection $\phi : A \to B$, where A is the set of Motzkin paths ending at $(m, s-1)$ with at least one horizontal step on the x-axis and B is the set of Riordan paths ending at (m,s) so that $|A| = \overline{\mathsf{R}}_{(m,s-1)}$ and $|B| = \mathsf{R}_{(m,s)}$. Let $T = t_1 t_2 \dots t_m \in A$, where $t_1, t_2, \dots, t_m$ are the steps of T in this order. Let t_i be the first horizontal step on the x-axis. Then we define $\phi(T) = t_1 \dots t_{i-1}(1,1)t_{i+1} \dots t_m$. It is easy to see that ϕ is a bijection from A to B. $\square$

There is a simple relation between Motzkin numbers and Riordan numbers.

LEMMA 4.10. *For $m \geqslant 0$, we have*

$$\mathsf{M}_m = \mathsf{R}_m + \mathsf{R}_{m+1}.$$

PROOF. By definition, we have $\mathsf{M}_m = \mathsf{R}_m + \overline{\mathsf{R}}_m$. By Lemma 4.9, we have

$$\overline{\mathsf{R}}_m = \overline{\mathsf{R}}_{(m,0)} = \mathsf{R}_{(m,1)} = \mathsf{R}_{(m+1,0)} = \mathsf{R}_{m+1}. \qquad \square$$

Note that $\mathsf{R}_{(m,s)} = \mathsf{M}_{(m,s)} = 0$ if $m < s$.

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PROPOSITION 4.11. *For $m, s \geqslant 1$, we have*

$$\mathsf{R}_{(m,s)} = \mathsf{M}_{(m-1,s)} + \mathsf{M}_{(m-1,s-1)} - \mathsf{R}_{(m-1,s)},$$

and

$$\mathsf{R}_{(m,s)} = \sum_{i=0}^{m-s} (-1)^i (\mathsf{M}_{(m+1-i,s)} + \mathsf{M}_{(m+1-i,s-1)}).$$

PROOF. The left side of the first equation is

$$\mathsf{R}_{(m,s)} = \mathsf{R}_{(m-1,s-1)} + \mathsf{R}_{(m-1,s)} + \mathsf{R}_{(m-1,s+1)}.$$

The right side is

$$\mathsf{R}_{(m-1,s)} + \overline{\mathsf{R}}_{(m-1,s)} + \mathsf{R}_{(m-1,s-1)} + \overline{\mathsf{R}}_{(m-1,s-1)} - \mathsf{R}_{(m-1,s)}.$$

By Lemma 4.9, these two quantities are equal.

Using the first identity iteratively, we obtain the second identity. $\square$

PROPOSITION 4.12. *For $m \geqslant 1$, we have*

$$\mathsf{R}_m = |\mathfrak{D}^{(3)}_{m-1}| = |\mathfrak{P}^{(3)}_m|.$$

PROOF. One can see that $\mathfrak{B}^{(3)}_m = \mathfrak{P}^{(3)}_m \bigsqcup \mathfrak{D}^{(3)}_m$. Consider the map $\phi : \mathfrak{P}^{(3)}_m \to \mathfrak{D}^{(3)}_{m-1}$ given by

$$T \longmapsto T_{>1},$$

where $T_{>1}$ is defined in Definition 3.4. Then it is easy to check that the map ϕ is a bijection. Thus we have $|\mathfrak{P}^{(3)}_m| = |\mathfrak{D}^{(3)}_{m-1}|$. Now we use an induction on m. If $m = 1$, then $|\mathfrak{P}^{(3)}_1| = \mathsf{R}_1 = 0$. Assume that $|\mathfrak{P}^{(3)}_m| = \mathsf{R}_m$. Since $\mathsf{M}_m = |\mathfrak{B}^{(3)}_m|$, we have

$$\begin{aligned} |\mathfrak{P}^{(3)}_{m+1}| = |\mathfrak{D}^{(3)}_m| &= \mathsf{M}_m - |\mathfrak{P}^{(3)}_m| \\ &= \mathsf{M}_m - \mathsf{R}_m = \mathsf{R}_{m+1} \qquad \text{by Lemma 4.10.} \end{aligned}$$

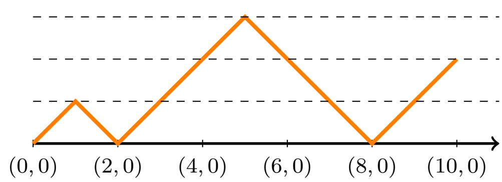
$\square$

REMARK 4.13. The set of parity tableaux $\mathfrak{P}^{(3)}_m$ and the set of almost even tableaux $\mathfrak{D}^{(3)}_{m-1}$ can be taken as tableaux models for the Riordan number R_m, so much as the set $\mathfrak{B}^{(3)}_m$ can be used to realize the Motzkin number M_m.

4.3. Catalan triangle

DEFINITION 4.14. A *Dyck path* is a Motzkin path without horizontal steps.

EXAMPLE 4.15. The following path is a Dyck path:



Note that the path in (4.4) is *not* a Dyck path.

DEFINITION 4.16. A *generalized Catalan number* $\mathsf{C}_{(m,s)}$ for $m \geqslant s \geqslant 0$ is the number of all Dyck paths ending at the lattice point (m, s). In particular, we write $\mathsf{C}_m = \mathsf{C}_{(2m,0)}$ which is known as the *m-th Catalan number.*

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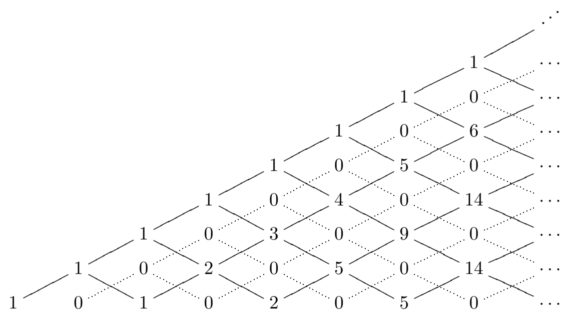
A recursive formula and a closed formula for $\mathsf{C}_{(m,s)}$ are also well-known:

$$\mathsf{C}_{(m,s)} = \delta(m \equiv_2 s)\, \frac{m!(s+1)}{(\frac{m+s+2}{2})!(\frac{m-s}{2})!}, \quad \mathsf{C}_{(m,s)} = \mathsf{C}_{(m-1,s+1)} + \mathsf{C}_{(m-1,s-1)}, \tag{4.7}$$

where we write $m \equiv_2 s$ for $m \equiv s \pmod 2$.

We have the following triangular array consisting of $\mathsf{C}_{(m,s)}$ and reflecting the recursive relation (4.7).



By the same argument as in Remark 4.4, the number $\mathsf{C}_{(m,s)}$ is equal to the multiplicity of V_s in the decomposition of $\mathbf{V}^{\otimes m}$.

Remark 4.17. Using the well known bijection [**37**, Exercise 6.19.xx] between the standard tableaux of shape (n, n) and the Dyck paths from $(0, 0)$ to $(2n, 0)$, one can easily see that the number of standard tableaux of shape $\lambda = (m + s, m)$ coincides with the number $\mathsf{C}_{(2m+s,s)}$.

4.4. Pascal Triangle

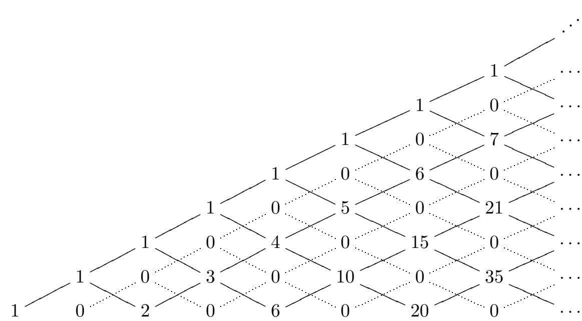
If we consider lattice paths from $(0, 0)$ to (m, s) for $m \geqslant s \geqslant 0$, having $U = (1, 1)$ and $D = (1, -1)$, that may go below the x-axis, then the number $\mathsf{B}_{(m,s)}$ of such paths is given by

$$\mathsf{B}_{(m,s)} = \delta(m \equiv_2 s) \binom{m}{\frac{m-s}{2}}.$$

Clearly, we have $\mathsf{B}_{(m,s)} = \mathsf{B}_{(m-1,s+1)} + \mathsf{B}_{(m-1,s-1)}$ and the corresponding triangular array is the (half of the) Pascal triangle. The number $\mathsf{B}_{(m,s)}$ is also equal to the multiplicity of V_{m+s} in the composition series of $V_m \otimes \mathbf{V}^{\otimes m}$ where $\mathbf{V}$ is the standard module over $\mathfrak{sl}_2$ as before.

We present the following triangular array consisting of $\mathsf{B}_{(m,s)}$ for reference.

(4.8)



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CHAPTER 5

Dominant maximal weights

In this chapter, we investigate the set of dominant maximal weights of highest weight modules $V(\Lambda)$ over affine Kac–Moody algebras of classical types. We will see that most of the dominant maximal weights of levels 2 and 3 are essentially finite, and will classify them into the corresponding finite types. Then, by Theorem 2.9, the multiplicities of distinct dominant maximal weights of the same finite type can be determined simultaneously even though they appear in highest weight modules over different affine Kac–Moody algebras. In other words, the multiplicities of essentially finite dominant maximal weights depend only on their finite types.

Another goal of this chapter is to determine certain families of dominant maximal weights of all levels, which can be associated with pairs $(\lambda(m), \lambda(s))$ of staircase partitions and are essentially finite of type B_n or D_n. Again, applying Theorem 2.9, we see the following:

(5.1) For two essentially finite dominant maximal weights of the same finite type, which are associated with the same $(\lambda(m), \lambda(s))$, their multiplicities coincide with each other, *even when their affine types are different.*

Throughout this chapter, the (fundamental) weights $\boldsymbol{\Lambda}$ of level 1 will be written in boldface; the weights Λ of level 2 in regular; the weights $\mathrm{\Lambda}$ of level $\geqslant 3$ in upright. As arguments and techniques are similar, some details are omitted for other types after we consider type $B_n^{(1)}$ thoroughly.

5.1. Type $A_{n-1}^{(1)}$

This case was studied in [**13**, **14**, **38**, **39**]. In this section, we briefly review their results and show that the dominant maximal weights obtained in [**38**, **39**] are essentially finite. Hence we can reduce them as dominant weights for some $L(\omega)$ over A_{n-1}.

For $0 \leqslant s < n$ and $1 \leqslant \ell \leqslant \left\lfloor \dfrac{n-s}{2} \right\rfloor$ and $1 \leqslant u \leqslant \left\lfloor \dfrac{s}{2} \right\rfloor$, we define $\Lambda := \boldsymbol{\Lambda}_0 + \boldsymbol{\Lambda}_s$ and

$$\lambda^n_{\ell,s} := \sum_{k=n-\ell+1}^{n-1} (k-n+\ell)\alpha_k + \ell \sum_{i=0}^{s} \alpha_i + \sum_{j=s+1}^{\ell+s-1} (\ell - j + s)\alpha_j,$$

$$\mu^n_{u,s} := \sum_{k=s-u+1}^{s-1} (k-s+u)\alpha_k + u \sum_{i=s}^{n-1} \alpha_i + \sum_{j=0}^{u-1} (u-j)\alpha_j.$$

LEMMA 5.1. [**38**, Theorem 1.4 (i)] *For $V(\Lambda)$ over $A_{n-1}^{(1)}$,*

$$\max{}^+(\Lambda|2) = \{\Lambda\} \bigsqcup \left\{ \Lambda - \lambda^n_{\ell,s} \;\middle|\; 1 \leqslant \ell \leqslant \left\lfloor \frac{n-s}{2} \right\rfloor \right\} \bigsqcup \left\{ \Lambda - \mu^n_{u,s} \;\middle|\; 1 \leqslant u \leqslant \left\lfloor \frac{s}{2} \right\rfloor \right\}.$$

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The above lemma tells us that every element in $\max^+(\Lambda|2)$ is essentially finite, since

$$\ell + s < n - \ell + 1 \quad \text{and} \quad u < s - u + 1. \tag{5.2}$$

Now we show that we obtain all the dominant weights of $L(\omega_t + \omega_{t+s})$ from $\max^+(\Lambda|2)$, where $t := \left\lfloor \dfrac{n-s}{2} \right\rfloor$. Since

$$J = [0, \ell+s-1] \bigsqcup [n-\ell+1, n-1] := \mathrm{Supp}(\lambda^n_{\ell,s}) \subsetneq I := [0, n-1]$$

and $\ell + s < n - \ell + 1$ for all ℓ, we can choose $s + t$ as an extremal vertex (see (1.4)). Thus

$$\Omega_1 := \{\Lambda\} \bigsqcup \left\{ \Lambda - \lambda^n_{\ell,s} \;\middle|\; 1 \leqslant \ell \leqslant t \right\}$$

can be considered as a subset of dominant maximal weights of $L(\omega_t + \omega_{t+s})$ over A_{n-1} via the embedding

$[0, t+s-1] \sqcup [t+s+1, n-1] \hookrightarrow \{1, 2, \ldots, n-1\}$ such that $x \longmapsto a \equiv s+t-x \pmod n$.

Hence Ω_1 can be identified with

$$\{\omega_{t-r} + \omega_{t+s+r} \mid 0 \leqslant r \leqslant t\} \tag{5.3}$$

which is a subset of dominant weights of $L(\omega_t + \omega_{t+s})$. (Here we set $\omega_0 := 0$.) By [**12**, §13], $L(\omega_t + \omega_{t+s})$ has $(t+1)$-many dominant weights and hence the set in (5.3) indeed coincides with the set of dominant weights of $L(\omega_t + \omega_{t+s})$.

By a similar argument, the set

$$\Omega_2 := \{\Lambda\} \bigsqcup \left\{ \Lambda - \mu^n_{u,s} \;\middle|\; 1 \leqslant u \leqslant \left\lfloor \frac{s}{2} \right\rfloor \right\}$$

can be identified with the dominant weights

$$\{\omega_{t'-r} + \omega_{n-s+t'-r} \mid 0 \leqslant r \leqslant t'\} \tag{5.4}$$

of $L(\omega_{t'} + \omega_{n-s+t'})$ over A_{n-1}, where $t' := \left\lfloor \dfrac{s}{2} \right\rfloor$.

5.2. Type $B_n^{(1)}$

Assume that $\mathfrak{g} = B_n^{(1)}$. If $\Lambda = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_n$, one can check that there are only two maximal weights Λ and $\mathbf{\Lambda}_1 + \mathbf{\Lambda}_n - \delta$, and their multiplicities are 1 and n, respectively. When $\Lambda = \mathbf{\Lambda}_1 + \mathbf{\Lambda}_n$, the same is true with $\mathbf{\Lambda}_0$ replaced by $\mathbf{\Lambda}_1$.

Assume that Λ is of level 2, other than $\mathbf{\Lambda}_0 + \mathbf{\Lambda}_n$ and $\mathbf{\Lambda}_1 + \mathbf{\Lambda}_n$; that is, for $0 \leqslant s \leqslant n$,

$$\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \delta_{s,n}\mathbf{\Lambda}_n + \Lambda_s = \begin{cases} 2\mathbf{\Lambda}_0 & \text{if } s = 0, \\ \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 & \text{if } s = 1, \\ 2\mathbf{\Lambda}_n & \text{if } s = n, \\ \Lambda_s & \text{otherwise.} \end{cases}$$

Recall that

$$\delta = \alpha_0 + \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n) \quad \text{and} \quad c = h_0 + h_1 + 2(h_2 + \cdots + h_{n-1}) + h_n,$$

and we have

$$2\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{\mathsf{Q}}) = \left\{ \lambda = \overline{\Lambda} + \sum_{i=1}^{n} m_i \alpha_i \;\middle|\; \langle h_i, \lambda \rangle \geqslant 0 \ (1 \leqslant i \leqslant n),\ (\lambda|\theta) \leqslant 2 \right\}, \tag{5.5}$$

where $\theta = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_n)$.

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LEMMA 5.2. *Let* $\Lambda = (\delta_{s,0}+\delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$. *Then the following weights are in* $\max^+(\Lambda|2)$, *i.e., they are dominant maximal weights of* $V(\Lambda)$:

$$(1+\delta_{2u-1+s,n})\Lambda_{2u-1+s} - u\delta =$$

$$\Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{(n)*\lambda(2u-2+s)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(s-1)}\right) \quad \text{for} \quad 1+\delta_{s,0} \leqslant u \leqslant \lfloor (n-s+1)/2 \rfloor, \tag{5.6a}$$

$$(1+\delta_{2u+s,n})\Lambda_{2u+s} - u\delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(2u-1+s)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(s-1)}\right) \quad \text{for} \quad 1 \leqslant u \leqslant \lfloor (n-s)/2 \rfloor. \tag{5.6b}$$

For $\Lambda = 2\mathbf{\Lambda}_0$, *we have* $\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 - \delta \in \max^+(\Lambda|2)$, *which is not of the form in* (5.6).

PROOF. The equalities in (5.6a) and (5.6b) can be checked by direct computations. In each equation in (5.6a) and (5.6b), the RHS shows that the image of the weight under the orthogonal projection is in $\overline{\Lambda} + \overline{\mathsf{Q}}$, and the LHS shows that the image of the orthogonal projection belongs to $2\mathcal{C}_{\mathrm{af}}$. Thus the weights are in $\max^+(\Lambda|2)$ by Proposition 1.4 $\square$

Let $\mathfrak{g}_n$ be the finite dimensional subalgebra of $\mathfrak{g}$, generated by e_i, h_i, f_i for $i \in I_n := I\backslash\{n\}$, as in Section 1.1. Then $\mathfrak{g}_n$ is of type D_n. For each dominant maximal weight $\mu = \Lambda - \sum_{i\in I} k_i\alpha_i$ in (5.6b), we have $k_n = 0$ and so μ is essentially finite of type D_n. Denote by ω the dominant integral weight of $\mathfrak{g}_n$ corresponding to Λ and consider the highest weight module $L(\omega)$ of $\mathfrak{g}_n$ with highest weight ω.

PROPOSITION 5.3. *Let* $\Lambda = (\delta_{s,0}+\delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$, *and consider the correspondences*

$$\Lambda \longmapsto \omega := (\delta_{s,0}+\delta_{s,1})\omega_n + \omega_{n-s} \ \textit{and}\ \Lambda - \sum_{i\in I_n} k_i\alpha_i \longmapsto \omega - \sum_{i\in I_n} k_i\alpha_{n-i}.$$

Then all the dominant weights of $L(\omega)$ *over* $\mathfrak{g}_n$ *of type* D_n *are obtained from* Λ *and the weights in* (5.6b).

PROOF. Since

$$n \notin \mathrm{Supp}\left(\mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(2u-1+s)}\right) - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(s-1)}\right)\right),$$

we can take n as an extremal vertex. Thus we can identify the weights in (5.6b) with

$$\{\omega_{n-s-2k} \mid 1 \leqslant k \leqslant \lfloor (n-s)/2 \rfloor\} \bigsqcup \{(\delta_{s,0}+\delta_{s,1})\omega_n + \omega_{n-s}\},$$

which is the subset of dominant weights of $L((\delta_{s,0}+\delta_{s,1})\omega_n + \omega_{n-s})$ over $\mathfrak{g}_n$ via the embedding

$$I_n = [0, n-1] \to [1, n] \quad \text{such that} \quad i \longmapsto n-i.$$

By [**27**, Lemma 2.6], $L((\delta_{s,0}+\delta_{s,1})\omega_n + \omega_{n-s})$ has $(\lfloor (n-s)/2 \rfloor + 1)$-many dominant weights and hence the weights in (5.6b) along with Λ coincides with the set of dominant weight of $L((\delta_{s,0}+\delta_{s,1})\omega_n + \omega_{n-s})$. $\square$

REMARK 5.4. The above proposition shows that the weights in (5.6b) are essentially finite of type D_n. In Proposition 6.7, we will show that $\dim V(\Lambda)_\mu$ for $\mu = \Lambda_{2u-1+s} - u\delta$ in (5.6a) is equal to $\dim V(\Lambda)_{\mu'}$ for $\mu' = (1+\delta_{2u+s,n})\Lambda_{2u+s} - u\delta$ in (5.6b). Thus they coincide with the multiplicity of $L(\omega)_\eta$ for some ω and $\eta \in \mathrm{wt}(L(\omega))$.

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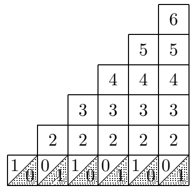
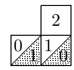
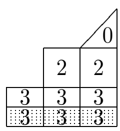
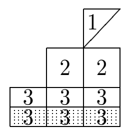
EXAMPLE 5.5. For $\mathfrak{g} = B_9^{(1)}$ and $\Lambda = \Lambda_3$, the dominant maximal weight $\Lambda_7 - 2\delta \in \max^+(\Lambda|2)$ can be written as follows:

$$\Lambda_7 - 2\delta = \Lambda - \left\{3\alpha_0 + 3\alpha_1 + \sum_{i=2}^{6}(7-i)\alpha_i\right\} + \{\alpha_0 + \alpha_1 + \alpha_2\}$$

$= \Lambda - \mathrm{cont}\Bigg($  $\Bigg) + \mathrm{cont}\Bigg($  $\Bigg)$

Define $\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda_\epsilon(n)}$ $(\epsilon = 0, 1)$ to be the Young wall determined by the staircase partition $\lambda(n)$ such that the top of the first column is the half-thickness block with color ϵ.

EXAMPLE 5.6. The $\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda_0(n)}$ and $\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda_1(n)}$ for $B_3^{(1)}$ are given as follows:

$\mathsf{Y}_{\mathbf{\Lambda}_3}^{\lambda_0(3)} =$  and $\mathsf{Y}_{\mathbf{\Lambda}_3}^{\lambda_1(3)} =$  .

LEMMA 5.7. *Let* $\Lambda = (1 + \delta_{s,n})\Lambda_s + \delta_{s,1}\mathbf{\Lambda}_0$ $(1 \leqslant s \leqslant n)$. *Then the following weights are in* $\max^+(\Lambda|2)$*:*

$$(1 + \delta_{u,n})\Lambda_u = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-u)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-s)}\right) \quad (2 \leqslant u \leqslant s), \tag{5.7a}$$

$$\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1 = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-1)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-s)}\right), \tag{5.7b}$$

$$2\mathbf{\Lambda}_1 - \delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda_0(n)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-s)}\right), \tag{5.7c}$$

$$2\mathbf{\Lambda}_0 = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda_1(n)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_n}^{\lambda(n-s)}\right). \tag{5.7d}$$

For $\Lambda = 2\mathbf{\Lambda}_0$, *we have* $2\mathbf{\Lambda}_0 \in \max^+(\Lambda|2)$ *and*

$$2\mathbf{\Lambda}_1 - 2\delta = 2\mathbf{\Lambda}_0 - 2\left(\alpha_0 + \sum_{i=2}^{n}\alpha_i\right) \in \max^+(\Lambda|2), \tag{5.8}$$

which is not of the form in (5.7).

PROOF. One can use the same argument as in Lemma 5.2. □

Let $\mathfrak{g}_1$ (resp. $\mathfrak{g}_0$) be the finite dimensional subalgebra of $\mathfrak{g}$, generated by e_i, h_i, f_i for $i \in I_1 := I\backslash\{1\}$ (resp. $i \in I_0 := I\backslash\{0\}$). Then $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are of type B_n. One can see that each dominant maximal weight $\mu = \Lambda - \sum_{i \in I} k_i\alpha_i$ in Lemma 5.7 is essentially finite of type B_n.

PROPOSITION 5.8. *For* $0 \leqslant s \leqslant n$, *through the correspondences*

$$\Lambda = (1 + \delta_{s,n})\Lambda_s + (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 \longmapsto \omega := (\delta_{s,n})\omega_n + (1 + \delta_{s,0})\omega_{s+\delta_{s,0}} \quad \textit{and}$$
$$\Lambda - \sum_{i \in I_\epsilon} k_i\alpha_i \longmapsto \omega - \sum_{i \in I_\epsilon} k_{i+\delta_{i+\epsilon,1}}\alpha_i (\epsilon = 0, 1),$$

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all the dominant weights of $L(\omega)$ over B_n are obtained from the weights in Lemma 5.7.

Proof. One can easily check that 0 (resp. 1) does not appear as an element of support for weights in (5.7a), (5.7b) and (5.7d) (resp. (5.7a), (5.7b), (5.7c) and (5.8)). Hence we can take 0 (resp. 1) as an extremal vertex. Thus we can identify the weights in (5.7a), (5.7b) and (5.7d) (resp. (5.7a), (5.7b), (5.7c) and (5.8)) with

$$\{\omega_k \mid 0 \leqslant k \leqslant s\},$$

which is the subset of dominant weights of $L((\delta_{s,n})\omega_n + (1+\delta_{s,0})\omega_{s+\delta_{s,0}})$ over B_n via the natural embedding

$$I_0 = [1,n] \to [1,n] \quad (\text{resp. } [0] \sqcup [2,n] \to [1,n]).$$

By [**27**, Lemma 2.4], $L((\delta_{s,n})\omega_n+(1+\delta_{s,0})\omega_{s+\delta_{s,0}})$ has $(s+1+\delta_{s,0})$-many dominant weights and hence the weights in (5.7a), (5.7b) and (5.7d) (resp. (5.7a), (5.7b), (5.7c) and (5.8)) coincides with the set of dominant weight of $L((\delta_{s,n})\omega_n + (1+\delta_{s,0})\omega_{s+\delta_{s,0}})$. □

Let $\max_i^+(\Lambda|2)$ be the set of the dominant maximal weights in Lemma 5.2 and $\max_{ii}^+(\Lambda|2)$ be the set of those in Lemma 5.7. Combining these two sets, we obtain the whole set of dominant maximal weights as stated in the following theorem.

Theorem 5.9. *Assume that $\mathfrak{g} = B_n^{(1)}$ and $\Lambda = (\delta_{s,0}+\delta_{s,1})\mathbf{\Lambda}_0 + \delta_{s,n}\mathbf{\Lambda}_n + \Lambda_s$ $(0 \leqslant s \leqslant n)$ is of level 2. Then we have*

$$\max{}^+(\Lambda|2) = \max{}_i^+(\Lambda|2) \bigsqcup \max{}_{ii}^+(\Lambda|2),$$

and the number of elements in $\max^+(\Lambda|2)$ is equal to $n+2$, since

$$|\max{}_i^+(\Lambda|2)| = n - s \quad \text{and} \quad |\max{}_{ii}^+(\Lambda|2)| = s+2.$$

Before we begin the proof of Theorem 5.9, we make some preparation. Recall that for a statement P, the number $\delta(P)$ is equal to 1 if P is true and 0 if P is false. Sometimes, we will write δ_P for $\delta(P)$.

