POSITIVITY OF FOURIER COEFFICIENTS OF WEAKLY MODULAR FORMS OF WEIGHT $\frac{1}{2}$

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ABSTRACT. In this short note, we will prove certain weakly modular forms of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$ have positive Fourier coefficients.

We use the notations from [8]. If d is an odd prime, let $(\frac{c}{d})$ be the usual Legendre symbol. For positive odd d, define $(\frac{c}{d})$ by multiplicativity. For negative odd d, let $\left(\frac{c}{d}\right) = \begin{cases} (\frac{c}{|d|}) & \text{if } d<0 \text{ and } c>0, \\ -(\frac{c}{|d|}) & \text{if } d<0 \text{ and } c<0. \end{cases}$

We let $(\frac{0}{\pm 1}) = 1$. Define ϵ_d, for odd d, by $\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \bmod 4, \\ i & \text{if } d \equiv 3 \bmod 4. \end{cases}$

Let $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ be the *Kohnen plus-space* of weakly homomorphic modular forms with integer coefficients of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$, namely, $f \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ if f is holomorphic on $\mathcal{H}$, and meromorphic at the cusps of $\Gamma_0(4)$, and

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{c}{d}\right)\epsilon_d^{-1}(cz+d)^{\frac{1}{2}} f(z),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, and it has a Fourier expansion of the form

$$f(z) = \sum_{\substack{n \geq n_0 \\ n \equiv 0,1 (\mathrm{mod} 4)}} a(n) q^n.$$

Given such an f, Borcherds [1] proved that for some h, the infinite product

$$q^{-h} \prod_{n=1}^{\infty} (1-q^n)^{a(n^2)},$$

is a meromorphic modular form with respect to $SL_2(\mathbb{Z})$. He also proved that for each nonnegative integer $d \equiv 0, 3 \pmod 4$, there exists a unique modular form $f_d(z) \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ with a Fourier expansion

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$$f_d(z) = q^{-d} + \sum_{\substack{n>0 \\ n\equiv 0,1 (\mathrm{mod} 4)}} a(n)q^n. \tag{0.1}$$

Here $f_d(z)$'s form a $\mathbb{Z}$-basis for $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$. Also $\mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$ is a free $\mathbb{Z}[j(4z)]$-module of rank 2 with generators $f_0(z), f_3(z)$, where

$$\begin{aligned} f_0(z) &= \theta(z) = 1 + 2q + 2q^4 + 2q^9 + \cdots, \\ f_3(z) &= F(z)\theta(z)(\theta(z)^4 - 2F(z))(\theta(z)^4 - 16F(z))\frac{E_6(4z)}{\Delta(4z)} + 56\theta(z) \\ &= q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 - \cdots. \end{aligned}$$

(See Section 1 for the definitions of θ and F.)

Now we consider f_d with $4|d$. In particular, we have

$$\begin{aligned} f_4(z) &= q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + 51180012q^9 + \cdots \\ f_8(z) &= q^{-8} + 7256q + 26124256q^4 + 190356480q^5 + 29071392966q^8 + \cdots. \end{aligned}$$

The numerical calculation shows that the Fourier coefficients are all positive. In this short note, we will prove the following result using the method of Hardy-Ramanujan-Rademacher.

Theorem 0.2. *In (0.1), suppose $4|d$ and $d > 0$. Then $a(n)$ is positive for all sufficiently large $n \equiv 0, 1$ (mod 4), and as $n \to \infty$,*

$$a(n) \sim \frac{2\cosh(\pi\sqrt{dn})}{\sqrt{n}}.$$

Similarly, we can prove that if $d \equiv 3 \pmod 4$, $(-1)^n a(n)$ is positive for all sufficiently large $n \equiv 0, 1 \pmod 4$, and as $n \to \infty$,

$$a(n) \sim (-1)^n \frac{2\cosh(\pi\sqrt{dn})}{\sqrt{n}}.$$

If we write $f_d(z) = q^{-d} + \sum_{D>0} A(D, d)q^D$, we can also show that given D with $4|D$, $A(D, d)$ is positive for all sufficiently large d. It is expected that $a(n)$ is positive for all n for any f_d, $4|d$. However, our method cannot prove it.

