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Equivalence of invariant metrics via Bergman kernel on complete noncompact Kähler manifolds

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Abstract

We study equivalence of invariant metrics on noncompact Kähler manifolds with a complete Bergman metric of bounded curvature. Especially only the boundedness of the ratio between Bergman kernel and the n-times wedge product of Bergman metric in any fundamental domain of such a Kähler manifold is required to obtain the equivalence of the Bergman metric and the complete Kähler–Einstein metric. To demonstrate the effectiveness of this method, we consider a two-parameter family of 3-dimensional bounded pseudoconvex domains

$$E_{p,\lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^2)^{1/\lambda} + |z|^2 < 1\}, \qquad p, \lambda > 0.$$

For this family, boundary limits of the holomorphic sectional curvature of the Bergman metric are not well-defined, and hence previously known methods for comparison of invariant metrics do not work. Lastly, we provide an estimate of lower bound of the integrated Carathéodory–Reiffen metric on complete noncompact simply-connected Kähler manifolds with negative sectional curvature.

1 Introduction

As the Bergman metric, the complete Kähler–Einstein metric of negative scalar curvature, the Kobayashi–Royden metric, and the Carathéodory–Reiffen metric are generalizations of the Poincaré–Bergman metric on the complex hyperbolic space,

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equivalence of these four invariant metrics on negatively curved complex manifolds has been studied in complex geometry. In addition, since these four metrics have the property that any automorphism becomes an isometry [31, 35], it makes sense to study them from the viewpoint of differential geometry. Hermitian metrics and Finsler metrics with this property are called *invariant metrics*. Some well-known classes having equivalence of these metrics are complex manifolds with uniform squeezing property, smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$, and weakly pseudoconvex domains of finite type in $\mathbb{C}^2$ [4, 36]. In complex dimension 3, the equivalence of these metrics breaks down for some weakly pseudoconvex domains with analytic boundary [16].

In this context, Wu and Yau proved the following remarkable theorems based on the quasi-bounded geometry and Shi's estimate [30] with Kähler–Ricci flow.

Theorem 1 ([33], Corollary 7) *Let (M, ω) be a complete simply-connected noncompact Kähler manifold whose Riemannian sectional curvature is negatively pinched. Then the base Kähler metric is uniformly equivalent to the Kobayashi–Royden metric, the Bergman metric and the complete Kähler–Einstein metric of negative scalar curvature.*

Theorem 2 ([33], Theorems 2, 3) *Let (M, ω) be a complete Kähler manifold whose holomorphic sectional curvature is negatively pinched. Then the base Kähler metric is uniformly equivalent to the Kobayashi–Royden metric and the complete Kähler–Einstein metric of negative scalar curvature.*

As an interesting application of equivalence of invariant metrics, it is recently showed by the first-named author that the non-equivalence of invariant metrics can be used to show the non-existence of complete Kähler metric whose holomorphic sectional curvature is negatively pinched on pseudoconex domains in $\mathbb{C}^n$ under some conditions (see [12]).

Based on Theorem 2, one possible method to show the equivalence of the invariant metrics on a complete Kähler manifold (M, ω) is to prove that the holomorphic sectional curvature of ω has a negative range. As explicit formulas are recently obtained for the Bergman kernels on certain weakly pseudoconvex domains (e.g., see [2, 3, 14, 28] and references therein), one could attempt to compute the holomorphic sectional curvature of the Bergman metric to establish the equivalence of the invariant metrics (for example, see [13]). However, in general, it seems to be a daunting task to compute the holomorphic sectional curvature for nontrivial pseudoconvex domains even with explicit formulas of the Bergman kernels.

Indeed, for the bounded pseudoconvex domains, even for the class of convex domains or strictly pseudoconvex domains, the curvature information of Bergman metric is known only near the boundary and not in the interior. The holomorphic sectional curvature of the Bergman metric has values between $-\infty$ and $+2$ [17, 24], but there is an example [21] of a semi-finite type pseudoconvex domain in which the holomorphic sectional curvature of Bergman metric blows up to $-\infty$.

Our main result in this paper is that, neither requiring the negative range of curvature as Wu–Yau theorems do, nor specifying the type of pseudoconvex domains, we provide a concrete approach to compare invariant metrics. Our method is based on knowledge of the Bergman kernel and can be applied to general bounded pseudoconvex domains Ω in $\mathbb{C}^n$ when an explicit description of the Bergman kernel near the boundary of Ω is available.

To state the main result (Theorem 1) below, we define the fundamental domain $\widetilde{M}$ of a complex manifold M to be the subset of M which contains exactly one point from each of the orbits of the group action by the automorphism group of M. An automorphism f of M means f and its inverse are holomorphic.

Theorem A *Let (M, ω_B) be an n-dimensional noncompact Kähler manifold with a complete Bergman metric ω_B of bounded curvature, where B denotes the Bergman kernel on M (as the (n, n)-form). Then the following statements hold:*

1. *Assume that $\frac{B}{\omega_B^n}$ is a bounded function for some fundamental domain $\widetilde{M}$. Here $\omega_B^n := \omega_B \wedge \cdots \wedge \omega_B$ (n-times). Then there exist a complete Kähler–Einstein metric ω_{KE} of negative scalar curvature and a constant $C_1 > 0$ such that ω_{KE} is uniformly equivalent to ω_B by C_1, i.e.,*

$$\frac{1}{C_1}\omega_{KE}(v, v) \le \omega_B(v, v) \le C_1\omega_{KE}(v, v) \qquad \textit{for all } v \in T'M.$$

2. *Assume that there exists a compact subset K in M such that the holomorphic sectional curvature of ω_B is negative outside of K, and that M is biholomorphically and properly embedded into B_N, $N \ge n$, where B_N is the unit ball in $\mathbb{C}^N$. Then the Carathéodory–Reiffen metric γ_M is not essentially zero, and the Bergman metric is uniformly equivalent to the Kobayashi–Royden metric, i.e., there exists $C_2 > 0$ such that*

$$\frac{1}{C_2}\chi_M(p; v) \le \sqrt{\omega_B(v, v)} \le C_2\chi_M(p; v) \qquad \textit{for all } v \in T'_pM,\ \ p \in M,$$

where χ_M is the Kobayashi–Royden metric on M. Moreover, if $N = n$, the Bergman metric is uniformly equivalent to the complete Kähler–Einstein metric of negative scalar curvature.

Remark 3 Under the same assumptions of Theorem A, but without additional assumptions of the first and second statements, we obtain the following from [32]: there exists $C_0 > 0$, which only depends on n and the curvature range of ω_B, such that

$$\chi_M(p; v) \le C_0\sqrt{\omega_B(v, v)} \qquad \text{for all } v \in T'_pM,\ \ p \in M.$$

(See Remark 11 for the details.)

The second statement of Theorem A differs from the Wu–Yau theorems (Theorems 1 and 2) in that the Bergman metric's holomorphic sectional curvature is not required to be everywhere negative, but it still ensures the equivalence of invariant metrics. For the other assumption, we note that every bounded strictly pseudoconvex domain in $\mathbb{C}^n$ admits a proper holomorphic embedding into a ball (for example, see [18, p.11]).

To demonstrate the effectiveness of our method, we consider invariant metrics on a two-parameter family of 3-dimensional bounded domains defined by

$$E_{p,\lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^2)^{1/\lambda} + |z|^2 < 1\}, \qquad p, \lambda > 0. \tag{1.1}$$

When $p = \lambda = 1$, the domain $E_{p,\lambda}$ is the unit ball in $\mathbb{C}^3$. When $\lambda = 1$ and $p \geq 1/2$, this reduces to the well-known convex egg (Thullen) domains whose invariant metrics are uniformly equivalent ([13, 23]). With other pairs of (p, λ) for (1.1), the boundary limits of the holomorphic sectional curvature of the Bergman metric are not well-defined, so neither squeezing functions nor the Wu–Yau theorems can be applied. However, we show that Theorem A can be applied. For this purpose, we use a concrete formula for the Bergman kernel of $E_{p,\lambda}$, which is obtained in [2]. We also verify the Cheng's conjecture on $E_{p,\lambda}$ in the process of calculation. Namely, we show that the Bergman metric and the complete Kähler–Einstein metric is the same on $E_{p,\lambda}$ if and only if $p = \lambda = 1$ (Proposition 25).

In the last section, we obtain a result on the Carathéodory–Reiffen metric which is missing in the Wu–Yau theorems. Classical invariant metrics include the Carathéodory–Reiffen metric whose definition is based on the existence of non-constant bounded holomorphic functions on noncompact complex manifolds. However, showing the existence of such functions still remains as a big challenge in hyperbolic complex geometry.

The upper bounds of the Carathéodory–Reiffen metric have been studied extensively. As for comparison between Carathéodory–Reiffen metric and the Bergman metric on the bounded domains, the first result is obtained by Qi-Keng Lu [26] and then on manifolds by Hahn [19, 20]. Further developments are made by Ahn, Gaussier and Kim [1]. Very recently, a comparison of Carathéodory distance and Kähler–Einstein distance of Ricci curvature -1 for certain weakly pseudoconvex domains is established by the first-named author [11].

Our result in the last section is a lower bound of the integrated Carathéodory–Reiffen metric (Theorem 7). The positive lower bound of the Carathéodory–Reiffen metric is important in that it is the smallest invariant metric among invariant metrics [11, 22], and it provides quantitative information about non-constant bounded holomorphic functions (also, see [5]).

The article is organized as follows: In Sect. 2, we review the definitions of the invariant metrics. In the next section, we recall the quasi-bounded geometry and a result on comparison with the Kobayashi–Royden metric. In Sect. 4, we apply Shi's estimate on Kähler–Ricci flow outside of a compact subset on noncompact Kähler manifold. In Sect. 5, we prove Theorem A by generating a complete Kähler metric with negatively pinched holomorphic sectional curvature and applying the Wu–Yau theorems. In Sect. 6, we perform explicit calculation on $E_{p,\lambda}$ for any (p, λ) to verify the bounded curvature of the complete Bergman metric, and the hypothesis of Theorem A-3. In the last section, we prove Theorem 7 to obtain an integrated lower bound of the Carathéodory–Reiffen metric in the setting of Theorem 1.

2 Preliminaries

Let M be an n-dimensional complex manifold equipped with a complex structure J and a Hermitian metric g. The complex structure $J : T_{\mathbb{R}}M \to T_{\mathbb{R}}M$ is a real linear endomorphism that satisfies for every $x \in M$, and $X, Y \in T_{\mathbb{R},x}M$, $g_x(J_xX, Y) = -g_x(X, J_xY)$, and for every $x \in M$, $J_x^2 = -\mathbf{Id}_{T_xM}$. We decompose the complexified tangent bundle $T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C} = T'M \oplus \overline{T'M}$, where $T'M$ is the eigenspace of J with respect to the eigenvalue $\sqrt{-1}$ and $\overline{T'M}$ is the eigenspace of J with respect to the eigenvalue $-\sqrt{-1}$. We can regard v, w as real tangent vectors, and η, ξ as corresponding holomorphic $(1, 0)$ tangent vectors under the $\mathbb{R}$-linear isomorphism $T_{\mathbb{R}}M \to T'M$, i.e. $\eta = \frac{1}{\sqrt{2}}(v - \sqrt{-1}Jv)$, $\xi = \frac{1}{\sqrt{2}}(w - \sqrt{-1}Jw)$.

