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A FAMILY OF GENERALIZED KAC–MOODY ALGEBRAS AND DEFORMATION OF MODULAR FORMS

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We develop an analogue of Gindikin–Karpelevich formula for a family of generalized Kac–Moody algebras, attached to Borcherds–Cartan matrices consisting of only one positive entry in the diagonal. As an application, we obtain a deformation of Fourier coefficients of modular forms such as the modular j-function and Ramanujan τ-function.

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0. Introduction

In this paper, we obtain an analogue of Gindikin–Karpelevich formula for a family of generalized Kac–Moody algebras (see Proposition 1.4). They are attached to Borcherds–Cartan matrices with only one positive entry in the diagonal. Here it is important for our purpose to take the definition of generalized Kac–Moody algebras as in [9] so that the imaginary simple roots have multiplicity one. Then we can apply our earlier technique [12] of writing the product over positive roots as the sum over certain PBW basis. (See also [9, Sec. 5.3, Remark].) Then we take a specialization of $\mathbb{Z} \times \mathbb{Z}$-grading to obtain our result (see Theorem 1.6). The resulting identities can also be considered as deformation of partial denominator identities.

Borcherds [2] obtained many weakly holomorphic modular forms as infinite products as an application of generalized Kac–Moody algebras. One striking example is

$$j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)} = q^{-1} - 984 + \sum_{n=1}^{\infty} c(n)q^n = q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{a(n^2)},$$

where $c(1) = 196884, c(2) = 21493760,\ldots$, and $\sum a(n)q^n = 2q^{-4} + 984q + 286752q^4 + \cdots$ is a weakly holomorphic modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$.

As an application of our Gindikin–Karpelevich formula, we obtain an expression of the form (Example 2.5)

$$\prod_{n=1}^{\infty} \left(\frac{1 - q^n}{1 - tq^n}\right)^{a(n^2)} = 1 + \sum_{n=0}^{\infty} c_t(n)q^{n+1},$$

where

$$c_t(0) = 984(t - 1), \quad c_t(1) = 484620t^2 - 681504t + 196884,\ldots.$$

So $c_t(n)$ can be considered as a t-deformation of $c(n)$.

In the same way, since we have

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{a(n^2)},$$

where $\sum a(n)q^n = q^{-4} + 6 + 504q + 143388q^4 + \cdots$ is another weakly holomorphic modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$, we can obtain t-deformation of the function $504\,\sigma_5(n)$ using our general result. This example is considered in Example 2.6.

Our deforming process of modular forms can be applied to any weakly holomorphic modular form f of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$ in the Kohnen plus space if the Fourier coefficients of f are positive. We prove in Sec. 2.1 that many such forms have positive Fourier coefficients. In particular we can prove that the Fourier coefficients of basis elements f_d of the Kohnen plus space are positive if $4|d$ (Theorem 2.1). The proof uses the explicit formula of the Fourier coefficients due to Bringmann and Ono [4]. This result not only provides an abundance of examples for our method but also makes quite a contrast to the class of holomorphic Hecke eigenforms with respect to $\Gamma_0(N)$, since such an eigenform has infinitely many positive and infinitely many negative coefficients. This result is of independent interest.

In the last section, we look at the special case of Ramanujan-type modular forms, namely, $\prod_{n=1}^{\infty}(1 - q^n)^k$, and their t-deformations. We start with the t-deformation $\prod_{n=1}^{\infty}(1 - t^{-1}q^n)^k = \sum_{n=0}^{\infty} \epsilon_{t,k}(n)q^n$ and derive some identities involving deformation of divisor-sum function and partition function (Propositions 3.1 and 3.6). The case $k = 1$ has long been studied in various contexts. Recently, a modular-type representation of the infinite product $\prod(1 - t^{-1}q^n)$ was investigated in [18].

The coefficients of t-deformation of arithmetical functions seem to carry important information. In the simplest case of the usual partition function, the first coefficient of the t-deformation $p_{t,1}(n)$ turns out to be equal to the number of divisors of n (Lemma 3.5(2)). This fact has an immediate application that involves a derivative with respect to t (Corollary 3.7).

As the last result of this paper, we obtain various expressions for the Ramanujan function $\tau(n)$ using the deformation $\epsilon_{t,24}(n)$ (Proposition 3.8). See also Remark 3.10.

1. Preliminaries

1.1. *Generalized Kac–Moody algebra*

Let I be a countable set of indices. Let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{R}$, satisfying the following conditions:

(a) A is symmetric,
(b) if $i \neq j$ then $a_{ij} \leq 0$,
(c) if $a_{ii} > 0$ then $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

Definition 1.1. The *generalized Kac–Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ associated to the matrix A is defined to be the Lie algebra with generators e_i, h_i, f_i $(i \in I)$ and the following defining relations:

(i) $[h_i, h_j] = 0$,
(ii) $[h_i, e_k] = a_{ik}e_k, [h_i, f_k] = -a_{ik}f_k$,
(iii) $[e_i, f_j] = \delta_{ij}h_i$,
(iv) $(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}}e_j = 0, (\operatorname{ad} f_i)^{1-2a_{ij}/a_{ii}}f_j = 0$ for $i \neq j$ and $a_{ii} > 0$,
(v) $[e_i, e_j] = 0, [f_i, f_j] = 0$ if $a_{ij} = 0$.

Let $\mathfrak{h} = \sum_{i \in I} \mathbb{R}h_i$. The Lie algebra $\mathfrak{g}$ has an automorphism η of order 2 which acts as -1 on $\mathfrak{h}$ and interchanges the elements e_i and f_i. We denote by $L(X)$ the free Lie algebra on a set X.

Theorem 1.2 ([9]). *Assume that the matrix A has only one positive diagonal entry, $a_{i_0 i_0} > 0$, and if $a_{mj} = 0$ then $m = i_0$ or $j = i_0$ or $m = j$. Let $\mathscr{S} = \{(\operatorname{ad} e_{i_0})^l e_j \mid 0 \leq l \leq -2a_{i_0 j}/a_{i_0 i_0}, j \neq i\}$. Then we have*

$$\mathfrak{g}(A) = L(\mathscr{S}) \oplus (\mathfrak{sl}_2 + \mathfrak{h}) \oplus L(\eta(\mathscr{S})).$$

We keep the assumption in the above theorem. Let the set of imaginary simple roots be denoted by

$$S' = \{\alpha_j \mid j \in I, j \neq i_0\} \subset \mathfrak{h}^*.$$

We consider the following subset of positive roots:

$$S = \{l\alpha_{i_0} + \alpha_j \mid l = 0, 1, \ldots, -2a_{i_0 j}/a_{i_0 i_0}, j \in I, j \neq i_0\}.$$

Let Δ'_+ be the additive monoid generated by S in $\mathfrak{h}^*$. Then $\Delta_+ = \{\alpha_{i_0}\} \cup \Delta'_+$ is the set of positive roots. Using the denominator identity for a free Lie algebra (see, e.g., [3]), we obtain the following corollary.

Corollary 1.3 ([9]). *We have the denominator identity for* $\mathfrak{g}(A)$:

$$\prod_{\alpha \in \Delta_+} (1 - \mathbf{z}^\alpha)^{\mathrm{mult}\,\alpha} = (1 - \mathbf{z}^{\alpha_{i_0}}) \prod_{\alpha \in \Delta'_+} (1 - \mathbf{z}^\alpha)^{\mathrm{mult}\,\alpha} = (1 - \mathbf{z}^{\alpha_{i_0}})\left(1 - \sum_{\alpha \in S} \mathbf{z}^\alpha\right).$$

1.2. *Deformation of partial denominator identities*

Now we begin to consider a certain family of generalized Kac–Moody algebras. First we assume that a sequence $b(n)$, $n \geq 1$, of positive integers is given and we set $b(-1) = 1$. Let M be the symmetric matrix of blocks indexed by $\{-1, 1, 2, \ldots\}$, where the block in position (i, j) has entries $-(i + j)$ and size $b(i) \times b(j)$. Let $\mathfrak{g}(M)$ be the generalized Kac–Moody algebra associated to the matrix M. The generators of $\mathfrak{g}(M)$ will be denoted by

$$f_{-1},\, f_{jk},\, h_{-1},\, h_{jk},\, e_{-1},\, e_{jk} \quad \text{for } j = 1, 2, 3, \ldots, \ k = 1, \ldots, b(j).$$

Then the set of imaginary simple roots are given by

$$S' = \{\alpha_{jk} \,|\, j = 1, 2, 3, \ldots, k = 1, \ldots, b(j)\},$$

and we set

$$\mathscr{S}' = \{e_{jk} \,|\, j = 1, 2, 3, \ldots, k = 1, \ldots, b(j)\}.$$

By Theorem 1.2, we have the decomposition

$$\mathfrak{g}(M) = L(\mathscr{S}) \oplus (\mathfrak{sl}_2 + \mathfrak{h}) \oplus L(\eta(\mathscr{S})),$$

where

$$\mathscr{S} = \bigcup_{j=1}^\infty \{(ade_{-1})^l e_{jk} \,|\, l = 0, 1, \ldots, j - 1, \ k = 1, \ldots, b(j)\}.$$