Now we consider the conditions on $\max^+(\Lambda|2)$ for $\Lambda = (\delta_{s,0}+\delta_{s,1})\mathbf{\Lambda}_0+\delta_{s,n}\mathbf{\Lambda}_n+\Lambda_s$ $(0 \leqslant s \leqslant n)$. For $\eta = \overline{\Lambda} + \sum_{i=1}^n x_i\alpha_i \in 2\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{\mathsf{Q}})$ (see (1.5)), the condition (5.5) tells us that

(1) $\eta(h_1) = 2x_1 - x_2 \geqslant -\delta_{1,s}$,
(i) $\eta(h_i) = -x_{i-1} + 2x_i - x_{i+1} \geqslant -\delta_{i,s}$ $(2 \leqslant i \leqslant n-1)$,
(n) $\eta(h_n) = -2x_{n-1} + 2x_n \geqslant -2\delta_{n,s}$,

and

$$(\eta|\theta) = x_2 + (2 - \delta_{s,1} - 2\delta_{s,0}) \leqslant 2.$$

Then by summing inequalities (2)∼(n−1) and $\frac{1}{2} \times$ (n), we have

$$-x_1 + x_2 \geqslant -\delta(s > 1). \tag{5.9}$$

We also have that

(a) for $s \leqslant i \leqslant n-1$,
$$x_{i+1} \geqslant x_i \text{ and } x_i = x_{i+1} \text{ imply } x_i = x_{i+1} = x_{i+2} = \cdots = x_n;$$
(b) for $1 \leqslant i \leqslant s-1$,
$$-x_i + x_{i+1} \geqslant -\delta(1 \leqslant i < s);$$

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(c) for all $2 \leqslant i \leqslant n$,

$$x_1 + x_i \geqslant x_{i+1} - \delta(i \geqslant s) \times \delta(s \geqslant 1).$$

With the inequality (1), the inequality (5.9) implies that

$$x_1 \geqslant -\delta(s \geqslant 1) \quad \text{and} \quad x_2 \geqslant -2\delta(s \geqslant 1).$$

Proof of Theorem 5.9. (a) Assume that $\Lambda = 2\mathbf{\Lambda}_0$. Then we have the inequalities

$$0 \leqslant x_1 \leqslant x_2 \leqslant 2 \quad \text{and} \quad 2x_1 - x_2 \geqslant 0.$$

Then $(x_1, x_2) = (0,0)$, $(1,1)$, $(1,2)$, or $(2,2)$. Now one can prove that, for $\eta = \sum_{i=1}^{n} x_i\alpha_i \in 2\mathcal{C}_{\mathrm{af}} \cap \overline{\mathrm{Q}}$ such that $\eta \neq 0$, we have

$$\eta = \begin{cases} \sum_{i=1}^{u} i\ \alpha_i + u \sum_{t=u}^{n} \alpha_t & \text{for some } 1 \leqslant u \leqslant n, \text{ or} \\ 2\sum_{i=1}^{n} \alpha_i = \overline{2\mathbf{\Lambda}_1 - 2\delta}. \end{cases}$$

Here $\{\sum_{i=1}^{u} i\ \alpha_i + u\sum_{t=u}^{n} \alpha_t\}$ contributes to (5.6a) and (5.6b).

(b) Assume that $\Lambda = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_1$. Then we have the inequalities

$$x_1 \geqslant -1,\ 1 \geqslant x_2 \geqslant -2,\ 2x_1 - x_2 \geqslant -1,\ -x_1 + 2x_2 - x_3 \geqslant 0 \text{ and } x_n \geqslant \cdots \geqslant x_2 \geqslant x_1.$$

Then $(x_1, x_2) = (0,0)$, $(0,1)$, $(1,1)$, or $(-1,-1)$. Now one can prove that, for $\eta = \overline{\mathbf{\Lambda}_1} + \sum_{i=1}^{n} x_i\alpha_i \in 2\mathcal{C}_{\mathrm{af}} \cap \overline{\mathrm{Q}}$ such that $\eta \neq \overline{\mathbf{\Lambda}_1}$, we have

$$\eta = \begin{cases} \sum_{i=1}^{u}(i-1)\ \alpha_i + (u-1)\sum_{t=u}^{n} \alpha_t & \text{for } 2 \leqslant u \leqslant n, \\ \sum_{i=1}^{n} \alpha_i = \overline{2\mathbf{\Lambda}_1 - \delta} \quad \text{or} \quad -\sum_{i=1}^{n} \alpha_i = \overline{2\mathbf{\Lambda}_0}. \end{cases}$$

Here $\{\sum_{i=1}^{u}(i-1)\ \alpha_i + (u-1)\sum_{t=u}^{n} \alpha_t\}$ contributes to (5.6a) and (5.6b).

(c) Assume that $\Lambda = \Lambda_s$ $(2 \leqslant s \leqslant n-1)$ or $2\mathbf{\Lambda}_n$. Then we have inequalities

$$\begin{aligned} &x_1 \geqslant -1, \quad 0 \geqslant x_2 \geqslant -2, \quad -x_1 + x_2 \geqslant -1, \quad x_n \geqslant x_{n-1} \geqslant \cdots \geqslant x_{s+1} \geqslant x_s, \\ &-x_{i-1} + 2x_i - x_{i+1} \geqslant 0 \quad \text{for } i < s, \quad 2x_1 - x_2 \geqslant 0, \\ &x_1 + x_i \geqslant x_{i+1} \text{ for } i \leqslant s \quad \text{and} \quad x_1 + x_i \geqslant x_{i+1} - 1 \text{ for } i > s. \end{aligned}$$

Then $(x_1, x_2) = (0,0)$, $(1,0)$, $(-1,-2)$, or $(0,-1)$.

(1) Assume $x_1 = 0$. Then, for $2 \leqslant i \leqslant s-1$, we have

$$x_i \geqslant x_{i+1} \geqslant x_i - 1. \tag{5.10}$$

(1-1) If there exists $1 \leqslant u \leqslant s-1$ such that $x_{i+1} = x_i - 1$, take t the smallest one; that is $x_{t+1} = -1$. Since

$$-x_t - 2x_{t+1} - x_{t+2} \geqslant 0,$$

the inequality (5.10) implies $x_{t+2} = -2$. Repeating this process, we obtain $x_{h+1} = x_h - 1$ for $t \leqslant h \leqslant s-1$. Since

$$-x_{s-1} - 2x_s - x_{s+1} \geqslant -1,$$

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we have $x_s = x_{s+1}$ and hence $x_s = x_{s+1} = \cdots = x_n$. Thus η is of the following form

$$\sum_{i=t}^{s}(i-t+1)\ \alpha_i + (s-t+1)\sum_{t=s}^{n}\alpha_t \quad \text{for } 1 \leqslant t \leqslant s-1,$$

which contributes to (5.7a) and (5.7b).

(1-2) Now we assume that $x_1 = x_2 = \cdots = x_s = 0$. Then we have, for $u > s$,

$$x_{u-1} \leqslant x_u \leqslant x_{u-1} + 1.$$

Then, by applying the same method as in (a), we see that η is of the following form:

$$\sum_{i=s+1}^{u}\ (i-s)\ \alpha_i + (u-s)\sum_{t=u}^{n}\alpha_t \quad \text{for } s+1 \leqslant u \leqslant n,$$

which contributes to (5.6a) and (5.6b).

(2) Assume $(x_1, x_2) = (1, 0)$. As in (1-1), we can conclude that

$$\eta = \alpha_1 - \sum_{i=3}^{s}(i-2)\alpha_i - (s-2)\sum_{j=s+1}^{n}\ \alpha_j = \overline{2\mathbf{\Lambda}_1 - \delta}.$$

(3) Assume $(x_1, x_2) = (-1, -2)$. As in (1-1), we can conclude that

$$\eta = -\sum_{i=1}^{s} i\ \alpha_i - s\sum_{j=s+1}^{n}\ \alpha_j = \overline{2\mathbf{\Lambda}_0}.$$

□

DEFINITION 5.10. Let $\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \delta_{s,n}\mathbf{\Lambda}_n + \Lambda_s$ $(0 \leqslant s \leqslant n)$ be of level 2. Suppose that $\eta \in \max^+(\Lambda|2)\backslash\{\Lambda\}$ is of the form

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda(m)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda(u)}\right) \quad \text{or} \quad \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}}^{(n)*\lambda(m-1)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda(u)}\right),$$

where $u \geqslant 0$ if $\Lambda \neq 2\mathbf{\Lambda}_0$ and $u = -1$ if $\Lambda = 2\mathbf{\Lambda}_0$. Then we define the *index* of the maximal weight η to be (m, u). Similarly, if $\eta \in \max^+(\Lambda|2)$ is of the form

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda_\epsilon(n)}\right) + \mathrm{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}}^{\lambda(u)}\right), \quad \epsilon = 0, 1,$$

then define the *index* of the maximal weight η to be (n, u).

REMARK 5.11. Though we have $\lambda(0) = \lambda(-1) = \varnothing$, we use $\lambda(-1)$ when $\Lambda = 2\mathbf{\Lambda}_0$.

Now we consider Λ of level $\geqslant 3$. The following lemma is useful:

LEMMA 5.12. *For any $\Lambda', \Lambda'' \in P^+$ with $\langle c, \Lambda'\rangle = k$ and $\langle c, \Lambda''\rangle = k'$, we have*

$$\Lambda'' + \max{}^+(\Lambda'|k) \subset \max{}^+(\Lambda'' + \Lambda'|k + k').$$

PROOF. Recall that $\theta = \delta - a_0\alpha_0$. For $\eta \in \max^+(\Lambda'|k)$ and $\Lambda'' \in P^+$ of level k', we have

- $(\overline{\eta}|\theta) \leqslant k$ and $(\overline{\eta}|\alpha_i) \geqslant 0$ for $i = 1, \ldots, n$,
- $(\overline{\Lambda''}|\theta) = \sum_{i=1}^{n}\left(a_i \times a_i^{-1}a_i^\vee\langle h_i, \overline{\Lambda''}\rangle\right) \leqslant \langle \Lambda'', c\rangle = k'$,
- $(\overline{\Lambda''}|\alpha_i) \geqslant 0$ for $i = 1, \ldots, n$.

Hence our assertion follows from Proposition 1.4. □

In the following lemma, we obtain maximal weights of level 3 that do not come from those of level 2.

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LEMMA 5.13. *Let* $\Lambda = (1 + \delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$. *Then the following weights are in* $\max^+(\Lambda|3)$*:*

(5.11a)

$$\mathbf{\Lambda}_1 + (1+\delta_{2u+s,n})\Lambda_{2u+s} - (u+1)\delta = \Lambda - \operatorname{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_1}^{(n)*\lambda(2u-1+s)}\right) + (\alpha_1 - \alpha_0) + \operatorname{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(s-1)}\right) \quad \text{for } \delta_{s,0} + \delta_{s,1} \leqslant u \leqslant \lfloor (n-s)/2 \rfloor,$$

(5.11b)

$$\mathbf{\Lambda}_1 + (1+\delta_{2u+1+s,n})\Lambda_{2u+1+s} - (u+1)\delta = \Lambda - \operatorname{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_1}^{\lambda(2u+s)}\right) + (\alpha_1 - \alpha_0) + \operatorname{cont}\left(\mathsf{Y}_{\mathbf{\Lambda}_0}^{\lambda(s-1)}\right) \quad \text{for } \delta_{s,0} \leqslant u \leqslant \lfloor (n-1-s)/2 \rfloor,$$

(5.11c)

$$3\mathbf{\Lambda}_1 - (2+\delta_{0,s})\delta = \Lambda - \begin{cases} \left(3\alpha_0 + 3\sum_{i=2}^{n} \alpha_i\right) & \text{if } s = 0, \\ \left(2\alpha_0 + 2\sum_{i=2}^{n} \alpha_i\right) & \text{if } s = 1, \\ \left(2\alpha_0 + \sum_{i=2}^{s} (i+1)\alpha_i + (s+1)\sum_{j=s+1}^{n} \alpha_j\right) & \\ \qquad\qquad \text{if } 2 \leqslant s \leqslant n-1, \end{cases}$$

(5.11d)

$$\mathbf{\Lambda}_1 + \Lambda_u - \delta = \Lambda - \left(\sum_{i=0}^{u} \alpha_i + \sum_{j=u+1}^{s} (j+1-u)\alpha_j + (s+1-u)\sum_{t=s+1}^{n} \alpha_t\right)$$

(5.11e)

$$(2 \leqslant u \leqslant s-1).$$

PROOF. The equalities can be checked through direct computations. Then, as in the proof of Lemma 5.2, we use Proposition 1.4 to show that the weights are dominant maximal. $\square$

We denote the set of weights in Lemma 5.13 by $\max^+_{iii}(\Lambda|3)$. By Lemma 5.12, we also have

$$\mathbf{\Lambda}_0 + \max^+(\Lambda|2) \subset \max^+(\mathbf{\Lambda}_0 + \Lambda|3) \quad \text{and} \quad \mathbf{\Lambda}_n + \max^+(\Lambda|2) \subset \max^+(\mathbf{\Lambda}_n + \Lambda|3),$$

where Λ is of level 2.

THEOREM 5.14. *We have*

$$\max^+(\mathbf{\Lambda}_0 + \Lambda|3) = (\mathbf{\Lambda}_0 + \max^+(\Lambda|2)) \bigsqcup \max^+_{iii}(\mathbf{\Lambda}_0 + \Lambda|3)$$

for $\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$, *and*

$$\max^+(\mathbf{\Lambda}_n + \Lambda|3) = \mathbf{\Lambda}_n + \max^+(\Lambda|2)$$

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for $\Lambda = (1+\delta_{s,n})\Lambda_s + \delta_{s,1}\mathbf{\Lambda}_0$ $(1 \leqslant s \leqslant n)$*. In particular, the number of elements in* $\max^+(\mathbf{\Lambda}_0 + \Lambda|3)$ *is equal to* $2(n+1)$*, and the number of elements in* $\max^+(\mathbf{\Lambda}_n + \Lambda|3)$ *is equal to* $n+2$*.*

PROOF. One can prove by applying a similar argument to that of the proof of Theorem 5.9. □

PROPOSITION 5.15. *For* $\Lambda := (1+\delta_{s,n})\Lambda_s + \delta_{s,1}\mathbf{\Lambda}_0$ $(1 \leqslant s \leqslant n)$*, the set* $\mathbf{\Lambda}_n + \max^+_{ii}(\Lambda|2)$ *of dominant maximal weights corresponds to the set of dominant weights of* $L((1+\delta_{s,n})\omega_n + \omega_s)$ *over* B_n*.*

PROOF. As in Proposition 5.8, one can show that the set $\mathbf{\Lambda}_n + \max^+_{ii}(\Lambda|2)$ corresponds to

$$\{\omega_n + \omega_k \mid 1 \leqslant k \leqslant s\} \bigsqcup \{(1+\delta_{s,n})\omega_n + \omega_s\},$$

which is a subset of dominant weights of $L((1+\delta_{s,n})\omega_n + \omega_s)$ over B_n. By [**27**, Lemma 2.4], $L((1+\delta_{s,n})\omega_n + \omega_s)$ has $(s+1)$-many dominant weights and hence our assertion follows. □

Define

$$\tilde{\omega}_s = \begin{cases} \omega_s & \text{if } 1 \leqslant s < n-1, \\ \omega_{n-1} + \omega_n & \text{if } s = n-1, \\ 2\omega_n & \text{if } s = n. \end{cases} \tag{5.12}$$

PROPOSITION 5.16. *Let* $\mathfrak{a}$ *be the set of dominant weights in* (5.11c) *and* $\mathfrak{b}$ *those in* (5.6b)*. Then the union of* $\mathfrak{a}$ *and* $\mathbf{\Lambda}_0 + \mathfrak{b}$ *corresponds to the set of dominant weights of* $L(\omega)$ *over* D_n*, where* $\omega := \omega_n + \tilde{\omega}_{n-s}$ *for* $0 \leqslant s \leqslant n-1$*.*

PROOF. Clearly, the sets $\mathfrak{a}$ and $\mathbf{\Lambda}_0 + \mathfrak{b}$ are disjoint. As in Proposition 5.3, one can show that the union of $\mathfrak{a}$ and $\mathbf{\Lambda}_0 + \mathfrak{b}$ corresponds to

$$\begin{cases} \{\tilde{\omega}_{s-i} + \omega_{n-\delta_i} \mid i = 0, 1, \ldots, s\} & \text{if } s \leqslant n-1, \\ \{\tilde{\omega}_{n-i} + \omega_{n-\delta_i} \mid i = 0, 2, 3, \ldots, s\} & \text{if } s = n, \end{cases} \tag{5.13}$$

which is a subset of dominant weights of $L(\omega)$. Here $\tilde{\omega}_0$ is to be understood as 0 and $\delta_i = 1$ if i is an odd integer and $\delta_i = 0$ otherwise. By [**27**, Lemma 2.6], $L(\omega)$ over D_n has $(n - s + \delta_{s\neq 0})$-many dominant weights and hence our assertion follows. □

DEFINITION 5.17. Assume that $\eta \in \max^+(\mathbf{\Lambda} + \Lambda|3)$, and set $\Lambda = \mathbf{\Lambda} + \Lambda$.

(1) If $\eta = \mathbf{\Lambda} + \mu$ with $\mu \in \max^+(\Lambda|2)$ of index (m, u), then we define the *index* of η to be (m, u).

(2) Assume that η is of the form

$$\begin{aligned} \eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}}\right) + (\alpha_1 - \alpha_0) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}'}\right) \\ \text{or} \quad \Lambda - \mathrm{cont}\left(\mathsf{Y}^{(n)*\lambda(m-1)}_{\mathbf{\Lambda}}\right) + (\alpha_1 - \alpha_0) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}'}\right) \end{aligned} \tag{5.14}$$

for some $\mathbf{\Lambda}'$, where $u \geqslant 0$ if $\Lambda \neq 3\mathbf{\Lambda}_0$ and $u = -1$ if $\Lambda = 3\mathbf{\Lambda}_0$. (cf. Remark 5.11) Then we define the *index* of the maximal weight η to be (m, u).

We will explain in Remark 6.18 (2) why the index (m, u) is well-defined. We generalize Definition 5.17 to higher levels.

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DEFINITION 5.18. Assume that $\eta \in (k-1)\mathbf{\Lambda} + \max^+(\mathbf{\Lambda} + \Lambda|3)$ for $k \geqslant 1$, and write $\eta = (k-1)\mathbf{\Lambda} + \mu$ with $\mu \in \max^+(\mathbf{\Lambda} + \Lambda|3)$. If μ is of index (m, u), then we define the *index* of η to be (m, u).

Whenever the index is defined for a maximal weight $\eta \in \max^+(k\mathbf{\Lambda} + \Lambda|k+2)$, $k \geqslant 0$, the weight η will be called a *staircase dominant maximal weight*. The set of staircase dominant maximal weights will be denoted by $\mathrm{smax}^+(k\mathbf{\Lambda} + \Lambda|k+2)$.

We close this section with a conjecture on the number of the dominant maximal weights.

CONJECTURE 5.19. *Assume that $\mathfrak{g} = B_n^{(1)}$, and let $\ell \geqslant 2$.*

(1) *The number of elements in $\max^+((\ell-2)\mathbf{\Lambda}_0 + \Lambda|\ell)$ is equal to*

$$\binom{n + \lfloor \ell/2 \rfloor}{\lfloor \ell/2 \rfloor} + \binom{n + \lfloor (\ell-1)/2 \rfloor}{\lfloor (\ell-1)/2 \rfloor}.$$

(2) *The number of elements in $\max^+((\ell-2)\mathbf{\Lambda}_n + \Lambda|\ell)$ is equal to*

$$\binom{n + \lfloor \ell/2 \rfloor}{\lfloor \ell/2 \rfloor} + \binom{n + \lfloor \ell/2 \rfloor - 1}{\lfloor \ell/2 \rfloor - 1}.$$

REMARK 5.20. The above conjecture is proved by the referee of this paper using computations similar to the proof of Theorem 5.9 in the referee's report. In [**26**], the cardinalities of maximal dominant weights for every Λ of level $k \geqslant 1$ of the affine Kac-Moody algebras are studied in connection with cyclic sieving phenomena.

5.3. Type $C_n^{(1)}$

Unlike other affine types, the set $\max^+(\mathbf{\Lambda}_s|1)$ is not trivial for any fundamental weight $\mathbf{\Lambda}_s$ of type $C_n^{(1)}$, $0 \leqslant s \leqslant n$.

For $0 \leqslant s \leqslant n$, we define

$$\begin{aligned}
\zeta^n_{\ell,s} &= \ell\alpha_0 + 2\ell \sum_{i=1}^{s} \alpha_i + \sum_{j=1}^{2\ell-1} (2\ell - j)\alpha_{s+j} \quad (1 \leqslant \ell \leqslant \lfloor (n-s)/2 \rfloor), \\
\xi^n_{u,s} &= \sum_{i=1}^{2u} i\alpha_{s-2u+i} + 2u \sum_{j=1}^{n-s-1} \alpha_{s+j} + u\alpha_n \quad (1 \leqslant u \leqslant \lfloor s/2 \rfloor).
\end{aligned}$$

Using a similar argument to that of the proof of Theorem 5.9, one can prove the following theorem:

THEOREM 5.21. *For $0 \leqslant s \leqslant n$, we have*

$$\begin{aligned}
&\max^+(\mathbf{\Lambda}_s|1) \\
&\quad = \{\mathbf{\Lambda}_s\} \bigsqcup \left\{ \mathbf{\Lambda}_s - \zeta^n_{\ell,s} \;\middle|\; 1 \leqslant \ell \leqslant \lfloor (n-s)/2 \rfloor \right\} \bigsqcup \left\{ \mathbf{\Lambda}_s - \xi^n_{u,s} \;\middle|\; 1 \leqslant u \leqslant \lfloor s/2 \rfloor \right\}.
\end{aligned}$$

Now we show that every element in $\max^+(\mathbf{\Lambda}_s|1)$ is essentially finite. Since

$$\mathrm{Supp}(\zeta^n_{\ell,s}) = [0, 2\ell - 1 + s] \subsetneq [0, n],$$

we can choose n as an extremal vertex. Then the set

$$\Omega_1 := \{\mathbf{\Lambda}_s\} \bigsqcup \left\{ \mathbf{\Lambda}_s - \zeta^n_{\ell,s} \;\middle|\; 1 \leqslant \ell \leqslant \lfloor (n-s)/2 \rfloor \right\}$$

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can be considered as a subset of dominant maximal weights of $L(\omega_{n-s})$ over C_n via the embedding

$$[0, n-1] \hookrightarrow [1, n] \quad \text{given by } i \mapsto n-i.$$

Hence Ω_1 can be identified with

$$\{\omega_{n-s-2k} \mid 0 \leqslant k \leqslant \lfloor (n-s)/2 \rfloor\} \tag{5.15}$$

which is a subset of dominant weights of $L(\omega_{n-s})$ (Here we set $\omega_0 := 0$). By [**27**, Lemma 2.5], $L(\omega_{n-s})$ has $(\lfloor (n-s)/2 \rfloor + 1)$-many dominant weights and the set in (5.15) coincides with the set of dominant weights of $L(\omega_{n-s})$ indeed.

In a similar way, the set

$$\Omega_2 := \{\mathbf{\Lambda}_s\} \bigsqcup \{\, \mathbf{\Lambda}_s - \xi^n_{u,s} \;\big|\; 1 \leqslant u \leqslant \lfloor s/2 \rfloor \}$$

can be identified with the set of dominant weights

$$\{\omega_{s-2k} \mid 0 \leqslant k \leqslant \lfloor s/2 \rfloor\}$$

of $L(\omega_s)$ over C_n (Here, again, we set $\omega_0 := 0$).

5.4. Type $D_n^{(1)}$

Recall that the affine type $D_n^{(1)}$ has fundamental weights $\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \mathbf{\Lambda}_{n-1}, \mathbf{\Lambda}_n$ of level 1. If $\Lambda = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_{n-1}$, one can check that there are only two maximal weights Λ and $\mathbf{\Lambda}_1 + \mathbf{\Lambda}_n - \delta$, and their multiplicities are 1 and $n-1$, respectively. When $\Lambda = \mathbf{\Lambda}_1 + \mathbf{\Lambda}_{n-1}$ (resp. $\mathbf{\Lambda}_0 + \mathbf{\Lambda}_{n-1}$, $\mathbf{\Lambda}_1 + \mathbf{\Lambda}_n$) , the same is true with $\mathbf{\Lambda}_0$ (resp. $\mathbf{\Lambda}_{n-1}$) replaced by $\mathbf{\Lambda}_1$ (resp. $\mathbf{\Lambda}_n$).

Since $(\mathbf{\Lambda}_0, \mathbf{\Lambda}_1)$ and $(\mathbf{\Lambda}_{n-1}, \mathbf{\Lambda}_n)$ are symmetric, we only consider the case when

$$\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s \quad (0 \leqslant s \leqslant n-2).$$

LEMMA 5.22.

(1) *If s is odd, we have*

$$\Lambda_0 + \Lambda_1, \quad \Lambda_{2u+1} \in \max{}^+(\Lambda|2) \quad \textit{for } 1 \leqslant u \leqslant \tfrac{s-1}{2},$$

and if s is even,

$$2\Lambda_0, \quad 2\Lambda_1 - (1+\delta_{s,0})\delta, \quad \Lambda_{2u} \in \max{}^+(\Lambda|2) \quad \textit{for } 1 \leqslant u \leqslant \tfrac{s}{2}.$$

(2) *For $1 \leqslant u \leqslant \lfloor (n-2-s)/2 \rfloor$, the following weights are in* $\max^+(\Lambda|2)$*:*

$$\Lambda_{s+2u} - u\delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(2u-1+s)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right).$$

(3) *Assume $n-s$ is an even integer. Then the following weights are in* $\max^+(\Lambda|2)$*:*

$$2\mathbf{\Lambda}_n - \frac{n-s}{2}\,\delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda_{n-1}(n-1)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right),$$

$$2\mathbf{\Lambda}_{n-1} - \frac{n-s}{2}\,\delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda_n(n-1)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right),$$

where $\mathsf{Y}^{\lambda_\epsilon(n-1)}_{\mathbf{\Lambda}_0}$ ($\epsilon = n-1, n$) is the Young wall whose top of the first column is the half-thickness block with color ϵ.

(4) *Assume $n-s$ is an odd integer. Then the following weight is in* $\max^+(\Lambda|2)$*:*

$$\mathbf{\Lambda}_{n-1} + \mathbf{\Lambda}_n - \frac{n-1-s}{2}\,\delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(n-2)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right).$$

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Proof. The lemma can be prove using direct computation as in Lemma 5.2, and we omit the details. □

Remark 5.23. We see that all the weights in Lemma 5.22 (2)-(4) are essentially finite of type D_n.

Theorem 5.24. *For* $\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-2)$ *of level* 2, *the set* $\max^+(\Lambda|2)$ *is completely given by the maximal weights in Lemma* 5.22. *In particular, we have*

$$|\max^+(\Lambda|2)| = \begin{cases} \frac{n+3}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2}+3 & \text{if } n \text{ is even and } s \text{ is even,} \\ \frac{n}{2} & \text{otherwise.} \end{cases}$$

Proof. One can prove the theorem by applying a similar strategy as in Theorem 5.9. □

We define the index of a maximal dominant weight in a similar way to Definition 5.10.

Definition 5.25. Assume that $\eta \in \max^+(\Lambda|2)\backslash\{\Lambda\}$ is of the form

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}}\right),$$

where $u \geqslant 0$ if $\Lambda \neq 2\mathbf{\Lambda}_0$ and $u = -1$ if $\Lambda = 2\mathbf{\Lambda}_0$ (see Remark 5.11). Then we define the *index* of the maximal weight η to be (m,u). Similarly, assume that $\eta \in \max^+(\Lambda|2)$ is of the form

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda_\epsilon(n-1)}_{\mathbf{\Lambda}}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}}\right), \quad \epsilon = n-1, n,$$

where $u \geqslant 0$ if $\Lambda \neq 2\mathbf{\Lambda}_0$ and $u = -1$ if $\Lambda = 2\mathbf{\Lambda}_0$. Then define the *index* of the maximal weight η to be $(n-1,u)$.

Now we consider highest weights of level 3.

Lemma 5.26.
(1) *The following weights are in* $\max^+(\mathbf{\Lambda}_0+\Lambda|3)$: *For* $0 \leqslant u \leqslant \lfloor (n-3-s)/2 \rfloor$,

$$\begin{aligned} &\mathbf{\Lambda}_1 + \Lambda_{s+2u+1} - (u+1)\delta \\ &\qquad = \mathbf{\Lambda}_0 + \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(2u+s)}_{\mathbf{\Lambda}_1}\right) + (\alpha_1 - \alpha_0) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right). \end{aligned}$$

(2) *Assume* $n-s$ *is an even integer. Then the following weight is in* $\max^+(\mathbf{\Lambda}_0+\Lambda|3)$:

$$\begin{aligned} &\mathbf{\Lambda}_1 + \mathbf{\Lambda}_{n-1} + \mathbf{\Lambda}_n - \frac{n-s}{2}\delta \\ &\qquad = \mathbf{\Lambda}_0 + \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(n-2)}_{\mathbf{\Lambda}_0}\right) + (\alpha_1 - \alpha_0) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right). \end{aligned}$$

(3) *Assume* $n-s$ *is an odd integer. Then the following weights are in* $\max^+(\mathbf{\Lambda}_0+\Lambda|3)$: $t \in \{n-1, n\}$

$$\begin{aligned} &\mathbf{\Lambda}_1 + 2\mathbf{\Lambda}_t - \frac{n-s+1}{2}\delta \\ &\qquad = \mathbf{\Lambda}_0 + \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda_t(n-1)}_{\mathbf{\Lambda}_0}\right) + \delta_{s\equiv_2 0}(\alpha_1 - \alpha_0) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right), \end{aligned}$$

where we write $s \equiv_2 0$ *for* $s \equiv 0$ *(mod 2).*

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REMARK 5.27. We see that all the weights in Lemma 5.26 are essentially finite of type D_n.