A Hermitian metric on M is a positive definite Hermitian inner product

$$g_p : T_p'M \otimes \overline{T_p'M} \to \mathbb{C}$$

which varies smoothly for each $p \in M$. The metric g can be decomposed into the real part denoted by $\mathrm{Re}(g)$, and the imaginary part denoted by $\mathrm{Im}(g)$. The real part $\mathrm{Re}(g)$ induces an inner product called the induced Riemannian metric of g, an alternating $\mathbb{R}$-differential 2-form. Define the $(1, 1)$-form $\omega := -\frac{1}{2}\,\mathrm{Im}(g)$, which is called the fundamental $(1, 1)$-form of g or the Kähler metric. In local coordinates this form can written as

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} g_{i\bar{j}}dz_i \wedge d\overline{z_j}.$$

The components of the curvature 4-tensor of the Chern connection associated with the Hermitian metric g are given by

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &:= R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i}) \\
&= g\left(\nabla^c_{\frac{\partial}{\partial z_i}}\nabla^c_{\frac{\partial}{\partial \overline{z_j}}}\frac{\partial}{\partial z_k} - \nabla^c_{\frac{\partial}{\partial \overline{z_j}}}\nabla^c_{\frac{\partial}{\partial z_i}}\frac{\partial}{\partial z_k} - \nabla^c_{[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z_j}}]}\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z_l}}\right) \\
&= -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \sum_{p,q=1}^{n} g^{q\bar{p}}\frac{\partial g_{i\bar{p}}}{\partial z_k}\frac{\partial g_{q\bar{j}}}{\partial \bar{z}_l},
\end{aligned}$$

where $i, j, k, l \in \{1, \ldots, n\}$.

The holomorphic sectional curvature with the unit direction η at $x \in M$ (i.e., $g_\omega(\eta, \eta) = 1$) is defined by

$$H(g)(x, \eta) = R(\eta, \overline{\eta}, \eta, \overline{\eta}) = R(v, Jv, Jv, v),$$

where v is the real tangent vector corresponding to η. We will often write $H(g)(x,\eta) = H(g)(\eta) = H(\eta)$. The Ricci tensor of a Kähler metric ω is defined by

$$\mathrm{Ric}(\omega) := -\sqrt{-1}\partial\overline{\partial}\log\det(g).$$

Given any complex manifold M, for each $p \in M$ and a tangent vector v at p, define the Carathéodory–Reiffen metric and the Kobayashi–Royden metric by

$$\gamma_M(p;v) := \sup\{|df(p)(v)|;\ f : M \to \mathbb{D},\ f(p)=0,\ f \text{ holomorphic}\},$$
$$\chi_M(p;v) := \inf\left\{\frac{1}{R};\ f : R\mathbb{D} \to M,\ f(0)=p,\ df(\tfrac{\partial}{\partial z}|_{z=0}) = v,\ f \text{ holomorphic}\right\},$$

respectively.

The Bergman metric is defined in terms of the Bergman kernel. Let $\Lambda^{(n,0)}M$ be the space of smooth complex differential $(n,0)$-forms on M. For $\varphi, \psi \in \Lambda^{(n,0)}M$, define

$$\langle\varphi,\psi\rangle = (-1)^{n^2/2}\int_M \varphi\wedge\overline{\psi},$$

and

$$||\varphi|| = \sqrt{\langle\varphi,\varphi\rangle}.$$

Let $L^2_{(n,0)}$ be the completion of

$$\left\{\varphi \in \Lambda^{(n,0)}M;\ ||\varphi|| < +\infty\right\}$$

with respect to $||\cdot||$. Then $L^2_{(n,0)}$ is a separable Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle$.

Define $\mathcal{H} = \left\{\varphi \in L^2_{(n,0)};\ \varphi \text{ is holomorphic}\right\}$. Suppose $\mathcal{H} \neq 0$. Let $\left\{e_j\right\}_{j\geq 0}$ be an orthonormal basis of $\mathcal{H}$ with respect to $\langle\cdot,\cdot\rangle$. Then the $2n$-form on $M \times M$, defined by

$$B(x,y) := \sum_{j\geq 0} e_j(x)\wedge\overline{e}_j(y), \qquad x,y \in M,$$

is called the Bergman kernel of M. Suppose for some point $p \in M$, we have $B(p,p) \neq 0$. Write $B(z,z) = b(z,z)dz_1\wedge\cdots\wedge dz_n\wedge d\overline{z}_1\wedge\cdots\wedge d\overline{z}_n$ in terms of local coordinates $(z_1,\cdots,z_n)$. Define

$$\omega_B(z) := \sqrt{-1}\partial\overline{\partial}\log b(z,z).$$

If the real $(1,1)$-form ω_B is positive definite, we call the corresponding Hermitian metric g^B_M the Bergman metric. By definition, g^B_M is Kähler.

Lastly, the Kähler–Einstein metric ω_{KE} means the Kähler metric which is also the Einstein metric, and the Kähler–Einstein metric of the negative scalar curvature becomes an invariant metric.

We will use the following lemma to prove Theorem A:

Lemma 4 ([33, Lemma 19]) *Let (M,ω) be a Hermitian manifold such that the holomorphic sectional curvature has the upper bound $-\kappa<0$. Then the Kobayashi–Royden metric satisfies*

$$\chi_M(x,v)\geq\sqrt{\frac{\kappa}{2}}|v|_\omega,$$

for each $x\in M$, $v\in T'_xM$.

3 Quasi-bounded geometry

In this section, we review some results from Sect. 2 in [33].

The notion of quasi-bounded geometry is introduced by Yau and Cheng ([9]). Let (M,ω) be an n-dimensional complete Kähler manifold. For a point $p\in M$, let $B_\omega(p;\rho)$ be the open geodesic ball centered at p in M of radius ρ; we omit the subscript ω if there is no peril of confusion. Denote by $B_{\mathbb{C}^n}(r)$ the open ball centered at the origin in $\mathbb{C}^n$ of radius r with respect to the standard metric $\omega_{\mathbb{C}^n}$.

An n-dimensional Kähler manifold (M,ω) is said to have *quasi-bounded geometry* if there exist two constants $r_2>r_1>0$ such that for each point $p\in M$, there is a domain $U\subset\mathbb{C}^n$ and a nonsingular holomorphic map $\psi:U\to M$ satisfying

(1) $B_{\mathbb{C}^n}(r_1)\subset U\subset B_{\mathbb{C}^n}(r_2)$ and $\psi(0)=p$;

(2) there exists a constant $C>0$ depending only on r_1,r_2,n such that

$$C^{-1}\omega_{\mathbb{C}^n}\leq\psi^*(\omega)\leq C\omega_{\mathbb{C}^n}\quad\text{on } U;\tag{3.1}$$

(3) for each integer $l\geq 0$, there exists a constant A_l depending only on l,n,r_1,r_2 such that

$$\sup_{x\in U}\left|\frac{\partial^{|\nu|+|\mu|}g_{i\bar{j}}}{\partial v^\mu\,\partial\overline{v}^\nu}\right|\leq A_l,\ \text{for all } |\mu|+|\nu|\leq l,\tag{3.2}$$

where $g_{i\bar{j}}$ are the components of $\psi^*\omega$ on U in terms of the natural coordinates $(v^1,\ldots,v^n)$, and μ,ν are multiple indices with $|\mu|=\mu_1+\cdots+\mu_n$. We call r_1 a *radius* of quasi-bounded geometry.

By applying the L^2-estimate, the following theorem is proved.

Theorem 5 ([33], Theorem 9) *Let (M,ω) be a complete Kähler manifold. Then the manifold (M,ω) has quasi-bounded geometry if and only if for each integer $q\geq 0$,* *there exists a constant $C_q > 0$ such that*

$$\sup_{p\in M} |\nabla^q R_m| \le C_q, \tag{3.3}$$

where $R_m = \{R_{i\bar{j}k\bar{l}}\}$ denotes the curvature tensor of ω. In this case, the radius of quasi-bounded geometry depends only on C_0 and the dimension of M.

Also, we will use the following lemma:

Lemma 6 ([33, Lemma 20]) *Suppose a complete Kähler manifold (M, ω) has quasi-bounded geometry. Then the Kobayashi–Royden metric satisfies*

$$\chi_M(x, v) \le C|v|_\omega,$$

for each $x \in M$, $v \in T'_x M$, where C depends only on the radius of quasi-bounded geometry of (M, ω).

4 The maximum principle and Shi's estimate on Kähler–Ricci flow

Let $(M, \widetilde{\omega})$ be an n-dimensional complete noncompact Kähler manifold. Suppose for some constant $T > 0$ there is a smooth solution $\omega(x, t) > 0$ for the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} g_{\alpha\overline{\beta}}(x, t) = -4R_{\alpha\overline{\beta}}(x, t) & \text{on } M \times [0, T], \\ g_{\alpha\overline{\beta}}(x, 0) = \widetilde{g}_{\alpha\overline{\beta}}(x) & x \in M, \end{cases} \tag{4.1}$$

where $g_{\alpha\overline{\beta}}(x, t)$ and $\widetilde{g}_{\alpha\overline{\beta}}$ are the metric components of $\omega(x, t)$ and $\widetilde{\omega}$, respectively. Assume that the curvature $R_m(x, t) = \left\{R_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)}\right\}$ of $\omega(x, t)$ satisfies

$$\sup_{M\times[0,T]} |R_m(x, t)|^2 \le k_0 \tag{4.2}$$

for some constant $k_0 > 0$.

The following lemma is an extension of Lemma 15 in [33] to the case of complement of compact subset. Though the proof is similar, we provide some details to indicate where modifications are needed for the complement.

Lemma 7 *With the above assumptions, suppose a smooth tensor $\left\{W_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)}\right\}$ on M with complex conjugation $W_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)} = W_{\beta\overline{\alpha}\delta\overline{\gamma}(x,t)}$ satisfies*

$$\left(\frac{\partial}{\partial t} W_{\alpha\overline{\beta}\gamma\overline{\delta}(x,t)}\right) \eta^\alpha \overline{\eta}^\beta \eta^\gamma \overline{\eta}^\delta \le (\triangle W_{\alpha\overline{\beta}\gamma\overline{\delta}}) \eta^\alpha \overline{\eta}^\beta \eta^\gamma \overline{\eta}^\delta + C_1 |\eta|^4_{\omega(x,t)}, \tag{4.3}$$

for all $x \in M$, $\eta \in T'_x M$, $0 \le t \le T$, where $\triangle \equiv 2\, g^{\alpha\overline{\beta}}(x, t)(\nabla_{\overline{\beta}} \nabla_\alpha + \nabla_\alpha \nabla_{\overline{\beta}})$ and C_1 is a constant. Let

$$h(x, t) = \max\left\{W_{\alpha\overline{\beta}\gamma\overline{\delta}} \eta^\alpha \overline{\eta}^\beta \eta^\gamma \overline{\eta}^\delta;\ \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\right\},$$

for all $x \in M$ *and* $0 \le t \le T$. *For any compact subset* K *in* M, *suppose*

$$\sup_{x\in M, 0\le t\le T} |h(x,t)| \le C_0, \tag{4.4}$$

$$\sup_{M\backslash K} h(x,0) \le -\kappa, \tag{4.5}$$

for some constants $C_0 > 0$ *and* κ. *Then,*

$$h(x,t) \le (8C_0\sqrt{nk_0} + C_1)t - \kappa,$$

for all $x \in M\backslash K$ *and* $0 \le t \le T$.

Proof Denote

$$C = 8C_0\sqrt{nk_0} + C_1 > 0. \tag{4.6}$$

Suppose

$$h(x_1, t_1) - Ct_1 + \kappa > 0, \tag{4.7}$$

for some $(x_1, t_1) \in M\backslash K \times [0, T]$. Then by (4.4) we have $t_1 > 0$. Under the conditions (4.1) and (4.2), it follows from [30] that there exists a function θ such that

$$0 < \theta(x,t) \le 1, \text{ on } M \times [0, T], \tag{4.8}$$

$$\frac{\partial\theta}{\partial t} - \triangle_{\omega(x,t)}\theta + 2\theta^{-1}|\nabla\theta|^2_{\omega(x,t)} \le -\theta \text{ on } M \times [0, T], \tag{4.9}$$

$$\frac{C_2^{-1}}{1 + d_0(x_0, x)} \le \theta(x,t) \le \frac{C_2}{1 + d_0(x_0, x)} \text{ on } M \times [0, T], \tag{4.10}$$

where x_0 is a fixed point in M, $d_0(x, y)$ is the geodesic distance between x and y with respect to $\omega(x, 0)$, and $C_2 > 0$ is a constant depending only on n, k_0 and T.