As in Sec. 1.1, we consider the following subset of positive roots

$$S = \{l\alpha_{-1} + \alpha_{jk} \,|\, j = 1, 2, \ldots, l = 0, 1, \ldots, j - 1, \ k = 1, \ldots, b(j)\}.$$

Let Δ'_+ be the additive monoid generated by S. Then $\Delta_+ = \{\alpha_{-1}\} \cup \Delta'_+$ is the set of positive roots. It follows from Corollary 1.3 that we have the identity

$$\prod_{\alpha \in \Delta_+} (1 - \mathbf{z}^\alpha)^{\mathrm{mult}\,\alpha}$$

$$= (1 - \mathbf{z}^{\alpha_{-1}}) \prod_{\alpha \in \Delta'_+} (1 - \mathbf{z}^\alpha)^{\mathrm{mult}\,\alpha}$$

$$= (1 - \mathbf{z}^{\alpha_{-1}})\left(1 - \sum_{\alpha \in S} \mathbf{z}^\alpha\right)$$

$$= (1 - \mathbf{z}^{\alpha_{-1}})\left(1 - \sum_{\substack{j=1,2,\ldots \\ l=0,1,\ldots,j-1 \\ k=1,\ldots,b(j)}} \mathbf{z}^{l\alpha_{-1}+\alpha_{jk}}\right). \tag{1.1}$$

Let $U(\mathfrak{g}(M))$ be the universal enveloping algebra of $\mathfrak{g}(M)$. We define a linear ordering on the set S by

$$l\alpha_{-1} + \alpha_{jk} \prec l'\alpha_{-1} + \alpha_{j'k'} \quad \text{if } j < j', \quad \text{or} \quad j = j', k < k', \quad \text{or} \quad j = j', k = k', l < l'.$$

Since $S' \subset S$, this ordering is restricted to the set S'.

We consider the following subset of $U(\mathfrak{g}(M))$:

$$\mathcal{B}' = \left\{ \prod_{j=1}^{\infty} \prod_{k=1}^{b(j)} e_{jk}^{c_{jk}} \,\middle|\, c_{jk} \in \mathbb{Z}_{\geq 0}, \ c_{jk}\text{'s are zero for all but finitely many } (j,k) \right\}.$$

The set $\mathcal{B}'$ is nothing but the set of PBW-basis elements that are products of elements of $\mathscr{S}'$. We also consider the set $\mathcal{B}$ of PBW-basis elements that are products of elements of $\mathscr{S}$. That is, we set

$$\mathcal{B} = \left\{ \prod_{j,k,l} (ad(e_{-1})^l e_{jk})^{c_{jk}^l} \,\middle|\, c_{jk}^l \in \mathbb{Z}_{\geq 0}, \right.$$

$$\left. c_{jk}^l\text{'s are zero for all but finitely many } (j,k,l) \right\}.$$

Note that we have $\mathscr{S}' \subset \mathscr{S}$ and $\mathcal{B}' \subset \mathcal{B}$. For $b = \prod_{j,k,l}(ad(e_{-1})^l e_{jk})^{c_{jk}^l} \in \mathcal{B}$, we define $d(b)$ to be the number of non-zero c_{jk}^l's, and also define

$$\mathrm{wt}(b) = \sum_{j,k,l} c_{jk}^l (l\alpha_{-1} + \alpha_{jk}).$$

These definitions are naturally restricted to the set $\mathcal{B}'$.

Proposition 1.4. *We have*

$$\prod_{\alpha \in S} \frac{1 - t^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} = \sum_{b \in \mathcal{B}} (1 - t^{-1})^{d(b)} \mathbf{z}^{\mathrm{wt}(b)}.$$

The same identity holds if we replace S and $\mathcal{B}$ with S' and $\mathcal{B}'$, respectively.

Proof. We use induction on S with respect to the ordering $\prec$. For convenience, we enumerate the elements of $\mathscr{S}$ corresponding to elements of S by $x_1, x_2, \ldots$ according to the ordering $\prec$. We define

$$S(k) = \{\mathrm{wt}(x_1), \ldots, \mathrm{wt}(x_k)\} \quad \text{and} \quad \mathcal{B}(k) = \{x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k} \mid b_i \in \mathbb{Z}_{\geq 0}\}.$$

If $k = 1$, then $x_1 = e_{11}$ and we have

$$\frac{1 - t^{-1}\mathbf{z}^{\alpha_{11}}}{1 - \mathbf{z}^{\alpha_{11}}} = 1 + (1 - t^{-1})\mathbf{z}^{\alpha_{11}} + (1 - t^{-1})\mathbf{z}^{2\alpha_{11}} + \cdots$$

$$= \sum_{b \in \mathcal{B}(1)} (1 - t^{-1})^{d(b)} \mathbf{z}^{\mathrm{wt}(b)}.$$

Now, using an induction argument, we obtain

$$\prod_{\alpha \in S(k)} \frac{1 - t^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} = \left(\prod_{\alpha \in S(k-1)} \frac{1 - t^{-1}\mathbf{z}^\alpha}{1 - \mathbf{z}^\alpha} \right) \frac{1 - t^{-1}\mathbf{z}^{\mathrm{wt}(x_k)}}{1 - \mathbf{z}^{\mathrm{wt}(x_k)}}$$

$$= \left(\sum_{b \in \mathcal{B}(k-1)} (1 - t^{-1})^{d(b)}\mathbf{z}^{\mathrm{wt}(b)} \right) \left(1 + \sum_{j \geq 1} (1 - t^{-1})\mathbf{z}^{j\mathrm{wt}(x_k)} \right)$$

$$= \sum_{b \in \mathcal{B}(k-1)} (1 - t^{-1})^{d(b)}\mathbf{z}^{\mathrm{wt}(b)}$$

$$+ \sum_{j \geq 1} \sum_{b \in \mathcal{B}(k-1)} (1 - t^{-1})^{d(b)+1}\mathbf{z}^{\mathrm{wt}(b)+j\mathrm{wt}(x_k)}. \tag{1.2}$$

On the other hand, we can write $\mathcal{B}(k)$ as a disjoint union

$$\mathcal{B}(k) = \bigcup_{j \geq 0} \{x_1^{b_1} x_2^{b_2} \cdots x_{k-1}^{b_{k-1}} x_k^j \mid b_i \in \mathbb{Z}_{\geq 0}\}.$$

Then we have

$$\sum_{b \in \mathcal{B}(k)} (1 - t^{-1})^{d(b)}\mathbf{z}^{\mathrm{wt}(b)} = \sum_{b \in \mathcal{B}(k-1)} (1 - t^{-1})^{d(b)}\mathbf{z}^{\mathrm{wt}(b)}$$

$$+ \sum_{j \geq 1} \sum_{b \in \mathcal{B}(k-1)} (1 - t^{-1})^{d(b)+1}\mathbf{z}^{\mathrm{wt}(b)+j\mathrm{wt}(x_k)},$$

which is the same as (1.2). This completes the proof.

One sees that the case with S' and $\mathcal{B}'$ can be proved similarly. $\qquad\square$

Remark 1.5. Note that we considered a partial denominator identity in the above proposition. In particular, every element in the set S has multiplicity 1 as one can see from Theorem 1.2. In the whole denominator identity (1.1), an element of $\Delta_+ \backslash S$ may have multiplicity bigger than 1.

1.3. *Specialization*

We write $\mathbb{N} = \mathbb{Z}_{>0}$. We consider the specialization map $\mathrm{sp} : \Delta'_+ \to \mathbb{N}^2$ defined by

$$l\alpha_{-1} + \alpha_{jk} \mapsto l(1, -1) + (1, j) = (l + 1, j - l),$$

and write $u = \mathbf{z}^{(1,0)}$, $v = \mathbf{z}^{(0,1)}$. We define

$$p_\mathcal{B}(i, j; t) = \sum_{\substack{b \in \mathcal{B} \\ \mathrm{sp}(\mathrm{wt}(b)) = (i,j)}} (1 - t^{-1})^{d(b)}.$$

The function $p_{\mathcal{B}'}(i, j; t)$ is defined similarly using $\mathcal{B}'$. Note that $p_{\mathcal{B}'}(i, j; t) = 0$ for $i > j$.

We apply this specialization to the identity in Proposition 1.4 and obtain:

Theorem 1.6.

(1) *We have*

$$\prod_{j=1}^{\infty}\left(\frac{1-t^{-1}uv^j}{1-uv^j}\right)^{b(j)} = 1 + \sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\leq j}} p_{\mathcal{B}'}(i,j;t)u^iv^j.$$

(2) *We have*

$$\prod_{(i,j)\in\mathbb{N}^2}\left(\frac{1-t^{-1}u^iv^j}{1-u^iv^j}\right)^{b(i+j-1)} = 1 + \sum_{(i,j)\in\mathbb{N}^2} p_{\mathcal{B}}(i,j;t)u^iv^j.$$

Proof. (1) Since there are $b(j)$ elements of degree $(1,j)$ in the set S', the identity follows from Proposition 1.4.

