CONSTANT TERMS OF CERTAIN EISENSTEIN SERIES ON ARITHMETIC QUOTIENTS OF LOOP GROUPS

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Abstract. In this paper, we construct Eisenstein series on arithmetic quotients of loop groups for arbitrary standard parabolic subgroups, generalizing Garland's construction for minimal parabolic subgroups. We compute the constant terms to obtain a formula in the self-conjugate case and work out some examples.

Introduction

In his papers [6, 7], Garland defined and studied certain Eisenstein series associated with minimal parabolic subgroups on arithmetic quotients of loop groups. He computed the constant terms and showed the absolute convergence of the series. This can be considered as a generalization of Godement's work in [8] and Langlands' work in [11] to affine Kac-Moody groups.

The Kac-Moody theory has undergone tremendous developments in connections with diverse areas—number theory, geometry, combinatorics and mathematical physics. However, automorphic forms on Kac-Moody groups have not yet been well established. One of the main difficulties is that Kac-Moody groups are infinite dimensional groups which are not locally compact. Hence there are no Haar measures by Weil's Theorem. This fact precludes many of classical approaches and calls for new ideas.

On the other hand, there has been an increasing need and expectation for a theory of automorphic forms on Kac-Moody groups. As indicated in [12], a satisfactory theory of Eisenstein series on Kac-Moody groups would bring breakthroughs in Langlands' functoriality conjecture. Garland's works have made important first steps in this direction.

The purpose of this paper is to extend Garland's construction to arbitrary standard parabolic subgroups. More precisely, we define Eisenstein series associated with cusp forms on the Levi components of standard parabolic subgroups and compute their constant terms in the self-conjugate case. As a main result, we obtain a formula (Theorem 3.7), which is an analogue of the formula in the classical case. We also characterize the set of double coset representatives that

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appears in the formula of constant terms. We will consider the convergence of the Eisenstein series in a subsequent paper, following the framework given by Garland in [7], and do not discuss the issue in this paper.

The outline of this paper is as follows. In Section 1, we prove certain consequences of the Iwasawa decomposition and define the Eisenstein series on the arithmetic quotient of a loop group. In the next section, we make a lifting of a function on the arithmetic quotient to form a function on the corresponding adelic space and consider measures on the unipotent part of the adelic space. In Section 3, we compute the constant terms for self-conjugate parabolic subgroups. In the last section, we describe the set of double coset representatives in the formula of constant terms, and work out a couple of examples completely.

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1. Construction of Eisenstein Series

In this section, we define Eisenstein series on loop groups, associated with cusp forms on the Levi components of arbitrary standard parabolic subgroups. We keep all the notations in Section 1 of [6] or [7]. In particular, we have the loop group $\hat{G}_R$ for any commutative ring R. When $R = \mathbb{R}$, we drop R from the notation.

Let $\hat{H}$ (resp. $\hat{A}$) denote the subgroup of $\hat{G}$ generated by all $h_\alpha(s)$, $\alpha \in \hat{\Delta}_W$, $s \in \mathbb{R}^*$ (resp. $s \in \mathbb{R}_{>0}$). Suppose that $\nu = (\nu_i)_{i=1,\dots,l+1}, \nu_i \in \mathbb{C}$, is a family of complex numbers. We identify ν with an element ν of $(\hat{\mathfrak{h}}^e_{\mathbb{C}})^*$ by

$$\nu = \sum_{i=1}^{l+1} \nu_i \Lambda_{\alpha_i},$$

where Λ_{α_i} is the fundamental weight such that $\Lambda_{\alpha_i}(h_{\alpha_j}) = \delta_{ij}$. We also identify ν with a quasi-character $\nu : \hat{A} \to \mathbb{C}^*$ of $\hat{A}$ defined by

$$\nu(h) = \nu(h_{\alpha_1}(s_1) \dots h_{\alpha_{l+1}}(s_{l+1})) = s_1^{\nu_1} \dots s_{l+1}^{\nu_{l+1}}, \tag{1.1}$$

where $h = h_{\alpha_1}(s_1) \dots h_{\alpha_{l+1}}(s_{l+1}) \in \hat{A}$, and we write $h^\nu = \nu(h)$. More generally, for any $\nu \in (\hat{\mathfrak{h}}^e_{\mathbb{C}})^*$ and $h = h_{\alpha_1}(s_1) \dots h_{\alpha_{l+1}}(s_{l+1}) \in \hat{A}$, we define

$$h^\nu = \nu(h) = s_1^{\nu(h_{\alpha_1})} \cdots s_{l+1}^{\nu(h_{\alpha_{l+1}})}.$$

Recall that we have the Iwasawa decomposition

$$\hat{G} = \hat{K}\hat{A}\hat{U}.$$

For a subset $\theta \subsetneq \{1,2,\dots,l+1\}$, we let $\hat{P}_\theta$ be the subgroup of $\hat{G}$ generated by $\hat{B}$ and $w_{\alpha_i}(s), i \in \theta,\ s \in \mathbb{R}^*$. Let L_θ be the subgroup of $\hat{G}$ generated by $\chi_{\pm\alpha_i}(s),\ i \in \theta,\ s \in \mathbb{R}$. We also define $A_\theta \subseteq \hat{A}$ to be the subgroup of all $h \in \hat{A}$ such that $h^{\alpha_i} = 1$ for all $i \in \theta$. Let $\hat{N}$ be the subgroup of $\hat{G}$ generated by the elements $w_{\alpha_i}(s), s \in \mathbb{R}^*, i = 1, \dots, l+1$, and let $\hat{N}_\theta$ be the subgroup of $\hat{N}$ generated by $\hat{H}$ and $w_{\alpha_i}(s), s \in \mathbb{R}^*, i \in \theta$. The group W_θ is defined by

$$W_\theta = \hat{N}_\theta / \hat{H}.$$

Then we have $\hat{P}_\theta = \hat{B} W_\theta \hat{B}$. Let $w_\theta \in W_\theta$ be the element of maximal length in W_θ, and set

$$\hat{U}_\theta = w_\theta \hat{U} w_\theta \cap \hat{U}.$$

We have, from [4], the decomposition

$$\hat{G} = \hat{K} L_\theta A_\theta \hat{U}_\theta. \tag{1.2}$$

Using the decomposition (1.2) and the Iwasawa decomposition $\hat{G} = \hat{K}\hat{A}\hat{U}$, we obtain

$$\hat{G} = \hat{K}(L_\theta \cap \hat{K})(L_\theta \cap \hat{A}\hat{U}) A_\theta \hat{U}_\theta = \hat{K} L_{\theta,\hat{A}\hat{U}} A_\theta \hat{U}_\theta, \tag{1.3}$$

where $L_{\theta,\hat{A}\hat{U}} := L_\theta \cap \hat{A}\hat{U}$.

Lemma 1.1. *Each component of an element $g \in \hat{G}$ in the decomposition*

$$\hat{G} = \hat{K} L_{\theta,\hat{A}\hat{U}} A_\theta \hat{U}_\theta,$$

is uniquely determined.

Proof. It follows from the uniqueness of expression in the Iwasawa decomposition $\hat{G} = \hat{K}\hat{A}\hat{U}$ and from [4, Theorem 6.1] that

$$\hat{K} \cap L_{\theta,\hat{A}\hat{U}} A_\theta \hat{U}_\theta = \{1\} \quad \text{and} \quad L_{\theta,\hat{A}\hat{U}} A_\theta \cap \hat{U}_\theta = \{1\}.$$

Moreover, the element $h \in L_{\theta,\hat{A}\hat{U}} \cap A_\theta$ can be written as

$$h = \prod_{i\in\theta} h_{\alpha_i}(s_i) \qquad \text{with } s_i \in \mathbb{R}_{>0}.$$

From the definition of A_θ, we obtain

$$h^{\alpha_j} = \prod_{i\in\theta} s_i^{\alpha_j(h_{\alpha_i})} = 1 \quad \text{for each } j \in \theta.$$

Taking logarithms, we get

$$\sum_{i\in\theta} \alpha_j(h_{\alpha_i}) \log s_i = 0, \qquad j \in \theta.$$

Since θ is a proper subset of $\{1, 2, \ldots, l+1\}$, the Cartan matrix corresponding to θ is positive-definite, and we obtain $s_i = 1$ for all $i \in \theta$, which implies that $h = 1$. Therefore,

$$L_{\theta,\hat{A}\hat{U}} \cap A_\theta = \{1\}.$$

□

We consider a family of complex numbers $\nu = (\nu_i)_{i=1,\ldots,l+1}$, $\nu_i \in \mathbb{C}$, such that $\nu_i = 0$ for $i \in \theta$ and the induced quasi-character $\nu : A_\theta \subset \hat{A} \to \mathbb{C}^*$ as in (1.1). Let f be a cusp form on L_θ satisfying

$$f(kg\gamma) = f(g) \quad \text{for any } k \in \hat{K} \cap L_\theta \text{ and } \gamma \in \hat{\Gamma} \cap L_\theta. \tag{1.4}$$

We define a function $\Phi_{f,\nu} : \hat{G} \to \mathbb{C}$ by

$$\Phi_{f,\nu}(g) = \Phi_{f,\nu}(kmau) = f(m)a^\nu,$$

where the decomposition $g = kmau$, $k \in \hat{K}$, $m \in L_{\theta,\hat{A}\hat{U}}$, $a \in A_\theta$, $u \in \hat{U}_\theta$, is given in Lemma 1.1. We fix an element $D_\theta \in \mathfrak{h}^e_{\mathbb{C}}$ such that

$$\alpha_i(D_\theta) = 0 \quad \text{for all } i \in \theta \quad \text{and} \quad \iota(D_\theta) = 1.$$

Then we set

$$\Phi_{f,\nu}(ge^{-rD_\theta}) = \Phi_{f,\nu}(g) \qquad \text{for } g \in \hat{G}, \quad r \in \mathbb{R}_{>0}.$$

Recall that we have

$$\hat{P}_\theta = R_\theta \hat{U}_\theta,$$

where $R_\theta = L_\theta \hat{H} = \hat{H} L_\theta$. (See [4, Theorem 6.1].)