The positivity of Fourier coefficients plays an important role in our work on Gindikin-Karpelevich formula for generalized Kac-Moody algebras and deformation of modular forms [3]. Also, the result is quite a contrast to some results in the literature. For example, it is proved (for example, [4]) that for any holomorphic Hecke eigenforms with respect to $\Gamma_0(N)$, there are infinitely many coefficients which are positive, and infinitely many coefficients which are negative. In higher level case, the positivity of Fourier coefficients of weakly modular forms does not seem to be true ([2]).

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1. Preliminaries on modular forms

Here we collect several modular forms we will use in this note. Let $q = e^{2\pi i z} = e(z)$. Let $j(z)$ be the modular j-function:

$$j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n, \quad c(1) = 196884,\, c(2) = 21493760, \cdots .$$

We let $\sqrt{z}$ be the branch of the square root having argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$. Hence for $z \in \mathcal{H}$, $(-z)^{\frac{1}{2}} = (-\frac{1}{z})^{-\frac{1}{2}} = -iz^{\frac{1}{2}}$. In particular, $(-i)^{\frac{1}{2}} = \frac{1-i}{\sqrt{2}}$. Recall the following from [9], page 177:

$$\begin{aligned} \theta(z) &= \sum_{n\in\mathbb{Z}} q^{n^2}, \quad \theta_2(z) = \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}, \quad \theta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}, \\ \theta(-\tfrac{1}{z}) &= \sqrt{-\tfrac{i}{2}}\sqrt{z}\,\theta(\tfrac{z}{4}), \quad \theta_4(-\tfrac{1}{z}) = \sqrt{-\tfrac{i}{2}}\sqrt{z}\,\theta_2(\tfrac{z}{4}). \end{aligned} \tag{1.1}$$

Hence we have

$$\theta(-\tfrac{1}{z} + \tfrac{1}{2}) = \theta_4(-\tfrac{1}{z}) = \sqrt{-\tfrac{i}{2}}\sqrt{z}\,\theta_2(\tfrac{z}{4}). \tag{1.2}$$

Recall the following from [5], pages 113 and 145. Let $E_2(z)$ be the Eisenstein series of weight 2. Then

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad z^{-2}E_2(-\tfrac{1}{z}) = E_2(z) + \frac{12}{2\pi i z}.$$

Let

$$\begin{aligned} F(z) &= -\frac{1}{24}(E_2(z) - 3E_2(2z) + 2E_2(4z)) = -\frac{1}{48}(E_2(z) - E_2(z + \tfrac{1}{2})) \\ &= \sum_{\text{odd}\, n>0} \sigma_1(n)q^n = q + 4q^3 + 6q^5 + \cdots . \end{aligned}$$

Then $F(z)$ is a modular form of weight 2 with respect to $\Gamma_0(4)$. We have

$$z^{-2}F(-\tfrac{1}{z}) = -z^{-2}F(-\tfrac{1}{z} + \tfrac{1}{2}) = -\frac{1}{192}(8E_2(z) - 6E_2(\tfrac{z}{2}) + E_2(\tfrac{z}{4})). \tag{1.3}$$

2. Method of Hardy-Ramanujan-Rademacher

In order to demonstrate our method, we first prove the Fourier coefficients $c(n)$ of the modular j-function are all positive for all $n > 0$. This is clear from the identity $j(z) = \frac{E_4(z)^3}{\Delta(z)}$, $E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$, $\frac{1}{\Delta(z)} = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-24}$, but we want to show how our method works in this example.