(2) By definition, the element $l\alpha_{-1}+\alpha_{jk}$ has degree $l(1,-1)+(1,j)=(l+1,j-l)$. So the number of elements of degree (i,j) in S is $b(i+j-1)$. Then we obtain the identity again from Proposition 1.4. □

Example 1.7. Let $j(z)$ be the modular j-function. We write $j(z) = q^{-1} + 744 + \sum_{i\geq 1} c(i)q^i$, $q = e^{2\pi i z}$. Then we have $c(i) > 0$ for $i \geq 1$ and we set $c(-1) = 1$. Let M be the symmetric matrix of blocks indexed by $\{-1,1,2,\ldots\}$, where the block in position (i,j) has entries $-(i+j)$ and size $c(i)\times c(j)$. The *Monster Lie algebra* is the generalized Kac–Moody algebra $\mathfrak{g}(M)$ associated to this matrix M. Using the same notations $S',S,\mathcal{B}',\mathcal{B}$ as before, we obtain from Theorem 1.6 the following identities:

$$\prod_{j=1}^{\infty}\left(\frac{1-t^{-1}uv^j}{1-uv^j}\right)^{c(j)} = 1 + \sum_{\substack{(i,j)\in\mathbb{N}^2\\ i\leq j}} p_{\mathcal{B}'}(i,j;t)u^iv^j$$

and

$$\prod_{(i,j)\in\mathbb{N}^2}\left(\frac{1-t^{-1}u^iv^j}{1-u^iv^j}\right)^{c(i+j-1)} = 1 + \sum_{(i,j)\in\mathbb{N}^2} p_{\mathcal{B}}(i,j;t)u^iv^j.$$

These are deformation of partial denominator identities. The whole denominator identity of the Monster Lie algebra is due to Borcherds [1] and can be written as

$$u^{-1}\prod_{i\in\mathbb{N},\, j\in\mathbb{Z}} (1-u^iv^j)^{c(ij)} = j(u) - j(v),$$

or equivalently,

$$\prod_{(i,j)\in\mathbb{N}^2} (1-u^iv^j)^{c(ij)} = 1 - \sum_{(i,j)\in\mathbb{N}^2} c(i+j-1)u^iv^j.$$

(See [10, (4.11)].)

In the above example, the whole denominator identity of the Monster Lie algebra is a modular function, while the partial denominator identity is not. In Sec. 2.2, we will construct many examples in which partial denominator identities represent modular forms and obtain deformation of those modular forms by applying Theorem 1.6.

2. Deformation of Modular Forms

In this section we will consider generalized Kac–Moody algebras whose partial denominator identities represent modular forms. In particular, the sequence $b(n)$ in Sec. 1.2 will be given by Fourier coefficients of certain weakly holomorphic modular forms. We need to show that each $b(n)$ is positive for these weakly holomorphic modular forms, and this will be accomplished in Sec. 2.1. In Sec. 2.2, we obtain deformation of modular forms using the results established in Sec. 1.

2.1. *Positivity of Fourier coefficients*

Let $\mathcal{M}_{\frac{1}{2}}^{+}(\Gamma_0(4))$ be the *Kohnen plus-space* of weakly homomorphic modular forms with integer coefficients of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$. See [14] for definitions. Assume that $f \in \mathcal{M}_{\frac{1}{2}}^{+}(\Gamma_0(4))$. Then the function f has a Fourier expansion

$$f(z) = \sum_{\substack{n \geq n_0 \\ n \equiv 0,1 \,(\mathrm{mod}\,4)}} a(n)q^n,$$

where $q = e^{2\pi i z}$ and $z \in \mathcal{H}$, the upper half-plane. For each non-negative integer $d \equiv 0,3 \pmod 4$ we let $f_d(z) \in \mathcal{M}_{\frac{1}{2}}^{+}(\Gamma_0(4))$ be the unique modular form with a Fourier expansion of the form

$$f_d(z) = q^{-d} + \sum_{n>0} a(n)q^n.$$

For the existence of the modular form $f_d(z)$, we refer the reader to [2, 14]. Here $f_d(z)$'s form a basis for $\mathcal{M}_{\frac{1}{2}}^{+}(\Gamma_0(4))$. It follows from [2, Lemma 14.2] that the space $\mathcal{M}_{\frac{1}{2}}^{+}(\Gamma_0(4))$ is a rank-2 free module over the polynomial ring $\mathbb{Z}[j(4z)]$ with generators $f_0(z)$ and $f_3(z)$. See also [14, Remark 4.3].

In Sec. 2.2, we will assume that $b(n) := a(n^2) > 0$ ($n \geq 1$) for $f(z) = \sum a(n)q^n \in \mathcal{M}_{\frac{1}{2}}^{+}(\Gamma_0(4))$. This assumption is not very restrictive, which we will justify in several ways.

First, let $f(z) = f_0(z)P(z)$, where $P(z) = c_0 + c_1 j(4z) + c_2 j(4z)^2 + \cdots + c_m j(4z)^m$, for $c_0, \ldots, c_m$ non-negative integers. We write $f(z) = \sum_{n=-4m}^{\infty} a(n)q^n$. We claim that $a(n^2)$ is a positive integer for all $n \geq 1$. Since $j(z)$ has positive Fourier coefficients, it is clear that $P(z)$ has non-negative integer Fourier coefficients. Obviously, $f_0(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ has non-negative Fourier coefficients; in particular, the coefficient of q^{n^2} is 2 for $n \geq 1$. Then we have $a(n^2) \geq 2p_0$, where p_0 is the constant term of $P(z)$, which is positive.

Second, we can use the denominator identity of a free Lie algebra to see the positivity $b(n) = a(n^2) > 0$ $(n \geq 1)$ for some weakly holomorphic modular forms $f(z)$. Suppose that we have a product identity

$$\prod_{n=1}^{\infty} (1 - q^n)^{b(n)} = 1 - \sum_{n=1}^{\infty} s(n) q^n. \tag{2.1}$$

If $s(n) \in \mathbb{Z}_{\geq 0}$ for all n, the product identity (2.1) can be interpreted as the denominator identity for the free Lie algebra generated by $s(n)$-generators of degree n for each n. Then $b(n)$ is the homogeneous dimension of degree n of the free Lie algebra. In particular, we have $b(n) \geq s(n)$. See [11] for more details and general results.

In Example 2.3(4) below, we will see that $E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$ can be written as a product of the form (2.1). Then we have $a(n^2) = b(n) \geq s(n) = 504\sigma_5(n)$ for the corresponding $f(z) = \sum a(n)q^n$. Since $f_4(z) = f(z) - 6f_0(z)$, we also see that the n^2th coefficient of $f_4(z)$ is $\geq 504\sigma_5(n) - 12 > 0$ for each $n \geq 1$.

Third, we can prove that $f_d(z)$, $4|d$, have all positive Fourier coefficients. This is quite a contrast to some results in the literature. For example, it is proved (for example, [13]) that for any holomorphic Hecke eigenforms with respect to $\Gamma_0(N)$, there are infinitely many coefficients which are positive, and infinitely many coefficients which are negative. Therefore the following theorem may be of independent interest.

Theorem 2.1. *Let* $f_d(z) = q^{-d} + \sum_{\substack{n > 0 \\ n \equiv 0,1 \,(\mathrm{mod}\,4)}} a(n)q^n \in \mathcal{M}_{\frac{1}{2}}^+(\Gamma_0(4))$. *Then if* $d \equiv 0$ (mod 4), *we have* $a(n) > 0$ *for all* $n > 0$, *and as* $n \to \infty$, *we obtain* $a(n) \sim \frac{2}{\sqrt{n}} \sinh(\pi\sqrt{dn})$. *Similarly, if* $d \equiv 3$ (mod 4), $(-1)^n a(n) > 0$ *for all* $n > 0$, *and as* $n \to \infty$, $a(n) \sim (-1)^n \frac{2}{\sqrt{n}} \sinh(\pi\sqrt{dn})$.

Proof. We use the explicit formula for $a(n)$ due to Bringmann and Ono [4, p. 599]:

$$a(n) = -24\delta_{\square,n} H(-d) + \pi\sqrt{2} \left(\frac{d}{n}\right)^{\frac{1}{4}} (1 - i)$$

$$\times \sum_{c > 0,\, 4|c} \left(1 + \delta_{\mathrm{odd}}\left(\frac{c}{4}\right)\right) \frac{K_0(-d, n; c)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{dn}}{c}\right), \tag{2.2}$$

where

$$\delta_{\square,n} = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise,} \end{cases} \qquad \delta_{\mathrm{odd}}(\nu) = \begin{cases} 1 & \text{if } \nu \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}} \frac{\sinh(z)}{\sqrt{z}}$, and

$$K_0(-d, n; c) = \sum_{\nu \,(\mathrm{mod}\, c)^*} \left(\frac{c}{\nu}\right) \epsilon_\nu e^{2\pi i \frac{-d\bar{\nu} + n\nu}{c}}, \qquad \epsilon_\nu = \begin{cases} 1 & \text{if } \nu \equiv 1 \pmod 4, \\ i & \text{if } \nu \equiv 3 \pmod 4. \end{cases}$$

In the sum, ν runs through the primitive residue classes modulo c, and $\bar{\nu}$ denotes the multiplicative inverse of ν modulo c. Then we can easily see

$$K_0(-d, n; 4) = \begin{cases} 1 + i & \text{if } d \equiv 0 \pmod 4, \\ (-1)^n(1 + i) & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Then $c = 4$ gives rise to the main term. It is

$$\begin{cases} \dfrac{2}{\sqrt{n}} \sinh(\pi\sqrt{dn}) & d \equiv 0 \pmod 4, \\ (-1)^n \dfrac{2}{\sqrt{n}} \sinh(\pi\sqrt{dn}) & d \equiv 3 \pmod 4. \end{cases} \tag{2.3}$$

Now we show that the sum of the remaining terms is smaller than the main term. First, by [5, p. 236], note that

$$H(-d) \leq \begin{cases} \dfrac{\sigma(d)}{3} & \text{if } d \equiv 3 \pmod 4, \\ \dfrac{\sigma(d+1)}{6} & \text{if } d \equiv 0 \pmod 4. \end{cases}$$

where $\sigma(d)$ is the sum of positive divisors of d. Since $\sigma(n) \leq e^\gamma n \log\log n + \frac{0.6482n}{\log\log n}$, where γ is the Euler constant, we have $H(-d) \leq 2d\log\log d$.