Lemma 1.2. *Suppose that $g, \gamma \in \hat{G}$ and $\beta \in \hat{\Gamma} \cap \hat{P}_\theta$. Then we have*

1. $\Phi_{f,\nu}(g\beta) = \Phi_{f,\nu}(g)$ *and*
2. $\Phi_{f,\nu}(ge^{-rD_\theta}\gamma\beta) = \Phi_{f,\nu}(ge^{-rD_\theta}\gamma)$.

Proof. (1) Write $\beta = \gamma_1 u_1$, $\gamma_1 \in \hat{\Gamma} \cap R_\theta$, $u_1 \in \hat{U}_\theta$, and $g = kmau$, $k \in \hat{K}$, $m \in L_\theta$, $a \in A_\theta$, $u \in \hat{U}_\theta$. Since R_θ normalizes $\hat{U}_\theta$, we have

$$g\beta = kmau\,\gamma_1 u_1 = kma\,\gamma_1\,u_2$$

for some $u_2 \in \hat{U}_\theta$, and since $\gamma_1 \in \hat{\Gamma} \cap R_\theta$, we can write $\gamma_1 = k_1 m_1$, $k_1 \in \hat{K} \cap \hat{H}$ and $m_1 \in \hat{\Gamma} \cap L_\theta$. Note that $mk_1 = k_1 m$. Then we obtain

$$g\beta = kmak_1m_1u_2 = kk_1mm_1au_2$$

and

$$\Phi_{f,\nu}(g\beta) = \Phi_{f,\nu}(kk_1mm_1au_2) = f(mm_1)a^\nu = f(m)a^\nu = \Phi_{f,\nu}(g).$$

(2) We write $\beta = \gamma_1 u_1$ as before and observe that

$$e^{-rD_\theta}\beta e^{rD_\theta} = e^{-rD_\theta}\gamma_1 u_1 e^{rD_\theta} = \gamma_1 u_3,$$

with $u_3 \in \hat{U}_\theta$. Since $e^{-rD_\theta}\gamma = \gamma' e^{-rD_\theta}$ for some $\gamma' \in \hat{G}$, we obtain

$$\begin{aligned}\Phi_{f,\nu}(ge^{-rD_\theta}\gamma\beta) &= \Phi_{f,\nu}(g\gamma' e^{-rD_\theta}\beta) = \Phi_{f,\nu}(g\gamma'\gamma_1 u_3) \\ &= \Phi_{f,\nu}(g\gamma') = \Phi_{f,\nu}(g\gamma' e^{-rD_\theta}) = \Phi_{f,\nu}(ge^{-rD_\theta}\gamma).\end{aligned}$$

□

Definition 1.3. For $g \in \hat{G}$, $r \in \mathbb{R}_{>0}$, we define

$$E(f,\nu,ge^{-rD_\theta}) = \sum_{\gamma \in \hat{\Gamma}/\hat{\Gamma}\cap\hat{P}_\theta} \Phi_{f,\nu}(ge^{-rD_\theta}\gamma).$$

Note that there is no ambiguity in the sum thanks to Lemma 1.2. In the rest of this paper, we will be mainly interested in the constant term in the Fourier expansion of the series.

2. Measure and Lifting

In this section, we consider a measure on the adelic space corresponding to the unipotent part of an arithmetic quotient and liftings of functions from a real group to an adelic group, which will be used later in Section 3.

The quotient space $\hat{U}_\theta/\hat{\Gamma}\cap\hat{U}_\theta$ is the projective limit of compact nilmanifolds and hence inherits both a compact, Hausdorff, projective limit topology, and a projective limit, probability measure, which is invariant with respect to left translation by elements of $\hat{U}_\theta$. We denote this measure on $\hat{U}_\theta/\hat{\Gamma}\cap\hat{U}_\theta$ by du_∞. See [6] for details.

Let $\mathcal{V}$ be the set of all primes $p \in \mathbb{Z}_{>0}$. We set $\mathcal{V}^e = \mathcal{V}\cup\{\infty\}$. We have the group $\hat{G}_p := \hat{G}_{\mathbb{Q}_p} \subseteq \mathrm{Aut}(V^\lambda_{\mathbb{Q}_p})$ for each $p \in \mathcal{V}^e$. In particular, $\hat{G}_\infty = \hat{G}_{\mathbb{Q}_\infty} = \hat{G}_\mathbb{R} = \hat{G}$. For $p \in \mathcal{V}$, let $\hat{K}_p \subseteq \hat{G}_p$ be the subgroup

$$\hat{K}_p = \{g \in \hat{G}_p \,|\, g\cdot V^\lambda_{\mathbb{Z}_p} = V^\lambda_{\mathbb{Z}_p}\}.$$

We define the adele ring $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ and the finite adele ring $\mathbb{A}_f = \mathbb{A}_{\mathbb{Q},f}$ in the usual way, and also define

$$\hat{G}_\mathbb{A} = \prod_{p\in\mathcal{V}^e}{}' \hat{G}_p \quad \text{and} \quad \hat{G}_{\mathbb{A}_f} = \prod_{p\in\mathcal{V}}{}' \hat{G}_p$$

to be the restricted products with respect to $\{\hat{K}_p\}_{p\in\mathcal{V}}$. We have the diagonal embedding

$$i : \hat{G}_\mathbb{Q} \hookrightarrow \prod_{p\in\mathcal{V}^e} \hat{G}_p,$$

and set

$$\hat{\Gamma}_{\mathbb{Q}} = i^{-1}(\hat{G}_{\mathbb{A}}).$$

Since $i(\hat{G}_{\mathbb{Q}}) \not\subseteq \hat{G}_{\mathbb{A}}$, the group $\hat{\Gamma}_{\mathbb{Q}}$ is important to obtain suitable restrictions of $\mathbb{Q}$-groups.

Fix $\theta \subsetneq \{1, 2, \dots, l+1\}$, and let $\hat{U}_{\theta,\mathbb{A}} \subseteq \hat{G}_{\mathbb{A}}$ and $\hat{U}_{\theta,\mathbb{A}_f} \subseteq \hat{G}_{\mathbb{A}_f}$ be the subgroups

$$\hat{U}_{\theta,\mathbb{A}} = \prod_{p\in\mathcal{V}^e}{}' \, \hat{U}_{\theta,\mathbb{Q}_p} \quad \text{and} \quad \hat{U}_{\theta,\mathbb{A}_f} = \prod_{p\in\mathcal{V}}{}' \, \hat{U}_{\theta,\mathbb{Q}_p},$$

where we take the restricted direct products with respect to $\{\hat{K}_p \cap \hat{U}_{\theta,\mathbb{Q}_p}\}_{p\in\mathcal{V}}$. We set

$$\hat{\mathbb{K}} = \prod_{p\in\mathcal{V}^e} \hat{K}_p \subset \hat{G}_{\mathbb{A}} \quad \text{and} \quad \hat{\mathbb{K}}_f = \prod_{p\in\mathcal{V}} \hat{K}_p \subset \hat{\mathbb{K}} \subset \hat{G}_{\mathbb{A}}.$$

Then we have

$$\left((\hat{\mathbb{K}}_f \cap \hat{U}_{\theta,\mathbb{A}_f}) \times \hat{U}_\theta \right) / \hat{\Gamma} \cap \hat{U}_\theta \cong \hat{U}_{\theta,\mathbb{A}} / \hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta,\mathbb{Q}},$$

the identification being induced by the inclusion

$$(\hat{\mathbb{K}}_f \cap \hat{U}_{\theta,\mathbb{A}_f}) \times \hat{U}_\theta \hookrightarrow \hat{U}_{\theta,\mathbb{A}},$$

and $\hat{\Gamma} \cap \hat{U}_{\theta,\mathbb{Q}}$ being diagonally embedded in $(\hat{\mathbb{K}}_f \cap \hat{U}_{\theta,\mathbb{A}_f}) \times \hat{U}_\theta$.