By [10], page 510,

$$c(n) = \frac{2\pi}{\sqrt{n}}\sum_{k=1}^{\infty}\frac{A_k(n)}{k}I_1\left(\frac{4\pi\sqrt{n}}{k}\right).$$

Here $I_1(z) \sim \frac{e^z}{\sqrt{2\pi z}}$ and

$$A_k(n) = \sum_{\substack{h(\text{mod }k),\,(h,k)=1\\ hh'\equiv -1(\text{mod }k)}} e\left(-\frac{nh+h'}{k}\right).$$

We have ([9], page 291)

$$|A_k(n)| < 2k^{\frac{3}{4}}.$$

When $k = 1$, it gives rise to $\dfrac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{\frac{3}{4}}}$. It is the main term. We show that the sum of the other terms is smaller. We first have

$$\frac{2\pi}{\sqrt{n}}\sum_{k=\sqrt{n}}^{\infty}2k^{-\frac{1}{4}}I_1\left(\frac{4\pi\sqrt{n}}{k}\right) \le \frac{2\pi}{\sqrt{n}}\int_{\sqrt{n}}^{\infty}2x^{-\frac{1}{4}}I_1\left(\frac{4\pi\sqrt{n}}{x}\right)dx < 4000\pi^2 n^{-\frac{1}{8}}.$$

Next we use the trivial estimate $|A_k(n)| \le k$ and obtain

$$\left|\frac{2\pi}{\sqrt{n}}\sum_{k=2}^{\sqrt{n}}\frac{1}{k}A_k(n)I_1\left(\frac{4\pi\sqrt{n}}{k}\right)\right| \le 2\pi I_1(2\pi\sqrt{n}).$$

Since $e^{2\pi\sqrt{n}} > 2\sqrt{\pi n}$, we have $c(n) > 0$.

Now we recall the result of J. Lehner [7] on Fourier coefficients of modular forms using the method of Hardy-Ramanujan-Rademacher. We refer to [7] for unexplained notations: Let $f(z)$ be a weakly homomorphic modular form of weight $r > 0$ with respect to Γ. Let $p_0 = \infty, p_1, ..., p_{s-1}$ be the cusps of Γ, and

$$A_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad A_j = \begin{pmatrix}0 & -1\\ 1 & -p_j\end{pmatrix},\ j > 0.$$

Let $M^* = A_jM = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ for $M \in \Gamma$. Let

$$\begin{aligned} C_{j0} &= \{c \,|\, \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix} \in A_j\Gamma\}, \\ D_c &= \{d \,|\, \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \in A_j\Gamma,\, 0 < d \le c\}. \end{aligned}$$

It can be shown ([6], page 313) that given such c, d, there is a unique a such that $-c\lambda_j \le a < 0$. For $k = 1, ..., s-1$, let

$$(z-p_k)^r e(-\kappa_k \tfrac{A_k z}{\lambda_k}) f(z) = \sum_{n=-\mu_k}^{\infty} a(n)^{(k)} q_k^n, \quad q_k = e(\tfrac{A_k z}{\lambda_k}),$$

where κ_k, λ_k are defined as in [7], page 398. By replacing $A_k z$ by z, this can be written as

$$f(p_k - \tfrac{1}{z}) = (-z)^r q^{\frac{\kappa_k}{\lambda_k}} \sum_{n=-\mu_k}^{\infty} a(n)^{(k)} q^{\frac{n}{\lambda_k}}.$$

For $k = 0$, we have the usual Fourier expansion: (We assume that $\lambda_0 = 1, \kappa_0 = 0$ for Γ.)

$$f(z) = \sum_{n=-\mu_0}^{\infty} a(n) q^n.$$

Theorem 2.1. [7] *For $n > 0$,*

$$a(n) = 2\pi i^{-r} \sum_{j=0}^{s-1} \sum_{\nu=1}^{\mu_j} a(-\nu)^{(j)} \sum_{\substack{c \in C_{j0} \\ 0 < c < \sqrt{n}}} c^{-1} A(c, n, \nu_j) M(c, n, \nu_j, r) + E(n, r), \tag{2.2}$$

where $\nu_j = \frac{\nu - \kappa_j}{\lambda_j}$, and

$$\begin{aligned} A(c, n, \nu_j) &= \sum_{d \in D_c} v^{-1}(M)\, e\left(\frac{nd - \nu_j a}{c}\right), \quad M = A_j^{-1} M^*, \\ M(c, n, \nu_j, r) &= \left(\frac{n}{\nu_j}\right)^{\frac{r-1}{2}} I_{r-1}\left(\frac{4\pi\sqrt{n\nu_j}}{c}\right). \end{aligned}$$

Here if $r = \frac{1}{2}$, then

$$E(n, \tfrac{1}{2}) = O(n^{\frac{1}{4}}). \tag{2.3}$$

Here the implied constant is independent of n.