We divide the sum in (2.2) into two regions: $4 < c \leq 4\pi\sqrt{dn}$ and $c > 4\pi\sqrt{dn}$. By Weil's bound (cf. [8, p. 403; 15, p. 26]), $|K_0(-d, n; c)| \leq (d, n, c)^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c)$, where $\tau(c)$ is the number of positive divisors of c.

In the region $c \leq 4\pi\sqrt{dn}$, using the fact that $(d, n, c) \leq (d, n) \leq (dn)^{\frac{1}{2}}$, and $\sum_{n < x} \tau(n) \leq 2x\log x$, the sum over $4 < c \leq 4\pi\sqrt{dn}$ is less than

$$2\sqrt{2}\left(\frac{d}{n}\right)^{\frac{1}{4}} \sum_{c \leq 4\pi\sqrt{dn}, \, 4|c} \tau(c) \sinh\left(\frac{4\pi\sqrt{dn}}{c}\right)$$

$$\leq 16\sqrt{2}\pi d^{\frac{3}{4}} n^{\frac{1}{4}} \log 4\pi\sqrt{dn} \, \sinh\left(\frac{\pi}{2}\sqrt{dn}\right). \tag{2.4}$$

In the region $c > 4\pi\sqrt{dn}$, we use the fact that for $0 < z < 1$, $\sinh(z) = z + h(z)$, where $|h(z)| \leq z^3$. The error term $h(z)$ gives rise to

$$2^7\sqrt{2}\pi^3 d^{\frac{7}{4}} n^{\frac{5}{4}} \sum_{c > 4\pi\sqrt{dn}} \frac{\tau(c)}{c^3} \leq 8\sqrt{2}\pi d^{\frac{3}{4}} n^{\frac{1}{4}} \log 4\pi\sqrt{dn}. \tag{2.5}$$

The term z gives rise to

$$4\pi d^{\frac{1}{2}}(1 - i) \sum_{c > 4\pi\sqrt{dn}, \, 4|c} \left(1 + \delta_{\text{odd}}\left(\frac{c}{4}\right)\right) \frac{K_0(-d, n; c)}{c^{\frac{3}{2}}}. \tag{2.6}$$

The above inner sum can be written as

$$2 \sum_{c > 4\pi\sqrt{dn}, \, 4|c} \frac{K_0(-d, n; c)}{c^{\frac{3}{2}}} - \sum_{c > \pi\sqrt{dn}, \, 8|c} \frac{K_0(-d, n; c)}{c^{\frac{3}{2}}}.$$

The first sum is Selberg–Kloosterman sum for $\Gamma_0(4)$, the second for $\Gamma_0(8)$. For $q = 4, 8$, let $\Gamma = \Gamma_0(q)$, and

$$Z_q(-m, n, s) = \sum_{c>0,\, q|c} \frac{K_0(-m, n; c)}{c^{2s}}. \tag{2.7}$$

This series occur in the Fourier expansion of the Poincaré series in [6]. In their notation, $\chi(\gamma) = \chi_\theta^3(\gamma) = (\frac{c}{d})\epsilon_d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $q = 1, \alpha = 0, k = \frac{3}{2}$, since $\chi(-1) = -i$ in our case. Here $\chi_\theta(\gamma) = (\frac{c}{d})\epsilon_d^{-1}$ is the usual θ-multiplier, and $j(\gamma, z) = \chi_\theta(\gamma)(cz + d)^{\frac{1}{2}}$. (Note that there is a misprint in [6, p. 244]: $\chi(-1) = e^{ik\pi}$ should be $\chi(-1) = e^{-ik\pi}$.)

In order to apply Theorem 2 in [6, p. 245], we need to show that there are no exceptional eigenvalues in our case: For $k = \frac{3}{2}$, there is no residual spectrum (cf. [17, p. 21]). For cuspidal spectrum, there are no exceptional eigenvalues: In [16, p. 304], Sarnak associated to each cusp form of type $(\Gamma_0(4N), \frac{3}{2}, \chi)$ with eigenvalue μ, a Maass form of type $(\Gamma_0(2N), 2, 1)$ with eigenvalue $4\mu - \frac{3}{4}$. When $N = 1, 2$, there are no exceptional eigenvalues for $\Gamma_0(2N)$. Hence $4\mu - \frac{3}{4} > \frac{1}{4}$, and $\mu > \frac{1}{4}$. Therefore, by [6, p. 245],

$$\sum_{c \le x} \frac{K_0(-d, n; c)}{c} = O(x^{\frac{1}{6}+\epsilon}).$$

Hence the series (2.7) converge at $s = \frac{3}{4}$. However, for our purpose, we need a precise estimate.

Consider the Poincaré series $P_m(z, s, \chi)$ in [16, p. 291]:

$$P_m(z, s, \chi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \overline{\chi(\gamma)} \left(\frac{cz + d}{|cz + d|} \right)^{-k} e^{2\pi i m \mathrm{Re}(\gamma z) - 2\pi |m| \mathrm{Im}(\gamma z)} \frac{y^s}{|cz + d|^{2s}},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. It satisfies

$$P_m(\gamma z, s, \chi) = \chi(\gamma) \left(\frac{cz + d}{|cz + d|} \right)^k P_m(z, s, \chi),$$

and let $\Delta_k = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - iky\frac{\partial}{\partial x}$ and $R_{s(1-s)} = (\Delta_k + s(1-s))^{-1}$. Then for $m > 0$,

$$P_m(z, s, \chi) = -4\pi m \left(s - \frac{k}{2} \right) R_{s(1-s)}(P_m(z, s+1, \chi)),$$

and

$$P_{-m}(z, s, \chi) = -4\pi m \left(s + \frac{k}{2} \right) R_{s(1-s)}(P_{-m}(z, s+1, \chi)). \tag{2.8}$$

Since there are no exceptional eigenvalues, $R_{s(1-s)}$ is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$ (see [6, p. 247]). By [15, p. 38],

$$\|R_\lambda\| \le (\mathrm{distance}(\lambda, \mathrm{Spec}(\Delta_k)))^{-1}.$$

Since $\mathrm{Spec}(\Delta_k) \geq \frac{1}{4}$, if $s = \frac{3}{4}$, $\|R_{s(1-s)}\| \leq 16$.

Observe by [4, p. 605; 6, p. 244] that $K_0(-m, n, c) = iS(-n, m, c, \chi)$ in the notation of [6].

In [6, p. 248], the inner product $\langle P_m(\cdot, s, \chi), P_n(\cdot, \bar{s} + 2, \chi) \rangle$ is computed for $m, n > 0$. In the exactly same way (cf. [15, p. 23]), we can compute the inner product $\langle P_{-n}(\cdot, s, \chi), P_m(\cdot, \bar{s} + 2, \chi) \rangle$ for $m, n > 0$:

$$\langle P_{-n}(\cdot, s, \chi), P_m(\cdot, w, \chi) \rangle$$

$$= \iint_{D_\Gamma} P_{-n}(\cdot, s, \chi) \overline{P_m(\cdot, w, \chi)} \frac{dx\,dy}{y^2}$$

$$= \int_0^\infty \int_0^1 P_{-n}(\cdot, s, \chi) e^{-2\pi i m x - 2\pi m y} y^{\bar{w}} \frac{dx\,dy}{y^2}$$

$$= \sum_{c \neq 0,\, \gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \int_0^\infty \int_{-\infty}^\infty j(\gamma, z)^{-3}$$

$$\times |j(\gamma, z)|^3 e^{-2\pi i n \mathrm{Re}(\gamma z) - 2\pi n \mathrm{Im}(\gamma z)} \mathrm{Im}(\gamma z)^s e^{-2\pi i m x - 2\pi m y} y^{\bar{w}} \frac{dx\,dy}{y^2}$$

$$= 2 \sum_{c > 0,\, q | c} \frac{S(-n, m, c, \chi)}{c^{2s}} \int_0^\infty \int_{-\infty}^\infty \frac{y^{\bar{w} - s}}{(x^2 + 1)^s}$$

$$\times \left(\frac{x + i}{\sqrt{x^2 + 1}} \right)^{-\frac{3}{2}} e^{\frac{2\pi i n}{c^2 y (x + i)} - 2\pi i m (xy - iy)} \frac{dx\,dy}{y}.$$