Let

$$m_0 = \sup_{M\backslash K, 0\le t\le T} ([h(x,t) - Ct + \kappa]\theta(x,t)).$$

Then $0 < m_0 \le C_0 + |\kappa|$ by (4.4),(4.7), and (4.8). Denote

$$\Lambda = \frac{2C_2(C_0 + CT + |\kappa|)}{m_0} > 0.$$

Then, for any $x \in M\backslash K$ with $d_0(x_0, x) \ge \Lambda$, we have

$$|(h(x,t) - Ct + \kappa)\theta(x,t)| \le \frac{C_2(C_0 + CT + |\kappa|)}{1 + d_0(x, x_0)} \le \frac{m_0}{2}.$$

It follows that the function $(h - Ct + \kappa)\theta$ must attain its supremum m_0 on the compact set $\overline{B(x_0; \Lambda)} \times [0, T] \subset M\backslash K \times [0, T]$, where $\overline{B(x_0; r)}$ denotes the closure of the geodesic ball with respect to $\omega(x, 0)$ centered at x_0 of radius r. Let

$$f(x, \eta, t) = \frac{W_{\alpha\overline{\beta}\gamma\overline{\delta}}\eta^{\alpha}\overline{\eta}^{\beta}\eta^{\gamma}\overline{\eta}^{\delta}}{|\eta|^4_{\omega(x,t)}} - Ct + \kappa,$$

for all $(x, t) \in M\backslash K \times [0, T]$, $\eta \in T'_x M\backslash \{0\}$. Then there exist x_*, η_*, t_* with $x_* \in \overline{B(x_0; r)}$, $0 \le t_* \le T$, $\eta_* \in T'_{x_*} M$ and $|\eta_*|_{\omega(x_*,t_*)} = 1$, such that

$$m_0 = f(x_*, \eta_*, t_*)\theta(x_*, t_*) = \max_{\mathcal{S}_t \times [0,T]} (f\theta),$$

where $\mathcal{S}_t = \left\{(x, \eta) \in T'M; x \in M, \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\right\}$. Since $h(., 0)$ is a continuous function on M, either $x_* \in M\backslash K$ or $x_* \in \partial K$, $t_* > 0$ by (4.5). Now we extend η_* to a smooth vector field using the same argument as in the proof of Lemma 15 in [33]. Since $f\theta = f(x, \eta(x), t)\theta(x, t)$ attains its maximum at (x_*, t_*), we have

$$\tfrac{\partial}{\partial t}(f\theta) \ge 0, \;\; \nabla(f\theta) = 0, \;\; \triangle(f\theta) \le 0 \quad \text{at } (x_*, t_*). \tag{4.11}$$

From (4.11) and (4.9), one can see that at the point (x_*, t_*), we have

$$0 \le \tfrac{\partial}{\partial t}(f\theta) = -m_0 < 0$$

(for details, see [33]). This yields a contradiction and the proof is completed.

The following lemma is an extension of Lemma 13 in [33] to the case of complement of a compact subset.

Lemma 8 *Let (M, ω) be an n-dimensional complete noncompact Kähler manifold. Let K be a compact set in M such that*

$$-\kappa_2 \le H(\omega) \le -\kappa_1 < 0 \textit{ on } M\backslash K, \tag{4.12}$$

where $H(\omega)$ is the holomorphic sectional curvature and κ_1, κ_2 are positive constants. Then there exists another Kähler metric $\widetilde{\omega}$ such that

$$C^{-1}\omega \le \widetilde{\omega} \le C\omega \qquad \textit{on } M, \tag{4.13}$$

$$-\widetilde{\kappa_2} \le H(\widetilde{\omega}) \le -\widetilde{\kappa_1} < 0 \qquad \textit{on } M\backslash K, \tag{4.14}$$

$$\sup_{p\in M} |\widetilde{\nabla}^q \widetilde{R_m}| \le C_q \qquad \textit{on } M, \tag{4.15}$$

where $\widetilde{\nabla}^q$ denotes the q-th order covariant derivative of $\widetilde{R_m}$ with respect to $\widetilde{\omega}$, and the positive constants $C = C(n)$, $\widetilde{\kappa_j} = \widetilde{\kappa_j}(n, \kappa_1, \kappa_2)$, $j = 1, 2$, $C_q = C_q(n, q, \kappa_1, \kappa_2)$ depend only on the parameters in their parentheses.

The conditions (4.13) and (4.15) appear in [30, 33]. We provide below details for the pinching estimate.

Proof From the short time existence of the Kähler–Ricci flow [30], the equation (4.1) admits a smooth solution $\left\{g_{\alpha\overline{\beta}}(x,t)\right\}$ for all $0 \le t \le T$. The curvature $R_m(x,t)$ satisfies

$$\sup_{x\in M} |\nabla^q R_m(x,t)|^2 \le \frac{C(q,n,K)(\kappa_2-\kappa_1)^2}{t^q}, \qquad 0 < t \le \frac{\theta_0(n,K)}{\kappa_2-\kappa_1} \equiv T, \tag{4.16}$$

for each nonnegative integer q, where $C(q,n,k) > 0$ is a constant depending only on q, K and n, and $\theta_0(n,K) > 0$ is a constant depending only on n and K.

From the evolution equation of the curvature tensor (see [30, 33]), we have

$$\begin{aligned}\frac{\partial}{\partial t} R_{\alpha\overline{\beta}\gamma\overline{\delta}} &= 4\triangle R_{\alpha\overline{\beta}\gamma\overline{\delta}} + 4g^{\mu\overline{\nu}}g^{\rho\overline{\tau}}(R_{\alpha\overline{\beta}\mu\overline{\tau}}R_{\gamma\overline{\delta}\rho\overline{\nu}} + R_{\alpha\overline{\delta}\mu\overline{\tau}}R_{\gamma\overline{\beta}\rho\overline{\nu}} - R_{\alpha\overline{\nu}\gamma\overline{\tau}}R_{\mu\overline{\beta}\rho\overline{\delta}})\\ &\quad - 2g^{\mu\overline{\nu}}(R_{\alpha\overline{\nu}}R_{\mu\overline{\beta}\rho\overline{\tau}} + R_{\mu\overline{\beta}}R_{\alpha\overline{\nu}\rho\overline{\tau}} + R_{\gamma\overline{\nu}}R_{\alpha\overline{\beta}\mu\overline{\tau}} + R_{\mu\overline{\delta}}R_{\alpha\overline{\beta}\rho\overline{\nu}}),\end{aligned}$$

where $\triangle \equiv \triangle_{\omega(x,t)} = \frac{1}{2}g^{\alpha\overline{\beta}}(x,t)(\nabla_{\overline{\beta}}\nabla_\alpha + \nabla_\alpha\nabla_{\overline{\beta}})$. It follows that

$$\left(\frac{\partial}{\partial t} R_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta \tag{4.17}$$

$$\le 4(\triangle R_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta + C_1(n)|\eta|^4_{g_{\alpha\overline{\beta}}}(x,t)|R_m(x,t)|^2_{\omega(x,t)} \tag{4.18}$$

$$\le 4(\triangle R_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta + \widetilde{C_1}(n,K)(\kappa_2-\kappa_1)^2|\eta|^4_{\omega(x,t)},$$

by (4.16) with $q = 0$. Let

$$H(x,\eta,t) = \frac{R_{\alpha\overline{\beta}\gamma\overline{\delta}})\eta^\alpha\overline{\eta}^\beta\eta^\gamma\overline{\eta}^\delta}{|\eta|^4_{\omega(x,t)}}.$$

Then by (4.12) and (4.16),

$$\begin{aligned}&H(\widetilde{\omega}) \le -\widetilde{\kappa_1} < 0 \text{ on } M\backslash K,\\ &|H(x,\eta,t)| \le |R_m(x,t)|_{\omega(x,t)} \le C_0(n,K)(\kappa_2-\kappa_1).\end{aligned}$$

To apply the maximum principle, let us denote

$$h(x,t) = \max\left\{H(x,\eta,t);\, |\eta|_{\omega(x,t)=1}\right\},$$

for all $x \in M$ and $0 \le t \le \frac{\theta(n,K)}{\kappa_2-\kappa_1}$. Then h with (4.17) satisfies the three conditions in Lemma 7. Then

$$H(x,\eta,t) \le h(x,t) \le -\frac{\kappa_1}{2} < 0,$$

for all $0 < t \le t_0 := \min\left\{\frac{\kappa_1}{2\widetilde{C_1}(n,K)(\kappa_2-\kappa_1)^2}, \frac{\theta_0(n,K)}{\kappa_2-\kappa_1}\right\}$. Since the curvature tensor is bounded by (4.16) with $q = 0$, the complete Kähler metric $\omega(x,t) = \frac{\sqrt{-1}}{2} g_{\alpha\overline{\beta}}(x,t) dz^\alpha \wedge d\overline{z}^\beta$ is a desired metric for an arbitrary $t \in (0, t_0]$.

5 Generation of Kähler metrics with negative holomorphic sectional curvature

In this section, after establishing a proposition below, we prove Theorem 1.

Proposition 9 *Given an n-dimensional Kähler manifold (M, ω), assume that there exists a compact subset K in M such that the holomorphic sectional curvature of ω is negative outside of K, and M is biholomorphically and properly embedded into B_N, $N \ge n$, where B_N is the unit ball in $\mathbb{C}^N$. Then there exists a complete Kähler metric $\widetilde{\omega}$ whose holomorphic sectional curvature has a negative upper bound and $\widetilde{\omega} \ge \omega$.*

Proof From the holomorphic embedding $M \hookrightarrow B_N$, consider a Kähler metric of the form

$$\omega_m := m\omega_P + \omega, \quad m > 0,$$

where ω_P is the Poincaré metric of the unit ball B_N in $\mathbb{C}^N$. It is clear that $\omega_m \ge \omega$ for each $m > 0$. From the decreasing property of the holomorphic sectional curvature, ω_P restricted to M has a negative holomorphic sectional curvature [34]. From Lemma 4 of [34], we may assume that the holomorphic sectional curvature of ω_m is the Gaussian curvature on some embedded Riemann surfaces in M. Recall that for a Hermitian metric G on a Riemann surface, the holomorphic sectional curvature of G is the Gaussian curvature $H(g) = -\frac{1}{g}\frac{\partial^2 \log g}{\partial z \partial \overline{z}}$ of G for some positive smooth function $g = g(z, \overline{z})$. In this case, the holomorphic sectional curvature $H(G,t)$ becomes a real-valued function independent of the unit vector t. Thus we write $H(G)$ instead of $H(G,t)$.

From [25, Proposition 3.1], for any positive functions f and g with $m > 0$,

$$\begin{aligned} H(f + mg) &\le \frac{f^2}{(f+mg)^2} H(f) + \frac{m^2 g^2}{(f+mg)^2} H(mg) \\ &= \frac{f^2}{(f+mg)^2} H(f) + \frac{m g^2}{(f+mg)^2} H(g). \end{aligned}$$

From here, we can deduce that $H(\omega_m)$ becomes negative on K by taking sufficiently large m. Since $H(\omega_m)$ is negative on $M \backslash K$, we are done.