The compact group $\hat{U}_{\theta,\mathbb{Z}_p}$, $p \in \mathcal{V}$, has a unique Haar measure du_p with total measure one. Set

$$du_f = \prod_{p\in\mathcal{V}} du_p \quad \text{on} \quad \hat{\mathbb{K}}_f \cap \hat{U}_{\theta,\mathbb{A}_f} = \prod_{p\in\mathcal{V}} \hat{U}_{\theta,\mathbb{Z}_p}.$$

Through the identification $\hat{U}_{\theta,\mathbb{A}} / \hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta,\mathbb{Q}} \cong \left(\prod_{p\in\mathcal{V}} \hat{U}_{\theta,\mathbb{Z}_p} \times \hat{U}_\theta \right) / \hat{\Gamma} \cap \hat{U}_\theta$, we have the induced measure $du = du_f \times du_\infty$ on $\hat{U}_{\theta,\mathbb{A}} / \hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta,\mathbb{Q}}$; more precisely, if f is a continuous function on $\hat{U}_{\theta,\mathbb{A}} / \hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta,\mathbb{Q}}$ then we have

$$\int_{\hat{U}_{\theta,\mathbb{A}} / \hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta,\mathbb{Q}}} f(u)du = \int_{\hat{U}_\theta / \hat{\Gamma} \cap \hat{U}_\theta} \left[\int_{\prod_{p\in\mathcal{V}} \hat{U}_{\theta,\mathbb{Z}_p}} f(u) du_f \right] du_\infty.$$

We define the conjugation by e^{-rD_θ} on $\hat{G}_{\mathbb{A}}$ to be its usual action on the factor $\hat{G}$ at ∞ and the trivial action on each factor $\hat{G}_p$ at $p \in \mathcal{V}$. Consider the natural maps

$$\hat{G}e^{-rD_\theta} \hookrightarrow \hat{G}_{\mathbb{A}} e^{-rD_\theta} \to \hat{\mathbb{K}}_f \backslash \hat{G}_{\mathbb{A}} e^{-rD_\theta} / \hat{\Gamma}_{\mathbb{Q}},$$

where the first map is an injection and the second map is a projection. Then the composite of the two maps induces a bijection

$$\theta : \hat{G}e^{-rD_\theta} / \hat{\Gamma} \to \hat{\mathbb{K}}_f \backslash \hat{G}_{\mathbb{A}} e^{-rD_\theta} / \hat{\Gamma}_{\mathbb{Q}}.$$

On the other hand, we have the bijection

$$\beta : (\hat{\mathbb{K}}_f \times \hat{G} e^{-rD_\theta})/\hat{\Gamma} \to \hat{G}_{\mathbb{A}} e^{-rD_\theta}/\hat{\Gamma}_{\mathbb{Q}}$$

induced by the injection $(\hat{\mathbb{K}}_f \times \hat{G} e^{-rD_\theta}) \hookrightarrow \hat{G}_{\mathbb{A}} e^{-rD_\theta}$, and we also have the projection

$$\pi : (\hat{\mathbb{K}}_f \times \hat{G} e^{-rD_\theta})/\hat{\Gamma} \to \hat{G} e^{-rD_\theta}/\hat{\Gamma}.$$

We denote by ω the projection

$$\omega : \hat{G}_{\mathbb{A}} e^{-rD_\theta}/\hat{\Gamma}_{\mathbb{Q}} \to \hat{\mathbb{K}}_f \backslash \hat{G}_{\mathbb{A}} e^{-rD_\theta}/\hat{\Gamma}_{\mathbb{Q}}.$$

Then we obtain the following commutative diagram:

$$\begin{array}{ccc} \hat{G}_{\mathbb{A}} e^{-rD_\theta}/\hat{\Gamma}_{\mathbb{Q}} & \xrightarrow{\beta^{-1}} & (\hat{\mathbb{K}}_f \times \hat{G} e^{-rD_\theta})/\hat{\Gamma} \\ \omega \downarrow & & \downarrow \pi \\ \hat{\mathbb{K}}_f \backslash \hat{G}_{\mathbb{A}} e^{-rD_\theta}/\hat{\Gamma}_{\mathbb{Q}} & \xrightarrow{\theta^{-1}} & \hat{G} e^{-rD_\theta}/\hat{\Gamma} \end{array} .$$

If F is a function on $\hat{G} e^{-rD_\theta}/\hat{\Gamma}$, then we have a lifting $\check{F}$ on $\hat{G}_{\mathbb{A}} e^{-rD_\theta}/\hat{\Gamma}_{\mathbb{Q}}$ defined by

$$\check{F}(g) = F((\pi \circ \beta^{-1})(g)). \tag{2.1}$$

Let f be a cusp form on L_θ satisfying (1.4). We obtain a lifting $\check{f}$ on $L_{\theta,\mathbb{A}}/L_{\theta,\mathbb{Q}}$ using a similar process as in the above constructions. We further assume that $\check{f}$ belongs to the representation space of a cuspidal representation $\Pi = \otimes \Pi_p$ of $L_{\theta,\mathbb{A}}$ and that $\check{f}$ can be written as $\check{f} = \otimes_{p \in \mathcal{V}^e} f_p$, where f_p is a $(\hat{K}_p \cap L_{\theta,\mathbb{Q}_p})$-fixed vector of the representation space of Π_p for each $p \in \mathcal{V}^e$. Recall that we have the Iwasawa decomposition

$$\hat{G}_p = \hat{K}_p L_{\theta,\mathbb{Q}_p} \hat{H}_{\mathbb{Q}_p} \hat{U}_{\theta,\mathbb{Q}_p} \qquad \text{for each } p \in \mathcal{V}.$$

Set

$$L_{\theta,\mathbb{A}} = \prod_{p \in \mathcal{V}^e}{}' L_{\theta,\mathbb{Q}_p} \quad \text{and} \quad L_{\theta,\mathbb{A}_f} = \prod_{p \in \mathcal{V}}{}' L_{\theta,\mathbb{Q}_p}$$

$$(\text{ resp. } \; A_{\theta,\mathbb{A}} = \prod_{p \in \mathcal{V}^e}{}' A_{\theta,\mathbb{Q}_p} \quad \text{and} \quad A_{\theta,\mathbb{A}_f} = \prod_{p \in \mathcal{V}}{}' A_{\theta,\mathbb{Q}_p} \;)$$

with respect to $\{\hat{K}_p \cap L_{\theta,\mathbb{Q}_p}\}_{p \in \mathcal{V}}$ (resp. $\{\hat{K}_p \cap A_{\theta,\mathbb{Q}_p}\}_{p \in \mathcal{V}}$). One obtains from the Iwasawa decomposition of $\hat{G}_p$ that

$$\hat{G}_{\mathbb{A}} = \hat{\mathbb{K}} L_{\theta,\mathbb{A}} \hat{A}_{\mathbb{A}} \hat{U}_{\theta,\mathbb{A}} \quad \text{and} \quad \hat{G}_{\mathbb{A}_f} = \hat{\mathbb{K}}_f L_{\theta,\mathbb{A}_f} \hat{A}_{\mathbb{A}_f} \hat{U}_{\theta,\mathbb{A}_f}.$$

We define the function $\Phi_{f_p,\nu}$ on $\hat{G}_p$ by

$$\Phi_{f_p,\nu}(g_p) = \Phi_{f_p,\nu}(k_p m_p a_p u_p) = f_p(m_p)|a_p|^\nu,$$

where the decomposition $g_p = k_p m_p a_p u_p$ is given by the Iwasawa decomposition, and we define the function $\Phi_{\check{f},\nu}$ on $\hat{G}_{\mathbb{A}}$ by

$$\Phi_{\check{f},\nu}(g) = \prod_{p\in\mathcal{V}^e} \Phi_{f_p,\nu}(g_p).$$

Consider an element $a = \prod_{p\in\mathcal{V}^e} a_p \in A_{\theta,\mathbb{A}}$ where $a_p = \prod_{i=1}^{l+1} h_{\alpha_i}(s_{p,i}) \in A_{\theta,\mathbb{Q}_p}$ for some $s_{p,i} \in \mathbb{Q}_p$. Let $s_i = \prod_{p\in\mathcal{V}^e} s_{p,i}$ for $i = 1, \ldots, l+1$. Then we have

$$\begin{aligned} |a|^\nu &= \left(\prod_{i=1}^{l+1} h_{\alpha_i}(|s_i|)\right)^\nu = \prod_{i=1}^{l+1} |s_i|^{\nu(h_i)} \\ &= \prod_{i=1}^{l+1}\left(\prod_{p\in\mathcal{V}^e} |s_{p,i}|_p\right)^{\nu(h_i)} = \prod_{p\in\mathcal{V}^e}\left(\prod_{i=1}^{l+1}(|s_{p,i}|_p)^{\nu(h_i)}\right) \\ &= \prod_{p\in\mathcal{V}^e} |a_p|^\nu. \end{aligned}$$

We obtain:

Lemma 2.1.

$$\Phi_{\check{f},\nu}(g) = \check{\Phi}_{f,\nu}(g).$$

Proof. Let $g \in \hat{G}_{\mathbb{A}}$ with the Iwasawa decomposition $g = kmau$. By definition, g can be written as $g = \prod_{p\in\mathcal{V}^e} g_p$ where $g_p \in \hat{G}_{\mathbb{Q}_p}$ with the Iwasawa decomposition $g_p = k_p m_p a_p u_p$. Note that $|a|^\nu = (\pi\circ\beta^{-1}(a))^\nu$ for any $a \in A_{\theta,\mathbb{A}}$(see [6, Section 5] for details). Thus we have

$$\begin{aligned} \Phi_{\check{f},\nu}(g) &= \prod_{p\in\mathcal{V}^e} \Phi_{f_p,\nu}(g_p) = \prod_{p\in\mathcal{V}^e} f_p(m_p)|a_p|^\nu = \prod_{p\in\mathcal{V}^e} f_p(m_p) \prod_{p\in\mathcal{V}^e} |a_p|^\nu \\ &= \check{f}(m)|a|^\nu = f((\pi\circ\beta^{-1}(m)))(\pi\circ\beta^{-1}(a))^\nu = \Phi_{f,\nu}(\pi\circ\beta^{-1}(g)) \\ &= \check{\Phi}_{f,\nu}(g). \end{aligned}$$

□

Remark. From now on, if there is no peril of confusion, we will omit ˇ from the notations for simplicity and for consistency with the notations in [6, 7].