Use the fact that

$$\int_{-\infty}^\infty \frac{(x + i)^{-\frac{3}{2}}}{(x^2 + 1)^{s - \frac{3}{4}}} e^{-2\pi i m x y}\, dx = \frac{-\pi (-i)^{\frac{3}{2}} (\pi y m)^{s-1}}{\Gamma\left(s + \frac{3}{4}\right)} W_{\frac{3}{4}, s - \frac{1}{2}}(4\pi m y).$$

Therefore by setting $w = \bar{s} + 2$,

$$\langle P_{-n}(\cdot, s, \chi), P_m(\cdot, \bar{s} + 2, \chi) \rangle$$

$$= (-i)^{\frac{3}{2}} 4^{-s-1} \pi^{-1} m^{-2} \frac{\Gamma(2s + 1)}{\Gamma\left(s + \frac{3}{4}\right) \Gamma\left(s - \frac{3}{4} + 2\right)} (-i) Z_q(-m, n, s)$$

$$+ 2 \sum_{c > 0} \frac{S(-n, m, c, \chi)}{c^{2s}} R_{m,n}(s, c), \tag{2.9}$$

where

$$R_{m,n}(s, c) = \int_0^\infty \int_{-\infty}^\infty \frac{y^2}{(x^2 + 1)^s} \left(\frac{x + i}{\sqrt{x^2 + 1}} \right)^{-\frac{3}{2}} (e^{\frac{2\pi i n}{c^2 y (x + i)}} - 1) e^{-2\pi i m (xy - iy)} \frac{dx\,dy}{y}$$

$$= \int_0^\infty \int_{-\infty}^\infty \frac{y^2}{(x^2 + 1)^s} \left(\frac{x + i}{\sqrt{x^2 + 1}} \right)^{-\frac{3}{2}} (e^{\frac{2\pi i n}{y c^2} \cdot \frac{x - i}{x^2 + 1}} - 1) e^{-2\pi i m (xy - iy)} \frac{dx\,dy}{y}.$$

Consider the integral

$$\int_0^\infty y|e^{\frac{2\pi i n}{yc^2} \cdot \frac{x-i}{x^2+1}} - 1|e^{-2\pi m y}\, dy.$$

As in [15, p. 40], divide the integral into $\int_0^{\frac{2\pi n}{c^2\sqrt{x^2+1}}}$ and $\int_{\frac{2\pi n}{c^2\sqrt{x^2+1}}}^\infty$. For the integral $\int_0^{\frac{2\pi n}{c^2\sqrt{x^2+1}}}$, use the trivial estimate, and the fact that $1 - e^{-x} \le x$ if $x \ge 0$:

$$\int_0^{\frac{2\pi n}{c^2\sqrt{x^2+1}}} \cdot \le \int_0^{\frac{2\pi n}{c^2\sqrt{x^2+1}}} 2y e^{-2\pi m y}\, dy \le \frac{2n}{c^2 m\sqrt{x^2+1}}.$$

For the integral $\int_{\frac{2\pi n}{c^2\sqrt{x^2+1}}}^\infty$, use the fact that $|e^{-z} - 1| \le 2|z|$ if $|z| \le 1$. Hence

$$\int_{\frac{2\pi n}{c^2\sqrt{x^2+1}}}^\infty \cdot \le \int_{\frac{2\pi n}{c^2\sqrt{x^2+1}}}^\infty y \frac{4\pi n}{c^2 y\sqrt{x^2+1}} e^{-2\pi m y}\, dy \le \frac{2n}{c^2 m\sqrt{x^2+1}}.$$

Therefore, for $\sigma = \mathrm{Re}(s)$,

$$|R_{m,n}(s,c)| \le \frac{4n}{c^2 m} \int_{-\infty}^\infty \frac{dx}{(x^2+1)^{\sigma+\frac{1}{2}}}.$$

Now we use the fact that $\int_{-\infty}^\infty (x^2+1)^{-s}\, dx = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$. Then

$$|R_{m,n}(s,c)| \le \frac{4n\sqrt{\pi}}{c^2 m} \frac{\Gamma(\sigma)}{\Gamma\left(\sigma+\frac{1}{2}\right)}.$$

Let $s = \frac{3}{4}$. Then

$$\left|R_{m,n}\left(\frac{3}{4},c\right)\right| \le \frac{9.6n}{c^2 m}.$$

Therefore,

$$\left|2\sum_{c>0} \frac{S(-n,m,\chi)}{c^{\frac{3}{2}}} R_{m,n}\left(\frac{3}{4},c\right)\right| \le \frac{19.2n}{m} \sum_{c=1}^\infty \frac{|S(-n,m,c,\chi)|}{c^{\frac{7}{2}}}.$$

By Weil's bound (cf. [8, p. 403; 15, p. 26]), $|S(-n,m,c,\chi)| \le (m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}}\tau(c)$. By using $(m,n,c) \le (m,n) \le (mn)^{\frac{1}{2}}$, and $\sum_{c=1}^\infty \tau(c)c^{-s} = \zeta(s)^2$, the above term is less than $28 n^{\frac{5}{4}} m^{-\frac{3}{4}}$.

By Cauchy–Schwarz inequality,

$$\left|\left\langle P_{-n}\left(\cdot,\frac{3}{4},\chi\right), P_m\left(\cdot,\frac{11}{4},\chi\right)\right\rangle\right|^2$$

$$\le \left\langle P_{-n}\left(\cdot,\frac{3}{4},\chi\right), P_{-n}\left(\cdot,\frac{3}{4},\chi\right)\right\rangle \left\langle P_m\left(\cdot,\frac{11}{4},\chi\right), P_m\left(\cdot,\frac{11}{4},\chi\right)\right\rangle.$$

Here by (2.8),

$$\left\langle P_{-n}\left(\cdot,\frac{3}{4}\right), P_{-n}\left(\cdot,\frac{3}{4}\right)\right\rangle \le (96\pi n)^2 \left\langle P_{-n}\left(\cdot,\frac{7}{4}\right), P_{-n}\left(\cdot,\frac{7}{4}\right)\right\rangle.$$

In order to compute $\langle P_{-n}(\cdot,\frac{7}{4}), P_{-n}(\cdot,\frac{7}{4})\rangle$, use the formula [6, p. 248]:

$$\langle P_m(\cdot, s, \chi), P_m(\cdot, \bar{s}, \chi)\rangle$$

$$= (4\pi m)^{1-2s}\Gamma(2s-1) + 2\sum_{c>0}\frac{S(-n, m, c, \chi)}{c^{2s}}\int_0^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)^s}$$

$$\times \left(\frac{x+i}{\sqrt{x^2+1}}\right)^{-\frac{3}{2}} e^{-\frac{2\pi i m}{c^2 y(x+i)} - 2\pi i m(xy-iy)}\frac{dxdy}{y}.$$

For $P_{-m}(\cdot, s, \chi)$, we have a similar formula. Then the inner integral is less than

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)^\sigma}e^{-\frac{2\pi m}{c^2 y(x^2+1)} - 2\pi my}\frac{dxdy}{y}$$

$$= \int_{-\infty}^\infty (x^2+1)^{-\sigma}2K_0\left(\frac{4\pi m}{c\sqrt{x^2+1}}\right)dx.$$

Here we used the fact that $\int_0^\infty e^{-\frac{a}{x} - bx}\frac{dx}{x} = 2K_0(2\sqrt{ab})$. Now we use the fact that

$$K_0(x) = \frac{1}{4\pi i}\int_{\text{Re}(s)=\alpha}\Gamma(s)^2\left(\frac{x}{2}\right)^{-2s}ds.$$

Then the above integral is

$$\frac{1}{2\pi i}\int_{\text{Re}(s)=\alpha}\sqrt{\pi}(2\pi m)^{-2s}c^{2s}\Gamma(s)^2\frac{\Gamma\left(\sigma-s-\frac{1}{2}\right)}{\Gamma(\sigma-s)}ds.$$

Take $\alpha = \frac{1}{2}$. If $\sigma = \frac{7}{4}$, it is less than

$$\frac{\sqrt{\pi}c}{(2\pi)^2 m}\int_{-\infty}^\infty \left|\Gamma\left(\frac{1}{2}+it\right)\right|^2 \left|\frac{\Gamma\left(\frac{3}{4}-it\right)}{\Gamma\left(\frac{5}{4}-it\right)}\right|dt.$$

By using Mathematica, it is easy to see that it is less than $\frac{1.1c}{m}$. If $\sigma = \frac{11}{4}$, it is less than $\frac{0.71c}{m}$. Hence

$$\left\langle P_{-n}\left(\cdot,\frac{7}{4},\chi\right), P_{-n}\left(\cdot,\frac{7}{4},\chi\right)\right\rangle \le (4\pi n)^{-\frac{5}{2}}\Gamma\left(\frac{5}{2}\right) + 2.2n^{-1}\sum_{c>0}\frac{|S(-n, n, c, \chi)|}{c^{\frac{5}{2}}}$$

$$\le (4\pi n)^{-\frac{5}{2}}\Gamma\left(\frac{5}{2}\right) + 2.2n^{-\frac{1}{2}}\zeta(2)^2 \le 7n^{-\frac{1}{2}}.$$