Proof of Theorem 1 For the first statement, we fix a fundamental domain $\widetilde{M}$ and define a function $f : M \to \mathbb{C}$ by $f(z) := \frac{B(z)}{\omega_B^n(z)}$. Since the numerator and the denominator are smooth (n,n)-forms, the function f is well-defined and clearly smooth. Note that the

Bergman kernel and the Bergman metric are invariant under the automorphism group of M. Thus the boundedness assumption of f on $\widetilde{M}$ implies the boundedness of f on M, and we have a function f which is smooth and bounded on M satisfying $\mathrm{Ric}_{i\bar{j}} + g_{i\bar{j}} = f_{i\bar{j}}$ for each i, j, where we denote the Bergman metric in local coordinates by $(g_{i\bar{j}})$. Now we apply the main theorem in [6], and the conclusion follows.

The first part of the second statement follows from Lemma 4, Lemma 6 and Proposition 9 with the fact that for each $m > 0$,

$$\omega_B \le \widetilde{\omega},$$

where $\widetilde{\omega}$ is defined in Proposition 9. For the second part of the case $N = n$, the metric $\widetilde{\omega}$ has the bounded curvature. Then one can solve the complex Monge–Ampere equation by following Wu–Yau's approach (see Lemma 31 and Theorem 3 in [33]).

Remark 10 When $N > n$, the holomorphic sectional curvature $\widetilde{\omega}$ does not need to be bounded below because of the presence of the second fundamental form (see [34]).

Remark 11 If (4.12) is replaced by

$$-\kappa_2 \le H(\omega) \le -\kappa_1 \text{ on } M \quad \text{for } \kappa_1 \in \mathbb{R},$$

then (4.13) and (4.15) still follow from the original Shi's argument. Combining it with Lemmas 6 and 8, we obtain a proof of the statement in Remark 3. Indeed, by applying Shi's estimate on Kähler–Ricci flow with the short-time existence, we can generate a complete Kähler metric ω such that any order of covariant derivatives of the curvature tensor is bounded, and ω is equivalent to the Bergman metric ω_B. Then by the characterization of quasi-bounded geometry of Wu–Yau [33], ω admits a quasi-bounded geometry, and the statement in Remark 3 follows from Lemma 6.

6 Domain $E_{p,\lambda}$

In this section, we consider the domain

$$E_{p,\lambda} = \{(x, y, z) \in \mathbb{C}^3; (|x|^{2p} + |y|^2)^{1/\lambda} + |z|^2 < 1\}, \qquad p, \lambda > 0,$$

and perform necessary computations to examine the comparison of invariant metrics through verification of the hypotheses in Theorem A.

First, we take a suitable compact set $K \subset E_{p,\lambda} \cup \partial E_{p,\lambda}$ that satisfies the conditions in Theorem A. Since any point $(x, y, z) \in \mathbb{C}^3$ can be realized as

$$|x| < r(z, y) = \left((1 - |z|^2)^\lambda - |y|^2\right)^{\frac{1}{2p}},$$

with a fixed pair (y, z), the point (x, y, z) can be mapped biholomorphically onto the form $(0, y, z)$ through the automorphism of one-dimensional disc with the radius $r(y, z)$ centered at the origin. Then using rotations, we can make the other two entries to have non-negative real-values. Since all these transformations are automorphisms of $E_{p,\lambda}$, we take the compact set:

$$K_1 = \overline{\{(0, y, z) \in E_{p,\lambda}; 0 \le x, y < 1\}},$$

where the closure is taken with respect to the usual topology of $\mathbb{C}^3$.

An explicit formula of Bergman kernel B on $E_{p,\lambda}$ is computed in [2]:

$$\begin{aligned} B((x,y,z),\overline{(x,y,z)}) &= \frac{\left((1-\nu_3)^\lambda - \nu_2\right)^{\frac{1}{p}-3} {\nu_1}^2 (p-1)(\lambda(p-1)+p)}{(1-\nu_3)^{2-2\lambda}\pi^3 p^2 \left(\nu_1 - ((1-\nu_3)^\lambda - \nu_2)^{1/p}\right)^4} \\ &+ \frac{(1-\nu_3)^{\lambda-2}\left((1-\nu_3)^\lambda - \nu_2\right)^{\frac{1}{p}-3} {\nu_1}^2 (p-1)(\lambda-1)\nu_2 p}{\pi^3 p^2 \left(\nu_1 - ((1-\nu_3)^\lambda - \nu_2)^{1/p}\right)^4} \\ &+ \frac{\left((1-\nu_3)^\lambda - \nu_2\right)^{\frac{3}{p}-3} (p+1)\left((1-\nu_3)^\lambda(\lambda+\lambda p+p) + (\lambda-1)\nu_2 p\right)}{(1-\nu_3)^{2-\lambda}\pi^3 p^2 \left(\nu_1 - ((1-\nu_3)^\lambda - \nu_2)^{1/p}\right)^4} \\ &- \frac{\left((1-\nu_3)^\lambda - \nu_2\right)^{\frac{2}{p}-3} 2\nu_1 \left((1-\nu_3)^\lambda(\lambda(p^2-2)+p^2) + (\lambda-1)\nu_2 p^2\right)}{(1-\nu_3)^{2-\lambda}\pi^3 p^2 \left(\nu_1 - ((1-\nu_3)^\lambda - \nu_2)^{1/p}\right)^4}, \end{aligned} \tag{6.1}$$

where we set $\nu_1 := x\overline{x}$, $\nu_2 := y\overline{y}$ and $\nu_3 := z\overline{z}$.

We write

$$a = 1 - \nu_3, \quad b = (1-\nu_3)^\lambda - \nu_2, \quad c = ((1-\nu_3)^\lambda - \nu_2)^{1/p} - \nu_1.$$

Then

$$\begin{aligned} B &= \frac{b^{\frac{1}{p}-3}{\nu_1}^2(p-1)(\lambda(p-1)+p)}{a^{2-2\lambda}\pi^3 p^2 c^4} + \frac{a^{\lambda-2} b^{\frac{1}{p}-3}{\nu_1}^2(p-1)(\lambda-1)\nu_2 p}{\pi^3 p^2 c^4} \\ &+ \frac{b^{\frac{3}{p}-3}(p+1)\left(a^\lambda(\lambda+\lambda p+p) + (\lambda-1)\nu_2 p\right)}{a^{2-\lambda}\pi^3 p^2 c^4} \\ &- \frac{b^{\frac{2}{p}-3} 2\nu_1 \left(a^\lambda(\lambda(p^2-2)+p^2) + (\lambda-1)\nu_2 p^2\right)}{a^{2-\lambda}\pi^3 p^2 c^4}. \end{aligned} \tag{6.2}$$

Write $D = a^2 c^4$ and

$$\begin{aligned} N &= a^{2\lambda} b^{\frac{1}{p}-3}{\nu_1}^2(p-1)(\lambda(p-1)+p) + a^\lambda b^{\frac{1}{p}-3}{\nu_1}^2(p-1)(\lambda-1)\nu_2 p \\ &+ a^\lambda b^{\frac{3}{p}-3}(p+1)\left(a^\lambda(\lambda+\lambda p+p) + (\lambda-1)\nu_2 p\right) \\ &- a^\lambda b^{\frac{2}{p}-3} 2\nu_1 \left(a^\lambda(\lambda(p^2-2)+p^2) + (\lambda-1)\nu_2 p^2\right). \end{aligned}$$

Then

$$B = \frac{N}{\pi^3 p^2 D}. \tag{6.3}$$

Write

$$N_1 = a^{2\lambda} b^{\frac{1}{p}-3} v_1^2, \qquad N_2 = a^{\lambda} b^{\frac{1}{p}-3} v_1^2 v_2, \qquad N_3 = a^{2\lambda} b^{\frac{3}{p}-3},$$

$$N_4 = a^{\lambda} b^{\frac{3}{p}-3} v_2, \qquad N_5 = a^{2\lambda} b^{\frac{2}{p}-3} v_1, \qquad N_6 = a^{\lambda} b^{\frac{2}{p}-3} v_1 v_2,$$

$$u_1 = (p-1)(\lambda(p-1)+p), \qquad u_2 = p(p-1)(\lambda-1), \qquad u_3 = (p+1)(\lambda+\lambda p+p),$$

$$u_4 = p(p+1)(\lambda-1), \qquad u_5 = -2(\lambda(p^2-2)+p^2), \qquad u_6 = -2(\lambda-1)p^2.$$

Then

$$N = \sum_{i=1}^{6} u_i N_i.$$

Note that we have

$$u_1 + u_3 + u_5 = 6\lambda \quad \text{and} \quad u_2 + u_4 + u_6 = 0.$$

From the description of the Bergman kernel, we can check the pseudoconvexity of $E_{p,\lambda}$ for each $p, \lambda > 0$.

Proposition 12 *$E_{p,\lambda}$ is a pseudoconvex domain for each $p, \lambda > 0$.*

Proof [1] To show that $u = u_{p,\lambda} := \left(|x|^{2p} + |y|^2\right)^{\frac{1}{\lambda}} + |z|^2$ is a (bounded) plurisubharmonic exhaustion function of $E_{p,\lambda}$, it suffices to show that $v = v_{p,\lambda} := \left(|x|^{2p} + |y|^2\right)^{\frac{1}{\lambda}}$ is plurisubharmonic. To this end, consider

$$\log v = \frac{1}{\lambda} \log\left(e^{\psi_1} + e^{\psi_2}\right), \qquad \text{where } \psi_1 := 2p \log|x| \quad \text{and} \quad \psi_2 := 2\log|y|.$$

Now the plurisubharmonicity of $\log v$ follows from the fact that $\log\left(e^{\psi_1} + e^{\psi_2}\right)$ is always plurisubharmonic whenever ψ_1 and ψ_2 are plurisubharmonic, since we have

$$\begin{aligned} &\frac{\partial^2}{\partial z \partial \overline{z}} \log\left(e^{\psi_1} + e^{\psi_2}\right) \\ &= \frac{1}{\left(e^{\psi_1} + e^{\psi_2}\right)^2} \left(e^{\psi_1+\psi_2} \left(\frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_2}{\partial z} \right)^2 + e^{\psi_1} \frac{\partial^2 \psi_1}{\partial z \partial \overline{z}} + e^{\psi_2} \frac{\partial^2 \psi_2}{\partial z \partial \overline{z}} \right) \geq 0. \end{aligned}$$

From the plurisubharmonicity of $\log v$ it follows that $v = e^{\log v}$ is plurisubharmonic, as desired.

We are interested in behaviours of the metric and curvature components on the compact set $K_1 = \overline{\{(0, y, z) \in E_{\lambda,p}; 0 \leq y, z < 1\}}$. In what follows, we compute those components.

[1] This proof is suggested by an anonymous referee and replaces our original proof. We are grateful to the referee.