3. Constant Term

In this section, we prove some properties of affine root systems and obtain a formula for the constant term of the Eisenstein series, which is the main result of this paper.

For each $i = 1, ..., l+1$, we define the simple reflection $w_i \in \mathrm{Aut}(\hat{\mathfrak{h}^e})$ by

$$w_i(h) = h - \alpha_i(h)h_i \qquad \text{for } h \in \hat{\mathfrak{h}^e}.$$

The affine Weyl group $\hat{W} \subset \mathrm{Aut}(\hat{\mathfrak{h}^e})$ is the group generated by w_i, $i = 1, ..., l+1$. Note that the group W_θ is identified with the subgroup of $\hat{W}$ generated by w_i, $i \in \theta$. We denote by Δ_θ^+ the set of elements in $\hat{\Delta}_W^+$ that are linear combinations of α_i with $i \in \theta$, and $\Delta_\theta^- = -\Delta_\theta^+$. For $w \in \hat{W}$, we define

$$\Delta_{\theta,w} = \{\alpha \in \hat{\Delta}_W^+ \setminus \Delta_\theta^+ \mid w^{-1}\alpha \in \hat{\Delta}_W^- \setminus \Delta_\theta^-\}. \tag{3.1}$$

For simplicity, we write Δ_w for $\Delta_{\emptyset,w}$. It is well-known that

$$\Delta_w = \{w_{i_r}w_{i_{r-1}}\cdots w_{i_{j+1}}\alpha_{i_j} \mid j = 1\dots r\},$$

where $w_{i_r}w_{i_{r-1}}\cdots w_{i_1}$ is a reduced expression of w in terms of simple reflections.

Let $\theta' \subsetneq \{1, 2, \dots, l+1\}$ be a subset with $\mathrm{Card}(\theta') = \mathrm{Card}(\theta)$. There exists a set $W(\theta', \theta)$ of double coset representatives of $W_{\theta'}\backslash\hat{W}/W_\theta$ such that, for each $w \in W(\theta', \theta)$, the length of w is minimal in $W_{\theta'}wW_\theta$. (See, for example, [1].) Moreover, the set $W(\theta', \theta)$ is uniquely determined. When $\theta = \theta'$, we simply write $W(\theta', \theta) = W(\theta)$. Then we have $W(\theta) = W(\theta)^{-1}$. For each $w \in \hat{W}$, there exist $\mathring{w} \in W(\theta)$ and $w_1, w_2 \in W_\theta$ such that $w = w_1\mathring{w}w_2$ and $l(w) = l(w_1) + l(\mathring{w}) + l(w_2)$, and the decomposition is unique.

Lemma 3.1. (1) *For $w \in W(\theta)$ and $\alpha \in \Delta_\theta^+$, we obtain*

$$w\alpha \in \hat{\Delta}_W^+.$$

(2) *For any $w \in W(\theta)$, we have*

$$\Delta_{\theta,w} = \Delta_w.$$

Proof. (1) Assume that $w \in W(\theta)$ and $\alpha \in \Delta_\theta^+$. Then there extists $w_\alpha \in W_\theta$ such that $\alpha = w_\alpha\alpha_i$ for some $i \in \theta$. Since $l(w_\alpha w_i) = l(w_\alpha) + 1$ (see [9, Lemma 3.11]),

$$l(ww_\alpha w_i) = l(w) + l(w_\alpha w_i) > l(w) + l(w_\alpha) = l(ww_\alpha).$$

Therefore,

$$w\alpha = ww_\alpha\alpha_i \in \hat{\Delta}_W^+. \tag{3.2}$$

(2) We fix $w \in W(\theta)$ and consider $\alpha \in \Delta_w$. By definition, $\alpha \in \hat{\Delta}_W^+$ and $w^{-1}\alpha \in \hat{\Delta}_W^-$. Suppose that $w^{-1}\alpha \in \Delta_\theta^-$. Then there exists $\beta \in \Delta_\theta^+$ such that

$$w^{-1}\alpha = -\beta,$$

and we obtain from (3.2)

$$\alpha = -w\beta \in \hat{\Delta}_W^-,$$

which is a contradiction. Thus

$$w^{-1}\alpha \in \hat{\Delta}_W^- \setminus \Delta_\theta^-.$$

Since $w^{-1} \in W(\theta)$, a similar argument yields

$$\alpha = w(w^{-1}\alpha) \in \hat{\Delta}_W^+ \setminus \Delta_\theta^+.$$

Thus we have shown that $\Delta_w \subseteq \Delta_{\theta,w}$. Clearly, $\Delta_{\theta,w} \subseteq \Delta_w$. Therefore,

$$\Delta_{\theta,w} = \Delta_w.$$

□

Let R be a commutative ring. As before, whenever the subscript R is omitted, it will be understood that $R = \mathbb{R}$. For any subgroup H of $\hat{G}_R$ and $g \in \hat{G}_R$, we will write

$${}^gH = gHg^{-1} \qquad \text{and} \qquad H^g = g^{-1}Hg.$$

We define $\hat{U}_{-,\theta,R}$ to be the subgroup generated by $\chi_\alpha(s)$, $s \in R$, $\alpha \in \hat{\Delta}_W^- \setminus \Delta_\theta^-$. For $w \in W(\theta', \theta)$, we define

$$U_{w,R} = \hat{U}_{\theta',R} \cap {}^wU_{-,\theta,R}, \qquad U'_{w,R} = \hat{U}_{\theta',R} \cap {}^wL_{\theta,R} \quad \text{and} \quad \hat{U}''_{w,R} = \hat{U}_{\theta',R} \cap {}^w\hat{U}_{\theta,R}.$$

Then we have

$$\hat{U}_{\theta',R} = U_{w,R}\, U'_{w,R}\, \hat{U}''_{w,R}. \tag{3.3}$$

Recall that the group $\hat{G}_R$ has the *Bruhat decomposition*

$$\hat{G}_R = \bigcup_{w \in W(\theta',\theta)} \hat{P}_{\theta',R}\, w\, \hat{P}_{\theta,R}.$$

Definition 3.2. Suppose that $g \in \hat{G}$ and $r \in \mathbb{R}_{>0}$. The *constant term* $E_{\theta'}(f, \nu, ge^{-rD_\theta})$ of the series $E(f, \nu, ge^{-rD_\theta})$ is defined to be

$$E_{\theta'}(f, \nu, ge^{-rD_\theta}) = \int_{\hat{U}_{\theta'}/\hat{U}_{\theta'} \cap \hat{\Gamma}} E(f, \nu, ge^{-rD_\theta}u)du_\infty.$$

From now on in this section, we assume that ν is real; i.e., $\nu : A_\theta \to \mathbb{R}_{>0}$. For each $w \in W(\theta', \theta)$, we set

$$\hat{\Gamma}(w) = \hat{\Gamma} \cap \hat{P}_{\theta'} w \hat{P}_\theta,$$

and define

$$E_w(f, \nu, ge^{-rD_\theta}) = \sum_{\gamma \in \hat{\Gamma}(w)/\hat{\Gamma} \cap \hat{P}_\theta} \Phi_{f,\nu}(ge^{-rD_\theta}\gamma).$$

It follows from the Bruhat Decomposition that

$$E(f,\nu,ge^{-rD_\theta}) = \sum_{w\in W(\theta',\theta)} E_w(f,\nu,ge^{-rD_\theta}).$$

By the monotone convergence theorem, we have

$$E_{\theta'}(f,\nu,ge^{-rD_\theta}) = \sum_{w\in W(\theta',\theta)} \int_{\hat{U}_{\theta'}/\hat{U}_{\theta'}\cap\hat{\Gamma}} E_w(f,\nu,ge^{-rD_\theta}u)du_\infty. \tag{3.4}$$

Remark. The assumption that ν is real should not be considered essential. After we prove that the constant term is convergent for real ν (in a subsequent paper), we will have the equation (3.4) for any complex ν.