The following definition is an analogue of Definition 5.17.

DEFINITION 5.28. Assume that $\eta \in \max^+(\mathbf{\Lambda}_0 + \Lambda|3)$, and set $\wedge = \mathbf{\Lambda}_0 + \Lambda$.

(1) If $\eta = \mathbf{\Lambda}_0 + \mu$ with $\mu \in \max^+(\Lambda|2)$ of index (m, u), then we define the *index* of η to be (m, u).

(2) Assume that η is of the form

$$\eta = \wedge - \operatorname{cont}\left(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}_0}\right) + (\alpha_1 - \alpha_0) + \operatorname{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}_0}\right),$$

where $u \geqslant 0$ if $\Lambda \neq 3\mathbf{\Lambda}_0$ and $u = -1$ if $\Lambda = 3\mathbf{\Lambda}_0$. Then define the *index* of the maximal weight η to be (m, u).

(3) Assume that η is of the form

$$\eta = \wedge - \operatorname{cont}\left(\mathsf{Y}^{\lambda_\epsilon(n-1)}_{\mathbf{\Lambda}_0}\right) + \delta_{s\equiv_2 0}(\alpha_1 - \alpha_0) + \operatorname{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}_0}\right), \quad \epsilon = n-1, n,$$

where $u \geqslant 0$ if $\Lambda \neq 3\mathbf{\Lambda}_0$ and $u = -1$ if $\Lambda = 3\mathbf{\Lambda}_0$. We define the *index* of the maximal weight η to be $(n-1, u)$.

Similarly, we consider higher levels to make the following definition.

DEFINITION 5.29. Assume that $\eta \in (k-1)\mathbf{\Lambda} + \max^+(\mathbf{\Lambda} + \Lambda|3)$ for $k \geqslant 1$, and write $\eta = (k-1)\mathbf{\Lambda} + \mu$ with $\mu \in \max^+(\mathbf{\Lambda} + \Lambda|3)$. If μ is of index (m, u), then we define the *index* of η to be (m, u).

Whenever the index is defined for a maximal weight $\eta \in \max^+(k\mathbf{\Lambda} + \Lambda|k+2)$, $k \geqslant 0$, the weight η will be called a *staircase dominant maximal weight*. The set of staircase dominant maximal weights will be denoted by $\operatorname{smax}^+(k\mathbf{\Lambda} + \Lambda|k+2)$.

5.5. Type $A^{(2)}_{2n-1}$

Recall that the affine type $A^{(2)}_{2n-1}$ has the fundamental weights $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_1$ of level 1. Let us take a level 2 dominant integral weight Λ of the form

$$\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s \quad (0 \leqslant s \leqslant n).$$

LEMMA 5.30.

(1) *For* $0 \leqslant u \leqslant \lfloor (n-s)/2 \rfloor$, *we have*

$$\begin{aligned}(5.16)\qquad &(\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_{s+2u} - u\delta \\ &\quad = \Lambda - \operatorname{cont}\left(\mathsf{Y}^{\lambda(2u-1+s)}_{\mathbf{\Lambda}_0}\right) + \operatorname{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}\right) \in \max^+(\Lambda|2).\end{aligned}$$

(2) *For* $1 \leqslant u \leqslant \left\lfloor \dfrac{s}{2} \right\rfloor$, *we have*

(5.17)
$$\begin{aligned}&(1 + \delta_{s-2u,0})\Lambda_{s-2u} + \delta_{s-2u,1}\mathbf{\Lambda}_1 \\ &\quad = \Lambda_s - \left(\sum_{i=s-2u+1}^{\max(s,n-1)} (i - s + 2u)\alpha_i + 2u \sum_{j=s+1}^{n-1} \alpha_j + u\alpha_n \right) \in \max^+(\Lambda|2).\end{aligned}$$

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(3) *If $s \geqslant 2$ is even, then we have*

$$2\boldsymbol{\Lambda}_1 - \delta = \Lambda_s - \left(\sum_{i=2}^{\max(s,n-1)} i\ \alpha_i + \alpha_0 + s\sum_{j=s+1}^{n-1}\alpha_j + \frac{s}{2}\alpha_n\right) \in \max{}^+(\Lambda|2). \tag{5.18}$$

(4) *When $s = 0$, we have*

$$2\boldsymbol{\Lambda}_1 - 2\delta = \Lambda_s - \left(2\sum_{i=2}^{n-1}\alpha_i + 2\alpha_0 + \alpha_n\right) \in \max{}^+(\Lambda|2). \tag{5.19}$$

REMARK 5.31. We see that the weights in (5.16) are essentially finite of type D_n, and that those in (5.17), (5.18) and (5.19) are essentially finite of type C_n.

THEOREM 5.32. *For $\Lambda = (\delta_{s,0}+\delta_{s,1})\boldsymbol{\Lambda}_0+\Lambda_s$ $(0 \leqslant s \leqslant n)$ of level 2, the maximal weights in Lemma 5.30 exhaust the whole set $\max^+(\Lambda|2)$. Hence the number of elements in $\max^+(\Lambda|2)$ is $\lfloor n/2\rfloor + 2$ if s is even, and $\lfloor (n-1)/2\rfloor + 1$ if s is odd.*

PROOF. One can prove the theorem by applying a similar argument as in Theorem 5.9. □

Now we consider highest weights of level 3. Recall $\Lambda := (\delta_{s,0} + \delta_{s,1})\boldsymbol{\Lambda}_0 + \Lambda_s$ for $0 \leqslant s \leqslant n$.

LEMMA 5.33. *The following weights are in $\max^+(\boldsymbol{\Lambda}_0 + \Lambda|3)$: For $0 \leqslant u \leqslant \lfloor (n-s)/2\rfloor$,*

$$\begin{aligned} &\boldsymbol{\Lambda}_1 + \Lambda_{s+2u+1} - (u+1)\delta \\ &\qquad = \boldsymbol{\Lambda}_0 + \Lambda - \operatorname{cont}\left(\mathsf{Y}^{\lambda(2u+s)}_{\boldsymbol{\Lambda}_1}\right) + (\alpha_1 - \alpha_0) + \operatorname{cont}\left(\mathsf{Y}^{\lambda(s-1)}_{\boldsymbol{\Lambda}_0}\right). \end{aligned} \tag{5.20}$$

We define the index of the weights in (5.16) and (5.20) as we did in Definition 5.10 and 5.17, respectively, and we extend it to higher levels as in Definition 5.18. Similarly, whenever the index is defined for a maximal weight $\eta \in \max^+(k\boldsymbol{\Lambda}+\Lambda|k+2)$, $k \geqslant 0$, the weight η will be called a *staircase dominant maximal weight*. The set of staircase dominant maximal weights will be denoted by $\operatorname{smax}^+(k\boldsymbol{\Lambda} + \Lambda|k+2)$.

5.6. Type $A^{(2)}_{2n}$

Recall that the affine type $A^{(2)}_{2n}$ has the only fundamental weight $\boldsymbol{\Lambda}_0$ of level 1. Let us take level 2 dominant integral weights Λ as follows:

$$\Lambda = \delta_{s,0}\boldsymbol{\Lambda}_0 + \Lambda_s \quad (0 \leqslant s \leqslant n).$$

LEMMA 5.34.

(1) *For $0 \leqslant u \leqslant \lfloor (n-s)/2\rfloor$, we have*

$$(1+\delta_{s+2u,0})\Lambda_{s+2u} - 2u\delta = \Lambda - \operatorname{cont}\left(\mathsf{Y}^{\lambda(2u-1+s)}_{\boldsymbol{\Lambda}_0}\right) + \operatorname{cont}\left(\mathsf{Y}^{\lambda(s)}_{\boldsymbol{\Lambda}_0}\right) \in \max{}^+(\Lambda|2). \tag{5.21}$$

(2) *For $1 \leqslant u \leqslant \left\lfloor \dfrac{s}{2}\right\rfloor$, we have*

$$\begin{aligned} &(1+\delta_{s-2u,0})\Lambda_{s-2u} \\ &\qquad = \Lambda_s - \left(\sum_{i=s-2u+1}^{\max(s,n-1)} (i-s+2u)\alpha_i + 2u\sum_{j=s+1}^{n-1}\alpha_j + u\alpha_n\right) \in \max{}^+(\Lambda|2). \end{aligned} \tag{5.22}$$

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REMARK 5.35. We see that the weights in (5.21) are essentially finite of type B_n, and that those in (5.22) are essentially finite of type C_n.

THEOREM 5.36. *For* $\Lambda = \delta_{s,0}\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n)$ *of level* 2*, the maximal weights in* Lemma 5.34 *exhaust the whole set* $\max^+(\Lambda|2)$*. Hence the number of elements in* $\max^+(\Lambda|2)$ *is* $(n+1)/2$ *if* n *is odd and* $n/2 + \delta_{s\equiv_2 0}$ *if* n *is even.*

PROOF. A similar argument as in Theorem 5.9 can be used. □

DEFINITION 5.37. Assume that $\eta \in \max^+(\Lambda|2)$ is of the form

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}_0}\right),$$

where $s \geqslant 0$. Then we define the *index* of the maximal weight η to be (m,u).

In the case of type $A_{2n}^{(2)}$, one can easily check the following:

- every level 3 dominant integral weight Λ is of the form $\mathbf{\Lambda}_0 + \Lambda$,
- $|\max^+(\Lambda|2)| = |\max^+(\mathbf{\Lambda}_0 + \Lambda|3)|$,
- $\max^+(\mathbf{\Lambda}_0 + \Lambda|3) = \{\mathbf{\Lambda}_0 + \eta \mid \eta$ is of the form (5.21) or (5.22) $\}$.

We extend the above definition to higher levels as before. Whenever the index is defined for a maximal weight $\eta \in \max^+(k\mathbf{\Lambda}_0 + \Lambda|k+2)$, $k \geqslant 0$, the weight η will be called a *staircase dominant maximal weight.* The set of staircase dominant maximal weights will be denoted by $\mathrm{smax}^+(k\mathbf{\Lambda}_0 + \Lambda|k+2)$.

5.7. Type $D_{n+1}^{(2)}$

Recall that the affine type $D_{n+1}^{(2)}$ has the fundamental weights $\mathbf{\Lambda}_0, \mathbf{\Lambda}_n$ of level 1. If $\Lambda = \mathbf{\Lambda}_0 + \mathbf{\Lambda}_n$, one can check that there are only one maximal weights Λ itself and hence its multiplicity are 1.

Let us consider level 2 dominant integral weights Λ:

$$\Lambda = (\delta_{s,0} + \delta_{s,n})\mathbf{\Lambda}_0 + \Lambda_s \quad (0 \leqslant s \leqslant n-1). \tag{5.23}$$

LEMMA 5.38. *The following weights are in* $\max^+(\Lambda|2)$*:*

(5.24)
$$(1 + \delta_{s+u,n})\Lambda_{s+u} - u\delta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(u-1+s)}_{\mathbf{\Lambda}_0}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s)}_{\mathbf{\Lambda}_0}\right) \ (0 \leqslant u \leqslant n-s),$$

$$(1 + \delta_{u,0})\Lambda_u = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(n-u)}_{\mathbf{\Lambda}_n}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(n-s)}_{\mathbf{\Lambda}_n}\right) \ (1 \leqslant u \leqslant s). \tag{5.25}$$

REMARK 5.39. We see that the weights in (5.24) and (5.25) are essentially finite of type B_n.

THEOREM 5.40. *For* $\Lambda = (\delta_{s,0} + \delta_{s,n})\mathbf{\Lambda}_0 + \Lambda_s$ $(0 \leqslant s \leqslant n-1)$ *of level* 2*, the maximal weights in Lemma* 5.38 *exhaust the whole set* $\max^+(\Lambda|2)$*. The number of elements in* $\max^+(\Lambda|2)$ *is* $n+1$*.*

DEFINITION 5.41. Assume that $\eta \in \max^+(\Lambda|2)$ is of the form

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(u)}_{\mathbf{\Lambda}}\right), \quad \mathbf{\Lambda} = \mathbf{\Lambda}_0, \mathbf{\Lambda}_n,$$

where $u \geqslant 0$. Then we define the *index* of the maximal weight η to be (m,u).

As in the case of type $A_{2n}^{(2)}$, one can easily check the following for $D_{n+1}^{(2)}$:

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- every level 3 dominant integral weight Λ is of the form $\mathbf{\Lambda}_0 + \Lambda$ or $\mathbf{\Lambda}_n + \Lambda$ for a Λ of the form (5.23),
- $|\max^+(\Lambda|2)| = |\max^+(\mathbf{\Lambda}_\epsilon + \Lambda|3)|$ $(\epsilon = 0, n)$,
- $\max^+(\mathbf{\Lambda}_\epsilon + \Lambda|3) = \{\mathbf{\Lambda}_\epsilon + \eta \mid \eta$ is of the form (5.24) or (5.25) $\}$.

We extend the above definition to higher levels as before. The set of staircase dominant maximal weights is defined in a similar way as in the previous sections.

5.8. Classification of staircase dominant maximal weights

As we have observed in the previous sections, the staircase maximal weights in $\mathrm{smax}^+(\Lambda)$ are essentially finite of type B_n or D_n. Hence we classify the staircase dominant maximal weights into two classes according to their finite types, and make the following definition.

DEFINITION 5.42. Define $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ (resp. $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$) to be the set of *staircase dominant maximal weights* of Λ of level $k \geqslant 2$ that are essentially finite of type B_n (resp. D_n).

Note that the staircase dominant maximal weights of Λ do *not* exhaust the set $\max^+(\Lambda|k)$ in general.

REMARK 5.43 (Indices for $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ and $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$).

(1) For $k \geqslant 2$, the indices for $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ are given as follows (see Lemma 5.7):

$$\{(m, s) \mid n \geqslant m \geqslant s \geqslant 0\}.$$

(2) For $k \geqslant 2$, the indices for $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ are given as follows (see Lemma 5.2, 5.13 and (5.13)):

$$\begin{cases} \{(m, s-1) \mid s \geqslant 0,\ n \geqslant m \geqslant s-1 \text{ and } m \not\equiv_2 s\} \backslash \{(0,-1)\} & \text{if } k = 2, \\ \{(m, s-1) \mid s \geqslant 0 \text{ and } n \geqslant m \geqslant s-1\} \backslash \{(0,-1)\} & \text{if } k \geqslant 3. \end{cases} \tag{5.26}$$

The following table shows which affine types are related to each type of staircase dominant maximal weights.

Staircase Type	$\mathrm{smax}^+_{\mathfrak{B}}(\Lambda\|k)$	$\mathrm{smax}^+_{\mathfrak{D}}(\Lambda\|k)$
Affine Types	$B_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$	$B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

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CHAPTER 6

Weight multiplicities and (spin) rigid Young tableaux

In this chapter, we will introduce the notion of *(spin) rigid Young tableaux*, and show that the set of these tableaux is equinumerous to the set of crystal basis elements in $\mathbf{B}(\Lambda)_\eta$ for a dominant integral weight $\Lambda = (k-2)\boldsymbol{\Lambda} + \Lambda$ of level k and its staircase dominant maximal weights $\eta \in \mathrm{smax}^+(\Lambda|k)$, $k \geqslant 2$. As noted in (5.1), it suffices to consider their finite types. Hence, in this chapter we only consider affine type $B_n^{(1)}$ and the sets $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ and $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$.

Considering the crystal rules for Young walls, one can prove the following lemma.

LEMMA 6.1. *For strict partitions $\lambda^{(1)}, \dots, \lambda^{(k)}$ with $\max\{\lambda_1^{(1)}, \dots, \lambda_1^{(k)}\} \leqslant n$, if the Young wall $\mathbb{Y}^{(\lambda^{(1)},\dots,\lambda^{(k)})}_{(\boldsymbol{\Lambda}^{(1)},\dots,\boldsymbol{\Lambda}^{(k)})}$ corresponds to a highest weight crystal vector then $\lambda^{(1)} = \varnothing$ and $\lambda^{(2)} = \lambda(s)$ for some $s \in \mathbb{Z}_{\geqslant 0}$.*

PROOF. The tensor product rule of crystals in Definition 2.4 implies that, for crystals $\mathbf{B}(\Lambda)$ and $\mathbf{B}$, every highest weight crystal in $\mathbf{B}(\Lambda) \otimes \mathbf{B}$ is $u_\Lambda \otimes b$ for some $b \in \mathbf{B}$ where u_Λ is the highest weight crystal of $\mathbf{B}(\Lambda)$. Thus $\lambda^{(1)} = \varnothing$. By the assumption that $\lambda_1^{(i)} \leqslant n$, $\lambda^{(2)}$ must be $\lambda(s)$ for some $s \in \mathbb{Z}_{\geqslant 0}$ from the ground-state pattern of the Young wall and the tensor product rule. □

DEFINITION 6.2. For strict partitions $\lambda^{(1)}$ and $\lambda^{(2)}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ of the same type, we define $s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda^{(1)}, \lambda^{(2)})$ to be the smallest nonnegative integer s satisfying

$$(\mathsf{Y}^{\lambda^{(1)}}_{\boldsymbol{\Lambda}})_{\geqslant 1} \supset (\mathsf{Y}^{\lambda^{(2)}}_{\boldsymbol{\Lambda}'})_{\geqslant s+1}, \tag{6.1}$$

where the containment in (6.1) is defined in Definition 2.16.

The following lemma implies that the quantity $s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda^{(1)}, \lambda^{(2)})$ is invariant under application of $\tilde{e}_i$'s.

PROPOSITION 6.3. *For strict partitions $\lambda^{(1)}, \lambda^{(2)}$ with $\max\{\lambda_1^{(1)}, \lambda_1^{(2)}\} \leqslant n$ and $i \in I$, suppose that*

$$\tilde{e}_i(\mathbb{Y}^{(\lambda^{(1)},\lambda^{(2)})}_{(\boldsymbol{\Lambda},\boldsymbol{\Lambda}')}) = \mathbb{Y}^{(\lambda',\lambda'')}_{(\boldsymbol{\Lambda},\boldsymbol{\Lambda}')}.$$

Then $s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda^{(1)}, \lambda^{(2)}) = s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda', \lambda'') \leqslant n$.

PROOF. Obviously, $s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda^{(1)}, \lambda^{(2)}), s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda', \lambda'') \leqslant n$. Let $s = s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda^{(1)}, \lambda^{(2)})$ and $s' = s_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(\lambda', \lambda'')$. Let $\epsilon = 0$ if $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are of type $\mathfrak{B}$, and $\epsilon = 1$ if they are of type $\mathfrak{D}$. The assumption implies that we have either

(1) $\lambda' = \lambda^{(1)}$ and $\|\lambda^{(2)}/\lambda''\| = 1$ or
(2) $\lambda'' = \lambda^{(2)}$ and $\|\lambda^{(1)}/\lambda'\| = 1$.

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Since the second case can be proved similarly, we will only consider the first case. Since $\lambda' = \lambda^{(1)} \supset \lambda^{(2)}_{\geqslant s+1} \supset \lambda''_{\geqslant s+1}$, if $s \leqslant \epsilon$, then it is the smallest possible and we have $s' = s$. Now assume that $s \geqslant 1+\epsilon$. Let j be the unique integer such that $\lambda^{(2)}_j = \lambda''_j + 1$. In order to show $s = s'$, it suffices to show $\lambda' \not\supset \lambda''_{\geqslant s-\epsilon}$. For a contradiction, suppose that $\lambda' \supset \lambda''_{\geqslant s-\epsilon}$. Then we have $\lambda^{(1)} = \lambda' \supset \lambda''_{\geqslant s-\epsilon}$ and $\lambda^{(1)} \not\supset \lambda^{(2)}_{\geqslant s-\epsilon}$. Since $\lambda^{(2)}$ and λ'' differ by only one part, we obtain that $\lambda^{(1)}$ must have a part equal to $t-1$, where $t := \lambda^{(2)}_j = \lambda''_j + 1$. Moreover, by considering the Young diagrams of $\lambda^{(1)}$, $\lambda^{(2)}_{\geqslant s-\epsilon}$, and $\lambda''_{\geqslant s-\epsilon}$, one can see that the position of the part $t-1$ in $\lambda^{(1)}$ is equal to the position of the part t in $\lambda^{(2)}_{\geqslant s-\epsilon}$. Therefore, we have $j \geqslant s-\epsilon$ and

$$\lambda^{(1)}_{j-s+\epsilon+1} = (\lambda^{(2)}_{\geqslant s-\epsilon})_{j-s+\epsilon+1} - 1 = \lambda^{(2)}_j - 1 = t-1.$$

If $j = s-\epsilon$, then $\lambda^{(1)}_1 = t-1$ and $\mathrm{sig}_i(\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}}) = (+)$. If $j \geqslant s-\epsilon+1$, then by the assumption $\lambda^{(1)} = \lambda' \supset \lambda''_{\geqslant s-\epsilon}$, we have

$$\lambda^{(1)}_{j-s+\epsilon} \geqslant (\lambda''_{\geqslant s-\epsilon})_{j-s+\epsilon} = \lambda''_{j-1} = \lambda^{(2)}_{j-1} > \lambda^{(2)}_j = t.$$

Thus we also have $\mathrm{sig}_i(\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}}) = (+)$. This means that

$$\tilde{e}_i(\mathbb{Y}^{(\lambda^{(1)},\lambda^{(2)})}_{(\mathbf{\Lambda},\mathbf{\Lambda}')}) = \tilde{e}_i(\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}}) \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}'} = 0,$$

which is a contradiction. Therefore, we must have $\lambda' \not\supset \lambda''_{\geqslant s-\epsilon}$, which implies $s = s'$. □

6.1. Case $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$

In this section, we assume that η is an element of $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|k)$ and that $\boxed{\mathbf{\Lambda}}$ is of type $\mathfrak{B}$.

Let $k \in \mathbb{Z}_{\geqslant 1}$ and $s \in \mathbb{Z}_{\geqslant 0}$. A skew Young tableau T of shape $\mu/(s^{k-1})$ with m cells for a partition μ of length k is naturally identified with a sequence of strict partitions

$$(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k-1)}, \lambda^{(k)})$$

such that $\lambda^{(k)} \neq \varnothing$, $\lambda^{(1)} * \lambda^{(2)} * \cdots * \lambda^{(k-1)} * \lambda^{(k)} = \lambda(m)$, $\lambda^{(i)} \supset \lambda^{(i+1)}$ for $1 \leqslant i \leqslant k-2$ and $\lambda^{(k-1)} \supset \lambda^{(k)}_{\geqslant s+1}$. For example, take $k = 3$ and $s = 1$ and we identify the following skew Young tableau with the corresponding sequence of partitions

$$\begin{array}{|c|c|c|c|}\hline \cdot & 7 & 5 & 4 \\ \hline \cdot & 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 6 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \longleftrightarrow ((7,5,4),(3,1),(6,2)).$$

From now on, we will freely use this identification of skew tableaux and sequences of strict partitions.

Definition 6.4. For $k \in \mathbb{Z}_{\geqslant 1}$ and $s, m \in \mathbb{Z}_{\geqslant 0}$, let $T = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k-1)}, \lambda^{(k)})$ be a skew Young tableau of shape $\mu/(s^{k-1})$ with m cells for a partition μ of length at most k. Then T is called a *rigid Young tableau of index* (m, s) *with* k *rows* if $s = 0$, or $s \geqslant 1$ and

$$\lambda^{(k-1)} \not\supset \lambda^{(k)}_{\geqslant s}. \tag{6.2}$$

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We denote by ${}_s\mathfrak{B}^{(k)}_m$ the set of all rigid Young tableaux of index (m,s) with k rows. In particular, we have ${}_0\mathfrak{B}^{(k)}_m = \mathfrak{B}^{(k)}_m$.

Note that if $T = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k-1)}, \lambda^{(k)})$ is a rigid tableau of index (m,s), then $\ell(\lambda^{(k)}) \geqslant s$. The condition (6.2) says that a shift of the last row to the right by 1 makes the tableau violate the column-strictness.

EXAMPLE 6.5.

(1) $T = ((432),(51)) \in {}_1\mathfrak{B}^{(2)}_5$ since

$\begin{array}{|c|c|c|c|}\hline \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$ is a skew Young tableau but $\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$ is not a Young tableau.

On the other hand, $((532),(41)) \notin {}_1\mathfrak{B}^{(2)}_5$ since

$\begin{array}{|c|c|c|c|}\hline \cdot & 5 & 3 & 2 \\ \hline 4 & 1 \\ \cline{1-2}\end{array}$ is a skew Young tableau and $\begin{array}{|c|c|c|}\hline 5 & 3 & 2 \\ \hline 4 & 1 \\ \cline{1-2}\end{array}$ is also a Young tableau.

(2) $\begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-6} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5}\end{array} \in {}_3\mathfrak{B}^{(3)}_{12}$ since $\begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-5} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5}\end{array}$ is not a skew Young tableau.

(3) We also have $T = (\varnothing, \varnothing, (2,1)) \longleftrightarrow \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline 2 & 1 \\ \hline\end{array} \in {}_2\mathfrak{B}^{(3)}_2$.

PROPOSITION 6.6. *For strict partitions $\lambda^{(1)}$ and $\lambda^{(2)}$ with $\max\{\lambda^{(1)}_1, \lambda^{(2)}_1\} \leqslant n$, the Young wall $\mathbb{Y}^{(\lambda^{(1)},\lambda^{(2)})}_{(\mathbf{\Lambda},\mathbf{\Lambda})}$ is connected to $\boxed{\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda(s)}_{\mathbf{\Lambda}}$ for a unique integer s and we have $s = s_{\mathbf{\Lambda},\mathbf{\Lambda}}(\lambda^{(1)},\lambda^{(2)})$.*

PROOF. If we apply $\tilde{e}_i$'s to $\mathbb{Y}^{(\lambda^{(1)},\lambda^{(2)})}_{(\mathbf{\Lambda},\mathbf{\Lambda})}$ until no longer possible, we obtain a Young wall corresponding to a highest weight vector. By Lemma 6.1, the resulting Young wall is of the form $\boxed{\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda(r)}_{\mathbf{\Lambda}}$ for some $r \geqslant 0$. By Proposition 6.3, we have

$$s = s_{\mathbf{\Lambda},\mathbf{\Lambda}}(\lambda^{(1)},\lambda^{(2)}) = s_{\mathbf{\Lambda},\mathbf{\Lambda}}(\varnothing, \lambda(r)) = r.$$

Therefore $r = s$ and such an integer is unique. $\square$

As in Introduction, define

$$\tilde{\omega}_s := \begin{cases} 2\omega_n & \text{if } s = n, \\ \omega_s & \text{otherwise.} \end{cases} \tag{6.3}$$

Let $L(\omega)$ be the highest weight module with highest weight ω over the finite dimensional Lie algebra of type B_n.

We have the following result:

PROPOSITION 6.7. *For $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2)$ of index (m,s), we have*

$$\dim(V(\Lambda)_\eta) = |{}_s\mathfrak{B}^{(2)}_m| = \dim\left(L(\tilde{\omega}_{n-s})_{\tilde{\omega}_{n-m}}\right).$$

PROOF. Recall from Definition 5.10 that

$$\eta = \Lambda - \mathrm{cont}\left(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s)}_{\mathbf{\Lambda}}\right) \quad \text{or} \quad \Lambda - \mathrm{cont}\left(\mathsf{Y}^{(n)*\lambda(m-1)}_{\mathbf{\Lambda}}\right) + \mathrm{cont}\left(\mathsf{Y}^{\lambda(s)}_{\mathbf{\Lambda}}\right). \tag{6.4}$$

Note that

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(i) $\mathrm{cont}(\mathbb{Y}^{T}_{(\mathbf{\Lambda},\mathbf{\Lambda})}) = \mathrm{cont}(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}})$ for any $T \in {}_s\mathfrak{B}^{(2)}_m$,

(ii) $\mathrm{cont}(\mathbb{Y}^{T'}_{(\mathbf{\Lambda},\mathbf{\Lambda})}) = \mathrm{cont}(\mathsf{Y}^{(n)*\lambda(m-1)}_{\mathbf{\Lambda}})$ for the tableau T' obtained by replacing m by n in the filling of $T \in {}_s\mathfrak{B}^{(2)}_m$ (see Remark 5.4).