Recall the formula for the components of the Bergman metric

$$g_{i\bar{j}} = \frac{\partial^2 \log B}{\partial z_i \partial \overline{z_j}}, \quad i, j = 1, 2, 3,$$

where we set $(z_1, z_2, z_3) = (x, y, z)$. For $i = 1, 2, 3$, we write

$$\partial_i = \frac{\partial}{\partial z_i} \quad \text{and} \quad \overline{\partial}_i = \frac{\partial}{\partial \overline{z_i}}.$$

Proposition 13 *Each component of the Bergman metric $g_{i\bar{j}}$ at $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, is given as follows:*

$$\begin{aligned}
g_{1\bar{1}} &= \frac{1}{c} \cdot \frac{u_5 + u_6\delta}{u_3 + u_4\delta} + \frac{4}{c}, \\
g_{2\bar{2}} &= \frac{a^\lambda}{b^2}\left(\frac{1}{p} + 3\right) + \frac{a^\lambda}{b^2} \cdot \frac{u_3u_4(1-\delta)^2}{(u_3 + u_4\delta)^2}, \\
g_{2\bar{3}} = g_{3\bar{2}} &= \frac{\lambda yz}{a^{1-\lambda}b^2} \cdot \left(\frac{1}{p} + 3\right) + \frac{\lambda yz}{a^{1-\lambda}b^2} \cdot \frac{u_3u_4(1-\delta)^2}{(u_3 + u_4\delta)^2}, \\
g_{3\bar{3}} &= \frac{1 + \delta(\lambda z^2 - 1)}{a^{2-2\lambda}b^2} \cdot \frac{\lambda}{p} + \frac{\delta^2(2 - 2\lambda) + \delta(2\lambda^2z^2 - 4) + \lambda + 2}{a^{2-2\lambda}b^2} \\
&\quad + \frac{\lambda\delta}{a^{2-2\lambda}b^2} \cdot \frac{u_3u_4(1+\delta^2)(1+\lambda z^2) + u_4^2\delta(1 + (\lambda z^2 - 1)\delta + \delta^2) + u_3^2(1 + \lambda z^2)}{(u_3 + u_4\delta)^2}, \\
g_{i\bar{j}} &= 0 \ \ \textit{otherwise},
\end{aligned}$$

where we write $\delta := y^2/a^\lambda = y^2/(1 - z^2)^\lambda$.

Proof All the formulas for $g_{i\bar{j}}$ are obtained from direct computations. For example, since

$$\begin{aligned}
&\overline{\partial}_1 D = -4a^2c^3x, && \overline{\partial}_1 N_1 = 2a^{2\lambda}b^{\frac{1}{p}-3}\nu_1 x, && \overline{\partial}_1 N_2 = 2a^{\lambda}b^{\frac{1}{p}-3}\nu_1 x\nu_2, \\
&\overline{\partial}_1 N_3 = 0, \quad \overline{\partial}_1 N_4 = 0, && \overline{\partial}_1 N_5 = a^{2\lambda}b^{\frac{2}{p}-3}x, && \overline{\partial}_1 N_6 = a^{\lambda}b^{\frac{2}{p}-3}x\nu_2,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_1\overline{\partial}_1 D = -4a^2c^3 + 12a^2c^2\nu_1, && \partial_1\overline{\partial}_1 N_1 = 4a^{2\lambda}b^{\frac{1}{p}-3}\nu_1, && \partial_1\overline{\partial}_1 N_2 = 4a^{\lambda}b^{\frac{1}{p}-3}\nu_1\nu_2, \\
&\partial_1\overline{\partial}_1 N_3 = 0, \quad \partial_1\overline{\partial}_1 N_4 = 0, && \partial_1\overline{\partial}_1 N_5 = a^{2\lambda}b^{\frac{2}{p}-3}, && \partial_1\overline{\partial}_1 N_6 = a^{\lambda}b^{\frac{2}{p}-3}\nu_2,
\end{aligned}$$

we have

$$\begin{aligned}
g_{1\bar{1}} &= \frac{N(\partial_1\overline{\partial}_1 N) - (\partial_1 N)(\overline{\partial}_1 N)}{N^2} - \frac{D(\partial_1\overline{\partial}_1 D) - (\partial_1 D)(\overline{\partial}_1 D)}{D^2} \\
&\overset{(0,y,z)}{\rightsquigarrow} \frac{\partial_1\overline{\partial}_1 N}{N} - \frac{\partial_1\overline{\partial}_1 D}{D} = \frac{u_5a^{2\lambda}b^{\frac{2}{p}-3} + u_6a^{\lambda}b^{\frac{2}{p}-3}y^2}{u_3a^{2\lambda}b^{\frac{3}{p}-3} + u_4a^{\lambda}b^{\frac{3}{p}-3}y^2} + \frac{4a^2c^3}{a^2c^4} \\
&= \frac{1}{c} \cdot \frac{u_5 + u_6\delta}{u_3 + u_4\delta} + \frac{4}{c},
\end{aligned}$$

where we use $c = b^{\frac{1}{p}}$ at $(0, y, z)$.

The other $g_{i\bar{j}}$ can be computed similarly, and we omit the details.

Remark 14 When $(0, y, z)$ approaches the boundary of K_1, we find that the limits of the metric components and those of curvature components cannot be determined. However, using δ introduced in the above proposition, we will be able to control the limit behaviors.

Write

$$g_{1\bar{1}} = \frac{1}{c} \cdot A_1, \quad g_{2\bar{2}} = \frac{a^\lambda}{b^2} \cdot A_2, \quad g_{2\bar{3}} = \frac{\lambda yz}{a^{1-\lambda} b^2} \cdot A_2, \quad g_{3\bar{3}} = \frac{1}{a^{2-2\lambda} b^2} \cdot A_3, \tag{6.4}$$

where

$$\begin{aligned} A_1 &= \frac{u_5 + u_6\delta}{u_3 + u_4\delta} + 4, \qquad A_2 = \frac{1}{p} + 3 + \frac{u_3u_4(1-\delta)^2}{(u_3 + u_4\delta)^2}, \\ A_3 &= (1 + \delta(\lambda z^2 - 1)) \cdot \frac{\lambda}{p} + \delta^2(2 - 2\lambda) + \delta(2\lambda^2 z^2 - 4) + \lambda + 2 \\ &\quad + \lambda\delta \cdot \frac{u_3u_4(1+\delta^2)(1+\lambda z^2) + u_4^2\delta(1 + (\lambda z^2 - 1)\delta + \delta^2) + u_3^2(1 + \lambda z^2)}{(u_3 + u_4\delta)^2}. \end{aligned}$$

Then

$$g_{2\bar{2}}g_{3\bar{3}} - g_{2\bar{3}}g_{3\bar{2}} = \frac{1}{a^{2-3\lambda}b^4} A_2(A_3 - \lambda^2\delta z^2 A_2) = \frac{1-\delta}{a^{2-3\lambda}b^4} \cdot A_2A_4 = \frac{A_2A_4}{a^{2-2\lambda}b^3}, \tag{6.5}$$

where we put $A_4 := (A_3 - \lambda^2\delta z^2 A_2)/(1-\delta)$ and use $1 - \delta = b/a^\lambda$. More explicitly, we have

$$A_4 = \frac{\delta^2 p^2(r-2)(r-1) + \delta p(r-1)(4pr + 4p + 3r) + p^2r^2 + 3p^2r + 2p^2 + 2pr^2 + 3pr + r^2}{p(\delta p(r-1) + pr + p + r)}.$$

Note that $0 \le \delta < 1$. Furthermore, as $(0, y, z) \in E_{p,\lambda}$ approaches the boundary, we have $\delta \to 1^-$. One sees that

$$\lim_{\delta \to 1^-} A_1 = \frac{4(2+p)}{1+2p}, \qquad \lim_{\delta \to 1^-} A_2 = 3 + \frac{1}{p} \quad \text{and} \quad \lim_{\delta \to 1^-} A_4 = \lambda\left(3 + \frac{1}{p}\right). \tag{6.6}$$

Lemma 15 *At $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, the ratio $\dfrac{\det g_B}{B}$ is bounded.*

Proof From (6.3), (6.4) and (6.5), we obtain

$$\frac{\det g_B}{B} = \frac{\frac{1}{c}A_1 \frac{A_2 A_4}{a^{2-2\lambda}b^3}}{\frac{N}{\pi^3 p^2 D}} = \frac{\pi^3 p^2 A_1 A_2 A_4 a^2 c^4}{ca^{2-2\lambda}b^3 \cdot a^\lambda b^{\frac{3}{p}-3}(p+1)\left(a^\lambda(\lambda+\lambda p+p)+(\lambda-1)y^2 p\right)}$$
$$= \frac{\pi^3 p^2 A_1 A_2 A_4}{(p+1)\left((\lambda+\lambda p+p)+(\lambda-1)p\delta\right)},$$

which is bounded.

Proposition 16 *The inverse metric of the Bergman metric* $g_{i\bar{j}}$ *at* $(0, y, z) \in E_{p,\lambda}$*,* $0 \le y, z < 1$*, are given as follows:*

$$g^{1\bar{1}} = \frac{c}{A_1}, \qquad g^{2\bar{2}} = \frac{b^2}{a^\lambda} \cdot \frac{A_3}{(1-\delta)A_2A_4} = \frac{bA_3}{A_2A_4},$$
$$g^{2\bar{3}} = g^{3\bar{2}} = -\frac{\lambda yza^{1-2\lambda}b^2}{(1-\delta)A_4} = -\frac{\lambda yza^{1-\lambda}b}{A_4}, \quad g^{3\bar{3}} = \frac{a^{2-2\lambda}b^2}{(1-\delta)A_4} = \frac{a^{2-\lambda}b}{A_4},$$
$$g^{i\bar{j}} = 0 \;\; \textit{otherwise.}$$

Proof The formulas are obtained by taking the inverse matrix of the 3×3 matrix $(g_{i\bar{j}})_{i,j=1,2,3}$ calculated in Proposition 13. In particular, the determinant of the 2×2 block $(g_{i\bar{j}})_{i,j=2,3}$ is computed in (6.5). Also recall $1 - \delta = b/a^\lambda$.

Through direct computations, we obtain the following for $(0, y, z) \in K_1$:

Here G_i are set to be the remaining factors after pulling out the factors involving a, b, c, y, z. Explicitly, we have

$$G_1 = \frac{4}{p} - \frac{(u_5+u_6\delta)((2p-3)u_4\delta+3(p-1)u_3+pu_4)}{p(u_3+\delta u_4)^2}$$
$$+ \frac{2(p-1)u_6\delta+(3p-2)u_5+pu_6}{p(u_3+u_4\delta)},$$
$$G_2 = \frac{4\lambda}{p} + \frac{\lambda}{p} \cdot \frac{u_5+u_6\delta}{u_3+u_4\delta} - \frac{\lambda\delta(1-\delta)(u_4u_5-u_3u_6)}{(u_3+u_4\delta)^2}.$$

For simplicity, we do not present expressions for the other G_i's. Since $u_3 + u_4\delta > 0$, one can see that G_i are bounded for $i = 1, 2, \ldots, 8$ as $\delta \to 1^-$.

Lemma 17 *We have*

$$G_4 = \lambda G_3.$$

If we define F_1 *and* F_2 *by*

$$F_1 := \frac{z^2}{1-\delta}\left(G_6 - \lambda\delta G_5\right) \quad \textit{and} \quad F_2 := \frac{1}{1-\delta}\left(G_8 - \lambda\delta z^2 G_7\right),$$

then

$$\lim_{\delta\to 1^-} F_1 = \lambda\left(3+\frac{1}{p}\right) \quad \textit{and} \quad \lim_{\delta\to 1^-} F_2 = \frac{2\lambda^2(1+3p)}{p}.$$

Proof We verify the identities through direct computations with help of a computer algebra system.

Similarly, we obtain
Here H_i are the remaining factors; in particular, we have

$$H_1 = 8 + 4\cdot\frac{u_1+u_2\delta}{u_3+u_4\delta} - 2\cdot\frac{(u_5+u_6\delta)^2}{(u_3+u_4\delta)^2}.$$

We do not present explicit expressions for the other H_i's. Using $0 \le \delta < 1$ and $u_3 + u_4\delta > 0$, one can check that H_i are bounded for $i = 1, 2, \ldots, 10$ as $\delta \to 1^-$.