We need to fix measures on various coset spaces. The measure du on $\hat{U}_{\theta',\mathbb{A}}/(\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}\cap{}^w\hat{P}_{\theta,\mathbb{Q}})$ is obtained from the measure du on $\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}$ through the projection

$$\hat{U}_{\theta',\mathbb{A}}/(\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}\cap{}^w\hat{P}_{\theta,\mathbb{Q}}) \to \hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}.$$

We obtain from (3.3) the decomposition

$$\hat{U}_{\theta',\mathbb{A}} = U_{w,\mathbb{A}}\,U'_{w,\mathbb{A}}\,\hat{U}''_{w,\mathbb{A}}$$

and by the definition, we have

$$U_{w,\mathbb{Q}}\cap{}^w\hat{P}_{\theta,\mathbb{Q}} = \emptyset \quad \text{and} \quad U'{}_{w,\mathbb{Q}}\,\hat{U}''_{w,\mathbb{Q}} \subseteq {}^w\hat{P}_{\theta,\mathbb{Q}}.$$

We define left-invariant probability measures du_2 and du_3 on

$$U'_{w,\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}} \quad \text{and} \quad \hat{U}''_{w,\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}},$$

respectively, using the same construction as in Section 2. Since $U_{w,\mathbb{A}}$ is locally compact, there exists a left Haar measure du_1 on $U_{w,\mathbb{A}}$ such that

$$\begin{aligned} &\int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}\cap{}^w\hat{P}_{\theta,\mathbb{Q}}} F(u)du \\ =\ &\int_{U_{w,\mathbb{A}}}\int_{U'_{w,\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}}\int_{\hat{U}''_{w,\mathbb{A}}/\hat{\Gamma}_\mathbb{Q}\cap\hat{U}_{\theta',\mathbb{Q}}} F(u_1u_2u_3)du_3du_2du_1. \end{aligned} \tag{3.5}$$

The following lemma will be useful in later calculations.

Lemma 3.3. *Assume that $g\in\hat{G}$ and $r\in\mathbb{R}_{>0}$.*

(1) *For any unipotent radical $\tilde{U}$ of a parabolic subgroup of $L_{\theta,\mathbb{A}}$, we have*

$$\int_{\tilde{U}} \Phi_{f,\nu}(ge^{-rD_\theta}\tilde{u})d\tilde{u} = 0.$$

(2) *If* $a \in A_\theta$ *then* $\Phi_{f,\nu}(ge^{-rD_\theta}a) = a^\nu \Phi_{f,\nu}(ge^{-rD_\theta})$.

Proof. (1) Write $g = k_g m_g a_g u_g$ according to the Iwasawa decomposition. Then we have

$$\begin{aligned}\Phi_{f,\nu}(ge^{-rD_\theta}\tilde{u}) &= \Phi_{f,\nu}(g\tilde{u}) = \Phi_{f,\nu}(k_g m_g a_g u_g \tilde{u}) \\ &= \Phi_{f,\nu}(k_g m_g a_g \tilde{u}\, u') = f(m_g a_g \tilde{u})\end{aligned}$$

for some $u' \in \hat{U}_\theta$, since e^{-rD_θ} commutes with $\tilde{u}$ and L_θ normalizes $\hat{U}_\theta$. Now the assertion follows from the cuspidality of f.

(2) Using the same notations, we obtain

$$\begin{aligned}\Phi_{f,\nu}(ge^{-rD_\theta}a) &= \Phi_{f,\nu}(m_g a_g u_g e^{-rD_\theta} a) = \Phi_{f,\nu}(m_g a_g u_g a) \\ &= \Phi_{f,\nu}(m_g a_g a) = f(m_g a_g^\nu) a^\nu = a^\nu \Phi_{f,\nu}(ge^{-rD_\theta}).\end{aligned}$$

□

We define

$$C(\theta', \theta) = \{w \in W(\theta', \theta) \mid L_{\theta'} = {}^w L_\theta\} = \{w \in W(\theta', \theta) \mid \Delta^{\pm}_{\theta'} = w\Delta^{\pm}_\theta\}.$$

Lemma 3.4. *Assume that* $g \in \hat{G}$ *and* $w \in W(\theta', \theta)$*. If* $w \in C(\theta', \theta)$*, we have*

$$\int_{\hat{U}_{\theta'}/\hat{U}_{\theta'}\cap\hat{\Gamma}} E_w(f, \nu, ge^{-rD_\theta}u)du_\infty = \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(ge^{-rD_\theta}u_1 w)du_1.$$

Otherwise,

$$\int_{\hat{U}_{\theta'}/\hat{U}_{\theta'}\cap\hat{\Gamma}} E_w(f, \nu, ge^{-rD_\theta}u)du_\infty = 0.$$

Proof. We have the projection

$$\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta',\mathbb{Q}} \cong ((\mathbb{K}_f \cap \hat{U}_{\theta',\mathbb{A}_f}) \times \hat{U}_{\theta'})/\hat{\Gamma} \cap \hat{U}_{\theta'} \xrightarrow{\pi} \hat{U}_{\theta'}/\hat{\Gamma} \cap \hat{U}_{\theta'}.$$

If we consider $E_w(f, \nu, ge^{-rD_\theta}u)$ as a function on $\hat{U}_{\theta'}/\hat{\Gamma} \cap \hat{U}_{\theta'}$, we can pull it back through the projection π to obtain a function on $\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}} \cap \hat{U}_{\theta',\mathbb{Q}}$, which we will denote by the same notation $E_w(f, \nu, ge^{-rD_\theta}u)$. Then we have

$$\int_{\hat{U}_{\theta'}/\hat{\Gamma}\cap\hat{U}_{\theta'}} E_w(f, \nu, ge^{-rD_\theta}u)du_\infty = \int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} E_w(f, \nu, ge^{-rD_\theta}u)du. \tag{3.6}$$

We define

$$\hat{\Gamma}_{\mathbb{Q}}(w) = \hat{\Gamma}_{\mathbb{Q}} \cap (\hat{P}_{\theta',\mathbb{Q}}\, w\, \hat{P}_{\theta,\mathbb{Q}}).$$

Then we have

$$\hat{\Gamma}_{\mathbb{Q}}(w) = \hat{\Gamma}(w)(\hat{P}_{\theta,\mathbb{Q}} \cap \hat{\Gamma}_{\mathbb{Q}}),$$

since $\hat{\Gamma}_{\mathbb{Q}} = \hat{\Gamma}(\hat{P}_{\theta,\mathbb{Q}} \cap \hat{\Gamma}_{\mathbb{Q}})$ and $\hat{\Gamma}(w) = \hat{\Gamma} \cap \hat{P}_{\theta'} w \hat{P}_{\theta} = \hat{\Gamma} \cap \hat{P}_{\theta',\mathbb{Q}} w \hat{P}_{\theta,\mathbb{Q}}$. So we obtain

$$E_w(f,\nu,ge^{-rD_\theta}) = \sum_{\gamma \in \hat{\Gamma}(w)/\hat{\Gamma}\cap\hat{P}_\theta} \Phi_{f,\nu}(ge^{-rD_\theta}\gamma) = \sum_{\gamma \in \hat{\Gamma}_{\mathbb{Q}}(w)/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{P}_{\theta,\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}\gamma).$$

Now we have

$$\begin{aligned}
&\int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} E_w(f,\nu,ge^{-rD_\theta}u)\,du \\
&= \int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} \sum_{\gamma\in\hat{\Gamma}_{\mathbb{Q}}(w)/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{P}_{\theta,\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}u\gamma)\,du \\
&= \sum_{m\in R_{\theta',\mathbb{Q}}\cap\hat{\Gamma}_{\mathbb{Q}}/R_{\theta',\mathbb{Q}}\cap\hat{\Gamma}_{\mathbb{Q}}\cap\,{}^{w}\hat{P}_{\theta,\mathbb{Q}}} \quad \sum_{n\in\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}\cap\,{}^{mw}\hat{P}_{\theta,\mathbb{Q}}} \\
&\qquad\qquad \int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}unmw)\,du \\
&= \sum_{m\in R_{\theta',\mathbb{Q}}\cap\hat{\Gamma}_{\mathbb{Q}}/R_{\theta',\mathbb{Q}}\cap\hat{\Gamma}_{\mathbb{Q}}\cap{}^{w}\hat{P}_{\theta,\mathbb{Q}}} \int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}\cap\,{}^{mw}\hat{P}_{\theta,\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}umw)\,du \\
&= \sum_{m\in R_{\theta',\mathbb{Q}}\cap\hat{\Gamma}_{\mathbb{Q}}/R_{\theta',\mathbb{Q}}\cap\hat{\Gamma}_{\mathbb{Q}}\cap{}^{w}\hat{P}_{\theta,\mathbb{Q}}} \int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}\cap{}^{w}\hat{P}_{\theta,\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}muw)\,du, \qquad (3.7)
\end{aligned}$$