By Proposition 6.6, the set $\{\mathbb{Y}^{T}_{(\mathbf{\Lambda},\mathbf{\Lambda})} \mid T \in {}_s\mathfrak{B}^{(2)}_m\}$ or $\{\mathbb{Y}^{T'}_{(\mathbf{\Lambda},\mathbf{\Lambda})} \mid T \in {}_s\mathfrak{B}^{(2)}_m\}$ (depending on η in (6.4)) forms the crystal basis for $V(\Lambda)_\eta$, which implies our assertion. The last equality follows from Proposition 5.8 and Theorem 2.7. $\square$

Now, we obtain the main theorem of this section:

THEOREM 6.8. *Assume that $k \geqslant 2$ and $0 \leqslant s \leqslant m$. Then, for $\eta \in \mathrm{smax}^{+}_{\mathfrak{B}}(\Lambda|k)$ of index (m, s), we have*

$$\dim V(\Lambda)_\eta = |{}_s\mathfrak{B}^{(k)}_m| = \dim L((k-2)\omega_n + \tilde{\omega}_{n-s})_{(k-2)\omega_n+\tilde{\omega}_{n-m}}. \tag{6.5}$$

PROOF. Since the case $k = 2$ is proved in Proposition 6.7, we may assume $k \geqslant 3$. Since $s \leqslant m \leqslant n$, a Young wall $\mathbb{Y} \in \mathbf{B}(\Lambda)_\eta$ connected to $\boxed{\Lambda} := \boxed{(k-1)\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda(s)}_{\mathbf{\Lambda}}$ cannot contain a removable δ. Thus, for each $\mathbb{Y} \in \mathbf{B}(\Lambda)_\eta$ connected to $\boxed{\Lambda}$, there exists a sequence of strict partitions $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)})$ satisfying $\lambda^{(1)} * \lambda^{(2)} * \cdots * \lambda^{(k-1)} * \lambda^{(k)} = \lambda(m)$ and $\mathbb{Y} = \mathbb{Y}^{\underline{\lambda}}_{\underline{\mathbf{\Lambda}}}$.

Let t be the smallest integer such that $t < k$ and $\lambda^{(t)} \not\supset \lambda^{(t+1)}$. If there is no such integer, we let $t = k$. If $t < k$, we also define u to be the smallest nonnegative integer satisfying

$$\lambda^{(t)} \supset \lambda^{(t+1)}_{\geqslant u+1}.$$

If $t < k-1$, the argument in Proposition 6.6 implies that $\mathbb{Y}$ cannot be connected to $\boxed{\Lambda}$. More precisely, if $t < k-1$, then $\mathbb{Y}$ is connected to $\boxed{t\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda(r)}_{\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda^{(t+2)}}_{\mathbf{\Lambda}} \otimes \cdots \otimes \mathsf{Y}^{\lambda^{(k)}}_{\mathbf{\Lambda}}$ for some $r > 0$, which implies that $\tilde{e}_i$ is not applicable to the $(t+1)$st component $\mathsf{Y}^{\lambda(r)}_{\mathbf{\Lambda}}$ for any $i \in I$ and hence $\mathbb{Y}$ cannot be connected to $\boxed{\Lambda}$.

Thus, if $t < k$,

$$\mathbb{Y} \text{ is connected to } \boxed{(k-1)\mathbf{\Lambda}} \otimes \mathsf{Y}^{\lambda(s)}_{\mathbf{\Lambda}} \iff t = k-1 \text{ and } u = s \iff \underline{\lambda} \in {}_s\mathfrak{B}^{(k)}_m,$$

and if $t = k$,

$$\mathbb{Y} \text{ is connected to } \boxed{k\mathbf{\Lambda}} \iff t = k \iff \underline{\lambda} \in {}_s\mathfrak{B}^{(k)}_m \ (s = 0).$$

The last equality in (6.5) follows from Proposition 5.8 and Theorem 2.7. $\square$

As a special case, when $s = 0$, the numbers $|\mathfrak{B}^{(k)}_m|$ for $m \leqslant n$ are the multiplicities of maximal weights of $V(k\mathbf{\Lambda})$. Explicit formulas for the numbers $|\mathfrak{B}^{(k)}_m|$ are given in Theorem 3.5 for $1 \leqslant k \leqslant 5$. We will obtain a closed formula for $|\mathfrak{B}^{(6)}_m|$ in Corollary 10.10. In [**39**], Tsuchioka and Watanabe studied the case $\Lambda = k\mathbf{\Lambda}_0$ for types $A^{(2)}_{2n}$ and $D^{(2)}_{n+1}$.

6.2. Case $\mathrm{smax}^{+}_{\mathfrak{D}}(\Lambda|k)$

In this section, we will deal with η in $\mathrm{smax}^{+}_{\mathfrak{D}}(\Lambda|k)$. Throughout this section, we assume that $\mathbf{\Lambda}$ is of type $\mathfrak{D}$.

PROPOSITION 6.9.

(1) *Let $\lambda^{(1)}, \lambda^{(2)}$ be strict partitions satisfying the following conditions:*

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(i) $\max\{\lambda^{(1)}_1, \lambda^{(2)}_1\} \leqslant n$,

(ii) $\lambda^{(1)} \supset \lambda^{(2)}_{\geqslant 2s}$, *and* (ii-1) $s = 1$ *or* (ii-1′) $\lambda^{(1)} \not\supset \lambda^{(2)}_{\geqslant 2s-2}$ *for some* $s \geqslant 2$.

Then the Young wall $\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}_1}$ *is connected to* $\boxed{\Lambda_{2s-1}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(2s-2)}_{\mathbf{\Lambda}_1}$.

(2) *Let* $\lambda^{(1)}, \lambda^{(2)}$ *be strict partitions satisfying the following conditions:*

(i) $\max\{\lambda^{(1)}_1, \lambda^{(2)}_1\} \leqslant n$,

(ii) $\lambda^{(1)} \supset \lambda^{(2)}_{\geqslant 2s+1}$ *and* $\lambda^{(1)} \not\supset \lambda^{(2)}_{\geqslant 2s-1}$ *for some* $s \geqslant 1$.

Then the Young wall $\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}_0}$ *is connected to* $\boxed{\Lambda_{2s}} := \boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(2s-1)}_{\mathbf{\Lambda}_0}$.

PROOF. By Remark 2.11, the patterns appearing in $\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}_0}$ and $(\mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}_1})_{\geqslant 2s}$ coincide with each other. By applying $\tilde{e}_i$'s until no longer possible, we obtain a Young wall corresponding to its highest weight vector. By Proposition 6.3, its highest weight vector is of the form $\boxed{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(2t)}_{\mathbf{\Lambda}_1}$ for some $t \geqslant 0$. By Lemma 6.1,

$$2s - 2 = s_{\mathbf{\Lambda}_0,\mathbf{\Lambda}_1}(\lambda^{(1)}, \lambda^{(2)}) = s_{\mathbf{\Lambda}_0,\mathbf{\Lambda}_1}(\varnothing, \lambda(2t)) = 2t.$$

This proves the first statement.

The second statement follows similarly with the consideration on patterns. $\square$

Recall that each $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2)$ is of index $(2m-1+s, s-1)$ (see (5.6b)).

THEOREM 6.10. *For* $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2)$ *of index* $(2m-1+s, s-1)$, *set* $\epsilon = 0$ *if* s *is even and* $\epsilon = 1$ *otherwise. Then*

$\mathbb{Y} \in \mathbf{B}((\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s)_\eta$ $(1 \leqslant s < n)$ *if and only if* $\mathbb{Y} = \mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}_\epsilon}$ *satisfies*

(a) $\lambda^{(1)} * \lambda^{(2)} = \lambda(2m-1+s)$,

$$(6.6)\quad \text{(b)} \begin{cases} \lambda^{(1)} \supset \lambda^{(2)}_{\geqslant s+1} \text{ and } \lambda^{(1)} \not\supset \lambda^{(2)}_{\geqslant s-1} & \text{if } s \geqslant 2, \\ \lambda^{(1)} \supset \lambda^{(2)}_{\geqslant 2} & \text{if } s = 1, \\ \lambda^{(1)} \supset \lambda^{(2)} & \text{if } s = 0. \end{cases}$$

PROOF. The "if" part follows from Proposition 6.9. Now it suffices to prove the "only if" part. Since η corresponds to $(\lambda(2m-1+s), \lambda(s-1))$ for $2m-1+s \leqslant n$, $\mathbb{Y}$ should be of the form $\mathsf{Y}^{\lambda^{(1)}}_{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda^{(2)}}_{\mathbf{\Lambda}_\epsilon}$ for some pair of strict partitions $(\lambda^{(1)}, \lambda^{(2)})$. Note that any pair of strict partitions $(\lambda^{(1)}, \lambda^{(2)})$ has the largest t satisfying one of the three conditions in (b) of (6.6). One can also check that $\max\{\lambda^{(1)}_1, \lambda^{(2)}_1\} \leqslant n$. Then the "only if" part follows from the form of weight η and Proposition 6.9 again; that is, $s = t$ and $\lambda^{(1)} * \lambda^{(2)} = \lambda(2m-1+s)$ by (5.6b). $\square$

Let $k \in \mathbb{Z}_{\geqslant 1}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Recall that a skew Young tableau T of shape $\mu/(s^{k-1})$ with m cells for a partition μ of length k is identified with a sequence of strict partitions

$$(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)})$$

such that $\lambda^{(k)} \neq \varnothing$, $\lambda^{(1)} * \lambda^{(2)} * \cdots * \lambda^{(k-1)} * \lambda^{(k)} = \lambda(m)$, $\lambda^{(i)} \supset \lambda^{(i+1)}$ for $1 \leqslant i \leqslant k-2$ and $\lambda^{(k-1)} \supset \lambda^{(k)}_{\geqslant s+1}$.

Now we define a family of tableaux which will play an important role for type $\mathfrak{D}$ constructions.

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Definition 6.11. For $s, m \in \mathbb{Z}_{\geqslant 0}$ with $m \geqslant s-1$, let T be a skew Young tableau of shape $\mu/(s^{k-1})$ with m cells for a partition μ of at most length k, which is identified with the sequence of strict partitions

$$\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)}) \quad \text{with } \lambda_i := \ell(\lambda^{(i)}),\ i = 1, \ldots, k.$$

Then T is called a *spin rigid Young tableau of index* (m, s) *with* k *rows* if it satisfies the following conditions:

(a) $(\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + s) \Vdash_0 m + s$,

(b) if $s \geqslant 2$, then $\lambda^{(k-1)} \not\supset \lambda^{(k)}_{\geqslant s-1}$.

We denote by ${}_s\mathfrak{D}^{(k)}_m$ the set of all spin rigid Young tableaux of index (m, s) with k rows. In particular, ${}_0\mathfrak{D}^{(k)}_m = \mathfrak{D}^{(k)}_m$ and hence ${}_0\mathfrak{D}^{(2)}_{2m-1} = \mathfrak{B}^{(2)}_{2m-1}$. (See Remark 3.10.)

Note that the condition (b) implies $\ell(\lambda^{(k)}) \geqslant \max\{0, s-1\}$. The condition (b) says that a shift of the last row to the right by 2 makes the tableau violate the column-strictness. The condition (a) naturally arises when we connect a spin rigid tableau with a staircase dominant maximal weight through a tensor product of Young walls. See Lemma 6.13 below.

We will color the columns of a spin rigid Young tableau in white and gray as follows to indicate the corresponding columns of Young walls starting from 0-blocks and 1-blocks.

The first column of spin rigid Young tableaux $T \in {}_{2s}\mathfrak{D}^{(k)}_m$ is colored in white while the first column of spin rigid Young tableaux $T \in {}_{2s+1}\mathfrak{D}^{(k)}_m$ is colored in gray.

Example 6.12.

(1) We have

$$T = \begin{array}{|c|c|c|c|c|} \hline \cdot & \cdot & 4 & 2 & 1 \\ \hline \cdot & \cdot \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \in {}_2\mathfrak{D}^{(3)}_4, \qquad \text{since} \quad \begin{array}{|c|c|c|c|c|} \hline \cdot & \cdot & 4 & 2 & 1 \\ \hline \cdot & \cdot \\ \cline{1-2} \cdot & \cdot & 3 \\ \cline{1-3} \end{array} \quad \text{is not a skew tableau.}$$

Here T corresponds to $\underline{\lambda} = ((4, 2, 1), \varnothing, (3))$.

The set ${}_2\mathfrak{D}^{(3)}_4$ consists of the following 15 spin rigid Young tableaux:

$$\begin{array}{lllll} \cdot\ \cdot\ 3\ 2\ 1 \\ \cdot\ \cdot \\ 4 \end{array},\ \begin{array}{lllll} \cdot\ \cdot\ 4\ 2\ 1 \\ \cdot\ \cdot \\ 3 \end{array},\ \begin{array}{lllll} \cdot\ \cdot\ 4\ 3\ 1 \\ \cdot\ \cdot \\ 2 \end{array},\ \begin{array}{lllll} \cdot\ \cdot\ 4\ 3\ 2 \\ \cdot\ \cdot \\ 1 \end{array},\ \begin{array}{llll} \cdot\ \cdot\ 3\ 1 \\ \cdot\ \cdot\ 2 \\ 4 \end{array},\ \begin{array}{llll} \cdot\ \cdot\ 4\ 2 \\ \cdot\ \cdot\ 1 \\ 3 \end{array},\ \begin{array}{llll} \cdot\ \cdot\ 4\ 3 \\ \cdot\ \cdot\ 1 \\ 2 \end{array},$$

$$\begin{array}{llll} \cdot\ \cdot\ 3\ 2 \\ \cdot\ \cdot\ 1 \\ 4 \end{array},\ \begin{array}{llll} \cdot\ \cdot\ 4\ 1 \\ \cdot\ \cdot\ 2 \\ 3 \end{array},\ \begin{array}{lll} \cdot\ \cdot\ 4 \\ \cdot\ \cdot\ 3 \\ 2\ 1 \end{array},\ \begin{array}{lll} \cdot\ \cdot\ 4 \\ \cdot\ \cdot\ 2 \\ 3\ 1 \end{array},\ \begin{array}{lll} \cdot\ \cdot\ 3 \\ \cdot\ \cdot\ 2 \\ 4\ 1 \end{array},\ \begin{array}{lll} \cdot\ \cdot\ 2 \\ \cdot\ \cdot\ 1 \\ 4\ 3 \end{array},\ \begin{array}{lll} \cdot\ \cdot\ 4 \\ \cdot\ \cdot\ 1 \\ 3\ 2 \end{array},\ \begin{array}{lll} \cdot\ \cdot\ 3 \\ \cdot\ \cdot\ 1 \\ 4\ 2 \end{array}.$$

(2) The set ${}_3\mathfrak{D}^{(3)}_4$ consists of the following 10 spin rigid Young tableaux:

$$\begin{array}{l} \cdot\ \cdot\ \cdot\ 2\ 1 \\ \cdot\ \cdot\ \cdot \\ 4\ 3 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 3\ 2 \\ \cdot\ \cdot\ \cdot \\ 4\ 1 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 3\ 1 \\ \cdot\ \cdot\ \cdot \\ 4\ 2 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 4\ 1 \\ \cdot\ \cdot\ \cdot \\ 3\ 2 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 4\ 3 \\ \cdot\ \cdot\ \cdot \\ 2\ 1 \end{array},$$

$$\begin{array}{l} \cdot\ \cdot\ \cdot\ 4\ 2 \\ \cdot\ \cdot\ \cdot \\ 3\ 1 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 1 \\ \cdot\ \cdot\ \cdot \\ 4\ 3\ 2 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 2 \\ \cdot\ \cdot\ \cdot \\ 4\ 3\ 1 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 3 \\ \cdot\ \cdot\ \cdot \\ 4\ 2\ 1 \end{array},\ \begin{array}{l} \cdot\ \cdot\ \cdot\ 4 \\ \cdot\ \cdot\ \cdot \\ 3\ 2\ 1 \end{array}.$$

When $\Lambda = (k - 2 + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_{2s-1}$, the crystal $\mathbf{B}(\Lambda)$ is embedded into $\mathsf{Y}(\mathbf{\Lambda}_0)^{\otimes k-1} \otimes \mathsf{Y}(\mathbf{\Lambda}_1)$, and when $\Lambda = (k-2)\mathbf{\Lambda}_0 + (1 + \delta_{s,0})\Lambda_{2s}$, the crystal $\mathbf{B}(\Lambda)$ is embedded into $\mathsf{Y}(\mathbf{\Lambda}_0)^{\otimes k}$. Hence we use gray color to distinguish the columns of

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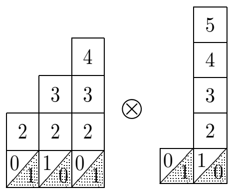
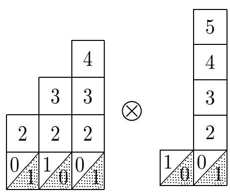
Young walls starting with 1-blocks with those starting with 0-blocks. For example, we have

(6.7) 

$$\longleftrightarrow \mathsf{Y}^{(4,3,2)}_{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{(5,1)}_{\mathbf{\Lambda}_1} \longleftrightarrow \begin{array}{|c|c|c|c|}\hline \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} \in {}_1\mathfrak{D}^{(2)}_5,$$

(6.8) 

$$\longleftrightarrow \mathsf{Y}^{(4,3,2)}_{\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{(5,1)}_{\mathbf{\Lambda}_0} \longleftrightarrow \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} \in {}_2\mathfrak{D}^{(2)}_5.$$

Note that the cells filled with white (resp. gray) color represent the columns starting with 0-blocks (resp. 1-blocks). In (6.8), we use

$$\begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array} \quad \text{instead of} \quad \begin{array}{|c|c|c|c|}\hline \cdot & 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}$$

so that each column of the tableau has the same color.

Let

$$\underline{\mathbf{\Lambda}} = \begin{cases} (\mathbf{\Lambda}_0, \ldots, \mathbf{\Lambda}_0, \mathbf{\Lambda}_0) & \text{if } s \text{ is even,} \\ (\mathbf{\Lambda}_0, \ldots, \mathbf{\Lambda}_0, \mathbf{\Lambda}_1) & \text{if } s \text{ is odd.} \end{cases}$$

The following lemma follows from the definitions of ${}_s\mathfrak{D}^{(k)}_m$ and $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$:

Lemma 6.13. *Let $s, m \in \mathbb{Z}_{\geqslant 0}$ with $n \geqslant m \geqslant s-1$, and $\Lambda = (k-2+\delta_{s,0}+\delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s$, $k \geqslant 2$. Then, for $T \in {}_s\mathfrak{D}^{(k)}_m$, we have*

$$\mathrm{cont}(\mathbb{Y}^T_{\underline{\mathbf{\Lambda}}}) = \begin{cases} \mathrm{cont}(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}_1}) - (\alpha_1 - \alpha_0) & \text{if } s \equiv_2 m, \\ \mathrm{cont}(\mathsf{Y}^{\lambda(m)}_{\mathbf{\Lambda}_0}) & \text{otherwise,} \end{cases}$$

and the tableau T is associated with $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ of index $(m, s-1)$ such that

$$\mathrm{cont}(\mathbb{Y}^T_{\underline{\mathbf{\Lambda}}}) - \mathrm{cont}(\mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0}) = \Lambda - \eta.$$

Recall the set of indices for $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ in (5.26). The following is the main theorem of this section:

Theorem 6.14. *Assume that $k \geqslant 2$. Then, for $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|k)$ of index $(m, s-1)$, we have*

$$\dim V(\Lambda)_\eta = |{}_s\mathfrak{D}^{(k)}_m| = \dim L\big((k-2)\omega_n + \tilde{\omega}_{n-s}\big)_\mu,$$

where the definition of $\tilde{\omega}_s$ is given in (5.12) and the weights μ are given by

$$\mu = \begin{cases} (k-2)\omega_n + \tilde{\omega}_{n-m-1} & \text{if } k = 2, \text{ or } k \geqslant 3 \text{ and } m \not\equiv_2 s, \\ (k-3)\omega_n + \omega_{n-1} + \tilde{\omega}_{n-m-1} & \text{if } k \geqslant 3 \text{ and } m \equiv_2 s. \end{cases}$$

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PROOF. Let $\eta, \eta' \in \operatorname{smax}^+_{\mathfrak{D}}(\Lambda|k)$ be of index $(m, s-1)$. If η is associated with $(\lambda(m), \lambda(s-1))$ and η' with $((n) * \lambda(m-1), \lambda(s-1))$, one can see that $\dim V(\Lambda)_\eta = \dim V(\Lambda)_{\eta'}$ by replacing the role of $(n) * \lambda(m-1)$ with that of $\lambda(m)$ to construct a one-to-one correspondence between the corresponding sets of tensor products of Young walls. Thus we only need to consider η associated with $(\lambda(m), \lambda(s-1))$.

Set

$$\boxed{\Lambda} := \begin{cases} \boxed{(k-1)\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_0} & \text{if } s \text{ is even,} \\ \boxed{(k-1)\mathbf{\Lambda}_0} \otimes \mathsf{Y}^{\lambda(s-1)}_{\mathbf{\Lambda}_1} & \text{if } s \text{ is odd.} \end{cases}$$

Since $m \leqslant n$, a Young wall $\mathbb{Y} \in \mathbf{B}(\Lambda)_\eta$ connected to $\boxed{\Lambda}$ cannot contain a removable δ. Hence Lemma 6.13 tells us that $\mathbb{Y} \in \mathbf{B}(\Lambda)_\eta$ corresponds to a sequence of strict partitions $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k-1)}, \lambda^{(k)})$ satisfying the condition (a) in Definition 6.11:

$$\mathbb{Y} = \mathbb{Y}^{\underline{\lambda}}_{\underline{\mathbf{\Lambda}}} \quad \text{where } \underline{\mathbf{\Lambda}} = \begin{cases} (\mathbf{\Lambda}_0, \ldots, \mathbf{\Lambda}_0, \mathbf{\Lambda}_0) & \text{if } s \text{ is even,} \\ (\mathbf{\Lambda}_0, \ldots, \mathbf{\Lambda}_0, \mathbf{\Lambda}_1) & \text{if } s \text{ is odd.} \end{cases}$$

Note that if $\ell(\lambda^{(k)}) < \max\{0, s-1\}$, then $\mathbb{Y}$ cannot be connected to $\boxed{\Lambda}$. Now the condition (b) in Definition 6.11 follows to represent the columns of Young walls starting with 1-blocks from Proposition 6.9 and Theorem 6.10.

The last equality follows from Proposition 5.3 and Theorem 2.7. □

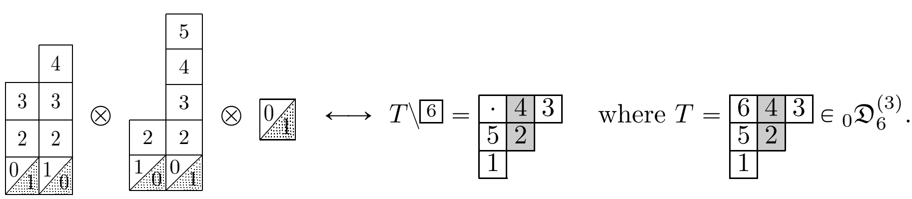
We record the special case $s = 0$ as a corollary for future reference.

COROLLARY 6.15. *The numbers* $|\mathfrak{D}^{(k)}_m|$ *of almost even tableaux of* m *with at most* k *rows are the multiplicities of dominant maximal weights for* $V(k\mathbf{\Lambda})$ *and hence the multiplicities of dominant weights for* $V(k\omega_n)$.

For the rest of this section, we investigate the relationship between ${}_0\mathfrak{D}^{(k)}_m$ and ${}_1\mathfrak{D}^{(k)}_{m-1}$, which will be used in chapter 8. Set $\Lambda = (k-1)\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1$ for $k \geqslant 3$. The crystal $\mathbf{B}(\Lambda)$ can also be realized by the subcrystal of $\mathcal{Y}(\mathbf{\Lambda}_1) \otimes \mathcal{Y}(\mathbf{\Lambda}_0)^{\otimes k-1}$ (as opposed to $\mathcal{Y}(\mathbf{\Lambda}_0)^{\otimes k-1} \otimes \mathcal{Y}(\mathbf{\Lambda}_1)$) connected to $\boxed{\mathbf{\Lambda}_1} \otimes \boxed{(k-1)\mathbf{\Lambda}_0}$. By applying the argument in this section, one can prove that the crystal basis of $V(\Lambda)_\eta$ for $\eta \in \operatorname{smax}^+_{\mathfrak{D}}((k-1)\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1|k)$ is realized by

$${}_0\mathfrak{D}^{(k)}_m \backslash \boxed{m} := \{T \backslash \boxed{m} \mid T \in {}_0\mathfrak{D}^{(k)}_m\},$$

where $\eta \in \operatorname{smax}^+(\Lambda|k)$ is of index $(m-1, 0)$ and $T \backslash \boxed{m}$ is the tableau obtained by removing the cell $\boxed{m}$ located in the position $(1,1)$. For example, when $m = 6$ and $k = 3$,



On the other hand, by Theorem 6.14, the crystal basis of $V(\Lambda)_\eta$ is also realized by the set ${}_1\mathfrak{D}^{(k)}_{m-1}$ of spin rigid Young tableaux.

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Hence we can conclude that

$$|{}_0\mathfrak{D}^{(k)}_m| = |{}_0\mathfrak{D}^{(k)}_m \backslash \boxed{m}| = |{}_1\mathfrak{D}^{(k)}_{m-1}|,$$

which will explain the correspondence with the equation $\mathsf{R}_{(m,0)} = \mathsf{R}_{(m-1,1)}$ in (4.5) (see chapter 8 below).

EXAMPLE 6.16. The set ${}_0\mathfrak{D}^{(3)}_4 \backslash \boxed{4}$ is given as follows:

$$\begin{smallmatrix} \cdot & 3 \\ 2 & \\ 1 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 2 \\ 3 & \\ 1 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 1 \\ 3 & \\ 2 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 2 & 1 \\ 3 & & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 3 & 1 \\ 2 & & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 3 & 2 \\ 1 & & \end{smallmatrix}.$$

On the other hand, the set ${}_1\mathfrak{D}^{(3)}_3$ is given as follows:

$$\begin{smallmatrix} \cdot & 3 & 1 \\ \cdot & 2 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 3 & 2 \\ \cdot & 1 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 3 & 2 & 1 \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 3 \\ \cdot & 2 \\ 1 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 3 \\ \cdot & 1 \\ 2 & \end{smallmatrix},\quad \begin{smallmatrix} \cdot & 2 \\ \cdot & 1 \\ 3 & \end{smallmatrix}.$$

The following corollary summarizes the above observations.

COROLLARY 6.17. *Set $\Lambda = (k-1)\mathbf{\Lambda}_0 + \mathbf{\Lambda}_1$ for $k \geqslant 2$. Then the number of the almost even tableaux of $m \geqslant 1$ with at most k rows appears as the multiplicity of a maximal weight $\eta \in \mathrm{smax}^+(\Lambda|k)$ of index $(m-1, 0)$. That is, we have*

$$|{}_0\mathfrak{D}^{(k)}_m| = |{}_1\mathfrak{D}^{(k)}_{m-1}| = \dim(V(\Lambda)_\eta).$$

REMARK 6.18.

(1) Explicit formulas for the numbers $|\mathfrak{D}^{(k)}_m|$ for $1 \leqslant k \leqslant 5$ will be given in Theorem 10.2. Thus we have explicit formulas for the multiplicities of $\eta \in \mathrm{smax}^+(\Lambda|k)$ of indices $(m, -1)$ and $(m-1, 0)$ for $1 \leqslant k \leqslant 5$.

(2) All the results in this section still hold when we replace the filling m by n in each tableau, since it does not affect the proofs and only affects the weight of a tableau (see (5.14)). This fact explains that the index (m, s) in Definition 5.17 is well-defined.