Proposition 18 *Each curvature components of the Bergman metric at* $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, *is given by*

$$R_{1\bar{1}1\bar{1}} = \frac{1}{c^2}(-H_1) = \frac{1}{c^2}\cdot\widetilde{H}_1,$$

$$R_{1\bar{1}2\bar{2}} = R_{2\bar{1}1\bar{2}} = R_{1\bar{2}2\bar{1}} = R_{2\bar{2}1\bar{1}} = \frac{a^\lambda}{b^2c}\cdot\left(-H_2+\frac{\delta {G_1}^2}{A_1}\right) = \frac{a^\lambda}{b^2c}\cdot\widetilde{H}_2,$$

$$\begin{aligned} R_{1\bar{1}2\bar{3}} &= R_{1\bar{1}3\bar{2}} = R_{2\bar{1}1\bar{3}} = R_{1\bar{2}3\bar{1}} = R_{1\bar{3}2\bar{1}} = R_{2\bar{3}1\bar{1}} = R_{3\bar{1}1\bar{2}} = R_{3\bar{2}1\bar{1}} \\ &= \frac{yza^{\lambda-1}}{b^2c}\cdot\left(-H_3+\frac{G_1G_2}{A_1}\right) = \frac{yza^{\lambda-1}}{b^2c}\cdot\widetilde{H}_3, \end{aligned}$$

$$R_{1\bar{1}3\bar{3}} = R_{1\bar{3}3\bar{1}} = R_{3\bar{1}1\bar{3}} = R_{3\bar{3}1\bar{1}} = \frac{a^{2\lambda-2}}{b^2c}\cdot\left(-H_4+\frac{z^2G_2^2}{A_1}\right) = \frac{a^{2\lambda-2}}{b^2c}\cdot\widetilde{H}_4,$$

$$R_{2\bar{2}2\bar{2}} = \frac{a^{2\lambda}}{b^4}\cdot\left(-H_5+\frac{\delta G_3^2}{A_2}\right) = \frac{a^{2\lambda}}{b^4}\cdot\widetilde{H}_5,$$

$$R_{2\bar{2}2\bar{3}} = R_{2\bar{2}3\bar{2}} = R_{2\bar{3}2\bar{2}} = R_{3\bar{2}2\bar{2}} = \frac{yza^{2\lambda-1}}{b^4}\cdot\left(-H_6+\frac{\delta G_3G_5}{A_2}\right) = \frac{yza^{2\lambda-1}}{b^4}\cdot\widetilde{H}_6,$$

$$\begin{aligned} R_{2\bar{2}3\bar{3}} &= R_{2\bar{3}3\bar{2}} = R_{3\bar{2}2\bar{3}} = R_{3\bar{3}2\bar{2}} \\ &= \frac{a^{3\lambda-2}}{b^4}\cdot\left(-H_7+\frac{\delta^2z^2G_5^2}{A_2}+\frac{\delta(1-\delta)F_1^2}{A_4}\right) = \frac{a^{3\lambda-2}}{b^4}\cdot\widetilde{H}_7, \end{aligned}$$

$$R_{2\bar{3}2\bar{3}} = R_{3\bar{2}3\bar{2}} = \frac{a^{2\lambda-2}y^2z^2}{b^4}\cdot\left(-H_8+\frac{G_3G_7}{A_2}\right) = \frac{a^{2\lambda-2}y^2z^2}{b^4}\cdot\widetilde{H}_8,$$

$$\begin{aligned} R_{2\bar{3}3\bar{3}} &= R_{3\bar{2}3\bar{3}} = R_{3\bar{3}2\bar{3}} = R_{3\bar{3}3\bar{2}} \\ &= \frac{a^{3\lambda-3}yz}{b^4}\cdot\left(-H_9+\frac{\delta z^2G_5G_7}{A_2}+\frac{(1-\delta)F_1F_2}{A_4}\right) = \frac{a^{3\lambda-3}yz}{b^4}\cdot\widetilde{H}_9, \end{aligned}$$

Table 1 Formulas for $\partial_i g_{j\bar{k}}$

$\partial_1 g_{2\bar{1}} = \partial_2 g_{1\bar{1}} = \bar{\partial}_1 g_{1\bar{2}} = \bar{\partial}_2 g_{1\bar{1}} = \frac{y}{bc} G_1,$
$\partial_1 g_{3\bar{1}} = \partial_3 g_{1\bar{1}} = \bar{\partial}_1 g_{1\bar{3}} = \bar{\partial}_3 g_{1\bar{1}} = \frac{z}{a^{1-\lambda} bc} G_2,$
$\partial_2 g_{2\bar{2}} = \bar{\partial}_2 g_{2\bar{2}} = \frac{y a^{\lambda}}{b^3} G_3,$
$\partial_2 g_{2\bar{3}} = \bar{\partial}_2 g_{3\bar{2}} = \frac{y^2 z}{a^{1-\lambda} b^3} G_4,$
$\partial_2 g_{3\bar{2}} = \partial_3 g_{2\bar{2}} = \bar{\partial}_2 g_{2\bar{3}} = \bar{\partial}_3 g_{2\bar{2}} = \frac{y^2 z}{a^{1-\lambda} b^3} G_5,$
$\partial_2 g_{3\bar{3}} = \partial_3 g_{2\bar{3}} = \bar{\partial}_2 g_{3\bar{3}} = \bar{\partial}_3 g_{3\bar{2}} = \frac{y z^2}{a^{2-2\lambda} b^3} G_6,$
$\partial_3 g_{3\bar{2}} = \bar{\partial}_3 g_{2\bar{3}} = \frac{y z^2}{a^{2-2\lambda} b^3} G_7,$
$\partial_3 g_{3\bar{3}} = \bar{\partial}_3 g_{3\bar{3}} = \frac{z}{a^{3-3\lambda} b^3} G_8,$
$\partial_i g_{j\bar{k}} = \bar{\partial}_i g_{j\bar{k}} = 0$ otherwise.

$$R_{3\bar{3}3\bar{3}} = \frac{a^{4\lambda-4}}{b^4} \cdot \left(-H_{10} + \frac{\delta z^4 G_7^2}{A_2} + \frac{z^2(1-\delta) F_2^2}{A_4} \right) = \frac{a^{4\lambda-4}}{b^4} \cdot \widetilde{H}_{10},$$

$$R_{i\bar{j}k\bar{l}} = 0 \ \ \textit{otherwise},$$

where we define $\widetilde{H}_i$ *for* $i = 1, 2, \ldots, 10$ *for later use.*

Proof Recall that the components of curvature tensor R associated with g is given by

$$R_{i\bar{j}k\bar{l}} = -\partial_k \bar{\partial}_l g_{i\bar{j}} + \sum_{p,q=1}^{3} g^{q\bar{p}} (\partial_k g_{i\bar{p}})(\bar{\partial}_l g_{q\bar{j}}).$$

Thus the results follow from Tables 1 and 2 and Proposition 16.

Lemma 19 *We have*

$$\widetilde{H}_3 = \lambda \widetilde{H}_2, \quad \widetilde{H}_6 = \lambda \widetilde{H}_5, \quad \widetilde{H}_8 = \lambda \widetilde{H}_6 \ \ \textit{and} \ \ \widetilde{H}_9 = 2\lambda \widetilde{H}_7 - \lambda^2 \delta z^2 \widetilde{H}_6.$$

If we define

$$\widetilde{F}_1 := \frac{1}{1-\delta} \left(\widetilde{H}_4 - \lambda \delta z^2 \widetilde{H}_3 \right), \qquad \widetilde{F}_2 := \frac{1}{1-\delta} \left(\widetilde{H}_7 - \lambda \delta z^2 \widetilde{H}_6 \right),$$

$$\widetilde{F}_3 = \frac{1}{(1-\delta)^2} \left(\widetilde{H}_{10} - 4\lambda^2 \delta z^2 \widetilde{H}_7 + 3\lambda^3 \delta^2 z^4 \widetilde{H}_6 \right),$$

then

$$\lim_{\delta \to 1^-} \widetilde{F}_1 = -\frac{4\lambda(2+p)}{p(1+2p)}, \quad \lim_{\delta \to 1^-} \widetilde{F}_2 = -\lambda \left(3 + \frac{1}{p} \right) \ \ \textit{and}$$

$$\lim_{\delta \to 1^-} \widetilde{F}_3 = -2\lambda^2 \left(3 + \frac{1}{p} \right). \tag{6.7}$$

Table 2 Formulas for $\partial_i\overline{\partial}_j g_{k\bar{l}}$

$\partial_1\overline{\partial}_1 g_{1\bar{1}} = \frac{1}{c^2}H_1,$
$\partial_1\overline{\partial}_1 g_{2\bar{2}} = \partial_1\overline{\partial}_2 g_{2\bar{1}} = \partial_2\overline{\partial}_1 g_{1\bar{2}} = \partial_2\overline{\partial}_2 g_{1\bar{1}} = \frac{a^\lambda}{b^2c}H_2,$
$\partial_1\overline{\partial}_1 g_{2\bar{3}} = \partial_1\overline{\partial}_3 g_{2\bar{1}} = \partial_2\overline{\partial}_1 g_{1\bar{3}} = \partial_2\overline{\partial}_3 g_{1\bar{1}} = \partial_1\overline{\partial}_1 g_{3\bar{2}} = \partial_1\overline{\partial}_2 g_{3\bar{1}} = \partial_3\overline{\partial}_1 g_{1\bar{2}} = \partial_3\overline{\partial}_2 g_{1\bar{1}} = \frac{yz}{a^{1-\lambda}b^2c}H_3,$
$\partial_1\overline{\partial}_1 g_{3\bar{3}} = \partial_1\overline{\partial}_3 g_{3\bar{1}} = \partial_3\overline{\partial}_1 g_{1\bar{3}} = \partial_3\overline{\partial}_3 g_{1\bar{1}} = \frac{1}{a^{2-2\lambda}b^2c}H_4,$
$\partial_2\overline{\partial}_2 g_{2\bar{2}} = \frac{a^{2\lambda}}{b^4}H_5,$
$\partial_2\overline{\partial}_2 g_{2\bar{3}} = \partial_2\overline{\partial}_3 g_{2\bar{2}} = \partial_2\overline{\partial}_2 g_{3\bar{2}} = \partial_3\overline{\partial}_2 g_{2\bar{2}} = \frac{yz}{a^{1-2\lambda}b^4}H_6,$
$\partial_2\overline{\partial}_2 g_{3\bar{3}} = \partial_2\overline{\partial}_3 g_{3\bar{2}} = \partial_3\overline{\partial}_2 g_{2\bar{3}} = \partial_3\overline{\partial}_3 g_{2\bar{2}} = \frac{1}{a^{2-3\lambda}b^4}H_7,$
$\partial_2\overline{\partial}_3 g_{2\bar{3}} = \partial_3\overline{\partial}_2 g_{3\bar{2}} = \frac{y^2z^2}{a^{2-2\lambda}b^4}H_8,$
$\partial_2\overline{\partial}_3 g_{3\bar{3}} = \partial_3\overline{\partial}_3 g_{2\bar{3}} = \partial_3\overline{\partial}_2 g_{3\bar{3}} = \partial_3\overline{\partial}_3 g_{3\bar{2}} = \frac{yz}{a^{3-3\lambda}b^4}H_9,$
$\partial_3\overline{\partial}_3 g_{3\bar{3}} = \frac{1}{a^{4-4\lambda}b^4}H_{10},$
$\partial_i\overline{\partial}_j g_{k\bar{l}} = 0$ otherwise.

Proof The identities are verified through direct computations and can be checked by a computer algebra system.