Now we apply the theorem to $f_k \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$, $k > 0$. The group $\Gamma_0(4)$ has three cusps: $p_0 = \infty$, $p_1 = 0$, $p_2 = \frac{1}{2}$ ([5], page 108).

First, $p_0=\infty$. In this case, $\lambda_0=1, \kappa_0=0$, and $A_0=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. If $c\in C_{00}$, then $4|c$, and the smallest $c\in C_{00}$ is 4, and $M=M^*=\begin{pmatrix}-3&-1\\4&1\end{pmatrix}, \begin{pmatrix}-1&-1\\4&3\end{pmatrix}$. Because of (0.1), we need only to consider $\nu_0=\mu_0$:

$$A(4,n,\mu_0)=e\left(\frac{n+3\mu_0}{4}\right)+ie\left(\frac{3n+\mu_0}{4}\right).$$

So if $k=4l$, $\mu_0=4l$, then $A(4,n,\mu_0)=1+i$ for any $n\equiv 0,1 \pmod 4$. If $k=4l+3$, $\mu_0=4l+3$, then $A(4,n,\mu_0)=(-1)^n(1+i)$ for $n\equiv 0,1 \pmod 4$.

Second, $p_1=0$. In this case, $\lambda_1=4, \kappa_1=0$, and $A_1=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. The smallest $c\in C_{10}$ is 1, and $M^*=\begin{pmatrix}-4&-5\\1&1\end{pmatrix}$, $M=\begin{pmatrix}1&1\\4&5\end{pmatrix}$. Hence

$$A(1,n,\nu_1)=e(n+\nu)=1, \quad \nu=1,...,\mu_1.$$

The Fourier expansion at 0 is

$$f_k(-\tfrac{1}{z})=-i\sqrt{z}\sum_{n=-\mu_1}^{\infty}a(n)^{(1)}q^{\frac{n}{4}}.$$

Third, $p_2=\frac{1}{2}$. In this case, $\lambda_2=4, \kappa_2=\frac{1}{4}$, and $A_2=\begin{pmatrix}0&-1\\1&-\frac{1}{2}\end{pmatrix}$. The smallest $c\in C_{20}$ is 1, and $M^*=\begin{pmatrix}-4&-3\\1&\frac{1}{2}\end{pmatrix}$, $M=\begin{pmatrix}3&2\\4&3\end{pmatrix}$. Hence

$$A(1,n,\nu_2)=i\,e\left(\tfrac{n}{2}+\nu-\tfrac{1}{4}\right)=e^{\pi in}=(-1)^n, \quad \nu=1,...,\mu_2.$$

The Fourier expansion at $\frac{1}{2}$ is

$$f_k(-\tfrac{1}{z}+\tfrac{1}{2})=-i\sqrt{z}q^{\frac{1}{16}}\sum_{n=-\mu_2}^{\infty}a(n)^{(2)}q^{\frac{n}{4}}.$$

For $f_k(z)$, we write the main term of (2.2) as follows:

$$2\pi i^{-\frac{1}{2}}\left(\tfrac{1}{4}A(4,n,\mu_0)M(4,n,k,\tfrac{1}{2})+\sum_{\nu=1}^{\mu_1}a(-\nu)^{(1)}M(1,n,\nu_1,\tfrac{1}{2})+\sum_{\nu=1}^{\mu_2}(-1)^n a(-\nu)^{(2)}M(1,n,\nu_2,\tfrac{1}{2})\right.$$
$$\left.+\sum_{\substack{c\in C_{00}\\ 4<c<\sqrt{n},\,4|c}}c^{-1}A(c,n,k)M(c,n,k,\tfrac{1}{2})+\sum_{j=1}^{2}\sum_{\substack{c\in C_{j0}\\ 1<c<\sqrt{n}}}\sum_{\nu=1}^{\mu_j}a(-\nu)^{(j)}c^{-1}A(c,n,\nu_j)M(c,n,\nu_j,\tfrac{1}{2})\right).$$