where the measure du on $\hat{U}_{\theta',\mathbb{A}}/(\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}\cap{}^{w}\hat{P}_{\theta,\mathbb{Q}})$ is explained above. Using the decomposition (3.5), the integral in (3.7) equals

$$\begin{aligned}
&\int_{U_{w,\mathbb{A}}}\int_{U'_{w,\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}}\int_{\hat{U}''_{w,\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}mu_1u_2u_3w)du_3du_2du_1 \\
&= \int_{U_{w,\mathbb{A}}}\int_{U'_{w,\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}}\int_{\hat{U}''_{w,\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}mu_1u_2w(w^{-1}u_3w))du_3du_2du_1 \\
&= \int_{U_{w,\mathbb{A}}}\int_{U'_{w,\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}mu_1u_2w)du_2du_1 \\
&= \int_{U_{w,\mathbb{A}}}\int_{U'_{w,\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}mu_1w(w^{-1}u_2w))du_2du_1 \\
&= \int_{U_{w,\mathbb{A}}}\int_{(U'_{w,\mathbb{A}})^w/L_{\theta,\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}mu_1wu'_2)du'_2du_1, \qquad (3.8)
\end{aligned}$$

where the measure du'_2 on $(U'_{w,\mathbb{A}})^w$ is the Haar measure induced from du_2 by conjugation. We also note that $(U'_{w,\mathbb{A}})^w \subset L_{\theta,\mathbb{A}}$. One can see that, unless $(U'_{w,\mathbb{A}})^w = \{1\}$,

$$\int_{(U'_{w,\mathbb{A}})^w/L_{\theta,\mathbb{Q}}} \Phi_{f,\nu}(ge^{-rD_\theta}mu_1wu'_2)du'_2 = 0$$

by Lemma 3.3 (1). Since

$$(U'_{w,\mathbb{A}})^w = \{1\} \quad \Longleftrightarrow \quad L_{\theta'} = {}^w L_\theta \quad \Longleftrightarrow \quad w \in C(\theta', \theta),$$

we have proven the second part of the assertion.

If $w \in C(\theta', \theta)$, then $R_{\theta',\mathbb{Q}} \subset {}^w\hat{P}_{\theta,\mathbb{Q}}$, and we have from (3.7) and (3.8)

$$\int_{\hat{U}_{\theta',\mathbb{A}}/\hat{\Gamma}_{\mathbb{Q}}\cap\hat{U}_{\theta',\mathbb{Q}}} E_w(f,\nu, ge^{-rD_\theta}u)\, du = \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(ge^{-rD_\theta}u_1 w) du_1,$$

which yields the first part of the proposition by (3.6). □

From now on, we mainly consider the case when $\theta' = \theta$, and write $C(\theta) = C(\theta,\theta)$. We define the shifted action of $\hat{W}$ on by

$$w \cdot \nu = w(\nu + \rho) - \rho,$$

and we write

$$m^w = w^{-1} m w.$$

In the following lemma, we further simplify the integral.

Lemma 3.5. *For $g \in \hat{G}$, $r \in \mathbb{R}_{>0}$ and $w \in C(\theta)$, we have*

$$\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(ge^{-rD_\theta}u_1 w) du_1 = (a_g e^{-rD_\theta})^{w\cdot\nu} f(m_g^w) \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(u_1 w) du_1,$$

where $a_g \in A_\theta$ and $m_g \in L_\theta$ are given by the Iwasawa decomposition of g.

Proof. We write $g = k_g m_g a_g u_g$ with respect to the Iwasawa decomposition. We set $\tilde{u}_g = e^{rD_\theta} u_g e^{-rD_\theta}$ and write $\tilde{u}_g = \tilde{u}_{g,1}\tilde{u}_{g,2}\tilde{u}_{g,3}$ with respect to the decomposition (3.3). Since $w \in C(\theta)$, we have $U'_{w,\mathbb{A}} = \{1\}$ and $\tilde{u}_{g,2} = 1$. Then we have

$$\begin{aligned}
\Phi_{f,\nu}(ge^{-rD_\theta}u_1 w) &= \Phi_{f,\nu}(m_g a_g u_g e^{-rD_\theta} u_1 w) = \Phi_{f,\nu}(m_g a_g e^{-rD_\theta} \tilde{u}_g u_1 w) \\
&= \Phi_{f,\nu}(m_g a_g e^{-rD_\theta} \tilde{u}_{g,1}\tilde{u}_{g,3} u_1 w) \\
&= \Phi_{f,\nu}(m_g a_g e^{-rD_\theta} \tilde{u}_{g,1}\, (\tilde{u}_{g,3} u_1 \tilde{u}_{g,3}^{-1})\, w\, (w^{-1}\tilde{u}_{g,3} w)\,) \\
&= \Phi_{f,\nu}(m_g a_g e^{-rD_\theta} \tilde{u}_{g,1} u_1 \hat{u}_{g,3} w) \\
&= \Phi_{f,\nu}(m_g a_g e^{-rD_\theta} \tilde{u}_{g,1} u_1 w\, (w^{-1}\hat{u}_{g,3} w)\,) \\
&= \Phi_{f,\nu}(m_g a_g e^{-rD_\theta} \tilde{u}_{g,1} u_1 w),
\end{aligned}$$

where $u_1\hat{u}_{g,3} = \tilde{u}_{g,3}u_1\tilde{u}_{g,3}^{-1}$ with $\hat{u}_{g,3} \in \hat{U}''_{w,\mathbb{A}}$. Now we obtain

$$\begin{aligned}
&\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(ge^{-rD_\theta}u_1w)du_1 \\
=\ &\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(m_g a_g e^{-rD_\theta}\tilde{u}_{g,1}u_1w)du_1 \\
=\ &\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(m_g a_g e^{-rD_\theta}u_1w)du_1 \qquad \text{(by the invariance of the measure)} \\
=\ &\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(m_g a_g e^{-rD_\theta}u_1 e^{rD_\theta}a_g^{-1}w\,(w^{-1}a_g e^{-rD_\theta}w)\,)du_1. \qquad (3.9)
\end{aligned}$$

We observe that, since $w \in C(\theta)$,

$$(w^{-1}a_g e^{-rD_\theta}w)^{\alpha_i} = (a_g e^{-rD_\theta})^{w\alpha_i} = 1 \qquad \text{for each } i \in \theta,$$

and so $w^{-1}a_g e^{-rD_\theta}w \in A_\theta$. Applying Lemma 3.3 (2) to (3.9), we obtain

$$\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(ge^{-rD_\theta}u_1w)du_1 = (a_g e^{-rD_\theta})^{w\nu} \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(m_g a_g e^{-rD_\theta}u_1 e^{rD_\theta}a_g^{-1}w)du_1.$$

Using Lemma 3.1 and the identity

$$\sum_{\alpha\in\Delta_w} \alpha = \rho - w\rho,$$

we continue to calculate and obtain

$$\begin{aligned}
\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(ge^{-rD_\theta}u_1w)du_1 &= (a_g e^{-rD_\theta})^{w\nu}\,(a_g e^{-rD_\theta})^{w\rho-\rho} \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(m_g u_1 w)du_1, \\
&= (a_g e^{-rD_\theta})^{w(\nu+\rho)-\rho} \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(m_g u_1 w)du_1 \\
&= (a_g e^{-rD_\theta})^{w(\nu+\rho)-\rho} \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(u_1 m_g w)du_1 \\
&= (a_g e^{-rD_\theta})^{w(\nu+\rho)-\rho} \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(u_1 w w^{-1} m_g w)du_1 \\
&= (a_g e^{-rD_\theta})^{w(\nu+\rho)-\rho} f(m_g^w) \int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(u_1 w)du_1.
\end{aligned}$$

This completes the proof. □

The following lemma can be proved in a standard way using the Gindikin-Karpelevich formula.

Lemma 3.6. [11, 6] *We have*

$$\int_{U_{w,\mathbb{A}}} \Phi_{f,\nu}(u_1 w) du_1 = \prod_{\alpha \in \Delta_{w^{-1}}} \frac{\Lambda(-(\nu+\rho)(h_\alpha))}{\Lambda(-(\nu+\rho)(h_\alpha)+1)},$$

where $\Lambda(s)$ is the completed L-function associated to f.

Now we obtain the following theorem from (3.4), Lemma 3.4, Lemma 3.5 and Lemma 3.6.

Theorem 3.7.

$$E_\theta(f,\nu,ge^{-rD_\theta}) = \sum_{w \in C(\theta)} (a_g e^{-rD_\theta})^{w\cdot\nu} f(m_g^w) \prod_{\alpha \in \Delta_{w^{-1}}} \frac{\Lambda(-(\nu+\rho)(h_\alpha))}{\Lambda(-(\nu+\rho)(h_\alpha)+1)}.$$

4. Examples

In this section, we characterize the set $C(\theta)$ of double coset representatives when $\theta \subseteq \{1,2,...,l\}$ and determine the constant terms explicitly in some non-maximal cases. It is known that there is no maximal self-conjugate parabolic subgroups (See [12]).