In order to see cancellations of factors involving a, b, c in the holomorphic sectional curvature, we apply the Gram–Schmidt process to determine an orthonormal frame X, Y, Z instead of using the global coordinate vector fields $\frac{\partial}{\partial z_i}$, $i = 1, 2, 3$. Indeed, let g be any Hermitian metric, and take the first unit vector field

$$X = \frac{\partial_1}{\sqrt{g_{1\bar{1}}}}. \tag{6.8}$$

Write $k_1 := \frac{1}{\sqrt{g_{1\bar{1}}}}$ so that $X = k_1\partial_1$. Then a vector field $\tilde{Y}$ which is orthogonal to X is given by

$$\tilde{Y} = \frac{\partial_2}{\sqrt{g_{2\bar{2}}}} - g\left(\frac{\partial_2}{\sqrt{g_{2\bar{2}}}}, X\right)X = a_1\partial_1 + a_2\partial_2,$$

where we put

$$a_1 := -\frac{g_{2\bar{1}}}{g_{1\bar{1}}\sqrt{g_{2\bar{2}}}} \quad \text{and} \quad a_2 := \frac{1}{\sqrt{g_{2\bar{2}}}}.$$

Since $g(\tilde{Y}, \tilde{Y}) = a_1\overline{a_1}g_{1\bar{1}} + a_1\overline{a_2}g_{1\bar{2}} + a_2\overline{a_1}g_{2\bar{1}} + a_2\overline{a_2}g_{2\bar{2}}$, we take

$$Y = \frac{\tilde{Y}}{\sqrt{g(\tilde{Y}, \tilde{Y})}} = \frac{a_1\partial_1 + a_2\partial_2}{\sqrt{a_1\overline{a_1}g_{1\bar{1}} + a_1\overline{a_2}g_{1\bar{2}} + a_2\overline{a_1}g_{2\bar{1}} + a_2\overline{a_2}g_{2\bar{2}}}} = t_1\partial_1 + t_2\partial_2, \tag{6.9}$$

where we put

$$t_i := \frac{a_i}{\sqrt{a_1\overline{a_1}g_{1\bar{1}} + a_1\overline{a_2}g_{1\bar{2}} + a_2\overline{a_1}g_{2\bar{1}} + a_2\overline{a_2}g_{2\bar{2}}}}, \quad i = 1, 2. \tag{6.10}$$

Similarly, consider

$$\tilde{Z} = p_1\partial_1 + p_2\partial_2 + p_3\partial_3,$$

where

$$p_1 := -\frac{g_{3\bar{1}}}{g_{1\bar{1}}\sqrt{g_{3\bar{3}}}} - \frac{t_1}{\sqrt{g_{3\bar{3}}}}(t_1 g_{3\bar{1}} + t_2 g_{3\bar{2}}),$$
$$p_2 := -\frac{t_2}{\sqrt{g_{3\bar{3}}}}(t_1 g_{3\bar{1}} + t_2 g_{3\bar{2}}), \qquad p_3 := \frac{1}{\sqrt{g_{3\bar{3}}}}.$$

Normalizing $\tilde{Z}$ yields

$$Z = s_1\partial_1 + s_2\partial_2 + s_3\partial_3, \tag{6.11}$$

where

$$s_i := \frac{p_i}{\sqrt{\sum_{k,l=1}^{3} p_k p_l g_{k\bar{l}}}}, \quad i = 1, 2, 3.$$

These X, Y, Z are used in the following proposition which is the main result of this section.

Proposition 20 *At* $(0, y, z) \in E_{p,\lambda}$, $0 \le y, z < 1$, *the components of the holomorphic sectional curvature* R *are given by as follows.*

$$H(X) = R(X, \bar{X}, X, \bar{X}) = \frac{\widetilde{H}_1}{A_1^2}, B(X, Y) = R(X, \bar{X}, Y, \bar{Y}) = \frac{\widetilde{H}_2}{A_1 A_2},$$
$$H(Y) = R(Y, \bar{Y}, Y, \bar{Y}) = \frac{\widetilde{H}_5}{A_2^2}, B(X, Z) = R(X, \bar{X}, Z, \bar{Z}) = \frac{\widetilde{F}_1}{A_1 A_4},$$
$$H(Z) = R(Z, \bar{Z}, Z, \bar{Z}) = \frac{\widetilde{F}_3}{A_4^2}, B(Y, Z) = R(Y, \bar{Y}, Z, \bar{Z}) = \frac{\widetilde{F}_2}{A_2 A_4},$$
$$R(X, \bar{X}, X, \bar{Y}) = R(Y, \bar{Y}, Y, \bar{X}) = R(Z, \bar{Z}, Z, \bar{Y}) = R(Y, \bar{X}, Y, \bar{X}) = 0,$$
$$R(X, \bar{X}, X, \bar{Z}) = R(Y, \bar{Y}, Y, \bar{Z}) = R(Z, \bar{Z}, Z, \bar{X}) = R(Z, \bar{X}, Z, \bar{X}) = 0,$$
$$R(X, \bar{X}, Y, \bar{Z}) = R(Y, \bar{Y}, X, \bar{Z}) = R(Z, \bar{Z}, X, \bar{Y}) = R(Z, \bar{Y}, Z, \bar{Y}) = 0.$$

Proof All the identities follow from Proposition 18 and Lemma 19. To illustrate the process, we compute $H(X)$, $B(X, Y)$ and $R(Y, \bar{Y}, Y, \bar{Z})$. Computations of the other components are similar.

Since $g_{2\bar{1}} = 0$ and $g_{3\bar{1}} = 0$, we have $a_1 = 0$, $t_1 = 0$, $p_1 = 0$ and $s_1 = 0$ on $(0, y, z)$. On the other hand,

$$t_2 = \frac{a_2}{\sqrt{a_2 \overline{a_2} g_{2\bar{2}}}} = \frac{1}{\sqrt{g_{2\bar{2}}}}.$$

Thus, using (6.4), we obtain

$$H(Y) = t_2^4 R_{2\bar{2}2\bar{2}} = \frac{b^4}{a^{2\lambda}} \frac{1}{A_2^2} \cdot \frac{a^{2\lambda}}{b^4} \widetilde{H}_5 = \frac{\widetilde{H}_5}{A_2^2}.$$

Similarly,

$$B(X, Y) = k_1^2 t_2^2 R_{1\bar{1}2\bar{2}} = \frac{1}{g_{1\bar{1}}} \frac{1}{g_{2\bar{2}}} \cdot \frac{a^\lambda}{b^2 c} \cdot \widetilde{H}_2 = \frac{c}{A_1} \frac{b^2}{a^\lambda A_2} \frac{a^\lambda}{b^2 c} \widetilde{H}_2 = \frac{1}{A_1 A_2} \widetilde{H}_2.$$

To compute $R(Y, \bar{Y}, Y, \bar{Z})$, first observe

$$s_2 = -s_3 t_2^2 g_{3\bar{2}} = -s_3 \frac{g_{3\bar{2}}}{g_{2\bar{2}}} = -s_3 \frac{\lambda y z}{a}.$$

Thus it follows from Proposition 18 and Lemma 19 that

$$\begin{aligned} R(Y, \bar{Y}, Y, \bar{Z}) &= t_2^3 s_2 R_{2\bar{2}2\bar{2}} + t_2^3 s_3 R_{2\bar{2}2\bar{3}} = t_2^3 \left(-s_3 \frac{\lambda y z}{a} \right) \frac{a^{2\lambda}}{b^4} \widetilde{H}_5 + t_2^3 s_3 \frac{y z a^{2\lambda - 1}}{b^4} \widetilde{H}_6 \\ &= \frac{t_2^3 s_3 a^{2\lambda-1} y z}{b^4} \left(-\lambda \widetilde{H}_5 + \widetilde{H}_6 \right) = 0. \end{aligned}$$

Corollary 21 *The holomorphic sectional curvature near ∂K_1 is bounded for any $p, \lambda > 0$.*

Proof The assertion follows from (6.6) and (6.7) and the fact that G_i and H_i are bounded as $\delta \to 1^-$.

It is known [10] that the curvature tensor of the Bergman metric is bounded for $\lambda = 1$ and $p > 0$. The following proposition tells us that the same is true for any $p, \lambda > 0$.

Proposition 22 *The curvature tensor of the Bergman metric on $E_{p,\lambda}$ is bounded for any $p, \lambda > 0$.*

Proof The curvature tensor can be explicitly expressed in terms of the holomorphic sectional curvature H_{g_B}. Using the invariance of the Bergman metric, it suffices to show $H_{g_B} \le C$ on ∂K_1 by some constant $C \in \mathbb{R}$. By Corollary 21, we are done.

Corollary 23 *For any $p, \lambda > 0$, there exist $C_0 > 0$ such that*

$$\chi_{E_{p,\lambda}}(p; v) \le C_0\sqrt{\omega_B(v, v)} \qquad \text{for all } v \in T'_p E_{p,\lambda},\ p \in M,$$

and $C_1 > 0$ such that

$$\frac{1}{C_1}\omega_{KE}(v, v) \le \omega_B(v, v) \le C_1\omega_{KE}(v, v) \qquad \text{for all } v \in T' E_{p,\lambda}.$$

Proof The assertion immediately follows from Proposition 22 and Lemma 15.

Remark 24 For the third statement of Theorem A, in general, the holomorphic sectional curvature is not negatively pinched for $E_{p,\lambda}$. For example, when $\lambda = 1$ and $p = 1/5$, we have $\lim_{\delta\to 1^-} H(X) \approx 0.033 > 0$.

Lastly, we obtain interesting rigidity in the following proposition from direct computation of the Ricci curvature of the Bergman metric and we omit the proof.

Proposition 25 *The Bergman metric g_B on $E_{p,\lambda}$ is a Kähler–Einstein metric if and only if $\lambda = p = 1$.*

7 A lower bound of the integrated Carathéodory–Reiffen metric

In this last section, we prove the following theorem.

Theorem B *Let (M, g) be a simply-connected complete noncompact n-dimensional Kähler manifold whose Riemannian sectional curvature k of g satisfies $k \le -a^2$ for some $a > 0$. We denote by d the geodesic distance on M, and by γ_M the Carathéodory–Reiffen metric on M. For any $p \ge 2$, the following are true.*

1. Let f be a holomorphic function from M to the unit disk $\mathbb{D}$ in $\mathbb{C}$. Then

$$\begin{aligned}\int_M \left|\int_M G(x, y)|\nabla f|^2(y)dy\right|^p dx \\ \le \left(\frac{p}{(2n-1)a}\right)^p \int_M |f(x)|^p \gamma_M(x; \nabla f(x))^{\frac{p}{2}} dx,\end{aligned} \tag{7.1}$$

where $G(x, y)$ is the minimal positive Green's function on M.

2. If the Riemannian sectional curvature k of g further satisfies $-b^2 \le k$ for some $b > 0$. Then there exists a constant $C(n) > 0$, which only depends on n, such that for any holomorphic function f from M to the unit disk $\mathbb{D}$, we have

$$\int_0^\infty \int_M \Bigg(\int_M t^{-n} \exp\left[-\frac{d(x, y)^2}{2t} - \frac{(2n-1)^2 b^2 t}{8} - \frac{(2n-1)bd(x, y)}{2}\right](1 + bd(x, y))|\nabla f|^2(y)dy\Bigg)^p dxdt$$

$$\le C(n)\left(\frac{2\pi p}{(2n-1)a}\right)^p \int_M |f(x)|^p \gamma_M(x;\nabla f(x))^{\frac{p}{2}}dx. \tag{7.2}$$

The inequalities (7.1) and (7.2) can be interpreted as integrated gradient estimates of bounded holomorphic functions.

Although the lemmas below are known, we prove them here for tracking explicit constants for the proof of Theorem 7.

Let M be an n-dimensional complete noncompact, simply connected Riemannian manifold, and let $L^2(M)$ be the space of L^2-functions on M. Denote by $W^1(M)$ the Hilbert space consisting of L^2-functions whose gradient are also L^2, and by $W_0^1(M)$ the subspace in $W^1(M)$ which is the completion of the space $C_0^\infty(M)$ under $W^1(M)$-norm. When M is complete, we have $W^1(M) = W_0^1(M)$.