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CHAPTER 7

Level 2 weight multiplicities: Catalan and Pascal triangles

In this chapter, we prove that all the multiplicities of the (staircase) dominant maximal weights of level 2 are generalized Catalan numbers or binomial coefficients. As will be indicated in Section 7.1, the results can be obtained through classical constructions. We will provide a different proof, which utilizes a new insertion scheme for (spin) rigid Young tableaux and makes the Catalan and Pascal triangles compatible with the insertion scheme. This insertion scheme will naturally generalize in the next chapter to the case of level 3 weights, where classical constructions do not easily generalize.

7.1. Classical realizations

Now we restate and give an alternative proof for [**38**, Theorem 1.4 (ii)], which was on the affine type $A^{(1)}_{n-1}$:

THEOREM 7.1. (*cf.* [**38**, Theorem 1.4 (ii)]) *For finite type A_{n-1}, we have*

$$\dim L(\omega_t + \omega_{t+s})_{\omega_{t-k}+\omega_{t+s+k}} = \mathsf{C}_{(s+2k,s)} \quad \text{for } 0 \leqslant k \leqslant t,$$

where $\mathsf{C}_{(m,s)}$ are the generalized Catalan numbers.

PROOF. By Kashiwara–Nakashima's realization ([**22**]) of the crystal basis for $\mathbf{B}(\omega_t + \omega_{t+s})$ via semi-standard tableaux filled with $1, 2, \ldots, n$, the dimension $\dim L(\omega_t+\omega_{t+s})_{\omega_{t-k}+\omega_{t+s+k}}$ is the same as the number of semi-standard tableaux T (the convention for semi-standard tableaux in [**22**] is different from ours) satisfying the following conditions:

- $\mathsf{Sh}(T) = (2^t, 1^s)$,
- for every $1 \leqslant i \leqslant t-k$, the two cells in the i-th row are filled with i,
- the remaining $2k+s$ cells are filled with the distinct numbers $t-k+1, t-k+2, \ldots, t+k+s$.

Hence Remark 4.17 implies our assertion. □

In Section 5.1, we showed that every dominant maximal weight of a highest weight Λ of level 2 is essentially finite of type A_{n-1}. Thus we obtain the following corollary:

COROLLARY 7.2. *For finite type A_{n-1}, assume that $\eta \in \max^+(\Lambda|2)$. Then the multiplicity of η is a generalized Catalan number.*

Generalized Catalan numbers also appear for type C_n as one can see in the following theorem.

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THEOREM 7.3. *For finite type C_n, $1 \leqslant s \leqslant n$ and $0 \leqslant i \leqslant \lfloor \frac{s}{2} \rfloor$, we have*

$$\dim L(\omega_s)_{\omega_{s-2i}} = \mathsf{C}_{(n-s+2i,n-s)}.$$

PROOF. This is a consequence of the exterior power realization of the fundamental representation (see [**8**, Theorem 17.5]) since

$$\mathsf{C}_{(n-s+2i,n-s)} = \binom{n-(s-2i)}{i} - \binom{n-(s-2i)}{i-1}. \qquad \square$$

In Section 5.3, we showed that every dominant maximal weight of a highest weight $\mathbf{\Lambda}_s$ of level 1 over type $C_n^{(1)}$ is essentially finite of type C_n. For types $A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$, we determined dominant maximal weights which are essentially finite of type C_n. See Remarks 5.31 and 5.35. Thus we obtain the following corollary:

COROLLARY 7.4. *Assume that η is a dominant maximal weight which is essentially finite of type C_n for a highest weight Λ of level 1 over type $C_n^{(1)}$ or of level 2 over type $A_{2n-1}^{(2)}$ or $A_{2n}^{(2)}$. Then the multiplicity of η is a generalized Catalan number.*

The following theorem shows that binomial coefficients appear as weight multiplicities for finite types B_n and D_n.

THEOREM 7.5. *For $1 \leqslant s \leqslant n$, we have*

$$\begin{cases} \dim L(\tilde{\omega}_s)_{\tilde{\omega}_k} = \dbinom{n-k}{\lfloor \frac{s-k}{2} \rfloor} & \text{if } L(\tilde{\omega}_s) \text{ is over } B_n, \\[2ex] \dim L(\tilde{\omega}_s)_{\tilde{\omega}_k} = \dbinom{n-k-\delta_{n,s}}{\frac{s-k}{2}} & \text{if } L(\tilde{\omega}_s) \text{ is over } D_n \text{ and } s \equiv_2 k. \end{cases}$$

PROOF. By the exterior power realization of the fundamental representation in [**8**, Theorem 19.2, Theorem 19.14], one can prove this assertion. $\square$

We remark here that it seems difficult in general to prove the above results using the Kashiwara–Nakashima realization for finite types B_n and D_n.

Though we can use Theorem 7.5 to describe the multiplicities of maximal weights in $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2)$ and $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2)$, we will develop a new method in the next sections for the reason mentioned at the beginning of this chapter.

7.2. Insertion of a box

DEFINITION 7.6. Let $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(k)})$ be a sequence of strict partitions with $\mathop{*}\limits_{j=1}^{k} \lambda^{(j)} = \mathsf{\lambda}(m-1)$. For $1 \leqslant u \leqslant k$, we define the *insertion of (m) into the u-th partition* by

$$\underline{\lambda} \mathop{*}_{u} (m) = (\lambda'^{(1)}, \dots, \lambda'^{(k)})$$

where

$$\begin{cases} \lambda'^{(j)} = \lambda^{(j)} & \text{if } j \neq u, \\ \lambda'^{(u)} = (m) * \lambda^{(u)} & \text{if } j = u. \end{cases}$$

Then $\underline{\lambda} \mathop{*}_{u} (m) = (\lambda'^{(1)}, \dots, \lambda'^{(k)})$ is a new sequence of strict partitions with $\mathop{*}\limits_{j=1}^{k} \lambda'^{(j)} = \mathsf{\lambda}(m)$.

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The operation $\underset{u}{*}(m)$ is to be understood as an *insertion* of the box $\boxed{m}$ into the u-th row of a skew-tableaux. For example, we have

$$\begin{array}{|c|c|c|c|}\hline 7&5&4&2\\\hline 6&3&1\\\cline{1-3}\end{array}\ \underset{1}{*}(8) = \begin{array}{|c|c|c|c|c|}\hline 8&7&5&4&2\\\hline 6&3&1\\\cline{1-3}\end{array}$$

7.3. Case $\mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2)$

We start with a simple observation. For $T = (\lambda^{(1)}, \dots, \lambda^{(k)}) \in {}_s\mathfrak{B}^{(k)}_m$, the number m can only appear as the first part of the first partition or as the first part of the last partition. That is, we have

$$m = \begin{cases} \lambda^{(1)}_1 \text{ or } \lambda^{(k)}_1 & \text{if } s \geqslant 1, \\ \lambda^{(1)}_1 & \text{if } s = 0. \end{cases} \tag{7.1}$$

EXAMPLE 7.7.

(1) ${}_1\mathfrak{B}^{(2)}_5$ consists of the following 10 rigid Young tableaux:

$$\begin{array}{|c|c|c|c|c|}\hline \cdot&4&3&2&1\\\hline 5\\\cline{1-1}\end{array},\ \begin{array}{|c|c|c|c|}\hline \cdot&4&2&1\\\hline 5&3\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|c|}\hline \cdot&4&3&1\\\hline 5&2\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|c|}\hline \cdot&4&3&2\\\hline 5&1\\\cline{1-2}\end{array},\ \begin{array}{|c|c|c|c|}\hline \cdot&5&2&1\\\hline 4&3\\\cline{1-2}\end{array},$$

$$\begin{array}{|c|c|c|}\hline \cdot&4&3\\\hline 5&2&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline \cdot&5&3\\\hline 4&2&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline \cdot&4&2\\\hline 5&3&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline \cdot&5&4\\\hline 3&2&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline \cdot&5&2\\\hline 4&3&1\\\hline\end{array}.$$

(2) ${}_3\mathfrak{B}^{(2)}_5$ consists of the following 5 rigid Young tableaux:

$$\begin{array}{|c|c|c|c|c|}\hline \cdot&\cdot&\cdot&2&1\\\hline 5&4&3\\\cline{1-3}\end{array},\ \begin{array}{|c|c|c|c|c|}\hline \cdot&\cdot&\cdot&\cdot&5\\\hline 4&3&2&1\\\cline{1-4}\end{array},\ \begin{array}{|c|c|c|c|c|}\hline \cdot&\cdot&\cdot&\cdot&4\\\hline 5&3&2&1\\\cline{1-4}\end{array},\ \begin{array}{|c|c|c|c|c|}\hline \cdot&\cdot&\cdot&\cdot&3\\\hline 5&4&2&1\\\cline{1-4}\end{array},\ \begin{array}{|c|c|c|c|c|}\hline \cdot&\cdot&\cdot&\cdot&2\\\hline 5&4&3&1\\\cline{1-4}\end{array},$$

LEMMA 7.8. *For $T = (\lambda, \mu) \in {}_s\mathfrak{B}^{(2)}_{m-1}$, we have*

$$T \underset{1}{*}(m) \in {}_{s-1}\mathfrak{B}^{(2)}_m \quad \textit{and} \quad T \underset{2}{*}(m) \in {}_{s+1}\mathfrak{B}^{(2)}_m.$$

PROOF. Recall that $(\lambda, \mu) \in {}_s\mathfrak{B}^{(2)}_{m-1}$ for $s \geqslant 1$ implies

(i) $\lambda_i < \mu_{s+i-1}$ and $\lambda_i > \mu_{s+i}$ for some $1 \leqslant i \leqslant \ell(\lambda)$ or (ii) $\ell(\mu) - s = \ell(\lambda)$.

Since $((m) * \lambda)_1 = m$, $((m) * \lambda)_{i+1} = \lambda_i$ and $\ell((m) * \lambda) = \ell(\lambda) + 1$, we can conclude that

$$T \underset{1}{*}(m) \in {}_{s-1}\mathfrak{B}^{(2)}_m.$$

Similarly, the facts that $((m) * \mu)_1 = m$, $((m) * \mu)_{i+1} = \mu_i$ and $\ell((m) * \mu) = \ell(\mu) + 1$ implies

$$T \underset{2}{*}(m) \in {}_{s+1}\mathfrak{B}^{(2)}_m.$$

□

REMARK 7.9. For $m \in \mathbb{Z}_{\geqslant 1}$, the sets ${}_m\mathfrak{B}^{(2)}_m$ and ${}_m\mathfrak{B}^{(2)}_{m+1}$ are described as follows:

$${}_m\mathfrak{B}^{(2)}_m = \{(\varnothing, \lambda(m))\} \quad \text{and} \quad {}_m\mathfrak{B}^{(2)}_{m+1} = \{((1), (m+1, m, \dots, 2))\}. \tag{7.2}$$

Hence $|{}_m\mathfrak{B}^{(2)}_m| = |{}_m\mathfrak{B}^{(2)}_{m+1}| = 1$.

Let $L(\omega)$ be the highest weight module with highest weight ω over the finite dimensional Lie algebra of type B_n. Recall the definition of $\tilde{\omega}_s$ in (6.3).

THEOREM 7.10. *Let $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|2)$ of index (m, s). For every $s \leqslant m$,*

$$|{}_s\mathfrak{B}^{(2)}_m| = \binom{m}{\lfloor \frac{m-s}{2} \rfloor} = \dim V(\Lambda)_\eta = \dim L(\tilde{\omega}_{n-s})_{\tilde{\omega}_{n-m}}.$$

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PROOF. By (7.1), for each $T = (\lambda, \mu) \in {}_s\mathfrak{B}_m^{(2)}$ with $s \geqslant 1$, we have

$$\lambda_1 = m \quad \text{or} \quad \mu_1 = m.$$

Thus

$$T = T_1 \underset{1}{*} (m) \quad \text{or} \quad T = T_2 \underset{2}{*} (m)$$

for some $T_1 \in {}_{s+1}\mathfrak{B}_{m-1}^{(2)}$ or $T_2 \in {}_{s-1}\mathfrak{B}_{m-1}^{(2)}$ respectively. Particularly, $T \in \mathfrak{B}_m^{(2)}$ is of the form $T' \underset{1}{*} (m)$ for some $T' \in \mathfrak{B}_{m-1}^{(2)} \sqcup {}_1\mathfrak{B}_{m-1}^{(1)}$. Since the sets $\left({}_{s+1}\mathfrak{B}_{m-1}^{(2)}\right) \underset{1}{*} (m)$ and $\left({}_{s-1}\mathfrak{B}_{m-1}^{(2)}\right) \underset{2}{*} (m)$ are distinct, our assertion follows from

$$|\mathfrak{B}_m^{(2)}| = |{}_0\mathfrak{B}_m^{(2)}| = \binom{m}{\lfloor \frac{m}{2} \rfloor}, \; |{}_m\mathfrak{B}_m^{(2)}| = \binom{m}{\lfloor \frac{m-m}{2} \rfloor} = 1 = \binom{m}{\lfloor \frac{m+1-m}{2} \rfloor} = |{}_m\mathfrak{B}_{m+1}^{(2)}|$$

and

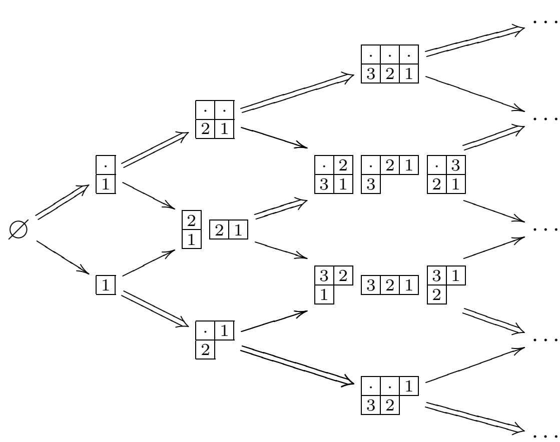
$$\left({}_{s-1}\mathfrak{B}_{m-1}^{(2)} \underset{2}{*} (m)\right) \bigsqcup \left({}_{s+1}\mathfrak{B}_{m-1}^{(2)} \underset{1}{*} (m)\right) = {}_s\mathfrak{B}_m^{(2)}$$

corresponding to $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

The last equality follows from Proposition 5.8 and Theorem 2.7. $\square$

The following lattice diagram illustrates the above theorem and realizes the Pascal triangle:

(7.3)



Here $\Rightarrow$ denotes insertion $\underset{2}{*}$ into the second row (or partition) and $\rightarrow$ denotes insertion $\underset{1}{*}$ into the first one. By taking the cardinality of the tableaux at each position, we obtain the Pascal triangle.

EXAMPLE 7.11. In Example 7.7, we can see that

$$|{}_1\mathfrak{B}_5^{(2)}| = \binom{5}{\lfloor \frac{5-1}{2} \rfloor} = \binom{5}{2} = 10 \quad \text{and} \quad |{}_3\mathfrak{B}_5^{(2)}| = \binom{5}{\lfloor \frac{5-3}{2} \rfloor} = \binom{5}{1} = 5.$$

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Furthermore, we get $|{}_2\mathfrak{B}^{(2)}_6| = 10 + 5 = \binom{6}{\lfloor\frac{6-2}{2}\rfloor}$ from the insertion scheme:

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 & 2 & 1 \\ \hline 6 & 5 \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 2 & 1 \\ \hline 6 & 5 & 3 \\ \cline{1-3}\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 & 1 \\ \hline 6 & 5 & 2 \\ \cline{1-3}\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 & 2 \\ \hline 6 & 5 & 1 \\ \cline{1-3}\end{array}, \quad \begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 5 & 2 & 1 \\ \hline 6 & 4 & 3 \\ \cline{1-3}\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 4 & 3 \\ \hline 6 & 5 & 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 5 & 3 \\ \hline 6 & 4 & 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 4 & 2 \\ \hline 6 & 5 & 3 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 5 & 4 \\ \hline 6 & 3 & 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 5 & 2 \\ \hline 6 & 4 & 3 & 1 \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|c|c|}\hline \cdot & \cdot & 6 & 2 & 1 \\ \hline 5 & 4 & 3 \\ \cline{1-3}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 6 & 5 \\ \hline 4 & 3 & 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 6 & 4 \\ \hline 5 & 3 & 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 6 & 3 \\ \hline 5 & 4 & 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 6 & 2 \\ \hline 5 & 4 & 3 & 1 \\ \hline\end{array}.$$

COROLLARY 7.12. *For $m \geqslant s \geqslant 0$, set*

$$a = \lfloor (m-s)/2 \rfloor \quad \textit{and} \quad b = m - a.$$

We have a bijective map between

$${}_s\mathfrak{B}^{(2)}_m \quad \textit{and} \quad \mathfrak{L}(a,b),$$

where $\mathfrak{L}(a,b)$ denotes the set of paths in the Pascal triangle (4.8) *starting from* $(0,0)$ *to $(m, b-a)$ using the vectors $(1,1)$ and $(1,-1)$.*

PROOF. For $T \in {}_s\mathfrak{B}^{(2)}_m$, we first assume that $s \equiv_2 m$. Then we record the vector v_m as

- $(1,1)$ if $\begin{cases} T = T' \underset{2}{*} (m) \text{ for some } T' \in {}_{s-1}\mathfrak{B}^{(2)}_{m-1} \text{ with } s \geqslant 1, \text{ or} \\ T = T' \underset{1}{*} (m) \text{ for some } T' \in \mathfrak{B}^{(2)}_{m-1}, \end{cases}$
- $(1,-1)$ if $T = T' \underset{1}{*} (m)$ for some $T' \in {}_{s+1}\mathfrak{B}^{(2)}_{m-1}$.

Now we assume that $s - 1 \equiv_2 m$. Then we record the vector v_m as

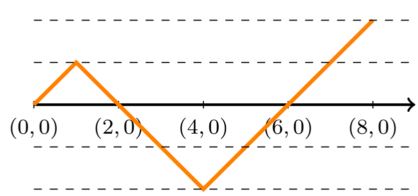
- $(1,-1)$ if $\begin{cases} T = T' \underset{2}{*} (m) \text{ for some } T' \in {}_{s-1}\mathfrak{B}^{(2)}_{m-1} \text{ with } s \geqslant 1, \text{ or} \\ T = T' \underset{1}{*} (m) \text{ for some } T' \in \mathfrak{B}^{(2)}_{m-1}, \end{cases}$
- $(1,1)$ if $T = T' \underset{1}{*} (m)$ for some $T' \in {}_{s+1}\mathfrak{B}^{(2)}_{m-1}$.

Then, by induction on m, we obtain the sequence of vectors $(v_1, v_2, \ldots, v_m)$ corresponding to a path in the Pascal triangle. $\square$

EXAMPLE 7.13. For

$$T = \begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 6 & 5 & 3 & 2 \\ \hline 8 & 7 & 4 & 1 \\ \cline{1-4}\end{array} \in {}_2\mathfrak{B}^{(2)}_8,$$

we have $a = 3$ and $b = 5$. Then the tableau T corresponds to the following lattice path:



7.4. Case $\mathrm{smax}^+_{\mathfrak{D}}(\Lambda|2)$

By Theorem 2.9, we may assume that $\mathfrak{g} = B^{(1)}_n$ and

$$\Lambda = (\delta_{s,0} + \delta_{s,1})\mathbf{\Lambda}_0 + \Lambda_s \quad (0 \leqslant s \leqslant n-1)$$

throughout this section.

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As in (7.1), the same property holds for $T = (\lambda^{(1)}, \dots, \lambda^{(k)}) \in {}_s\mathfrak{D}_m^{(k)}$ to have

$$m = \begin{cases} \lambda_1^{(1)} \text{ or } \lambda_1^{(k)} & \text{if } s \geqslant 1, \\ \lambda_1^{(1)} & \text{if } s = 0. \end{cases}$$

EXAMPLE 7.14.
(1) The set ${}_1\mathfrak{D}_4^{(2)}$ consists of the following 10 spin rigid Young tableaux:

$$\begin{array}{ccccc} \cdot & 4 & 3 & 2 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & 3 & 2 & 1 \\ 4 \end{array}, \quad \begin{array}{cccc} \cdot & 4 & 2 & 1 \\ 3 \end{array}, \quad \begin{array}{cccc} \cdot & 4 & 3 & 1 \\ 2 \end{array}, \quad \begin{array}{cccc} \cdot & 4 & 3 & 2 \\ 1 \end{array},$$

$$\begin{array}{ccc} \cdot & 4 & 3 \\ 2 & 1 \end{array}, \quad \begin{array}{ccc} \cdot & 4 & 2 \\ 3 & 1 \end{array}, \quad \begin{array}{ccc} \cdot & 3 & 2 \\ 4 & 1 \end{array}, \quad \begin{array}{ccc} \cdot & 4 & 1 \\ 3 & 2 \end{array}, \quad \begin{array}{ccc} \cdot & 3 & 1 \\ 4 & 2 \end{array}.$$

(2) The set ${}_3\mathfrak{D}_4^{(2)}$ consists of the following 5 spin rigid Young tableaux:

$$\begin{array}{ccccc} \cdot & \cdot & \cdot & 2 & 1 \\ 4 & 3 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \cdot & 1 \\ 4 & 3 & 2 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \cdot & 2 \\ 4 & 3 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \cdot & 3 \\ 4 & 2 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \cdot & 4 \\ 3 & 2 & 1 \end{array}.$$

LEMMA 7.15. *For any* $(\lambda, \mu) \in {}_s\mathfrak{D}_{m-1}^{(2)}$, *we have*

$$(\lambda, \mu) \underset{1}{*} (m) \in {}_{s-1}\mathfrak{D}_m^{(2)} \quad \text{and} \quad (\lambda, \mu) \underset{2}{*} (m) \in {}_{s+1}\mathfrak{D}_m^{(2)}.$$

PROOF. Recall Definition 6.11. In particular, since $k = 2$, we have $m \not\equiv_2 s$. Then one can use a similar argument to that of the proof of Lemma 7.8. $\square$

Let $L(\omega)$ be the highest weight module with highest weight ω over the finite dimensional Lie algebra of type D_n. Recall the definition of $\tilde{\omega}_s$ in (5.12).

THEOREM 7.16. *Let* $\eta \in \operatorname{smax}^+_{\mathfrak{D}}(\Lambda|2)$ *of index* $(2u-1+s, s-1)$. *For* $s \geqslant 0$ *and* $u \geqslant 0$,

$$|{}_s\mathfrak{D}_{2u-1+s}^{(2)}| = \binom{2u+s-\delta_{s,0}}{u} = \dim V(\Lambda)_\eta = \dim L(\tilde{\omega}_{n-s})_{\tilde{\omega}_{n-s-2u}}.$$

PROOF. With Corollary 6.17 and the fact that

$$|{}_s\mathfrak{D}_{s-1}^{(2)}| = |\{(\varnothing, \lambda(s-1))\}| = 1,$$

one can apply a similar argument to that of the proof of Theorem 7.10. The last equality follows from Proposition 5.3 and Theorem 2.7. $\square$

EXAMPLE 7.17. From Example 7.14, we see that

$$|{}_1\mathfrak{D}_4^{(2)}| = \binom{4+1}{2} = \binom{5}{2} = 10 \quad \text{and} \quad |{}_3\mathfrak{D}_4^{(2)}| = \binom{2+3}{1} = \binom{5}{1} = 5.$$

Furthermore, we get $|{}_2\mathfrak{D}_5^{(2)}| = 10 + 5 = \binom{4+2}{2}$ from the insertion scheme:

$$\begin{array}{cccccc} \cdot & \cdot & 4 & 3 & 2 & 1 \\ \mathbf{5} \end{array}, \quad \begin{array}{ccccc} \cdot & \cdot & 3 & 2 & 1 \\ \mathbf{5} & 4 \end{array}, \quad \begin{array}{ccccc} \cdot & \cdot & 4 & 2 & 1 \\ \mathbf{5} & 3 \end{array}, \quad \begin{array}{ccccc} \cdot & \cdot & 4 & 3 & 1 \\ \mathbf{5} & 2 \end{array}, \quad \begin{array}{ccccc} \cdot & \cdot & 4 & 3 & 2 \\ \mathbf{5} & 1 \end{array},$$

$$\begin{array}{cccc} \cdot & \cdot & 4 & 3 \\ \mathbf{5} & 2 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & 4 & 2 \\ \mathbf{5} & 3 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & 3 & 2 \\ \mathbf{5} & 4 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & 4 & 1 \\ \mathbf{5} & 3 & 2 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & 3 & 1 \\ \mathbf{5} & 4 & 2 \end{array},$$

$$\begin{array}{ccccc} \cdot & \cdot & \mathbf{5} & 2 & 1 \\ 4 & 3 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \mathbf{5} & 1 \\ 4 & 3 & 2 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \mathbf{5} & 2 \\ 4 & 3 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \mathbf{5} & 3 \\ 4 & 2 & 1 \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \mathbf{5} & 4 \\ 3 & 2 & 1 \end{array}.$$

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CHAPTER 8

Level 3 weight multiplicities: Motzkin and Riordan triangles

As a special case $k = 3$ in Theorems 6.8 and 6.14, the multiplicity of $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|3)$ of index (m, s) is equal to the number of rigid Young tableaux

$$\dim(V(\Lambda)_\eta) = |{}_s\mathfrak{B}^{(3)}_m| = \dim\left(L(\omega_n + \tilde{\omega}_{n-s})_{\omega_n+\tilde{\omega}_{n-m}}\right),$$

and the multiplicity of $\eta \in \mathrm{smax}^+_{\mathfrak{D}}(\Lambda|3)$ of index $(m, s-1)$ is equal to the number of spin rigid Young tableaux

$$\dim(V(\Lambda)_\eta) = |{}_s\mathfrak{D}^{(k)}_m| = \dim\left(L(\omega_n + \tilde{\omega}_{n-s})_\mu\right),$$

where $\mu = \omega_n + \tilde{\omega}_{n-m-1}$ if $m \not\equiv_2 s$ and $\mu = \omega_{n-1} + \tilde{\omega}_{n-m-1}$ if $m \equiv_2 s$.

In this chapter, we will prove that these multiplicities are equal to the generalized Motzkin numbers and the generalized Riordan numbers respectively.

THEOREM 8.1. *For $m \geqslant s \geqslant 0$, we have*

$$|{}_s\mathfrak{B}^{(3)}_m| = \mathsf{M}_{(m,s)}.$$

THEOREM 8.2. *For $m \geqslant s \geqslant 0$, we have*

$$|{}_s\mathfrak{D}^{(3)}_m| = \mathsf{R}_{(m+1,s)}.$$

REMARK 8.3.

(1) Note that $|{}_0\mathfrak{D}^{(3)}_0| = 0 = \mathsf{R}_{(1,0)}$. For $m \geqslant 1$, we have proved in Corollary 6.17 that

$$|{}_0\mathfrak{D}^{(3)}_m| = |{}_1\mathfrak{D}^{(3)}_{m-1}|.$$

Hence

$$|{}_1\mathfrak{D}^{(3)}_m| = \mathsf{R}_{(m+1,1)} = \mathsf{R}_{(m+2,0)} = |{}_0\mathfrak{D}^{(3)}_{m+1}|.$$

Thus, for Theorem 8.2, it is enough to prove when $s \geqslant 1$.

(2) Note that $\dim L(3\omega_n)_{3\omega_n} = 1 = \mathsf{R}_{(0,0)}$. In (5.13), we saw that $\tilde{\omega}_{n-1} + \omega_{n-1}$ is not a dominant weight of $L(3\omega_n)$. Then Theorem 8.2 can be restated as

$$\mathsf{R}_{(m,s)} = \dim L(\omega_n + \tilde{\omega}_{n-s})_{\tilde{\omega}_{n-m}+\omega_{n-\delta(m\not\equiv_2 s)}} \qquad \text{for any } m \geqslant s \geqslant 0,$$

which explains the relationship with Riordan triangle better.

In Section 8.1, we show Theorems 8.1 and 8.2 using the Robinson–Schensted algorithm. In Section 8.2 we prove Theorem 8.1 using a generalization of the insertion scheme in chapter 7.

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8.1. Proof by the RS algorithm

Up until now, in this paper, we have used reverse standard Young tableaux. However, in this section we will consider standard Young tableaux (or SYTs for short), which are more suitable for the usual Robinson–Schensted algorithm.