Let $\hat{\mathfrak{g}}$ be the affine Kac-Moody algebra and $\hat{W}$ the Weyl group corresponding to $\hat{\mathfrak{g}}$. It is well-known that

$$\hat{W} = W \ltimes T,$$

where W is the subgroup of $\hat{W}$ generated by $\{w_1,\dots,w_l\}$, which is the Weyl group corresponding to $\mathfrak{g}$, and T is the translation group which is isomorphic to some lattice $M \subset \mathring{\mathfrak{h}}_{\mathbb{R}}^*$ (see [9, Proposition 6.5]). In the case of type $A_l^{(1)}$, for example, $M = \oplus_{i=1}^l \mathbb{Z}\alpha_i$ and the translation t_ϑ by $\vartheta := \sum_{i=1}^l \alpha_i$ is written as

$$t_\vartheta = w_{l+1}w_1w_2\dots w_{l-1}w_lw_{l-1}\dots w_2w_1.$$

Assume that $\theta \subseteq \{1,\dots,l\}$ and we set

$$\hat{\mathfrak{C}}(\theta) := \{w \in \hat{W} |\ w\Delta_\theta^\pm = \Delta_\theta^\pm\},$$
$$\mathfrak{C}(\theta) := \{w \in W |\ w\Delta_\theta^\pm = \Delta_\theta^\pm\},$$
$$M(\theta) := \{\alpha \in M |\ \alpha(h_i) = 0\ {}^\forall i \in \theta\}.$$

By definition, $C(\theta)$ is the set of double coset representatives of $W_\theta\backslash\hat{\mathfrak{C}}(\theta)/W_\theta$ such that, for each $w \in \hat{\mathfrak{C}}(\theta)$, the length of w in $W_\theta w W_\theta$ is minimal in $W_\theta\backslash\hat{\mathfrak{C}}(\theta)/W_\theta$.

Proposition 4.1. *We have*

$$\hat{\mathfrak{C}}(\theta) = \{wt_\alpha |\ w \in \mathfrak{C}(\theta),\ \alpha \in M(\theta)\}.$$

Proof. Let $\hat{\mathfrak{C}} = \{wt_\alpha |\ w \in \mathfrak{C}(\theta),\ \alpha \in M(\theta)\}$. For any element $wt_\alpha \in \hat{\mathfrak{C}}$ and $a \in \Delta_\theta^\pm$, we have

$$wt_\alpha(a) = w(a - (\alpha|a)\iota) = w(a) \in \Delta_\theta^\pm,$$

which implies $\hat{\mathfrak{C}} \subseteq \hat{\mathfrak{C}}(\theta)$. Now we fix some element $x \in \hat{\mathfrak{C}}(\theta)$. Then x can be written as $x = wt_\alpha$ for some $w \in W$ and $\alpha \in M$. Since

$$wt_\alpha(a) = w(a - (\alpha|a)\iota) = w(a) - (\alpha|a)\iota \in \Delta_\theta^\pm$$

for all $a \in \Delta_\theta^\pm$, we have $\alpha(h_i) = 0$ and $w\alpha_i \in \Delta_\theta^+$ for all $i \in \theta$. Hence, $\hat{\mathfrak{C}}(\theta) \subseteq \hat{\mathfrak{C}}$. □

Corollary 4.2. [12] *If* $\theta = \{1, 2, ..., l\}$, *then* $C(\theta) = \{id\}$.

Proof. Note that $\mathfrak{C}(\theta) = W_\theta$ and $M(\theta) = \emptyset$ if $\theta = \{1, \dots, l\}$. Applying Proposition 4.1 to this case proves our assertion. □

Example 4.3. Let $\hat{\mathfrak{g}}$ be the affine Kac-Moody algebra of type $A_1^{(1)}$ and let $\theta = \emptyset$. Then the cusp form f on L_θ satisfying (1.4) must be constant. We take f to be 1, and set

$$\hat{\mathfrak{h}}_{\mathbb{C}}^e = \mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}D, \quad \left(\hat{\mathfrak{h}}_{\mathbb{C}}^e\right)^* = \mathbb{C}\Lambda_1 \oplus \mathbb{C}\Lambda_2 \oplus \mathbb{C}\Lambda_D,$$

where

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(D) = 0, \quad \Lambda_D(h_j) = 0 \quad \text{and} \quad \Lambda_D(D) = 1 \quad (i, j = 1, 2).$$

The simple roots are given by

$$\alpha_1 = 2\Lambda_1 - 2\Lambda_2, \quad \alpha_2 = -2\Lambda_1 + 2\Lambda_2 + \Lambda_D.$$

Set $\rho = \Lambda_1 + \Lambda_2$. The Weyl group $\hat{W}$ can be described as

$$\begin{aligned} \hat{W} &= \langle w_1, w_2 | w_1^2 = 1, w_2^2 = 1 \rangle \\ &= \{w_1^{\varepsilon_1}(w_2w_1)^n w_2^{\varepsilon_2} | \varepsilon_1, \varepsilon_2 = 0 \text{ or } 1,\ n \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

Let $w = w_1^{\varepsilon_1}(w_2w_1)^n w_2^{\varepsilon_2} \in \hat{W}$. Then

$$\begin{aligned} \hat{\Delta}_w &= \hat{\Delta}_+ \cap w\hat{\Delta}_- \\ &= \{\varepsilon_1\alpha_1,\ \varepsilon_2 w_1^{\varepsilon_1}(w_2w_1)^n\alpha_2,\ w_1^{\varepsilon_1}(w_2w_1)^i\alpha_2,\ w_1^{\varepsilon_1}(w_2w_1)^j w_2\alpha_1 \,|\, 0 \leq i, j \leq n-1\} \setminus \{0\}. \end{aligned}$$

We fix $\nu = a\Lambda_1 + b\Lambda_2 + c\Lambda_D$ with $\mathrm{Re}(a), \mathrm{Re}(b) < -2$. Recall the definition of $\tilde{c}(\nu, w)$:

$$\tilde{c}(\nu, w) = \prod_{\alpha \in \hat{\Delta}_{w^{-1}}} \frac{\xi(-(\nu+\rho)(h_\alpha))}{\xi(-(\nu+\rho)(h_\alpha)+1)}.$$

Let $\kappa = (\nu + \rho)(h_1 + h_2) = a + b + 2$.

(1) Assume that $w = (w_2w_1)^n$ $(n \geq 1)$. Then $w^{-1} = w_1(w_2w_1)^{n-1}w_2$,

$$\hat{\Delta}_{w^{-1}} = \{(2t+2)\alpha_1 + (2t+1)\alpha_2,\ (2s+1)\alpha_1 + 2s\alpha_2 \,|\, 0 \leq s, t \leq n-1\},$$

$$\tilde{c}(\nu, w) = \prod_{t=0}^{2n-1} \frac{\xi(-\kappa t - a - 1)}{\xi(-\kappa t - a)},$$

and we get

$$w(\nu+\rho) - \rho = (2\kappa n + a)\Lambda_1 + (-2\kappa n + b)\Lambda_2 + (-\kappa n^2 - (a+1)n + c)\Lambda_D.$$

(2) Assume that $w = w_1(w_2w_1)^n$ $(n \geq 0)$. Then $w^{-1} = w_1(w_2w_1)^n$,

$$\hat{\Delta}_{w^{-1}} = \{(2t+2)\alpha_1 + (2t+1)\alpha_2,\ (2s+1)\alpha_1 + 2s\alpha_2 \,|\, 0 \leq t \leq n-1,\ 0 \leq s \leq n\},$$

$$\tilde{c}(\nu, w) = \prod_{t=0}^{2n} \frac{\xi(-\kappa t - a - 1)}{\xi(-\kappa t - a)},$$

and we get

$$\begin{aligned} w(\nu+\rho) - \rho = (-2\kappa n - a - 2)\Lambda_1 + (2\kappa(n+1) - b - 2)\Lambda_2 \\ + (-\kappa n^2 - (a+1)n + c)\Lambda_D. \end{aligned}$$

(3) Assume that $w = (w_2w_1)^n w_2$ $(n \geq 0)$. Then $w^{-1} = (w_2w_1)^n w_2$,

$$\hat{\Delta}_{w^{-1}} = \{2t\alpha_1 + (2t+1)\alpha_2,\ (2s+1)\alpha_1 + (2s+2)\alpha_2 \,|\, 0 \leq t \leq n,\ 0 \leq s \leq n-1\},$$

$$\tilde{c}(\nu, w) = \prod_{t=0}^{2n} \frac{\xi(-\kappa t - b - 1)}{\xi(-\kappa t - b)},$$

and we get

$$\begin{aligned} w(\nu+\rho) - \rho = (2\kappa(n+1) - a - 2)\Lambda_1 + (-2\kappa n - b - 2)\Lambda_2 \\ + (-\kappa n^2 - (a+2b+3)n - b + c - 1)\Lambda_D. \end{aligned}$$