In the same way,

$$\left\langle P_m\left(\cdot, \frac{11}{4}, \chi\right), P_m\left(\cdot, \frac{11}{4}, \chi\right)\right\rangle \le (4\pi m)^{-\frac{9}{2}}\Gamma\left(\frac{9}{2}\right) + 1.42m^{-1}\sum_{c>0}\frac{|S(-m,m,c,\chi)|}{c^{\frac{9}{2}}}$$

$$\le (4\pi m)^{-\frac{9}{2}}\Gamma\left(\frac{9}{2}\right) + 1.42m^{-\frac{1}{2}}\zeta(4)^2 \le 2m^{-\frac{1}{2}}.$$

By combining all these estimates, by (2.9), we have

$$\left|Z\left(-d, n; \frac{3}{4}\right)\right| \le 26740d^{\frac{7}{4}}n^{\frac{3}{4}} + 664d^{\frac{5}{4}}n^{\frac{5}{4}}.$$

Here

$$\left|\sum_{c\le 4\pi\sqrt{dn}}\frac{K_0(-d,n;c)}{c^{\frac{3}{2}}}\right| \le \sum_{c\le 4\pi\sqrt{dn}}\frac{(d,n,c)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c)}{c^{\frac{3}{2}}}$$

$$\le (dn)^{\frac{1}{4}}\sum_{c\le 4\pi\sqrt{dn}}\frac{\tau(c)}{c} \le 2(dn)^{\frac{1}{4}}\log 4\pi\sqrt{dn}.$$

Hence

$$|(2.6)| \le 1425630d^{\frac{9}{4}}n^{\frac{3}{4}} + 35402d^{\frac{7}{4}}n^{\frac{5}{4}} + 107d^{\frac{3}{4}}n^{\frac{1}{4}}\log 4\pi\sqrt{dn}.$$

So the main term (2.3) is bigger than the sum of the remaining terms (2.4), (2.5) and (2.6) except for finitely many terms: For example, $\frac{2}{\sqrt{n}}\sinh(\pi\sqrt{dn}) > 1425630d^{\frac{9}{4}}n^{\frac{3}{4}}$ if $dn \ge 55$. If $dn < 55$, we can verify by looking at the tables that $a(n)$ is positive. □

2.2. Borcherds' correspondence

Suppose that $f(z) = \sum_{\substack{n \ge n_0 \\ n \equiv 0,1 \,(\mathrm{mod}\,4)}} a(n)q^n \in \mathcal{M}_{\frac{1}{2}}^+(\Gamma_0(4))$. In this section, we assume that $a(n^2)$ is positive for all $n \ge 1$. Let $H(-n)$ be the usual Hurwitz class number of discriminant $-n$ for $n > 0$, and define

$$\tilde{H}(z) = -\frac{1}{12} + \sum_{\substack{n > 1 \\ n \equiv 0,3 \,(\mathrm{mod}\,4)}} H(-n)q^n$$

$$= -\frac{1}{12} + \frac{q^3}{3} + \frac{q^4}{2} + q^7 + q^8 + q^{11} + \frac{4}{3}q^{12} + \cdots.$$

We set h to be the constant term of $f(z)\tilde{H}(z)$, and put

$$\Psi(z) = q^{-h}\prod_{n=1}^{\infty}(1-q^n)^{a(n^2)}.$$

Theorem 2.2 ([2]). *The function $\Psi(z)$ is a meromorphic modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$, of weight $a(0)$, with integer coefficients and leading coefficient 1.*

Example 2.3. (1) We have $f_0(z) = \theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$. Let $f(z) = 12\,\theta(z) = 12 + 24q + 24q^4 + 24q^9 + \cdots$. Then

$$\Psi(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \Delta(z).$$

(2) Let $\sigma_k(n) = \sum_{d|n} d^k$ and $\sigma(n) = \sigma_1(n)$. We set

$$F(z) = \sum_{\substack{n>0 \\ n\,\mathrm{odd}}} \sigma(n)\,q^n = q + 4q^3 + 6q^5 + \cdots.$$

Then we have

$$f_3(z) = F(z)\theta(z)(\theta(z)^4 - 2F(z))(\theta(z)^4 - 16F(z))\frac{E_6(4z)}{\Delta(4z)} + 56\,\theta(z)$$

$$= q^{-3} - 248q + 26752q^4 - \cdots,$$

where $E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)\,q^n$ is the Eisenstein series. Finally we set

$$f(z) = 3f_3(z) = \sum_n a(n)q^n = 3q^{-3} - 744q + 80256q^4 + \cdots.$$

Then we obtain

$$\Psi(z) = q^{-1}(1-q)^{-744}(1-q^2)^{80256}\cdots$$

$$= q^{-1} + 744 + 196884q + 21493760q^2 + \cdots = j(z).$$

(3) We have $f_4(z) = q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + 51180012q^9 + \cdots$. One can see that $f_4(z) = f_0(z)j(4z) - 2f_3(z) - 746f_0(z)$. Set $f(z) = 2f_4(z)$. Then we have

$$\Psi(z) = q^{-1}(1-q)^{984}(1-q^2)^{286752}(1-q^3)^{102360024}\cdots$$

$$= q^{-1} - 984 + 196884q + 21493760q^2 + \cdots = j(z) - 1728.$$

(4) We consider

$$f(z) = f_4(z) + 6f_0(z)$$

$$= q^{-4} + 6 + 504q + 143388q^4 + 565760q^5 + 18473000q^8 + 51180024q^9 + \cdots.$$

Then we obtain

$$\Psi(z) = (1-q)^{504}(1-q^2)^{143388}(1-q^3)^{51180024}\cdots$$

$$= 1 - 504q - 16632q^2 - 122976q^3 - \cdots$$

$$= 1 - 504\sum_{n>0}\sigma_5(n)q^n = E_6(z).$$

Since $2f_4(z) = 2f(z) - 12f_0(z)$, it follows from (1) and (3) that we recover the well-known identity $j(z) - 1728 = \frac{E_6(z)^2}{\Delta(z)}$.

Now we set $b(n) = a(n^2)$ as before and recall the developments in Sec. 1. We obtain from Theorem 1.6(1)

$$\prod_{j=1}^{\infty}\left(\frac{1 - t^{-1}uv^j}{1 - uv^j}\right)^{b(j)} = 1 + \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i\le j}} p_{\mathcal{B}'}(i, j; t)u^i v^j, \tag{2.10}$$

where the function $p_{\mathcal{B}'}(i, j; t)$ is the same as defined in Sec. 1.3. This identity should be considered as deformation of the modular form $\Psi(z)$ in Theorem 2.2. More precisely, we put $u = t$ and $v = q$ in (2.10) and obtain

$$\prod_{n=1}^{\infty}\left(\frac{1 - q^n}{1 - tq^n}\right)^{a(n^2)} = 1 + \sum_{n=1}^{\infty} s_t(n)q^n, \quad s_t(n) = \sum_{i=1}^{n} p_{\mathcal{B}'}(i, n; t)t^i. \tag{2.11}$$

As $t \to 0$, the product becomes $q^h\Psi(z)$ and the polynomial $s_t(n)$ becomes the Fourier coefficients $s(n)$ of the modular form $\Psi(z) = q^{-h}\sum s(n)q^n$. So $s_t(n)$ is a t-deformation of $s(n)$.

Example 2.4. Let $f(z) = 12f_0(z)$. Then we have $b(j) = 24$ for all $j \ge 1$, and obtain

$$p_{\mathcal{B}'}(1, 1; t) = 24(1 - t^{-1}), \quad p_{\mathcal{B}'}(1, 2; t) = 24(1 - t^{-1}) \quad \text{and}$$

$$p_{\mathcal{B}'}(2, 2; t) = \binom{24}{2}(1 - t^{-1})^2 + 24(1 - t^{-1}) = 300 - 576t^{-1} + 276t^{-2}.$$

Thus we have obtained the first few terms of the sum in the product identity

$$\prod_{j=1}^{\infty}\left(\frac{1 - t^{-1}uv^j}{1 - uv^j}\right)^{24} = 1 + \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i\le j}} p_{\mathcal{B}'}(i, j; t)u^i v^j$$

$$= 1 + 24(1 - t^{-1})uv + 24(1 - t^{-1})uv^2$$

$$+ (300 - 576t^{-1} + 276t^{-2})u^2 v^2 + \cdots. \tag{2.12}$$

Hence we have obtained a deformation of $\Delta(z)/q$. We will see applications of this formula in the next section.