Recall that a composition $\lambda = (\lambda_1, \dots, \lambda_k)$ is called *almost-even* if the number of odd parts is exactly 1 or 2. Note that for an almost-even composition λ of m, the number of odd parts is 1 if m is odd, and 2 if m is even. An *almost-even partition* is a partition that is almost-even when considered as a composition.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition. We say that λ is a *parity partition* if $\lambda_i \equiv_2 \lambda_j$ for all $1 \leqslant i, j \leqslant k$.

Definition 8.4.

(1) Let $\mathcal{S}_m^{(k)}$ be the set of SYTs of shape $\lambda \vdash m$ for some partition $\lambda = (\lambda_1, \dots, \lambda_k)$.
(2) Let ${}_s\mathcal{S}_m^{(k)}$ be the set of SYTs of shape $\lambda/(s^{k-1}) \vdash m$ for some partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of size $(m + s(k-1))$.
(3) Let ${}_s\mathcal{P}_m^{(k)}$ be the set of SYTs of shape $\lambda/(s^{k-1}) \vdash m$ for some parity partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of size $(m + s(k-1))$.
(4) Let ${}_s\mathcal{AE}_m^{(k)}$ be the set of SYTs of shape $\lambda/(s^{k-1}) \vdash m$ for some partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of size $(m+s(k-1))$ such that $(\lambda_1 - s, \dots, \lambda_{k-1} - s, \lambda_k + s)$ is almost-even.

Using the obvious bijection between the SYTs and the reverse standard Young tableaux, we obtain the following lemma.

Lemma 8.5. *We have*

$$|{}_s\mathfrak{B}_m^{(k)}| = |{}_s\mathcal{S}_m^{(k)}| - |{}_{s-1}\mathcal{S}_m^{(k)}|, \tag{8.1}$$

$$|{}_s\mathfrak{D}_m^{(k)}| = |{}_s\mathcal{AE}_m^{(k)}| - |{}_{s-2}\mathcal{AE}_m^{(k)}|, \tag{8.2}$$

where we define ${}_t\mathcal{S}_m^{(k)} = {}_t\mathcal{AE}_m^{(k)} = \varnothing$ *if* $t < 0$.

In order to prove Theorems 8.1 and 8.2, we will find formulas for $|{}_s\mathcal{S}_m^{(3)}|$ and $|{}_s\mathcal{AE}_m^{(3)}|$. We need the following lemma which can be taken as an equivalent definition of ${}_s\mathcal{AE}_m^{(3)}$. Notice that this lemma is not true for ${}_s\mathcal{AE}_m^{(k)}$ in general.

Lemma 8.6. *The set* ${}_s\mathcal{AE}_m^{(3)}$ *consists of the SYTs of shape* $\lambda/(s, s) \vdash m$ *for some almost-even partition* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *of size* $m + 2s$.

Proof. It is sufficient to show that $(\lambda_1, \lambda_2, \lambda_3)$ is almost-even if and only if $(\lambda_1 - s, \lambda_2 - s, \lambda_3 + s)$ is almost-even. This is trivial if s is even. Suppose that s is odd. Let t be the number of odd parts in $(\lambda_1, \lambda_2, \lambda_3)$. Then the number of odd parts in $(\lambda_1 - s, \lambda_2 - s, \lambda_3 + s)$ is $3 - t$. Since $t \in \{1, 2\}$ if and only if $3 - t \in \{1, 2\}$, we have that $(\lambda_1, \lambda_2, \lambda_3)$ is almost-even if and only if $(\lambda_1 - s, \lambda_2 - s, \lambda_3 + s)$ is almost-even. $\square$

Our main tool is the Robinson–Schensted algorithm. Let us first fix some notations. A *permutation* of $\{1, 2, \dots, n\}$ is a bijection $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. We denote by $\mathfrak{S}_n$ the set of permutations of $\{1, 2, \dots, n\}$. As usual, we will also write a permutation $\pi \in \mathfrak{S}_n$ as a word $\pi = \pi_1 \pi_2 \dots \pi_n$, where $\pi_i = \pi(i)$.

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DEFINITION 8.7. An *involution* is a permutation $\pi \in \mathfrak{S}_n$ such that π^2 is the identity permutation $12\dots n$. We denote by $\mathcal{I}_n$ the set of involutions in $\mathfrak{S}_n$. Let $\pi \in \mathcal{I}_n$. Then for every $1 \leqslant i \leqslant n$, we have either $\pi(i) = i$ or $\pi(i) = j$ and $\pi(j) = i$ for some $j \neq i$. If $\pi(i) = i$, we call i a *fixed point* of π. If $\pi(i) = j$ for $i \neq j$, we say that i and j are *connected* in π. If there are no four integers $a < b < c < d$ such that a and d are connected and b and c are connected in π, we say that π is *non-nesting*. We denote by $\mathcal{NI}_n$ the set of non-nesting involutions in $\mathcal{I}_n$.

DEFINITION 8.8. For a permutation $\pi \in \mathfrak{S}_n$ and an integer $0 \leqslant k \leqslant n$, we denote by $\pi_{\leqslant k}$ the permutation in $\mathfrak{S}_k$ obtained from π by removing every integer greater than k. Similarly, for a SYT T with n cells and an integer $0 \leqslant k \leqslant n$, we denote by $T_{\leqslant k}$ the SYT with k cells obtained from T by removing every cell with entry greater than k.

For a permutation $\pi \in \mathfrak{S}_n$, let $P(\pi)$ and $Q(\pi)$ be the insertion tableau and the recording tableau respectively via the Robinson–Schensted algorithm. The following properties of the Robinson–Schensted algorithm are well known, see [**35**].

- The map $\pi \mapsto (P(\pi), Q(\pi))$ is a bijection from $\mathfrak{S}_n$ to the set of pairs (P, Q) of SYTs of the same shape with n cells.
- For $\pi \in \mathfrak{S}_n$, we have $P(\pi^{-1}) = Q(\pi)$. Therefore, the map $\pi \mapsto P(\pi)$ gives a bijection from $\mathcal{I}_n$ to the set of SYTs with n cells.
- For $\pi \in \mathfrak{S}_n$ and $1 \leqslant k \leqslant n$, we have $P(\pi_{\leqslant k}) = P(\pi)_{\leqslant k}$.
- For $\pi = \pi_1 \dots \pi_n \in \mathfrak{S}_n$, the number of rows of $P(\pi)$ is equal to the length of a longest decreasing subsequence of $\pi_1 \dots \pi_n$.

These properties implies the following proposition.

PROPOSITION 8.9. *The map $\pi \mapsto P(\pi)$ is a bijection from $\mathcal{NI}_n$ to $\mathcal{S}_n^{(3)}$.*

The following lemma is the main lemma in this section.

LEMMA 8.10. *Let ${}_s\mathcal{NI}_m$ be the set of elements $\pi \in \mathcal{NI}_{2s+m}$ satisfying the following condition: there exists an integer $0 \leqslant t \leqslant s$ such that*

- *$2i - 1$ and $2i$ are connected in π for all $1 \leqslant i \leqslant t$,*
- *$2j - 1$ is connected to an integer greater than $2s$ and $2j$ is a fixed point for all $t + 1 \leqslant j \leqslant s$.*

Let ${}_s\overline{\mathcal{S}}_m^{(3)}$ be the set of elements $T \in \mathcal{S}_{2s+m}^{(3)}$ satisfying the following condition: $T_{\leqslant 2s}$ is the SYT of shape (s, s) such that the ith column consists of $2i - 1$ and $2i$ for all $1 \leqslant i \leqslant s$.

Then the map $\pi \mapsto P(\pi)$ is a bijection from ${}_s\mathcal{NI}_m$ to ${}_s\overline{\mathcal{S}}_m^{(3)}$.

PROOF. Let $\pi \in \mathcal{I}_{2s+m}$ and $T = P(\pi) \in \mathcal{S}_{2s+m}$. It is sufficient to show that $\pi \in {}_s\mathcal{NI}_m$ if and only if $T \in {}_s\overline{\mathcal{S}}_m^{(3)}$.

Suppose that $\pi \in {}_s\mathcal{NI}_m$. Then we have

$$T_{\leqslant 2s} = P(\pi)_{\leqslant 2s} = P(\pi_{\leqslant 2s}).$$

Since $\pi \in \mathcal{I}_{2s+m}$, we obtain that

$$\pi_{\leqslant 2s} = 2, 1, 4, 3, \dots, 2t - 1, 2t, 2t + 2, 2t + 4, \dots, 2s, 2t + 1, 2t + 3, \dots, 2s - 1.$$

Then $T_{\leqslant 2s} = P(\pi)_{\leqslant 2s} = P(\pi_{\leqslant 2s})$ is the desired SYT of shape (s, s) and we obtain $T \in {}_s\overline{\mathcal{S}}_m^{(3)}$.

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Now suppose that $T \in {}_s\overline{\mathcal{S}}_m^{(3)}$. Let t be the largest integer such that $2i-1$ and $2i$ are connected in π for all $1 \leqslant i \leqslant t$. If there is no such integer, we set $t = 0$. If $t \geqslant s$, we are done. Assume that $t < s$. By the definition of t, we have that $2t+1$ is connected to some integer $j > 2t+2$ in π. We claim that $2t+2$ is a fixed point. For a contradiction, suppose that $2t+2$ is connected to some integer $r > 2t+2$ in π. If $r < j$, then the four integers $2t+1 < 2t+2 < r < j$ violate the condition for a non-nesting involution, which is a contradiction. If $r > j$, then

$$\pi_{\leqslant 2t+2} = 2, 1, 4, 3, \ldots, 2t, 2t-1, 2t+1, 2t+2.$$

The insertion tableau of this permutation is not equal to $T_{\leqslant 2t+2}$, which is a contradiction to

$$P(\pi_{\leqslant 2t+2}) = P(\pi)_{\leqslant 2t+2} = T_{\leqslant 2t+2}.$$

Therefore, $2t+2$ must be a fixed point of π. Moreover, $2t+1$ is connected to an integer greater than $2s$. To see this suppose that $2t+1$ is connected to an integer $j \leqslant 2s$. Then $\pi_{\leqslant 2s}$ has a decreasing sequence $j, 2t+2, 2t+1$ of length 3. Then the insertion tableau of $\pi_{\leqslant 2s}$ would have at least 3 rows and it cannot be $T_{\leqslant 2s}$. Therefore, $2t+1$ must be connected to an integer greater than $2s$. By the same argument, we can show that $2i-1$ is connected to an integer greater than $2s$ and $2i$ is a fixed point for all $t \leqslant i \leqslant s$. This finishes the proof. $\square$

Now we recall a well-known bijection between the non-nesting involutions and the Motzkin paths. For $\pi \in \mathcal{NI}_n$, let $\phi(\pi)$ be the Motzkin path L constructed as follows. If i is a fixed point of π, the ith step of L is a horizontal step. If i and j are connected in π for $i < j$, the ith step of L is an up step and the jth step of L is a down step. It is easy to see that ϕ is a bijection from $\mathcal{NI}_n$ to the set of Motzkin paths of length n.

Proposition 8.11. *We have*

$$|{}_s\mathcal{S}_m^{(3)}| = \sum_{t=0}^{s} \mathsf{M}_{(m,t)}.$$

Proof. First, observe that there is a natural bijection from ${}_s\mathcal{S}_m^{(3)}$ to the set ${}_s\overline{\mathcal{S}}_m^{(3)}$ in Lemma 8.10. Such a bijection can be constructed as follows. For $T \in {}_s\mathcal{S}_m^{(3)}$, let T' be the SYT obtained from T by increasing every entry in T by $2s$ and filling the two empty cells in the ith column with $2i-1$ and $2i$ for all $1 \leqslant i \leqslant s$. Thus, by Lemma 8.10, we have

$$|{}_s\mathcal{S}_m^{(3)}| = |{}_s\overline{\mathcal{S}}_m^{(3)}| = |{}_s\mathcal{NI}_m|.$$

Now consider $\pi \in {}_s\mathcal{NI}_m$ and the corresponding Motzkin path $\phi(\pi)$ from $(0,0)$ to $(2s+m, 0)$. By definition of ${}_s\mathcal{NI}_m$ in Lemma 8.10, there is an integer $0 \leqslant t \leqslant s$ such that the first $2s$ steps of $\phi(\pi)$ are $(UD)^t(UH)^{s-t}$. Therefore if we take L to be the path consisting of the first m steps of the reverse path of $\phi(\pi)$, then L is a Motzkin path from $(0,0)$ to (m,t). It is easy to see that the map $\pi \mapsto L$ is a bijection from ${}_s\mathcal{NI}_m$ to the set of all Motzkin paths from $(0,0)$ to (m,t) for some $0 \leqslant t \leqslant m$. Thus we have

$$|{}_s\mathcal{NI}_m| = \sum_{t=0}^{s} \mathsf{M}_{(m,t)},$$

which completes the proof. $\square$

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Now we have all the ingredients to prove Theorem 8.1.

PROOF OF THEOREM 8.1. By (8.1) and Proposition 8.11, we have

$$|{}_s\mathfrak{B}_m^{(3)}| = |{}_s\mathcal{S}_m^{(3)}| - |{}_{s-1}\mathcal{S}_m^{(3)}| = \mathsf{M}_{(m,s)}. \qquad \square$$

In order to prove Theorem 8.2 we need two lemmas.

LEMMA 8.12. *For integers $m \geqslant 0$ and $s \geqslant 0$, we have*

$$|{}_s\mathcal{S}_m^{(3)}| = |{}_s\mathcal{P}_m^{(3)}| + |{}_s\mathcal{AE}_m^{(3)}|, \quad |{}_s\mathcal{AE}_m^{(3)}| = |{}_s\mathcal{P}_{m+1}^{(3)}| \quad \textit{and} \quad |{}_s\mathcal{S}_m^{(3)}| = |{}_s\mathcal{P}_m^{(3)}| + |{}_s\mathcal{P}_{m+1}^{(3)}|.$$

PROOF. For the first identity, consider a tableau $T \in {}_s\mathcal{S}_m^{(3)}$. Then the shape of T is $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda/(s,s) \vdash m$. It is easy to see that λ is either a parity partition or an almost-even partition. Thus we obtain the first identity.

For the second identity, consider a tableau $T \in {}_s\mathcal{AE}_m^{(3)}$. Then the shape of T is an almost-even partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda/(s,s) \vdash m$. If m is even, then only one of $\lambda_1, \lambda_2, \lambda_3$ is even, and if m is odd, only one of them is odd. Thus, in any case, one of $\lambda_1, \lambda_2, \lambda_3$ has a different parity than the others. Suppose that λ_i is the one with the different parity. Let T' be the tableau obtained from T by increasing every entry by 1 and add a new cell at the end of λ_i. Then $T' \in {}_s\mathcal{P}_{m+1}^{(3)}$. The map $T \mapsto T'$ gives a bijection from ${}_s\mathcal{AE}_m^{(3)}$ to ${}_s\mathcal{P}_{m+1}^{(3)}$. Thus we obtain the second identity.

The third identity follows from the first two identities. $\square$

LEMMA 8.13. *For integers $m \geqslant 0$ and $s \geqslant 1$, we have*

$$|{}_s\mathcal{P}_m^{(3)}| - |{}_{s-2}\mathcal{P}_m^{(3)}| = \mathsf{R}_{(m,s)}.$$

PROOF. We will prove this by induction on m when $s \geqslant 1$ is fixed. If $m = 0$, then both sides are zero. Now suppose that the statement

$$|{}_s\mathcal{P}_m^{(3)}| - |{}_{s-2}\mathcal{P}_m^{(3)}| = \mathsf{R}_{(m,s)} \tag{8.3}$$

is true for $m \geqslant 0$. By Lemma 8.12, we have

$$|{}_s\mathcal{S}_m^{(3)}| - |{}_{s-2}\mathcal{S}_m^{(3)}| = |{}_s\mathcal{P}_m^{(3)}| + |{}_s\mathcal{P}_{m+1}^{(3)}| - |{}_{s-2}\mathcal{P}_m^{(3)}| - |{}_{s-2}\mathcal{P}_{m+1}^{(3)}|.$$

By Proposition 8.11 and Proposition 4.11, we have

$$|{}_s\mathcal{S}_m^{(3)}| - |{}_{s-2}\mathcal{S}_m^{(3)}| = \mathsf{M}_{(m,s)} + \mathsf{M}_{(m,s-1)} = \mathsf{R}_{(m,s)} + \mathsf{R}_{(m+1,s)}.$$

Thus,

$$(|{}_s\mathcal{P}_m^{(3)}| - |{}_{s-2}\mathcal{P}_m^{(3)}|) + (|{}_s\mathcal{P}_{m+1}^{(3)}| - |{}_{s-2}\mathcal{P}_{m+1}^{(3)}|) = \mathsf{R}_{(m,s)} + \mathsf{R}_{(m+1,s)}. \tag{8.4}$$

By (8.3) and (8.4), we obtain that

$$|{}_s\mathcal{P}_{m+1}^{(3)}| - |{}_{s-2}\mathcal{P}_{m+1}^{(3)}| = \mathsf{R}_{(m+1,s)}.$$

Thus, by induction, the statement is true for all $m \geqslant 0$. $\square$

Now we give a proof of Theorem 8.2.

PROOF OF THEOREM 8.2. By (8.2), Lemmas 8.12 and 8.13, we have

$$|{}_s\mathfrak{D}_m^{(3)}| = |{}_s\mathcal{AE}_m^{(3)}| - |{}_{s-2}\mathcal{AE}_m^{(3)}| = |{}_s\mathcal{P}_{m+1}^{(3)}| - |{}_{s-2}\mathcal{P}_{m+1}^{(3)}| = \mathsf{R}_{(m+1,s)}.$$

Thus, we have $|{}_s\mathfrak{D}_m^{(3)}| = \mathsf{R}_{(m+1,s)}$. Our assertion for $s = 0$ follows from Corollary 6.17. $\square$

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8.2. Proof by insertion scheme

In this section, we will prove that all the multiplicities of $\eta \in \mathrm{smax}^+_{\mathfrak{B}}(\Lambda|3)$ are generalized Motzkin numbers $\mathsf{M}_{(m,s)}$ using an insertion scheme which generalizes the one in chapter 7. Namely, we will introduce a new kind of jeu du taquin which realizes the recursive formula (4.2):

$$\mathsf{M}_{(m,s)} = \mathsf{M}_{(m-1,s)} + \mathsf{M}_{(m-1,s-1)} + \mathsf{M}_{(m-1,s+1)}.$$

As its corollary, we have a bijective map between $\{{}_s\mathfrak{B}^{(3)}_m \mid 0 \leqslant s \leqslant m\}$ and the set of all Motzkin paths.

Note that, for $T = (\lambda, \mu, \nu) \in {}_s\mathfrak{B}^{(3)}_m$,

$$\begin{cases} \lambda_1 = m \text{ or } \nu_1 = m & \text{if } s > 0, \\ \lambda_1 = m & \text{if } s = 0. \end{cases}$$

Lemma 8.14. *For $T = (\lambda, \mu, \nu) \in {}_s\mathfrak{B}^{(3)}_{m-1}$, we have*

$$T \underset{1}{*} (m) \in {}_s\mathfrak{B}^{(3)}_m \quad \text{and} \quad T \underset{3}{*} (m) \in {}_{s+1}\mathfrak{B}^{(3)}_m.$$

Proof. In the definition of ${}_s\mathfrak{B}^{(3)}_{m-1}$ (Definition 6.4), the conditions are relevant only with μ and ν. Hence $T \underset{1}{*} (m) \in {}_s\mathfrak{B}^{(3)}_m$, since $(m) * \lambda \supset \mu$ and nothing happens to μ and ν. The second assertion follows from the second assertion of Lemma 7.8. $\square$

Example 8.15. The set ${}_0\mathfrak{B}^{(3)}_3$ consists of four tableaux

$$\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 3&2\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 3&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array},$$

and the set ${}_1\mathfrak{B}^{(3)}_3$ has five elements

$$\begin{array}{|c|c|}\hline \cdot&2\\\hline \cdot&1\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline \cdot&3\\\hline \cdot&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline \cdot&2&1\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline \cdot&3&1\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline \cdot&3&2\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}.$$

Using the operations $\underset{3}{*}(4)$ and $\underset{1}{*}(4)$, we get the elements in ${}_1\mathfrak{B}^{(3)}_4$ from ${}_0\mathfrak{B}^{(3)}_3$ and ${}_1\mathfrak{B}^{(3)}_3$ as follows:

$$\begin{array}{|c|c|c|c|}\hline \cdot&3&2&1\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline \cdot&3&2\\\hline \cdot&\multicolumn{1}{c|}{1}\\\cline{1-2} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline \cdot&3&1\\\hline \cdot&\multicolumn{1}{c|}{2}\\\cline{1-2} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline \cdot&3\\\hline \cdot&2\\\hline 4&1\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline \cdot&4&2\\\hline \cdot&\multicolumn{1}{c|}{1}\\\cline{1-2} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array},$$

$$\begin{array}{|c|c|c|}\hline \cdot&4&3\\\hline \cdot&\multicolumn{1}{c|}{1}\\\cline{1-2} \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot&4&2&1\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot&4&3&1\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \cdot&4&3&2\\\hline \multicolumn{1}{|c|}{\cdot}\\\cline{1-1} \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}.$$

Remark 8.16. One can observe that an element $(\lambda, \mu, \nu) \in {}_s\mathfrak{B}^{(3)}_m$ obtained from ${}_s\mathfrak{B}^{(3)}_{m-1}$ in the above way can be distinguished from others by the following characterization:

$$\lambda_1 = m \quad \text{and} \quad (\lambda_{\geqslant 2}, \mu, \nu) \in {}_s\mathfrak{B}^{(3)}_{m-1}.$$

Similarly, an element $(\lambda, \mu, \nu) \in {}_s\mathfrak{B}^{(3)}_m$ obtained from ${}_{s-1}\mathfrak{B}^{(3)}_{m-1}$ can be distinguished from others by the following characterization:

$$\nu_1 = m \quad \text{and} \quad (\lambda, \mu, \nu_{\geqslant 2}) \in {}_{s-1}\mathfrak{B}^{(3)}_{m-1}.$$

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But there are elements in ${}_s\mathfrak{B}_m^{(3)}$ which cannot be obtained from ${}_s\mathfrak{B}_{m-1}^{(3)}$ or ${}_{s-1}\mathfrak{B}_{m-1}^{(3)}$. For example, there are elements in ${}_1\mathfrak{B}_4^{(3)}$ which do not appear in Example 8.15:

$$\begin{array}{|c|c|}\hline \cdot & 4 \\ \hline \cdot & 3 \\ \hline 2 & 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline \cdot & 4 \\ \hline \cdot & 2 \\ \hline 3 & 1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|}\hline \cdot & 4 & 1 \\ \hline \cdot & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}\,.$$

LEMMA 8.17. *Let $T = (\lambda, \mu, \nu) \in {}_s\mathfrak{B}_m^{(3)}$ with $m \geqslant 1$. If $\nu_1 = m$, then*

$$s \geqslant 1 \quad \text{and} \quad T = T' \underset{3}{*} (m) \quad \text{for some } T' \in {}_{s-1}\mathfrak{B}_{m-1}^{(3)}.$$

PROOF. This assertion follows from the definition of ${}_s\mathfrak{B}_m^{(k)}$ directly. □

Now we will construct an algorithm to get elements (λ, μ, ν) of ${}_s\mathfrak{B}_m^{(3)}$ from ${}_{s+1}\mathfrak{B}_{m-1}^{(3)}$. By Remark 8.16 and Lemma 8.17, such an element in ${}_s\mathfrak{B}_m^{(3)}$ should satisfy the following conditions:

$$\lambda_1 = m \quad \text{and} \quad (\lambda_{\geqslant 2}, \mu, \nu) \notin {}_s\mathfrak{B}_{m-1}^{(3)} \text{ (or equivalently } \lambda_{\geqslant 2} \not\supset \mu). \tag{8.5}$$

In tableau notation, the construction of $T = (\lambda, \mu, \nu) \in {}_s\mathfrak{B}_m^{(3)}$ from $T' = (\lambda', \mu', \nu') \in {}_{s+1}\mathfrak{B}_{m-1}^{(3)}$ can be understood as filling the top-rightmost empty cell with m and performing jeu de taquin to fill the empty cell right below. For example, for given

$$T' = \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & \cdot & \cdot & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-6} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \in {}_3\mathfrak{B}_{12}^{(3)},$$

we put 13 in the top blue cell

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & \cdot & 13 & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-6} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array}\,. \tag{8.6}$$

Now we explain the jeu de taquin to fill the remaining blue cell.

ALGORITHM 8.18 (Rigid jeu de taquin). Assume that T' is given, and fill the top-rightmost empty cell with m as described above. Take the reference point to be the empty cell in the second row.

(1) Perform $\swarrow^1$ on the north-east cell in the first row and $\leftarrow_1$ on the other cells in the first row. If the resulting tableau is standard, terminate the process; otherwise (recover the original tableau and) go to (2).
(2) Perform $\underset{3}{\uparrow}$ on the south cell in the third row and $\leftarrow_3$ on the other cells in the third row. If the resulting tableau is standard, terminate the process; otherwise (recover the original tableau and) go to (3).
(3) Perform $\leftarrow_2$ on the east cell to switch the position of the empty cell and go to (1).

Denote the resulting tableau by T. We call this process the *rigid jeu de taquin* (*of level* 3).

By applying the operation (1) of Algorithm 8.18 to (8.6), we have

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 13 & 10 & 8 & 7 \\ \hline \cdot & \cdot & 12 & 11 & 9 & 1 \\ \hline 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array}\,.$$

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The cell $\boxed{12}$ moves from the first row to second row $\swarrow^1$ and the cells $\boxed{10}\boxed{8}\boxed{7}$ located on the right hand side of $\boxed{12}$ are shifted by 1 to the left $\leftarrow_1$. Thus we shall denote the operation (1) by $\swarrow^1 \leftarrow_1$. Clearly, the resulting tableau is not standard.

We apply the operation (2) in Algorithm 8.18 to (8.6) to obtain

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & 13 & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & 4 & 11 & 9 & 1 & \\ \hline 6 & 5 & 3 & 2 & & & \\ \hline \end{array}\ .$$

The cell $\boxed{4}$ moves from the third row to second row $\underset{3}{\uparrow}$ and the cells $\boxed{3}\boxed{2}$ located on the right hand side of $\boxed{4}$ are shifted by 1 to the left $\leftarrow_3$. Thus we shall denote the operation (2) by $\underset{3}{\uparrow}\leftarrow_3$. The resulting tableau is not standard either.

Now perform the operation (3) in Algorithm 8.18 to (8.6) and obtain

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & 13 & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 & \\ \hline 6 & 5 & 4 & 3 & 2 & & \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & 13 & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & 11 & \cdot & 9 & 1 & \\ \hline 6 & 5 & 4 & 3 & 2 & & \\ \hline \end{array}\ . \tag{8.7}$$

One can easily see that neither of the operations (1) and (2) performed on the new tableau in (8.7) produces a standard tableau. Thus we perform the operation (3) to obtain

$$\begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & 13 & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & 11 & 9 & \cdot & 1 & \\ \hline 6 & 5 & 4 & 3 & 2 & & \\ \hline \end{array}\ . \tag{8.8}$$

Now we perform the operation (1) on the tableau (8.8) and obtain

$$\begin{array}{|c|c|c|c|c|c|} \hline \cdot & \cdot & 13 & 12 & 10 & 7 \\ \hline \cdot & \cdot & 11 & 9 & 8 & 1 \\ \hline 6 & 5 & 4 & 3 & 2 & \\ \hline \end{array} \in {}_2\mathfrak{B}_{13}^{(3)},$$

which is standard. In this way, we have obtained a tableau $T \in {}_2\mathfrak{B}_{13}^{(3)}$ from $T' \in {}_3\mathfrak{B}_{12}^{(3)}$.

Clearly, the process terminates in finite steps, and one can check that the resulting tableau T in Algorithm 8.18 satisfies the conditions in (8.5) and is contained in ${}_s\mathfrak{B}_m^{(3)}$. Furthermore, we can construct the reverse of the rigid jeu de taquin easily.