Lemma 26 ([29, Poincaré inequality]) *Let M be an n-dimensional complete noncompact, simply connected Riemannian manifold with sectional curvature $k \le -a^2 < 0$. Then*

$$\int_M |u|^2 \le \frac{4}{(n-1)^2a^2}\int_M |\nabla u|^2, \qquad u \in W_0^1(M). \tag{7.3}$$

Proof Let $r(x) = d(p_0, x)$ be the distance function from a fixed point $p_0 \in M$. From the Rauch comparison theorem, we have

$$\triangle r \ge (n-1)a, \tag{7.4}$$

where $a > 0$.

Let Ω be the geodesic ball centered at p_0 with radius $R > 0$ in M. From the Green's theorem, we have for every $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |u|^2\triangle r - \int_\Omega \nabla(|u|^2)\cdot\nabla r = \int_{b\Omega} |u|^2 d\sigma = 0,$$

where $d\sigma$ is the surface measure on $b\Omega$. We remark that r may not be smooth at p_0, but we can apply the Green's theorem to Ω minus a small ball of radius $\epsilon > 0$ around p_0 and let $\epsilon \to 0$. From (7.4) and $|\nabla r| = 1$, we have

$$(n-1)a\|u\|^2 \le \int_\Omega |u|^2\triangle r = \int_\Omega \nabla(|u|^2)\cdot\nabla r \le \int_\Omega |\nabla(|u|^2)| \le 2\|u\|\,\|\nabla u\|.$$

This gives

$$\|u\| \le \frac{2}{(n-1)a}\|\nabla u\|, \qquad u \in C_0^\infty(\Omega).$$

Since $C_0^\infty(M)$ is dense in $W_0^1(M)$, we are done.

Let $\triangle_0$ denote the Laplace–Beltrami operator. We use Mckean's estimate [27] on the first eigenvalue of $\triangle_0$.

Lemma 27 ([27, Mckean's estimate]) *Let M be an n-dimensional complete noncompact, simply-connected Riemannian manifold with sectional curvature $k \le -a^2 < 0$. Then we have*

$$\lambda_1 \ge \frac{(n-1)^2 a^2}{4}, \tag{7.5}$$

where λ_1 is the smallest eigenvalue of $\triangle_0$.

Proof From Lemma 26, for every $u \in C_0^\infty(M)$,

$$(\triangle_0 u, u) = (du, du) = \int_\Omega |\nabla u|^2 \ge \frac{(n-1)^2 a^2}{4} \int_\Omega |u|^2.$$

The assertion follows.

Lemma 28 ([8, Cheng]) *Let M be an n-dimensional Riemannian manifold. Consider the first eigenvalue for the Dirichlet problem $\lambda_1(M) > 0$. Let Ω be a relatively compact domain of M such that $b\Omega$ is smooth. Let $f \in C^\infty(M)$ and let u be the solution of*

$$\begin{cases} \triangle u = \triangle f & \text{on } \Omega, \\ u = 0 & \text{on } b\Omega. \end{cases}$$

Then for any $p \ge 2$,

$$\int_\Omega |u|^p \le C_p \int_\Omega |\nabla f|^p, \tag{7.6}$$

where the constant C_p depends only on p and $\lambda_1(M)$.

Proof Assume that $p \ge 2$. Multiplying the equation by u^{p-1} and integrating it, we have

$$\begin{aligned}(p-1)\int_\Omega |\nabla u|^2 u^{p-2} &= (\nabla u, \nabla u^{p-1}) = (\nabla f, \nabla u^{p-1}) \\ &\le (p-1)\int_\Omega |\nabla f||\nabla u| u^{p-2} \\ &\le (p-1)\left(\int_\Omega |\nabla u|^2 u^{p-2}\right)^{1/2}\left(\int_\Omega |\nabla f|^2 u^{p-2}\right)^{1/2}.\end{aligned}$$

Thus we have

$$\frac{4}{p^2}\int_\Omega |\nabla u^{p/2}|^2 \le \int_\Omega |\nabla f|^2 u^{p-2} \le \left(\int_\Omega |u|^p\right)^{\frac{p-2}{p}}\left(\int_\Omega |\nabla f|^p\right)^{\frac{2}{p}}.$$

From (7.3), we obtain

$$\left(\frac{4\lambda_1}{p^2}\right)^{\frac{p}{2}}\int_\Omega |u|^p \leq \int_\Omega |\nabla f|^p.$$

The constant C_p depends only on p and λ_1. The general case can be proved similarly through multiplication by $(\operatorname{sgn} u)|u|^{p-1}$ and integration.

Proof of Theorem B From Lemma 27, M has the positive spectrum. It is a standard result that if the manifold has positive spectrum then there exists a positive symmetric Green's function G on M. Moreover, we can always take $G(x, y)$ to be the minimal Green's function constructed using exhaustion of compact subdomains. Hence

$$G(x, y) = \lim_{i\to\infty} G_i(x, y) > 0,$$

where G_i is the Dirichlet Green's function of a compact exhaustion $\{\Omega_i\}_i$ of M, and the limit is uniform on compact subsets of M.

Take any (bounded) holomorphic function $f : M \to \mathbb{D}$. For any relatively compact subdomain $\Omega \subset M$ with the smooth boundary $b\Omega$, we use f^2 in Lemma 28 and solving the Dirichlet boundary problem with the inequality

$$\Big(g(\nabla f^2, \nabla f^2)(x)\Big)^{\frac{p}{2}} = \Big(4|f(x)|^2 df(\nabla f)(x)\Big)^{\frac{p}{2}} \leq 2^p|f|^p(x)\gamma_M(x; \nabla f(x))^{\frac{p}{2}} \tag{7.7}$$

for any $x \in M$, and the condition $p \geq 2$ implies

$$\begin{aligned}\int_\Omega |u|^p &\leq \left(\frac{2p}{(2n-1)a}\right)^p \int_\Omega |f|^p\gamma_M(.; \nabla f)^{\frac{p}{2}}\\ &\leq \left(\frac{2p}{(2n-1)a}\right)^p \int_M |f|^p\gamma_M(.; \nabla f)^{\frac{p}{2}},\end{aligned} \tag{7.8}$$

where u is the solution of

$$\begin{cases}\triangle u = 2|\nabla f|^2 & \text{on } \Omega,\\ u = 0 & \text{on } b\Omega,\end{cases} \tag{7.9}$$

and $a > 0$ is for the upper bound of the Riemannian sectional curvature $\leq -a^2 < 0$.

From the hypothesis $|f|^p\gamma_M(.; \nabla f)^{\frac{p}{2}} \in L^1(M)$ and from the exhaustion of compact subdomains, there exists $u \in C^\infty(M, \mathbb{R})$ such that

$$\int_M |u|^p < \infty,$$

and $\triangle u = 2|\nabla f|^2$ on M. Furthermore, the fact $\inf_{x\in M} \operatorname{Vol} B(x,r) > 0$ for any $r > 0$ implies that $u(x) \to 0$ as $d(p,x) \to \infty$ from some fixed point $p \in M$. Thus the Dirichlet problem is solvable and u can be represented by

$$u(x) = 2\int_M G(x,y)|\nabla f|^2(y)dy, \tag{7.10}$$

which proves part (1).

For part (2), the positive minimal Green's function satisfies

$$G(x,y) = \int_0^\infty h_M(x,y,t)dt,$$

where we denote the heat kernel of the Laplace–Beltrami operator by $h_M(x,y,t)$. Hence (7.10) becomes

$$u(x) = 2\int_0^\infty \int_M h_M(x,y,t)|\nabla f|^2(y)dydt. \tag{7.11}$$

We use the Cheeger and Yau's heat kernel comparison theorem [7]:

$$h_M(x,y,t) \geq h_{M_k}(d(x,y)), \tag{7.12}$$

where M_k is the space form with constant sectional curvature equal to k. From the two-sided estimate of Davies and Mandouvalos [15],

$$c(n)^{-1}h(t,d(x,y)) \leq h_{M_k}(d(x,y)) \leq c(n)h(t,d(x,y)), \tag{7.13}$$

where $c(n)$ depends only on n and

$$h(t,r) = (2\pi t)^{-n}\exp\left[-\frac{r^2}{2t} - \frac{(2n-1)^2b^2t}{8} - \frac{(2n-1)br}{2}\right](1+br)\left(1+br+\frac{b^2t}{2}\right)^{\frac{2n-1}{2}-1} \tag{7.14}$$

for $t, r > 0$, where $b > 0$ is for the lower bound of the Riemannian sectional curvature $\geq -b^2$.

Now combining (7.8) with (7.11), (7.12), (7.13), and (7.14) gives the desired inequality (7.2). This completes the proof.

We end this paper with an example for Theorem B.

Proposition 29 *In the case of unit disk $\mathbb{D}$ in $\mathbb{C}$, for each $p \geq 2$, we have*

$$\begin{aligned} &2\pi\int_0^1 \left(\frac{1}{6} - \frac{R^2}{2}\ln R - \frac{R^4}{8}(4\ln R - 1) - \frac{R^6}{36}(6\ln R - 1)\right)^p R\,dR \\ &\leq p^p \int_{\mathbb{D}} |z|^p \gamma_{\mathbb{D}}(z;\nabla z)^{\frac{p}{2}}. \end{aligned}$$

Proof The Green function of the unit disk $\mathbb{D}$ in $\mathbb{C}$ has the following form:

$$G(x, y) = \frac{1}{2\pi} \ln \frac{|x-y|}{|x||y - \frac{x}{|x|^2}|}.$$

The function G satisfies $\triangle_x G(x, y) = \delta_y$ at fixed $y \in \mathbb{D}$ and $G(x, y) = 0$ when $|x| = 1$ and $|y| < 1$. Since the gradient vector of $z \in \mathbb{D}$ with respect to the Poincaré metric is $(1-|z|^2)\frac{\partial}{\partial z}$, the integrand of the left-hand side of (7.1) is

$$\int_{|y|<1} G(x, y)(1-|y|^2)^2 dy. \tag{7.15}$$

Rewrite $G(x, y) = \frac{1}{4\pi} \ln \left(\frac{|x|^2|y-x/|x|^2|^2}{|x-y|^2} \right)$ and choose coordinates $x = (R, 0)$ and $y = (r\cos\theta, r\sin\theta)$, then (7.15) becomes

$$\begin{aligned} &\frac{1}{4\pi} \int_0^1 \int_0^{2\pi} \ln \left(\frac{1 + r^2R^2 - 2rR\cos\theta}{R^2 + r^2 - 2rR\cos\theta} \right) r(1-r^2)^2 d\theta dr \\ &= \frac{1}{4\pi} \int_0^1 r(1-r^2)^2 \left(I(1, rR) - I(r, R) \right) dr, \end{aligned}$$

where $I(a, b) := \int_0^{2\pi} \ln(a^2 + b^2 - 2ab\cos\theta) d\theta$. It is well-known that

$$I(a, b) = 4\pi \max\{\ln|a|, \ln|b|\}.$$

Since $0 \le r, R \le 1$, we have $I(1, rR) = 0$. Thus the integral becomes

$$\begin{aligned} &-\int_0^1 r(1-r^2)^2 \max\{\ln|r|, \ln|R|\} dr \\ &= -\ln R \int_0^R r(1-r^2)^2 dr - \int_R^1 r(1-r^2)^2 \ln r dr \\ &= \frac{1}{6} - \frac{R^2}{2}\ln R - \frac{R^4}{8}(4\ln R - 1) - \frac{R^6}{36}(6\ln R - 1). \end{aligned}$$

Thus the left-hand side of (7.1) is

$$2\pi \int_0^1 \left(\frac{1}{6} - \frac{R^2}{2}\ln R - \frac{R^4}{8}(4\ln R - 1) - \frac{R^6}{36}(6\ln R - 1) \right)^p R\, dR.$$

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Declarations

Conflicts of interest The corresponding author states that there is no conflict of interest.

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