Example 2.5. Let $f(z) = 2f_4(z) = \sum a(n)q^n$. Then we have $a(n^2) > 0$ from the discussion in Sec. 2.1. The expression of $f_4(z)$ in Example 2.3(3) gives $a(1) = 984$, $a(4) = 286752$ and $a(9) = 102360024$. On the other hand, we obtain

$$p_{\mathcal{B}'}(1, 1; t) = 984(1 - t^{-1}), \quad p_{\mathcal{B}'}(1, 2; t) = 286752(1 - t^{-1}),$$

$$p_{\mathcal{B}'}(2, 2; t) = \binom{984}{2}(1 - t^{-1})^2 + 984(1 - t^{-1}) = 484620 - 968256t^{-1} + 483636t^{-2},$$

$$p_{\mathcal{B}'}(1, 3; t) = 102360024(1 - t^{-1}).$$

Thus we have

$$\left(\frac{1-t^{-1}uv}{1-uv}\right)^{984}\left(\frac{1-t^{-1}uv^2}{1-uv^2}\right)^{286752}\left(\frac{1-t^{-1}uv^3}{1-uv^3}\right)^{102360024}\cdots$$

$$= 1 + 984(1-t^{-1})uv + 286752(1-t^{-1})uv^2$$

$$+ (484620 - 968256t^{-1} + 483636t^{-2})u^2v^2 + 102360024(1-t^{-1})uv^3 + \cdots.$$

Comparing this with the formula of $\Psi(z)$ in Example 2.3(3), one sees that this is a deformation of the function $q(j(z) - 1728)$. We write $q(j(z) - 1728) = 1 + \sum_{n=0}^{\infty} c(n)q^{n+1}$. By putting $u = t, v = q$ in the deformation, we have

$$\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1-tq^n}\right)^{a(n^2)} = 1 + \sum_{n=0}^{\infty} c_t(n)q^{n+1},$$

where $c_t(n) = \sum_{i=1}^{n+1} p_{\mathcal{B}'}(i, n+1; t)t^i$. Note that

$$c_t(0) = 984(t-1), \quad c_t(1) = 484620t^2 - 681504t + 196884, \ldots.$$

So $c_t(n)$ can be considered as a t-deformation of $c(n)$.

Example 2.6. Let $f(z) = f_4(z) + 6f_0(z) = \sum a(n)q^n$. It follows from Example 2.3(4) that $a(1) = 504$, $a(4) = 143388$ and $a(9) = 51180024$. We calculate as in the previous example, and obtain

$$\left(\frac{1-t^{-1}uv}{1-uv}\right)^{504}\left(\frac{1-t^{-1}uv^2}{1-uv^2}\right)^{143388}\left(\frac{1-t^{-1}uv^3}{1-uv^3}\right)^{51180024}\cdots$$

$$= 1 + 504(1-t^{-1})uv + 143388(1-t^{-1})uv^2$$

$$+ (127260 - 254016t^{-1} + 126756t^{-2})u^2v^2 + 51180024(1-t^{-1})uv^3 + \cdots.$$

If we put $u = t, v = q$, then the sum becomes

$$1 - \sum_{n=1}^{\infty} s_t(n)q^n = 1 - 504(1-t)q - (16632 + 110628t - 127260t^2)q^2 + \cdots.$$

Hence $s_t(n)$ is a t-deformation of $504\,\sigma_5(n)$. Notice that the coefficients 110628 and -127260 are not divisible by 504. It is interesting that we obtain a t-deformation $504\,\sigma_5(n)$, not $\sigma_5(n)$.

3. Special Case of Ramanujan-Type Modular Forms

In this section, we study a special case of Borcherds product attached to $f(z) = lf_0(z)$ for $l = 1, 2, \ldots$ in Theorem 2.2. This gives rise to

$$\Psi(z) = q^{\frac{l}{12}} \prod_{n=1}^{\infty}(1-q^n)^{2l}.$$

We first consider the t-deformation $\prod(1-t^{-1}q^n)^k$ and then move on to the product $\prod(\frac{1-q^n}{1-tq^n})^k$, which is related to the product (2.11). The purpose of this section is to provide some evidences that the function $p_{\mathcal{B}'}(i,j;t)$ and its variants $\epsilon_{t,k}(n)$ and $p_{t,k}(n)$ defined below are natural deformation and contain interesting arithmetical information.

3.1. *Deformation of arithmetical functions*

For each $k \in \mathbb{Z}$, recall the function $\epsilon_{t,k} : \mathbb{N} \to \mathbb{Z}[t^{-1}]$ from [12]:

$$\prod_{n=1}^{\infty}(1-t^{-1}q^n)^k = \sum_{n=0}^{\infty} \epsilon_{t,k}(n)q^n.$$

Note that $\epsilon_{1,24}(n) = \tau(n+1)$. Here we switched q and t in the notations of [12], since we like to keep the conventional notation $q = e^{2\pi i z}$.

We define a t-deformation of the divisor-sum function by

$$\sigma_t(l) = \sum_{k|l} kt^{-\frac{l}{k}}.$$

When $t = 1$, we get $\sigma_t(l) = \sigma(l)$, the classical sum of divisors function.

We write $F(q) = \prod_{n=1}^{\infty}(1-t^{-1}q^n)^k$. Using the fact that $-\log(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m}$, we have

$$\log F(q) = k\sum_{n=1}^{\infty}\log(1-t^{-1}q^n) = -k\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{t^{-m}q^{nm}}{m}.$$

By taking the logarithmic derivative, we obtain

$$q\frac{F'(q)}{F(q)} = -k\sum_{n,m}nt^{-m}q^{nm} = -k\sum_{l=1}^{\infty}\left\{\sum_{n|l}nt^{-l/n}\right\}q^l.$$

Hence

$$qF'(q) = -kF(q)\sum_{l=1}^{\infty}\sigma_t(l)q^l$$

and we get

$$\sum_{m=1}^{\infty}m\epsilon_{t,k}(m)q^m = -k\left(\sum_{n=0}^{\infty}\epsilon_{t,k}(n)q^n\right)\left(\sum_{l=1}^{\infty}\sigma_t(l)q^l\right)$$

$$= -k\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\sigma_t(l)\epsilon_{t,k}(m-l)\right)q^m.$$

Therefore we have proved.

Proposition 3.1. *For $m > 0$, we have*

$$m\epsilon_{t,k}(m) = -k\sum_{l=1}^{m}\sigma_t(l)\epsilon_{t,k}(m-l), \tag{3.1}$$

and $\epsilon_{t,k}(0) = 1$.

Remark 3.2. When $t = 1$ and $k = 1$ or $k = -1$, we obtain the classical identities due to Euler. Note that $\epsilon_{1,-1}(m)$ is the usual partition function $p(m)$.

By induction, we can prove the following proposition.

Proposition 3.3. *Fix $m > 0$. Then we have*

$$m!\epsilon_{t,k}(m) = \sum_{i=1}^{m} \alpha_i^{(m)}(k)t^{-i},$$

where $\alpha_i^{(m)}(x)$ is a polynomial of degree i with integer coefficients and zero constant term for each i. In particular, we have

$$\alpha_1^{(m)}(x) = -m!x \quad and \quad \alpha_m^{(m)}(x) = (-1)^m x(x-1)(x-2)\cdots(x-m+1).$$

By direct computation, we can see

$$\epsilon_{t,k}(1) = -kt^{-1},$$

$$2\epsilon_{t,k}(2) = k(k-1)t^{-2} - 2kt^{-1},$$

$$3!\epsilon_{t,k}(3) = -k(k-1)(k-2)t^{-3} + 6k^2t^{-2} - 6kt^{-1},$$

$$4!\epsilon_{t,k}(4) = k(k-1)(k-2)(k-3)t^{-4} - 12k^2(k-1)t^{-3}$$
$$+ 12k(3k-1)t^{-2} - 24kt^{-1},$$

$$5!\epsilon_{t,k}(5) = -k(k-1)(k-2)(k-3)(k-4)t^{-5} + 4k(k-1)(5k^2 - 4k - 6)t^{-4}$$
$$- 120k^2(k-1)t^{-3} + 240k^2t^{-2} - 120kt^{-1}.$$

Proof of Proposition 3.3. Suppose that the assertion is true for all $n < m$ and assume $m > 0$. Then by (3.1),

$$m!\epsilon_{t,k}(m) = -k(m-1)!\sum_{l=1}^{m} \sigma_t(l)\epsilon_{t,k}(m-l)$$

$$= -k\sum_{l=1}^{m} \frac{(m-1)!}{(m-l)!}\sigma_t(l)(m-l)!\epsilon_{t,k}(m-l)$$

$$= -k\left((m-1)!\sigma_t(m) + \sum_{l=1}^{m-1} \frac{(m-1)!}{(m-l)!}\sigma_t(l)\sum_{i=1}^{m-l} \alpha_i^{(m-l)}(k)t^{-i}\right)$$

$$= -k\left((m-1)!\sigma_t(m) + \sum_{i=1}^{m-1} t^{-i}\left(\sum_{l=1}^{m-i} \frac{(m-1)!}{(m-l)!}\sigma_t(l)\alpha_i^{(m-l)}(k)\right)\right)$$

$$= \sum_{i=1}^{m} \alpha_i^{(m)}(k)t^{-i}.$$

By substituting $\sigma_t(l)$, we can see that all the coefficients are integers. The coefficient of t^{-1} comes only from $\sigma_t(m)$. So $\alpha_1^{(m)}(k) = -m!\,k$. On the other hand,

$$\alpha_m^{(m)}(k) = -k\left((m-1)! + \sum_{i=1}^{m-1} \frac{(m-1)!}{i!}\alpha_i^{(i)}(k)\right) = (-1)^m \frac{k!}{(k-m)!}.$$

Here we used the induction and the well-known formula

$$\sum_{j=0}^{n}(-1)^j \binom{k}{j} = (-1)^n \binom{k-1}{n}.$$

Similarly, one can see that the leading term of $\alpha_i^{(m)}(k)$ is

$$(-1)^i \frac{m!}{i!}\binom{m-1}{i-1}k^i$$

using the formula

$$\binom{m+1}{i+1} = \sum_{l=1}^{m-i+1} l\binom{m-l}{i-1},$$

which can be derived from the well-known formula

$$\sum_{j=n}^{k}\binom{j}{n} = \binom{k+1}{n+1}. \qquad \square$$

Remark 3.4. When $t = 1$, $\epsilon_{1,k}(m)$ becomes $P_{-k}(m)$ in [7, p. 332] and Proposition 3.3 above can be compared with Lemma 1.1 on [7, p. 332].