Algorithm 8.19 (Reverse rigid jeu de taquin). Assume that $T = (\lambda, \mu, \nu) \in {}_s\mathfrak{B}_m^{(3)}$ satisfies (8.5). Remove m from its cell. Take the reference point to be the leftmost non-empty cell, say $\mathfrak{c}$, in the second row.

(1) Perform $\rightarrow_3$ on the cells in the third row from the rightmost cell all the way to the south cell of $\mathfrak{c}$, and $\overset{2}{\downarrow}$ on the cell $\mathfrak{c}$, and $\rightarrow_2$ on the cells, if any, which were at the left-side of $\mathfrak{c}$. If the resulting tableau is standard, terminate the process; otherwise (recover the original tableau and) go to (2).

(2) Perform $\rightarrow_1$ on the cells in the first row from the rightmost cell all the way to the northeast cell of $\mathfrak{c}$, and $\nearrow_2$ on the cell $\mathfrak{c}$, and $\rightarrow_2$ on the cells, if any, which were at the left-side of $\mathfrak{c}$. If the resulting tableau is standard, terminate the process; otherwise (recover the original tableau and) go to (3).

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(3) Take the east cell to be new $\mathfrak{c}$ for the next round, and make it the reference point, and go to (1).

Denote the resulting tableau by T'. We call this process the *reverse rigid jeu de taquin* (*of level* 3).

One can check that the resulting tableau T' in Algorithm 8.19 is contained in ${}_{s+1}\mathfrak{B}^{(3)}_{m-1}$. It is also easy to see that Algorithm 8.19 is an inverse process of Algorithm 8.18.

EXAMPLE 8.20. For a given

$$T = \begin{array}{|c|c|c|c|c|c|}\hline \cdot & 13 & 12 & 10 & 7 & 5 \\ \hline \cdot & 11 & 9 & 8 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array} \in {}_1\mathfrak{B}^{(3)}_{13},$$

one can check that it satisfies the conditions in (8.5). Now we delete 13.

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 12 & 10 & 7 & 5 \\ \hline \cdot & 11 & 9 & 8 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array}$$

Since $\nu_1 = 6 < 11 = \mu_1$, (1) in Algorithm 8.19 fails, and since $\mu_1 = 11 < 12 = \lambda_1$, (2) fails. Hence we apply (3) to change the reference point (in blue color):

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 12 & 10 & 7 & 5 \\ \hline \cdot & 11 & 9 & 8 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array}$$

As (1) and (2) fail again, we apply (3) to obtain

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdot & \cdot & 12 & 10 & 7 & 5 \\ \hline \cdot & 11 & 9 & 8 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array}$$

Now (2) works to produce a standard tableau:

$$T' = \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & \cdot & 12 & 10 & 8 & 7 & 5 \\ \hline \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array} \in {}_2\mathfrak{B}^{(3)}_{12}.$$

To check that it is an inverse process, we add 13 again and see:

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & 13 & 12 & 10 & 8 & 7 & 5 \\ \hline \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|c|c|}\hline \cdot & 13 & 12 & 10 & 8 & 7 & 5 \\ \hline \cdot & 11 & 9 & \cdot & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|c|}\hline \cdot & 13 & 12 & 10 & 7 & 5 \\ \hline \cdot & 11 & 9 & 8 & 1 \\ \cline{1-5} 6 & 4 & 3 & 2 \\ \cline{1-4}\end{array} = T.$$

THEOREM 8.21. *The rigid-type jeu de taquin gives a bijection between*

$${}_{s+1}\mathfrak{B}^{(3)}_{m-1} \quad \textit{and} \quad {}_s\mathfrak{B}^{(3)}_m \setminus \left({}_s\mathfrak{B}^{(3)}_{m-1} \underset{1}{*} (m) \bigsqcup {}_{s-1}\mathfrak{B}^{(3)}_{m-1} \underset{3}{*} (m) \right).$$

PROOF. Our assertion follows from Algorithm 8.18 and Algorithm 8.19 which are inverses to each other. $\square$

Now we give another proof of Theorem 8.1.

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Proof of Theorem 8.1. From Theorem 8.21, we have

$$|_{s+1}\mathfrak{B}^{(3)}_{m-1}| + |_{s}\mathfrak{B}^{(3)}_{m-1}| + |_{s-1}\mathfrak{B}^{(3)}_{m-1}| = |_{s}\mathfrak{B}^{(3)}_{m}|,$$

which is the same as (4.2). Since we have $|_{m}\mathfrak{B}^{(3)}_{m}| = 1$, we are done. $\square$

Corollary 8.22. *We have a bijective map between* ${}_{s}\mathfrak{B}^{(3)}_{m}$ *and* $\mathfrak{M}_{(m,s)}$ *where* $\mathfrak{M}_{(m,s)}$ *is the set of Motzkin paths ending at* (m, s)

Proof. Assume that we have $T \in {}_{s}\mathfrak{B}^{(3)}_{m}$. For each step of the reverse rigid jeu de taquin (removing the cell $\boxed{m}$), we record the vector v_m as

- $(1, 0)$ if $T = T' \underset{1}{*} (m)$ for some $T' \in {}_{s}\mathfrak{B}^{(3)}_{m-1}$,
- $(1, 1)$ if $T = T' \underset{3}{*} (m)$ for some $T' \in {}_{s-1}\mathfrak{B}^{(3)}_{m-1}$,
- $(1, -1)$ if T can be obtained from $T' \in {}_{s+1}\mathfrak{B}^{(3)}_{m-1}$.

Then, by induction on m, we obtain the sequence of vectors corresponding to a Motzkin path. $\square$

Example 8.23. For

$$T = (\lambda, \mu, \nu) = \begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & 12 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 \\ \cline{1-6} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \in {}_{3}\mathfrak{B}^{(3)}_{12}$$

we see $\nu_1 \neq 12$ and

$$\begin{array}{|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & 11 & 9 & 1 \\ \hline 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \notin {}_{3}\mathfrak{B}^{(3)}_{11}.$$

Hence $v_{12} = (1, -1)$ and T can be obtained from

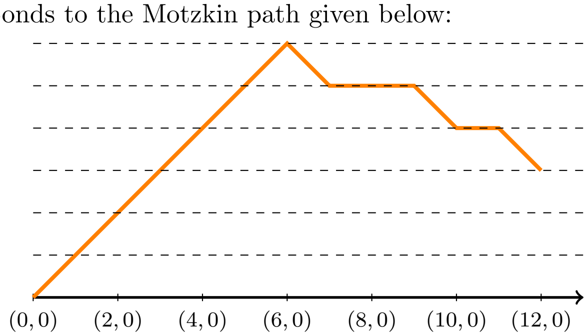
$$T' = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & 11 & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 9 & 1 \\ \cline{1-7} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \in {}_{4}\mathfrak{B}^{(3)}_{11}.$$

Now we have

$$\xleftarrow[(1,0)]{} \begin{array}{|c|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & 10 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & 9 & 1 \\ \cline{1-7} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \xleftarrow[(1,-1)]{} \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 9 & 8 & 7 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cline{1-7} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array}$$

$$\xleftarrow[(1,0)]{} \begin{array}{|c|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 8 & 7 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cline{1-7} 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \xleftarrow[(1,0)]{} \begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 7 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \hline 6 & 5 & 4 & 3 & 2 \\ \cline{1-5} \end{array} \xleftarrow[(1,-1)]{} \begin{array}{|c|c|c|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline 6 & 5 & 4 & 3 & 2 & 1 \\ \cline{1-6} \end{array} \xleftarrow[(1,1)^6]{} \varnothing.$$

Thus T corresponds to the Motzkin path given below:



Remark 8.24. In [4], Eu constructed a bijection between ${}_{0}\mathfrak{B}^{(3)}_{m}$ and $\mathfrak{M}_{(m,0)}$. His bijection gives paths different from those obtained by our bijection.

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CHAPTER 9

Some level k weight multiplicities when $k \to \infty$: Bessel triangle

In this chapter we will compute level k weight multiplicities $|{}_s\mathfrak{B}_m^{(k)}|$ and $|{}_s\mathfrak{D}_m^{(k)}|$ when k is as large as m (or $m/2$). Recall that we have ${}_0\mathfrak{B}_m^{(k)} = \mathfrak{B}_m^{(k)}$ and ${}_0\mathfrak{D}_m^{(k)} = \mathfrak{D}_m^{(k)}$. Let $\mathcal{R}_m$ be the set of reverse SYTs with m cells and $\mathcal{S}_m$ be the set of SYT with m cells.

First, observe that if $k \geqslant m$, the set $\mathfrak{B}_m^{(k)}$ is the same as the set $\mathcal{R}_m$. Since $|\mathcal{S}_m|$ is equal to the number of involutions in $\mathfrak{S}_m$, we have

$$\mathsf{B}_m^{(\infty)} := \lim_{k\to\infty} |\mathfrak{B}_m^{(k)}| = |\mathcal{R}_m| = |\mathcal{S}_m| = \sum_{s=0}^{\lfloor m/2 \rfloor} \binom{m}{2s} (2s-1)!!, \tag{9.1}$$

where $(2s-1)!! = 1 \cdot 3 \cdots (2s-1)$. Similarly, if $k \geqslant m$, the set $\mathfrak{D}_m^{(k)}$ becomes the set of Young tableaux with m cells that have exactly one or two rows of odd length depending on the parity of m. Using a well known property of the Robinson–Schensted algorithm we can deduce that $\lim_{k\to\infty} |\mathfrak{D}_m^{(k)}|$ is the number of involutions in $\mathcal{I}_m$ with one or two fixed points.

In Section 9.1 we find formulas for $|\mathfrak{D}_{2m}^{(k)}|$ when $k \geqslant m-1$ and for $|\mathfrak{D}_{2m-1}^{(k)}|$ when $k \geqslant m-2$. Our formulas (Theorems 9.2 and 9.3) imply that

$$\mathsf{D}_m^{(\infty)} := \lim_{k\to\infty} |\mathfrak{D}_m^{(k)}| = \begin{cases} m!! & \text{if } m \text{ is odd,} \\ \dfrac{m}{2} \times (m-1)!! & \text{if } m \text{ is even.} \end{cases} \tag{9.2}$$

In Section 9.2 we find a formula for $|{}_s\mathfrak{B}_m^{(k)}|$ when $k \geqslant m-s$ and compute the limit of $|{}_s\mathfrak{B}_m^{(k)}|$ as $k \to \infty$. In Section 9.3 we find a formula for $|{}_s\mathfrak{D}_m^{(k)}|$ when $k \geqslant m-s+1$ and compute the limit of $|{}_s\mathfrak{D}_m^{(k)}|$ as $k \to \infty$.

9.1. The limit of $|\mathfrak{D}_m^{(k)}|$ when $k \to \infty$

The following lemma is well-known ([**35**, Exercise 3.12]). Here we identify a reverse standard Young tableau with a standard Young tableau using the obvious bijection.

LEMMA 9.1. *The Robinson–Schensted algorithm gives a bijection between the set of Young tableaux of n cells with k columns of odd length and the set of involutions of $\{1, 2, \dots, n\}$ with k fixed points.*

Let $I(m,k)$ denote the number of involutions of $\{1, 2, \dots, m\}$ with k fixed points. It is easy to see that

$$I(2m,0) = I(2m-1,1) = (2m-1)!!, \qquad I(2m,2) = m \cdot I(2m,0) = m(2m-1)!!.$$

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THEOREM 9.2. *For an odd integer $2m-1$ and any $k \geqslant m$,*

$$|\mathfrak{D}_{2m-1}^{(k)}| = (2m-1)!!.$$

PROOF. Since $k \geqslant m$, any Young tableau of $2m-1$ cells has at most $m-1$ (nonzero) rows of even length. Thus $\mathfrak{D}_{2m-1}^{(k)}$ is the set of Young tableaux of $2m-1$ cells with exactly one row of odd length and there is no restriction on the number of rows. By taking the conjugate, this number is also equal to the number of Young tableaux of $2m-1$ cells with exactly one column of odd length. By Lemma 9.1, this is equal to the number of involutions of $\{1, 2, \ldots, 2m-1\}$ with one fixed point. Thus we get $|\mathfrak{D}_{2m-1}^{(k)}| = I(2m-1, 1) = (2m-1)!!$. $\square$

THEOREM 9.3. *For an even integer $2m$ and any $k \geqslant m+1$,*

$$|\mathfrak{D}_{2m}^{(k)}| = m(2m-1)!! = \frac{(2m)!}{(m-1)!2^m}.$$

PROOF. This can be shown by the same argument as in the proof of the previous theorem. $\square$

COROLLARY 9.4. *For each m,*

$$|\mathfrak{D}_{2m-1}^{(m-1)}| = (2m-1)!! - \mathsf{C}_m.$$

PROOF. Note that

$$\mathfrak{D}_{2m-1}^{(m)} \backslash \mathfrak{D}_{2m-1}^{(m-1)} = \mathcal{R}^\lambda$$

where $\lambda = (2, 2, \ldots, 2, 1) \vdash 2m-1$. Since $|\mathcal{R}^\lambda| = f^\lambda = \mathsf{C}_m$, our assertion follows. $\square$

By applying the same strategy as in Corollary 9.4, we have the following corollary:

COROLLARY 9.5. *For each m, we have*

(1) $|\mathfrak{D}_{2m}^{(m)}| = m(2m-1)!! - 3\dfrac{2m!}{(m-1)!(m+2)!}$.

(2) $|\mathfrak{D}_{2m-1}^{(m-2)}| = (2m-1)!! - \mathsf{C}_m - \dfrac{(2m-1)!}{m!(m-3)!} - \dbinom{2m-1}{m+1}$.

(3) $|\mathfrak{D}_{2m}^{(m-1)}| = m(2m-1)!! - 3\dfrac{2m!}{(m-1)!(m+2)!} - \dfrac{4}{m+2} \times \dfrac{(2m-1)!}{m!(m-2)!}$.

Since $\mathfrak{B}_m^{(k)}$ and $\mathfrak{D}_m^{(k)}$ can be understood as special cases of ${}_s\mathfrak{B}_m^{(k)}$ and ${}_s\mathfrak{D}_m^{(k)}$ respectively, in the next two sections we will investigate

$$\lim_{k\to\infty} |{}_s\mathfrak{B}_m^{(k)}| \quad \text{and} \quad \lim_{k\to\infty} |{}_s\mathfrak{D}_m^{(k)}|.$$

9.2. The limit of $|{}_s\mathfrak{B}_m^{(k)}|$ when $k \to \infty$

PROPOSITION 9.6. *Let $k \geqslant m - s + 2$. Then*

$$|{}_s\mathfrak{B}_m^{(k)}| = \binom{m}{s} \times \mathsf{B}_{m-s}^{(\infty)},$$

where $\mathsf{B}_m^{(\infty)}$ is the number defined in (9.1).

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PROOF. Let $T \in {}_s\mathfrak{B}_m^{(k)}$. Since the kth row of T has at least s cells, the first $k-1$ rows can have at most $m-s$ cells. Since $m-s \leqslant k-2$, the $(k-1)$st row must be empty. Thus the kth row of T has exactly s cells. Such a tableau can be constructed by selecting s integers from $\{1, 2, \dots, m\}$ for the kth row and filling the remaining $m-s$ integers in a Young diagram so that the entries are increasing in each row and column. The number of ways to do this is equal to $\binom{m}{s} \times \mathsf{B}_{m-s}^{(\infty)}$. $\square$

REMARK 9.7. By similar arguments, one can show the following identities:

$$|{}_s\mathfrak{B}_m^{(m-s+1)}| = \binom{m}{s} \times \mathsf{B}_{m-s}^{(\infty)} - \binom{m-1}{s-1}$$

and

$$|{}_s\mathfrak{B}_m^{(m-s)}| = \binom{m}{s} \times \mathsf{B}_{m-s}^{(\infty)} - \binom{m-1}{s-1}(m-s-1).$$

COROLLARY 9.8. *For positive integers $s \leqslant m$,*

$$ {}_s\mathsf{B}_m^{(\infty)} := \lim_{k\to\infty} |{}_s\mathfrak{B}_m^{(k)}| = \binom{m}{s} \times \mathsf{B}_{m-s}^{(\infty)}.$$

The triangular array consisting of $\{{}_s\mathsf{B}_m^{(\infty)}\}$ is given as follows:

					⋰	⋰	⋰	⋰	⋮
				1	5	30	140	700	⋯
			1	4	20	80	350	1456	⋯
		1	3	12	40	150	546	2128	⋯
	1	2	6	16	50	156	532	1856	⋯
1	1	2	4	10	26	76	232	764	⋯

where the bottom row is the number of involutions in $\mathfrak{S}_m$.

9.3. The limit of $|{}_s\mathfrak{D}_m^{(k)}|$ when $k \to \infty$

THEOREM 9.9. *Assume that we have a pair of positive integers $2 \leqslant s \leqslant m$ satisfying $s \not\equiv_2 m$. Then, for $k \geqslant m-s+3$, we have*

$$|{}_s\mathfrak{D}_m^{(k)}| = \binom{m+1}{s} \times (m-s)!!.$$

Therefore, we have a closed formula for the limit as follows:

$$ {}_s\mathsf{D}_m^{(\infty)} := \lim_{k\to\infty} |{}_s\mathfrak{D}_m^{(k)}| = \binom{m+1}{s} \times (m-s)!!. \tag{9.3}$$

PROOF. Let $T \in {}_s\mathfrak{D}_m^{(k)}$. By the same arguments as in the proof of Proposition 9.6, the kth row of T has $s-1$ or s cells. Now we consider the two cases separately.

(1) The kth row of T has s cells. Let T' be the tableau obtained from the first $k-1$ rows of T by relabeling the integers with $1, 2, \dots, m-s$ with respect to their relative order. Then T' is an almost even tableau of the odd number $m-s$. The number of such tableaux T' is $\mathsf{D}_{m-s}^{(\infty)} = (m-s)!!$. Since we can select the entries

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in the kth row of T freely, there are $\binom{m}{s}$ ways to do this. Thus, the number of tableaux T in this case is $\binom{m}{s}(m-s)!!$.

(2) The kth row of T has $s-1$ cells. Let T' be the tableau obtained from the first $k-1$ rows of T by relabeling the integers with $1, 2, \dots, m-s+1$ with respect to their relative order. Then all the rows of T' have even length. By the same arguments as in the proof of Theorem 9.2, the number of such tableaux T' is equal to $I(m-s+1, 0) = (m-s)!!$, the number of fixed-point free involutions. Similarly to the first case, the number of tableaux T in this case is $\binom{m}{s-1}(m-s)!!$.

By the above two cases, we have

$$|{}_s\mathfrak{D}_m^{(k)}| = \binom{m}{s}(m-s)!! + \binom{m}{s-1}(m-s)!! = \binom{m+1}{s}(m-s)!!. \qquad \square$$

THEOREM 9.10. *Assume that a given pair of positive integers $2 \leqslant s \leqslant m$ satisfies $s \equiv_2 m$. Then for a $k \geqslant m-s+3$, we have*

$$|{}_s\mathfrak{D}_m^{(k)}| = \binom{m}{s} \times \mathsf{D}_{m-s}^{(\infty)} + \binom{m}{s-1} \times \mathsf{D}_{m-s+1}^{(\infty)},$$

where $\mathsf{D}_m^{(\infty)}$ is given in (9.2). *Therefore, we have a closed formula for the limit as follows:*

$${}_s\mathsf{D}_m^{(\infty)} := \lim_{k\to\infty} |{}_s\mathfrak{D}_m^{(k)}| = \binom{m}{s} \times \mathsf{D}_{m-s}^{(\infty)} + \binom{m}{s-1} \times \mathsf{D}_{m-s+1}^{(\infty)}.$$

PROOF. The proof is almost identical to the proof of Theorem 9.9. $\square$

The closed formula (9.3) is known to compute the triangular array consisting of coefficients of Bessel polynomials ([**36**, A001497]):

$$\begin{array}{ccccccccccc}
 & & & & & & & & \iddots & \iddots & \cdots \\
 & & & & & & & 1 & 36 & 990 & \cdots \\
 & & & & & & 1 & 28 & 630 & 13860 & \cdots \\
 & & & & & 1 & 21 & 378 & 6930 & 135135 & \cdots \\
 & & & & 1 & 15 & 210 & 3150 & 51975 & 945945 & \cdots \\
 & & & 1 & 10 & 105 & 1260 & 17325 & 270270 & 4729725 & \cdots \\
 & & 1 & 6 & 45 & 420 & 4725 & 62370 & 945945 & 16216200 & \cdots \\
 & 1 & 3 & 15 & 105 & 945 & 10395 & 135135 & 2027025 & 34459425 & \cdots \\
1 & 1 & 3 & 15 & 105 & 945 & 10395 & 135135 & 2027025 & 34459425 & \cdots
\end{array} \tag{9.4}$$

where the lowest two rows are $\mathsf{D}_{2m-1}^{(\infty)} = (2m-1)!!$. We call this triangular array *Bessel triangle.*

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CHAPTER 10

Standard Young tableaux with a fixed number of rows of odd length

In this chapter we consider SYTs with a fixed number of rows of odd length. We denote by $\mathcal{S}_m$ the set of SYTs with m cells. Recall that $\mathcal{S}_m^{(k)}$ is the set of SYTs with m cells and at most k rows, and that there is an obvious bijection from $\mathcal{S}_m^{(k)}$ to $\mathfrak{B}_m^{(k)}$. The main objects in this section are the sets $\mathcal{S}_m^{(k)}$ and their subsets $\mathcal{S}_m^{(k,t)}$ defined below.

Definition 10.1. For $0 \leqslant t \leqslant k$, we denote by $\mathcal{S}_m^{(k,t)}$ the set of SYTs with m cells, at most k rows and exactly t rows of odd length.

Observe that by the obvious bijection between SYTs and reverse standard Young tableaux, we have

$$|\mathcal{S}_{2m}^{(k,2)}| = |\mathfrak{D}_{2m}^{(k)}| \quad \text{and} \quad |\mathcal{S}_{2m-1}^{(k,1)}| = |\mathfrak{D}_{2m-1}^{(k)}|. \tag{10.1}$$

Thus, $|\mathcal{S}_m^{(k,t)}|$ can be thought of as a generalization of $|\mathfrak{D}_m^{(k)}|$. In this section, we study the cardinalities of $\mathcal{S}_m^{(k)}$ and $\mathcal{S}_m^{(k,t)}$.

In Section 10.1, we express $|\mathcal{S}_m^{(k)}|$ in terms of $|\mathcal{S}_i^{(k,0)}|$ and $|\mathcal{S}_i^{(k,k)}|$ (Proposition 10.5). Using this relation and some known results, we find an explicit formula for $\mathcal{S}_m^{(k,t)}$ for every $0 \leqslant t \leqslant k \leqslant 5$ (Theorem 10.2). In Section 10.2, we express $|\mathcal{S}_m^{(k)}|$ as an integral over the orthogonal group $\mathrm{O}(k)$ with respect to the normalized Haar measure (Theorem 10.7). In Section 10.3, we evaluate this integral to find an explicit formula for $|\mathcal{S}_m^{(k)}| = |\mathfrak{B}_m^{(k)}|$ (Theorem 10.9).

10.1. The cardinality of $\mathcal{S}_m^{(k,t)}$ for $0 \leqslant t \leqslant k \leqslant 5$

In this section we give an explicit formula for $\mathcal{S}_m^{(k,t)}$ for every $0 \leqslant t \leqslant k \leqslant 5$. Note that $\mathcal{S}_m^{(k,t)} = \varnothing$ if $m \not\equiv_2 t$. Since it is trivial for $k = 0, 1$, we consider $k \geqslant 2$. Recall that

$$\mathsf{R}_m = \frac{1}{m+1} \sum_{i=1}^{\lfloor m/2 \rfloor} \binom{m+1}{i} \binom{m-i-1}{i-1}.$$

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THEOREM 10.2. *We have a formula for* $|\mathcal{S}_m^{(k,t)}|$ *for* $0 \leqslant t \leqslant k \leqslant 5$ *as follows:*

For $k = 2$, $|\mathcal{S}_{2m}^{(2,0)}| = |\mathcal{S}_{2m-1}^{(2,1)}| = |\mathcal{S}_{2m}^{(2,2)}| = \binom{2m-1}{m}$.

For $k = 3$, $|\mathcal{S}_{2m}^{(3,0)}| = |\mathcal{S}_{2m-1}^{(3,1)}| = |\mathfrak{D}_{2m-1}^{(3)}| = \mathsf{R}_{2m}$,

$|\mathcal{S}_{2m}^{(3,2)}| = |\mathcal{S}_{2m+1}^{(3,3)}| = |\mathfrak{D}_{2m}^{(3)}| = \mathsf{R}_{2m+1}$.

For $k = 4$, $|\mathcal{S}_{2m}^{(4,0)}| = |\mathcal{S}_{2m-1}^{(4,1)}| = |\mathfrak{D}_{2m-1}^{(4)}| = \binom{\mathsf{C}_m + 1}{2}$,

$$|\mathcal{S}_{2m}^{(4,2)}| = |\mathfrak{D}_{2m}^{(4)}| = \mathsf{C}_m\mathsf{C}_{m+1} - \mathsf{C}_m^2 = \frac{3(2m)!^2}{(m-1)!m!(m+1)!(m+2)!},$$

$$|\mathcal{S}_{2m-1}^{(4,3)}| = |\mathcal{S}_{2m}^{(4,4)}| = \binom{\mathsf{C}_m}{2}.$$

For $k = 5$, $|\mathcal{S}_{2m}^{(5,0)}| = |\mathcal{S}_{2m-1}^{(5,1)}| = |\mathfrak{D}_{2m-1}^{(5)}| = \sum_{i=0}^{m} \binom{2m}{2i} \mathsf{C}_i\mathsf{C}_{i+1} - \sum_{i=0}^{m-1} \binom{2m}{2i+1} \mathsf{C}_{i+1}^2$,

$$|\mathcal{S}_{2m}^{(5,2)}| = |\mathfrak{D}_{2m}^{(5)}| = \sum_{i=0}^{m} \frac{2i}{i+3} \binom{2m}{2i} \mathsf{C}_i\mathsf{C}_{i+1} - \sum_{i=0}^{m-1} \frac{2i}{i+3} \binom{2m}{2i+1} \mathsf{C}_{i+1}^2,$$

$$|\mathcal{S}_{2m-1}^{(5,3)}| = \sum_{i=0}^{m-1} \frac{2i}{i+3} \binom{2m-1}{2i} \mathsf{C}_i\mathsf{C}_{i+1} - \sum_{i=0}^{m-1} \frac{2i}{i+3} \binom{2m-1}{2i+1} \mathsf{C}_{i+1}^2,$$

$$|\mathcal{S}_{2m}^{(5,4)}| = |\mathcal{S}_{2m-1}^{(5,5)}| = \sum_{i=0}^{m-1} \binom{2m-1}{2i} \mathsf{C}_i\mathsf{C}_{i+1} - \sum_{i=0}^{m-1} \binom{2m-1}{2i+1} \mathsf{C}_{i+1}^2.$$

Before proving this theorem we first find some relations between the numbers $|\mathcal{S}_m^{(k,t)}|$ and $|\mathcal{S}_m^{(k)}|$.

LEMMA 10.3. *We have*

$$|\mathcal{S}_m^{(k,k)}| = |\mathcal{S}_{m-1}^{(k,k-1)}| \quad \textit{and} \quad |\mathcal{S}_m^{(k,0)}| = |\mathcal{S}_{m-1}^{(k,1)}|.$$

PROOF. The map deleting the cell with m gives a bijection from $\mathcal{S}_m^{(k,k)}$ to $\mathcal{S}_{m-1}^{(k,k-1)}$. The same map also gives a bijection from $\mathcal{S}_m^{(k,0)}$ to $\mathcal{S}_{m-1}^{(k,1)}$. □

The next lemma is the key lemma in this chapter. The proof is based on the Robinson–Schensted algorithm and a sign-reversing involution. Recall that an SYT is a filling of a Young diagram $\lambda \vdash m$ with integers $1, 2, \dots, m$. We need to extend this definition to a *partial SYT* which is a filling of a Young diagram with distinct integers such that the entries are increasing in each row and each column.