(4) Assume that $w = w_1(w_2w_1)^n w_2$ $(n \geq 0)$. Then $w^{-1} = (w_2w_1)^{n+1}$,

$$\hat{\Delta}_{w^{-1}} = \{2t\alpha_1 + (2t+1)\alpha_2,\ (2s+1)\alpha_1 + (2s+2)\alpha_2 \,|\, 0 \leq t, s \leq n\},$$

$$\tilde{c}(\nu, w) = \prod_{t=0}^{2n+1} \frac{\xi(-\kappa t - b - 1)}{\xi(-\kappa t - b)},$$

and we get

$$\begin{aligned} w(\nu+\rho) - \rho = (-2\kappa(n+1) + a)\Lambda_1 + (2\kappa(n+1) + b)\Lambda_2 \\ + (-\kappa n^2 - (a+2b+3)n - b + c - 1)\Lambda_D. \end{aligned}$$

Let $g \in \hat{G}$, $a_g = h_1(s_1)h_2(s_2)$, $s_1, s_2 \in \mathbb{R}_{>0}$. Then we obtain

$$\begin{aligned}
E_\theta(1,\nu,ge^{-rD}) &= \sum_{w\in\hat{W}} (a_g e^{-rD})^{w(\nu+\rho)-\rho}\tilde{c}(\nu,w) \\
&= \sum_{i=0}^{\infty} \Big\{ (a_g e^{-rD})^{(w_2w_1)^i(\nu+\rho)-\rho}\ \tilde{c}(\nu,(w_2w_1)^i) \\
&\qquad + (a_g e^{-rD})^{w_1(w_2w_1)^i(\nu+\rho)-\rho}\ \tilde{c}(\nu,w_1(w_2w_1)^i) \\
&\qquad + (a_g e^{-rD})^{(w_2w_1)^i w_2(\nu+\rho)-\rho}\ \tilde{c}(\nu,(w_2w_1)^i w_2) \\
&\qquad + (a_g e^{-rD})^{w_1(w_2w_1)^i w_2(\nu+\rho)-\rho}\ \tilde{c}(\nu,w_1(w_2w_1)^i w_2) \Big\} \\
&= \sum_{i=0}^{\infty} \left\{ s_1^a s_2^b e^{-rc} e^{r(\kappa i^2+(a+1)i)} \prod_{j=0}^{2i-1} \frac{s_1^\kappa \xi(-\kappa j-a-1)}{s_2^\kappa \xi(-\kappa j-a)} \right. \\
&\qquad + s_1^b s_2^a e^{-rc} e^{r(\kappa i^2+(a+1)i)} \prod_{j=0}^{2i} \frac{s_2^\kappa \xi(-\kappa j-a-1)}{s_1^\kappa \xi(-\kappa j-a)} \\
&\qquad + s_1^b s_2^a e^{-rc} e^{r(\kappa i^2+(a+2b+3)i+b+1)} \prod_{j=0}^{2i} \frac{s_1^\kappa \xi(-\kappa j-b-1)}{s_2^\kappa \xi(-\kappa j-b)} \\
&\qquad \left. + s_1^a s_2^b e^{-rc} e^{r(\kappa i^2+(a+2b+3)i+b+1)} \prod_{j=0}^{2i+1} \frac{s_2^\kappa \xi(-\kappa j-b-1)}{s_1^\kappa \xi(-\kappa j-b)} \right\}.
\end{aligned}$$

Example 4.4. Let $\hat{\mathfrak{g}}$ be the affine Kac-Moody algebra of type $A_2^{(1)}$ and let $\theta = \{2\}$. Choose a cusp form f on L_θ satisfying (1.4), and set

$$D_\theta := D + c_1(2h_1 + h_2) + c_2(h_2 + 2h_3)$$

for some $c_1, c_2 \in \mathbb{C}$. Since $t_{\alpha_1+\alpha_2} = w_3w_1w_2w_1$, we obtain

$$\begin{aligned}
t_{\alpha_1} &= t_{w_2(\alpha_1+\alpha_2)} = w_2 t_{\alpha_1+\alpha_2} w_2 = w_2w_3w_2w_1, \\
t_{\alpha_2} &= t_{w_1(\alpha_1+\alpha_2)} = w_1 t_{\alpha_1+\alpha_2} w_1 = w_1w_3w_1w_2.
\end{aligned}$$

Let $\sigma := 2\alpha_1 + \alpha_2$. Then, by Proposition 4.1, we have

$$\begin{aligned}
\mathfrak{C}(\theta) &= \{w \in W \mid \Delta_\theta^\pm = w\Delta_\theta^\pm\} = \{id, w_2\}, \\
M(\theta) &= \{\alpha \in M \mid \alpha(h_i) = 0\ {}^\forall i \in \theta\} = \{k\sigma \mid k \in \mathbb{Z}\},
\end{aligned}$$

which implies

$$\begin{aligned}\hat{\mathfrak{C}}(\theta) &= \{wt_\alpha|\ w \in \mathfrak{C}(\theta),\ \alpha \in M(\theta)\} \\ &= \{t_{k\sigma},\ w_2 t_{k\sigma}|\ k \in \mathbb{Z}\} \\ &= \{(t_{\alpha_1}^2 t_{\alpha_2})^k,\ w_2(t_{\alpha_1}^2 t_{\alpha_2})^k|\ k \in \mathbb{Z}\} \\ &= \{(w_3w_2w_1w_3w_2w_1)^k,\ w_2(w_3w_2w_1w_3w_2w_1)^k|\ k \in \mathbb{Z}\}.\end{aligned}$$

Therefore

$$C(\theta) = \{(w_3w_2w_1w_3w_2w_1)^k|\ k \in \mathbb{Z}\}.$$

Note that $(w_3w_2w_1w_3w_2w_1)^k$ is a reduced expression of $t_{k\sigma}$ since $l(t_{k\sigma}) = 6k$.

Set $\nu := a\Lambda_1 + b\Lambda_3$, $K := h_1 + h_2 + h_3$, $\rho := \Lambda_1 + \Lambda_2 + \Lambda_3$ and $\ell := a + b + 3$. Then

$$\begin{aligned}\Delta_{t_{k\sigma}^{-1}} = \{\ & t_{(i-1)\sigma}^{-1}\alpha_1,\ (w_2w_3w_1w_2w_3t_{i\sigma})^{-1}\alpha_2,\ (w_3w_1w_2w_3t_{i\sigma})^{-1}\alpha_3, \\ & (w_1w_2w_3t_{i\sigma})^{-1}\alpha_1,\ (w_2w_3t_{i\sigma})^{-1}\alpha_2,\ (w_3t_{i\sigma})^{-1}\alpha_3|\ 1 \le i \le k\},\end{aligned}$$

and, by using [9, (6.5.2)], we have

$$\begin{aligned}t_{k\sigma}(\nu+\rho) &= \nu + \rho + \langle \nu+\rho, K\rangle k\sigma - \left((\nu+\rho|k\sigma) + \frac{1}{2}k^2|\sigma|^2\langle \nu+\rho, K\rangle\right)\iota \\ &= \nu + \rho + \ell k\sigma - \left(\frac{3\ell}{2}k^2 + (2a+b+3)k\right)\iota,\end{aligned}$$

which yields

$$t_{k\sigma}(\nu+\rho)(h_i) = \begin{cases} a + 1 + 3\ell k & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ b + 1 - 3\ell k & \text{if } i = 3. \end{cases}$$

Since $(\nu+\rho)(h_{t_{k\sigma}^{-1}(\alpha)}) = t_{k\sigma}(\nu+\rho)(h_\alpha)$ for any $\alpha \in (\hat{\mathfrak{h}}^e)^*$, we obtain

$$\prod_{\alpha \in \Delta_{t_{k\sigma}^{-1}}} \frac{\Lambda(-(\nu+\rho)(h_\alpha))}{\Lambda(-(\nu+\rho)(h_\alpha)+1)} = \prod_{1 \le i \le |k|} \prod_{1 \le j \le 6} \frac{\Lambda(a_j - 3\mathrm{sgn}(k)\ell i)}{\Lambda(a_j + 1 - 3\mathrm{sgn}(k)\ell i)},$$

where $\Lambda(s)$ is the completed standard L-function associated to f, $\mathrm{sgn}(k) = \frac{k}{|k|}$ and

$$\begin{aligned}&a_1 = -a-1, && a_2 = 2a+3b+7, && a_3 = a+2b+5, \\ &a_4 = a+2b+4, && a_5 = b+2, && a_6 = b+1.\end{aligned}$$

Consequently,

$$E_\theta(f, \nu, ge^{-rD_\theta}) = \sum_{k \in \mathbb{Z}} (a_g)^{\nu + \ell k\sigma} e^{d_\theta} f(m_g^{w(k)}) \prod_{1 \le i \le |k|} \prod_{1 \le j \le 6} \frac{\Lambda(a_j - 3\mathrm{sgn}(k)\ell i)}{\Lambda(a_j + 1 - 3\mathrm{sgn}(k)\ell i)},$$

where

$$d_\theta = r\left(\frac{3\ell}{2}k^2 + (2a+b+3)k - (6\ell(c_1-c_2)k + 2ac_1 + 2bc_2)\right)$$

and $w(k) = (w_3w_2w_1w_3w_2w_1)^k$.

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