Let $\mathcal{P}$ be the set of partitions. For a partition $\mathbf{p} = (1^{m_1}2^{m_2}\cdots r^{m_r}\cdots) \in \mathcal{P}$, we set

$$d(\mathbf{p}) = \#\{r\,|\,m_r \neq 0\} \quad \text{and} \quad |\mathbf{p}| = m_1 + 2m_2 + 3m_3 + \cdots.$$

We define for $n \geq 1$

$$p_{t,1}(n) = \sum_{\substack{\mathbf{p}\in\mathcal{P}\\|\mathbf{p}|=n}} (1 - t^{-1})^{d(\mathbf{p})},$$

and we set $p_{t,1}(0) = 1$. We write for $n \geq 1$

$$p_{t,1}(n) = a_1(1 - t^{-1}) + a_2(1 - t^{-1})^2 + \cdots + a_m(1 - t^{-1})^m. \qquad (3.2)$$

Then we obtain the following properties of $p_{t,1}(n)$.

Lemma 3.5. *Assume that $n \geq 1$.*

(1) *We have $p_{\infty,1}(n) = a_1 + a_2 + \cdots + a_m = p(n)$, the usual partition function.*
(2) *The coefficient a_1 is equal to the number of divisors of n. In particular, n is a prime if and only if $a_1 = 2$.*

(3) *The degree m is given by the condition*

$$\frac{m(m+1)}{2} \leq n < \frac{(m+1)(m+2)}{2}.$$

In particular, n is a triangular number if and only if $a_m = 1$.

Proof. (1) We obtain the assertion by taking the limit $t \to \infty$ in the definition of $p_{t,1}(n)$ and (3.2).

(2) If $d(\mathbf{p}) = 1$, then $\mathbf{p} = (l, l, \ldots, l)$ and $l|n$. Thus a_1 counts the number of divisors of n.

(3) If n is a triangular number, the degree m is given by the partition $\mathbf{p} = (1^1 2^1 3^1 \cdots m^1)$, and this is the only partition with m distinct parts. If we have $\frac{m(m+1)}{2} < n < \frac{(m+1)(m+2)}{2}$, the number n has two distinct partitions

$$\mathbf{p}_1 = (1^r 2^1 3^1 \cdots m^1), \quad r \geq 2, \quad \text{and} \quad \mathbf{p}_2 = (1^{r-1} 2^2 3^1 \cdots (m-1)^1 (m+1)^1),$$

and we get $a_m \geq 2$.   □

Proposition 3.6 ([12]). *If $n > 0$, then*

$$\epsilon_{t,1}(n) - p_{t,1}(n) = \sum_{m=1}^{\infty} (-1)^m \left\{ p_{t,1}\left(n - \frac{1}{2}m(3m - 1)\right) \right.$$

$$\left. + p_{t,1}\left(n - \frac{1}{2}m(3m + 1)\right) \right\}, \tag{3.3}$$

where we define $p_{t,1}(M) = 0$ for all negative integer M.

Let $d(n)$ be the number of positive divisors of $n \in \mathbb{Z}_{>0}$ and we set $u = t^{-1}$. The following result follows from Lemma 3.5(2) and (3.3).

Corollary 3.7. *We have*

$$d(n) = -\frac{d}{du}\Big|_{u=1} \epsilon_{t,1}(n) + \sum_{m=1}^{\infty} (-1)^{m-1} \left\{ d\left(n - \frac{1}{2}m(3m - 1)\right) \right.$$

$$\left. + d\left(n - \frac{1}{2}m(3m + 1)\right) \right\},$$

where we define $d(M) = 0$ for $M \leq 0$.

3.2. *Deformation of Ramanujan-type modular forms*

Let $\mathbf{p} = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(k)})$ be a multi-partition with k components, i.e. each component $\rho^{(i)}$ is a partition. We denote by $\mathcal{P}(k)$ the set of all multi-partitions with k components. For a multi-partition $\mathbf{p} = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(k)}) \in \mathcal{P}(k)$, we set

$$d(\mathbf{p}) = d(\rho^{(1)}) + d(\rho^{(2)}) + \cdots + d(\rho^{(k)}) \quad \text{and} \quad |\mathbf{c}_0| = |\rho^{(1)}| + \cdots + |\rho^{(k)}|.$$

We define for $n \geq 1$

$$p_{t,k}(n) = \sum_{\substack{\mathbf{p} \in \mathcal{P}(k) \\ |\mathbf{p}| = n}} (1 - t^{-1})^{d(\mathbf{p})},$$

and set $p_{t,k}(0) = 1$. Notice that if $n > 0$, the function $p_{\infty,k}(n)$ is nothing but the multi-partition function with k-components.

The function $\epsilon_{t,k}$ and $p_{t,k}$ are closed related. More precisely, we proved in [12, Proposition 3.8] that if $n > 0$, then

$$\epsilon_{t,k}(n) = \sum_{r=0}^{n} \epsilon_{1,k}(r) p_{t,k}(n - r). \tag{3.4}$$

This recursive relation follows from the identity

$$\prod_{n=1}^{\infty} \left(\frac{1 - t^{-1} q^n}{1 - q^n} \right)^k = 1 + \sum_{n=1}^{\infty} p_{t,k}(n) q^n. \tag{3.5}$$

Recall that we obtained in (2.12)

$$\prod_{n=1}^{\infty} \left(\frac{1 - t^{-1} u v^n}{1 - u v^n} \right)^{24} = 1 + \sum_{\substack{(i,j) \in \mathbb{N}^2 \\ i \leq j}} p_{\mathcal{B}'}(i, j; t) u^i v^j. \tag{3.6}$$

Set $k = 24$ in (3.5) and make the substitution $u = 1$ and $v = q$ in (3.6). Then, comparing (3.6) with (3.5), we obtain

$$p_{t,24}(n) = \sum_{i=1}^{n} p_{\mathcal{B}'}(i, n; t). \tag{3.7}$$

Thus $p_{\mathcal{B}'}(i, n; t)$ is a refinement of $p_{t,24}(n)$. For example, one can check

$$p_{t,24}(2) = 324 - 600\, t^{-1} + 276\, t^{-2} = p_{\mathcal{B}'}(1, 2; t) + p_{\mathcal{B}'}(2, 2; t).$$

Now we make a specialization of $u = t, v = q$ in (3.6), and obtain

$$\prod_{n=1}^{\infty} \left(\frac{1 - q^n}{1 - t q^n} \right)^{24} = \sum_{n=0}^{\infty} \tau_t(n + 1) q^n, \tag{3.8}$$

where $\tau_t(n + 1) = \sum_{i=1}^{n} p_{\mathcal{B}'}(i, n; t) t^i$. Recall the definition

$$\prod_{n=1}^{\infty} (1 - t q^n)^{24} = \sum_{n=0}^{\infty} \epsilon_{t^{-1}, 24}(n) q^n.$$

Thus we have

$$\prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=0}^{\infty} \tau(n + 1) q^n = \left(\sum_{n=0}^{\infty} \tau_t(n + 1) q^n \right) \left(\sum_{l=0}^{\infty} \epsilon_{t^{-1}, 24}(l) q^l \right),$$

and obtain the first identity of the following proposition.

Proposition 3.8.

$$\tau(n+1) = \epsilon_{t^{-1},24}(n) + \sum_{j=1}^{n} \epsilon_{t^{-1},24}(n-j)\tau_t(j+1)$$

$$= \epsilon_{t,24}(n) - \sum_{j=1}^{n} \tau(n+1-j)\left(\sum_{i=1}^{j} p_{\mathcal{B}'}(i,j;t)\right).$$

Proof. We only need to prove the second identity. We set $n = 24$ in (3.4) and obtain

$$\epsilon_{t,24}(k) = \sum_{r=0}^{k} \tau(r+1)p_{t,24}(k-r).$$

We rewrite this identity as

$$\tau(n+1) = \epsilon_{t,24}(n) - \sum_{j=1}^{n} \tau(n+1-j)p_{t,24}(j).$$

Now we use (3.7) to obtain the second identity in the proposition.   □

From (3.8), we have the following corollary.

Corollary 3.9.

$$\tau(n+1) = \lim_{t\to 0} \sum_{i=1}^{n} p_{\mathcal{B}'}(i,n;t)t^i.$$

Remark 3.10. The famous Lehmer's conjecture predicts $\tau(n) \neq 0$ for all n. The conjecture has been verified for all $n < 22798241520242687999$. Suppose $\tau(n+1) = 0$. Since $p_{\mathcal{B}'}(i,j;t)$ is divisible by $1 - t^{-1}$ for all i,j, it follows from the second identity of the proposition that $\epsilon_{t,24}(n)$ is divisible by $1 - t^{-1}$. Our calculations show that it is unlikely. However, it is not clear whether it is useful to prove Lehmer's conjecture.

Acknowledgments

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