LEMMA 10.4. *For integers* $k \geqslant 1$ *and* $m \geqslant 0$, *we have*

$$|\mathcal{S}_m^{(k,0)}| - |\mathcal{S}_m^{(k,k)}| = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} |\mathcal{S}_i^{(k-1)}|.$$

PROOF. Let X be the set of pairs (T, A) of a partial SYT T and a subset A of $\{1, 2, \dots, m\}$ such that T has at most $k-1$ rows and the set of entries of T is $\{1, 2, \dots, m\} \setminus A$. Then we have

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} |\mathcal{S}_i^{(k-1)}| = \sum_{(T,A)\in X} (-1)^{|A|}.$$

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We define Y to be the set of pairs (P, H) of an SYT P and a sequence $H = (t_1, t_2, \dots, t_k)$ such that

- P has at most k rows, and
- if $\lambda = (\lambda_1, \dots, \lambda_k)$ is the shape of P (some λ_i can be zero), then $0 \leqslant t_i \leqslant \lambda_i - \lambda_{i+1}$ for all $1 \leqslant i \leqslant k-1$ and $t_k = \lambda_k$.

Note that if $\mu = (\mu_1, \dots, \mu_k)$ is defined by $\mu_i = \lambda_i - t_i$ for $1 \leqslant i \leqslant k$, then the second condition above means that $\mu \subset \lambda$ and λ/μ is a skew partition whose Young diagram contains at most one cell in each column. Such a skew partition is called a *horizontal strip*. By identifying the sequence H and the skew partition λ/μ, one can consider H as a horizontal strip of P which contains all cells in row k of P.

We claim that there is a bijection from X to Y such that if $(T, A) \in X$ corresponds to $(P, H) \in Y$, then $|A| = t_1 + t_2 + \cdots + t_k$. For $(T, A) \in X$, let P be the SYT obtained from T by inserting the elements of A in increasing order via the Robinson–Schensted algorithm and $H = (t_1, \dots, t_k)$ be the sequence of integers such that t_i is the number of newly added cells in row i. In other words, if $\mathsf{Sh}(P) = \lambda = (\lambda_1, \dots, \lambda_k)$ and $\mathsf{Sh}(T) = \mu = (\mu_1, \dots, \mu_k)$, then $t_i = \lambda_i - \mu_i$. It is well known that if $i < j$ and i is inserted to a partial SYT T and j is inserted to the resulting tableau via the Robinson–Schensted algorithm, then the newly added cell after inserting j is strictly to the right of the newly added cell after inserting i. This property implies that λ/μ is a horizontal strip and the cells in it have been added from left to right. Therefore, we can recover (T, A) from (P, H) using the inverse map of the Robinson–Schensted algorithm and this proves the claim.

By the above claim, we have

$$\sum_{(T,A)\in X} (-1)^{|A|} = \sum_{(P,H)\in Y} (-1)^{t_1+\cdots+t_k}.$$

Now we define a map ϕ on Y as follows. Suppose that $(P, H) \in Y$ and the shape of P is $\lambda = (\lambda_1, \dots, \lambda_k)$ and $H = (t_1, \dots, t_k)$. Find the smallest $i \leqslant k-1$ such that t_i is an odd integer or t_i is an even integer less than $\lambda_i - \lambda_{i-1}$. In this case we define $\phi(P, H) = (P, H')$, where $H' = (t'_1, \dots, t'_k)$ is obtained from H by replacing t_i by $t_i - 1$ if t_i is odd and by $t_i + 1$ if t_i is even. If there is no such integer i, then we define $\phi(P, H) = (P, H)$. It is easy to see that ϕ is an involution on Y such that if $\phi(P, H) = (P, H')$ and $H \neq H'$, then $(-1)^{t_1+\cdots+t_k} = -(-1)^{t'_1+\cdots+t'_k}$. Moreover, if $\phi(P, H) = (P, H)$, then $t_i = \lambda_i - \lambda_{i+1}$ is even for all $1 \leqslant i \leqslant k-1$. This can happen only if $P \in \mathcal{S}_m^{(k,0)}$ or $P \in \mathcal{S}_m^{(k,k)}$. If $\phi(P, H) = (P, H)$ for $P \in \mathcal{S}_m^{(k,0)}$, then $(-1)^{t_1+\cdots+t_k} = (-1)^{t_k} = 1$. If $\phi(P, H) = (P, H)$ for $P \in \mathcal{S}_m^{(k,k)}$, then $(-1)^{t_1+\cdots+t_k} = (-1)^{t_k} = -1$. Therefore, ϕ is a sign-reversing involution and we have

$$\sum_{(P,H)\in Y} (-1)^{t_1+\cdots+t_k} = |\mathcal{S}_m^{(k,0)}| - |\mathcal{S}_m^{(k,k)}|,$$

which finishes the proof. □

Applying the principle of inclusion and exclusion to Lemma 10.4, we obtain the following proposition.

PROPOSITION 10.5. *For integers $k \geqslant 1$ and $m \geqslant 0$, we have*

$$|\mathcal{S}_m^{(k-1)}| = \sum_{i=0}^{m} \binom{m}{i} \left(|\mathcal{S}_i^{(k,0)}| - |\mathcal{S}_i^{(k,k)}| \right).$$

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Now we prove Theorem 10.2.

Proof of Theorem 10.2. We have already proved the formulas for $k=2$ in (3.2) and for $k=3$ in Proposition 4.12. Now we consider the cardinality of $\mathcal{S}_m^{(k,t)}$ for $k=4$.

Recall that we have a formula for $|\mathcal{S}_m^{(4)}| = |\mathfrak{B}_m^{(4)}|$ in Theorem 3.5:

$$|\mathcal{S}_{2m}^{(4)}| = \mathsf{C}_m\mathsf{C}_{m+1} \quad \text{and} \quad |\mathcal{S}_{2m-1}^{(4)}| = \mathsf{C}_m\mathsf{C}_m.$$

Since $2m$ is even,

$$|\mathcal{S}_{2m}^{(4,0)}| + |\mathcal{S}_{2m}^{(4,2)}| + |\mathcal{S}_{2m}^{(4,4)}| = |\mathcal{S}_{2m}^{(4)}| = \mathsf{C}_m\mathsf{C}_{m+1}. \tag{10.2}$$

By Lemma 10.3, we have

$$|\mathcal{S}_{2m}^{(4,0)}| + |\mathcal{S}_{2m}^{(4,4)}| = |\mathcal{S}_{2m-1}^{(4,1)}| + |\mathcal{S}_{2m-1}^{(4,3)}| = |\mathcal{S}_{2m-1}^{(4)}| = \mathsf{C}_m^2. \tag{10.3}$$

By Lemma 10.4, we have

$$|\mathcal{S}_{2m}^{(4,0)}| - |\mathcal{S}_{2m}^{(4,4)}| = \sum_{i=0}^{2m}(-1)^i\binom{2m}{i}|\mathcal{S}_i^{(3)}| = \mathsf{C}_m. \tag{10.4}$$

In (10.4), we used the fact that $|\mathcal{S}_i^{(3)}| = \mathsf{M}_i$ and

$$\sum_{i=0}^{2m}(-1)^i\binom{2m}{i}\mathsf{M}_i = \mathsf{C}_m,$$

which can be obtained from the following identity using inclusion-exclusion:

$$\mathsf{M}_m = \sum_{i=0}^{\lfloor m/2\rfloor}\binom{m}{2i}\mathsf{C}_i.$$

By (10.2), (10.3) and (10.4), we obtain the formulas for $|\mathcal{S}_{2m}^{(4,0)}|$, $|\mathcal{S}_{2m}^{(4,2)}|$ and $|\mathcal{S}_{2m}^{(4,4)}|$. By Lemma 10.3, we obtain the formulas for $|\mathcal{S}_{2m-1}^{(4,1)}|$ and $|\mathcal{S}_{2m-1}^{(4,3)}|$.

Now we consider the cardinality of $\mathcal{S}_m^{(k,t)}$ for $k=5$. First, we have

$$|\mathcal{S}_{2m}^{(5,0)}| + |\mathcal{S}_{2m}^{(5,2)}| + |\mathcal{S}_{2m}^{(5,4)}| = |\mathcal{S}_{2m}^{(5)}| \quad \text{and} \quad |\mathcal{S}_{2m-1}^{(5,1)}| + |\mathcal{S}_{2m-1}^{(5,3)}| + |\mathcal{S}_{2m-1}^{(5,5)}| = |\mathcal{S}_{2m-1}^{(5)}|.$$

By Lemma 10.3, we have

$$|\mathcal{S}_{2m}^{(5,0)}| = |\mathcal{S}_{2m-1}^{(5,1)}| \quad \text{and} \quad |\mathcal{S}_{2m+1}^{(5,5)}| = |\mathcal{S}_{2m}^{(5,4)}|.$$

By Lemma 10.4, we have

$$|\mathcal{S}_{2m}^{(5,0)}| - |\mathcal{S}_{2m}^{(5,5)}| = |\mathcal{S}_{2m}^{(5,0)}| = \sum_{i=0}^{2m}(-1)^i\binom{2m}{i}|\mathcal{S}_i^{(4)}|,$$

$$|\mathcal{S}_{2m-1}^{(5,0)}| - |\mathcal{S}_{2m-1}^{(5,5)}| = -|\mathcal{S}_{2m-1}^{(5,5)}| = \sum_{i=0}^{2m-1}(-1)^i\binom{2m-1}{i}|\mathcal{S}_i^{(4)}|.$$

By solving the above equations, we obtain the desired formulas. □

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10.2. Traces of orthogonal matrices

There is an interesting integral representation of the number $|\mathcal{S}_{2m}^{(k,0)}|$ as follows, see Example 2 on page 423 in [**32**]:

$$\int_{\mathrm{O}(k)} \mathrm{Tr}(X)^m d\mu(X) = |\mathcal{S}_m^{(k,0)}|. \tag{10.5}$$

Here, the integral is taken with respect to the normalized Haar measure μ on the orthogonal group $\mathrm{O}(k)$ consisting of $k \times k$ orthogonal matrices. Note that if m is odd, we have $|\mathcal{S}_m^{(k,0)}| = 0$. Thus, by (10.1) and Lemma 10.3, we have

$$|\mathfrak{D}_{2m-1}^{(k)}| = |\mathcal{S}_{2m}^{(k,0)}| = \int_{\mathrm{O}(k)} \mathrm{Tr}(X)^{2m} d\mu(X).$$

In this section we show that $|\mathcal{S}_m^{(k,k)}|$ and $|\mathcal{S}_m^{(k)}|$ also have similar integral representations.

For a symmetric function $f(x_1, \dots, x_k)$ with k variables and $X \in \mathrm{O}(k)$, we define $f(X)$ by $f(X) = f(e^{i\theta_1}, \dots, e^{i\theta_k})$, where $e^{i\theta_1}, \dots, e^{i\theta_k}$ are the eigenvalues of X. Note that $\mathrm{Tr}(X^m) = p_m(X)$, where $p_m(x_1, \dots, x_k) = x_1^m + \cdots + x_k^m$ is the m-th power sum symmetric function.

We need the following known result, see [**32**, pp.420–421]:

$$\int_{\mathrm{O}(k)} s_\lambda(X) d\mu(X) = \begin{cases} 1 & \text{if every part of } \lambda \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{10.6}$$

where s_λ is the Schur function.

Proposition 10.6. *We have*

$$|\mathcal{S}_m^{(k,k)}| = \int_{\mathrm{O}(k)} \det(X) \mathrm{Tr}(X)^m d\mu(X).$$

Proof. Note that

$$\mathrm{Tr}(X)^m = p_1(X)^m = \sum_{\lambda \vdash m,\, \ell(\lambda) \leqslant k} f^\lambda s_\lambda(X),$$

where f^λ is the number of standard Young tableaux of shape λ. Since

$$x_1 \dots x_k s_\lambda(x_1, \dots, x_k) = s_{\lambda + (1^k)}(x_1, \dots, x_k)$$

for λ with at most k rows, we have $\det(X) s_\lambda(X) = s_{\lambda+(1^k)}(X)$. Thus,

$$\int_{\mathrm{O}(k)} \det(X) \mathrm{Tr}(X)^m d\mu(X) = \sum_{\lambda \vdash m,\, \ell(\lambda) \leqslant k} f^\lambda \int_{\mathrm{O}(k)} s_{\lambda+(1^k)}(X) d\mu(X).$$

By (10.6), this is equal to $|\mathcal{S}_m^{(k,k)}|$. $\square$

Now we give an integral expression for the number SYTs with m cells and at most k rows.

Theorem 10.7. *For integers $k, m \geqslant 0$, we have*

$$|\mathfrak{B}_m^{(k)}| = |\mathcal{S}_m^{(k)}| = \int_{\mathrm{O}(k+1)} (1 - \det(X))(1 + \mathrm{Tr}(X))^m d\mu(X).$$

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PROOF. By Proposition 10.5,

$$|\mathcal{S}_m^{(k)}| = \sum_{i=0}^{m} \binom{m}{i} \left(|\mathcal{S}_i^{(k+1,0)}| - |\mathcal{S}_i^{(k+1,k+1)}|\right).$$

By (10.5) and Proposition 10.6, we have

$$\begin{aligned} |\mathcal{S}_m^{(k)}| = \sum_{i=0}^{m} \binom{m}{i} \left(\int_{\mathrm{O}(k+1)} \mathrm{Tr}(X)^i d\mu(X) - \int_{\mathrm{O}(k+1)} \det(X)\mathrm{Tr}(X)^i d\mu(X)\right) \\ = \int_{\mathrm{O}(k+1)} (1-\det(X)) \left(\sum_{i=0}^{m} \binom{m}{i} \mathrm{Tr}(X)^i\right) d\mu(X). \end{aligned}$$

We then obtain the desired identity using the binomial theorem. □

10.3. Evaluation of integrals

In this section we obtain an explicit formula for the number of SYTs with m cells and at most k rows by evaluating the integral in Theorem 10.7. For the reader's convenience we recall a well-known fact on the normalized Haar measure on the orthogonal group $\mathrm{O}(k)$ due to Weyl [**40**], see also [**3**, Remarks 3 on p. 57].

For any orthogonal matrix $A \in \mathrm{O}(n)$, the eigenvalues of A lie on the unit circle. Let $P_n(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$ be the probability that a random matrix $A \in \mathrm{O}(n)$ has the given eigenvalues $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}$ for $\theta_1, \dots, \theta_n \in [0, 2\pi)$. Here, we assume that A is selected randomly with respect to the normalized Haar measure. Then this probability is given as follows.

PROPOSITION 10.8. *For $k \geqslant 1$, $\epsilon \in \{1, -1\}$ and $\theta_1, \dots, \theta_k \in [0, \pi]$ we have*

$$P_{2k}(e^{\pm i\theta_1}, e^{\pm i\theta_2}, \dots, e^{\pm i\theta_k}) = \frac{2^{k^2-2k+1}}{\pi^k k!} \prod_{1 \leqslant r < s \leqslant k} (\cos\theta_r - \cos\theta_s)^2,$$

$$P_{2k+2}(\pm 1, e^{\pm i\theta_1}, e^{\pm i\theta_2}, \dots, e^{\pm i\theta_k}) = \frac{2^{k^2-1}}{\pi^k k!} \prod_{t=1}^{k} (1-\cos^2\theta_t) \prod_{1 \leqslant r < s \leqslant k} (\cos\theta_r - \cos\theta_s)^2,$$

$$P_{2k+1}(\epsilon, e^{\pm i\theta_1}, e^{\pm i\theta_2}, \dots, e^{\pm i\theta_k}) = \frac{2^{k^2-k-1}}{\pi^k k!} \prod_{t=1}^{k} (1-\epsilon\cos\theta_t) \prod_{1 \leqslant r < s \leqslant k} (\cos\theta_r - \cos\theta_s)^2.$$

We denote by $\mathrm{O}_+(k)$ (resp. $\mathrm{O}_-(k)$) the set of matrices $A \in \mathrm{O}(k)$ with $\det(A) = 1$ (resp. $\det(A) = -1$).

Now we give an explicit formula for $|\mathcal{S}_m^{(k)}|$.

THEOREM 10.9. *For $k \geqslant 1$ and $m \geqslant 0$, we have*

$$|\mathcal{S}_m^{(2k)}| = \sum_{t_1+\cdots+t_k=m} \binom{m}{t_1, \dots, t_k} \det\left(\binom{t_i + 2k - i - j}{\lfloor \frac{t_i+2k-i-j}{2} \rfloor}\right)_{i,j=1}^{k},$$

$$|\mathcal{S}_m^{(2k+1)}| = \sum_{t_0+t_1+\cdots+t_k=m} \binom{m}{t_0, t_1, \dots, t_k} \det\left(C\left(\frac{t_i + 2k - i - j}{2}\right)\right)_{i,j=1}^{k},$$

where $C(x) = \frac{1}{x+1}\binom{2x}{x}$ if x is an integer and $C(x) = 0$ otherwise.

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PROOF. By Theorem 10.7 and Proposition 10.8, we have

$$
\begin{aligned}
&|\mathcal{S}_m^{(2k)}| \\
&\quad = 2\int_{\mathrm{O}_-(2k+1)} (1+\mathrm{Tr}(X))^m d\mu(X) \\
&\quad = \frac{2^{k^2-k}}{\pi^k k!}\int_{[0,\pi]^k} (2\cos\theta_1 + \cdots + 2\cos\theta_k)^m \\
&\qquad\qquad \prod_{1\leqslant r<s\leqslant k} (\cos\theta_r - \cos\theta_s)^2 \prod_{i=1}^{k} (1+\cos\theta_i) d\theta_i \\
&\quad = \frac{2^{k^2-k+m}}{\pi^k k!} \sum_{t_1+\cdots+t_k=m} \binom{m}{t_1,\dots,t_k} \int_{[0,\pi]^k} \det(x_i^{t_i+k-j})_{i,j=1}^k \det(x_i^{k-j})_{i,j=1}^k \\
&\qquad\qquad \prod_{i=1}^{k}(1+\cos\theta_i)d\theta_i \\
&\quad = \frac{2^{k^2-k+m}}{\pi^k k!} \sum_{t_1+\cdots+t_k=m} \binom{m}{t_1,\dots,t_k} \sum_{\sigma,\tau\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \int_{[0,\pi]^k} x_i^{t_i+2k-\sigma(i)-\tau(i)} \\
&\qquad\qquad \prod_{i=1}^{k}(1+\cos\theta_i)d\theta_i,
\end{aligned}
$$

where $x_i = \cos\theta_i$. When $\sigma \in \mathfrak{S}_n$ is fixed, since t_i's are symmetric, we can replace t_i by $t_{\sigma(i)}$. We can also replace τ by $\tau\sigma$. Then the resulting summand is independent of σ. Thus, we obtain

$$
\begin{aligned}
&|\mathcal{S}_m^{(2k)}| \\
&= \frac{2^{k^2-k+m}}{\pi^k} \sum_{t_1+\cdots+t_k=m} \binom{m}{t_1,\dots,t_k} \sum_{\tau\in\mathfrak{S}_n} \mathrm{sgn}(\tau) \prod_{i=1}^{k} \int_0^\pi x^{t_i+2k-i-\tau(i)}(1+\cos\theta)d\theta.
\end{aligned}
$$

By expressing the second sum as a determinant and using the fact

$$\int_0^\pi \cos^n\theta(1+\cos\theta)d\theta = \frac{\pi}{2^n}\binom{n}{\lfloor \frac{n}{2} \rfloor},$$

we obtain the desired formula. The second formula can be proved similarly. $\square$

In the literature there is an explicit formula for $|\mathcal{S}_m^{(k)}|$ for $k \leqslant 5$. As a corollary of Theorem 10.9, we obtain a double-sum formula for $|\mathcal{S}_m^{(6)}|$.

COROLLARY 10.10. *Letting* $\gamma_n = \binom{n}{\lfloor \frac{n}{2}\rfloor}$, *we have*

$$|\mathcal{S}_m^{(6)}| = \sum_{i+j+k=m} \binom{m}{i,j,k} \det\begin{pmatrix} \gamma_{i+4} & \gamma_{i+3} & \gamma_{i+2} \\ \gamma_{j+3} & \gamma_{j+2} & \gamma_{j+1} \\ \gamma_{k+2} & \gamma_{k+1} & \gamma_k \end{pmatrix}.$$

There is another way to compute $|\mathcal{S}_m^{(k)}|$ using symmetric functions due to Gessel [**9**, Section 6]. It would be interesting to find a connection between his result and Theorem 10.9. Eu et al. [**5**] found a bijection between $\mathcal{S}_m^{(k)}$ and the set of certain colored Motzkin paths.

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We also note that the integrals in the proof of Theorem 10.9 are Selberg-type integrals, see [**6**]. There is a combinatorial interpretation for Selberg-type integrals, see [**37**, Exercise 1.10 (b)]. Recently, a connection between SYTs and the Selberg integral was found in [**24**]. There is also a combinatorial interpretation for a q-analog of the Selberg integral, see [**25**]. It would be interesting to study the combinatorial aspects of the formulas in Theorem 10.9 and their q-analogs.

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Bibliography

[1] S. Ariki, *On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$*, J. Math. Kyoto Univ. **36** (1996), no. 4, 789–808, DOI 10.1215/kjm/1250518452. MR1443748

[2] G. Benkart and T. Halverson, *Motzkin algebras*, European J. Combin. **36** (2014), 473–502, DOI 10.1016/j.ejc.2013.09.010. MR3131911

[3] P. Diaconis and M. Shahshahani, *On the eigenvalues of random matrices*, J. Appl. Probab. **31A** (1994), 49–62, DOI 10.2307/3214948. Studies in applied probability. MR1274717

[4] S.-P. Eu, *Skew-standard tableaux with three rows*, Adv. in Appl. Math. **45** (2010), no. 4, 463–469, DOI 10.1016/j.aam.2010.03.004. MR2679926

[5] S.-P. Eu, T.-S. Fu, J. T. Hou, and T.-W. Hsu, *Standard Young tableaux and colored Motzkin paths*, J. Combin. Theory Ser. A **120** (2013), no. 7, 1786–1803, DOI 10.1016/j.jcta.2013.06.007. MR3092698

[6] P. J. Forrester and S. O. Warnaar, *The importance of the Selberg integral*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 4, 489–534, DOI 10.1090/S0273-0979-08-01221-4. MR2434345

[7] I. B. Frenkel and M. G. Khovanov, *Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$*, Duke Math. J. **87** (1997), no. 3, 409–480, DOI 10.1215/S0012-7094-97-08715-9. MR1446615

[8] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics, DOI 10.1007/978-1-4612-0979-9. MR1153249

[9] I. M. Gessel, *Symmetric functions and P-recursiveness*, J. Combin. Theory Ser. A **53** (1990), no. 2, 257–285, DOI 10.1016/0097-3165(90)90060-A. MR1041448

[10] D. Gouyou-Beauchamps, *Standard Young tableaux of height 4 and 5*, European J. Combin. **10** (1989), no. 1, 69–82, DOI 10.1016/S0195-6698(89)80034-4. MR977181

[11] J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, vol. 42, American Mathematical Society, Providence, RI, 2002, DOI 10.1090/gsm/042. MR1881971

[12] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978. Second printing, revised. MR499562

[13] R. L. Jayne and K. C. Misra, *On multiplicities of maximal weights of $\widehat{sl}(n)$-modules*, Algebr. Represent. Theory **17** (2014), no. 4, 1303–1321, DOI 10.1007/s10468-014-9470-2. MR3228490

[14] R. L. Jayne and K. C. Misra, *Lattice paths, Young tableaux, and weight multiplicities*, Ann. Comb. **22** (2018), no. 1, 147–156, DOI 10.1007/s00026-018-0374-4. MR3767671

[15] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990, DOI 10.1017/CBO9780511626234. MR1104219

[16] V. G. Kac and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), no. 2, 125–264, DOI 10.1016/0001-8708(84)90032-X. MR750341

[17] J. Kamnitzer, *The crystal structure on the set of Mirković-Vilonen polytopes*, Adv. Math. **215** (2007), no. 1, 66–93, DOI 10.1016/j.aim.2007.03.012. MR2354986

[18] S.-J. Kang, *Crystal bases for quantum affine algebras and combinatorics of Young walls*, Proc. London Math. Soc. (3) **86** (2003), no. 1, 29–69, DOI 10.1112/S0024611502013734. MR1971463

[19] S.-J. Kang and J.-H. Kwon, *Fock space representations of quantum affine algebras and generalized Lascoux-Leclerc-Thibon algorithm*, J. Korean Math. Soc. **45** (2008), no. 4, 1135–1202, DOI 10.4134/JKMS.2008.45.4.1135. MR2422732

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[20] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), no. 2, 249–260. MR1090425

[21] N. H. Xi, *Special bases of irreducible modules of the quantized universal enveloping algebra* $U_v(\mathrm{gl}(n))$, J. Algebra **154** (1993), no. 2, 377–386, DOI 10.1006/jabr.1993.1020. MR1206127

[22] M. Kashiwara and T. Nakashima, *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra **165** (1994), no. 2, 295–345, DOI 10.1006/jabr.1994.1114. MR1273277

[23] S. Kass, R. V. Moody, J. Patera, and R. Slansky, *Affine Lie algebras, weight multiplicities, and branching rules. Vols. 1, 2*, Los Alamos Series in Basic and Applied Sciences, vol. 9, University of California Press, Berkeley, CA, 1990. MR1117679

[24] J. S. Kim and S. Oh, *The Selberg integral and Young books*, J. Combin. Theory Ser. A **145** (2017), 1–24, DOI 10.1016/j.jcta.2016.07.005. MR3551643

[25] J. S. Kim and D. Stanton, *On q-integrals over order polytopes*, Adv. Math. **308** (2017), 1269–1317, DOI 10.1016/j.aim.2017.01.001. MR3600087

[26] Y.-H. Kim, S.-j. Oh, and Y.-T. Oh, *Cyclic sieving phenomenon on dominant maximal weights over affine Kac-Moody algebras*, Adv. Math. **374** (2020), 107336, 75, DOI 10.1016/j.aim.2020.107336. MR4133517

[27] K. Koike, *On new multiplicity formulas of weights of representations for the classical groups*, J. Algebra **107** (1987), no. 2, 512–533, DOI 10.1016/0021-8693(87)90100-1. MR885808

[28] A. Lascoux, B. Leclerc, and J.-Y. Thibon, *Hecke algebras at roots of unity and crystal bases of quantum affine algebras*, Comm. Math. Phys. **181** (1996), no. 1, 205–263. MR1410572

[29] C. Lecouvey and M. Shimozono, *Lusztig's q-analogue of weight multiplicity and one-dimensional sums for affine root systems*, Adv. Math. **208** (2007), no. 1, 438–466, DOI 10.1016/j.aim.2006.03.001. MR2304324

[30] J. Lepowsky and S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. in Math. **29** (1978), no. 1, 15–59, DOI 10.1016/0001-8708(78)90004-X. MR501091

[31] P. Littelmann, *Paths and root operators in representation theory*, Ann. of Math. (2) **142** (1995), no. 3, 499–525, DOI 10.2307/2118553. MR1356780

[32] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. With contributions by A. Zelevinsky; Oxford Science Publications. MR1354144

[33] S.-j. Oh, *The Andrews-Olsson identity and Bessenrodt insertion algorithm on Young walls*, European J. Combin. **43** (2015), 8–31, DOI 10.1016/j.ejc.2014.07.001. MR3266281

[34] A. Regev, *Asymptotic values for degrees associated with strips of Young diagrams*, Adv. in Math. **41** (1981), no. 2, 115–136, DOI 10.1016/0001-8708(81)90012-8. MR625890

[35] B. E. Sagan, *The symmetric group*, 2nd ed., Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001. Representations, combinatorial algorithms, and symmetric functions, DOI 10.1007/978-1-4757-6804-6. MR1824028

[36] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org.

[37] R. P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin, DOI 10.1017/CBO9780511609589. MR1676282

[38] S. Tsuchioka, *Catalan numbers and level 2 weight structures of* $A^{(1)}_{p-1}$, New trends in combinatorial representation theory, RIMS Kôkyûroku Bessatsu, B11, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009, pp. 145–154. MR2562491

[39] S. Tsuchioka and M. Watanabe, *Pattern avoidance seen in multiplicities of maximal weights of affine Lie algebra representations*, Proc. Amer. Math. Soc. **146** (2018), no. 1, 15–28, DOI 10.1090/proc/13597. MR3723117

[40] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N.J., 1939. MR0000255

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