Example 2.4. Consider $f_3(z)=q^{-3}+\sum_{n=1}^{\infty}a(n)q^n$. By using (1.1, 1.2, 1.3), we can show

$$\begin{aligned} z^{-\frac{1}{2}}f_3(-\tfrac{1}{z}) &= \frac{1-i}{2}(26752q^{\frac{1}{4}}+1707264q^{\frac{1}{2}}+\cdots),\\ z^{-\frac{1}{2}}f_3(-\tfrac{1}{z}+\tfrac{1}{2}) &= \frac{1-i}{2}q^{\frac{1}{16}}(q^{-\frac{1}{4}}-248-85995q^{\frac{1}{4}}-4096248q^{\frac{1}{2}}+\cdots). \end{aligned}$$

Here $I_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}\dfrac{\cosh z}{\sqrt{z}}$. If $n\equiv 0,1 \pmod 4$, the main term for $a(n)$ in (2.2) is

$$(-1)^n\frac{2\cosh(\pi\sqrt{3n})}{\sqrt{n}}.$$

If $n=9$, it is $-4096247.99\ldots$. It is very close to the actual value -4096248.

Example 2.5. Consider $f_4(z)=f_0(z)j(4z)-2f_3(z)-746f_0(z)=q^{-4}+\sum_{n=1}^{\infty}a(n)q^n$. In this case, we can show that

$$\begin{aligned} z^{-\frac{1}{2}}f_4(-\tfrac{1}{z}) &= \frac{1-i}{2}(q^{-\frac{1}{4}}+143376q^{\frac{1}{4}}+18473000q^{\frac{1}{2}}+\cdots),\\ z^{-\frac{1}{2}}f_4(-\tfrac{1}{z}+\tfrac{1}{2}) &= \frac{1-i}{2}q^{\frac{1}{16}}(492+565760q^{\frac{1}{4}}+51180012q^{\frac{1}{2}}+\cdots). \end{aligned}$$

Hence if $n\equiv 0,1 \pmod 4$, the main term for $a(n)$ is

$$\frac{2\cosh(\pi\sqrt{4n})}{\sqrt{n}}.$$

If $n=9$, it is $51184311.8\ldots$. It is very close to the actual value 51180012.

We consider $f_{4k}(z)$. Let $J(z)=j(z)-744=q^{-1}+\sum_{n=1}^{\infty}c(n)q^n$. Then

$$f_4(z)=f_0(z)J(4z)-2f_3(z)-2f_0(z),\quad f_8(z)=f_4(z)J(4z)-492f_3(z)-340260f_0(z),$$

and for $k\geq 3$, if $f_{4(k-1)}(z)=q^{-4(k-1)}+a(1)q+a(4)q^4+\cdots$, then

$$f_{4k}(z)=f_{4(k-1)}(z)J(4z)-\sum_{m=1}^{k-2}c(m)f_{4(k-m-1)}(z)-a(1)f_3(z)-(c(k-1)+a(4))f_0(z).$$

Now we can prove the following either by induction or by imitating the proof of Lemma 14.2 in [1].

Lemma 2.6. *Suppose* $f_{4k}(z) = q^{-4k} + \sum_{n=1}^{\infty} a(n)q^n$. *Then*

$$\begin{aligned} z^{-\frac{1}{2}} f_{4k}(-\tfrac{1}{z}) &= \frac{1-i}{2}(q^{-\frac{k}{4}} + \sum_{n=1}^{\infty} a(4n)q^{\frac{n}{4}}), \\ z^{-\frac{1}{2}} f_{4k}(-\tfrac{1}{z} + \tfrac{1}{2}) &= \frac{1-i}{2} q^{\frac{1}{16}} \sum_{n=0}^{\infty} a(4n+1)q^{\frac{n}{4}}. \end{aligned}$$

Proof. Let

$$h_0(z) = q^{-k} + \sum_{n=1}^{\infty} a(4n)q^n, \quad h_1(z) = \sum_{n=0}^{\infty} a(4n+1)q^{n+\frac{1}{4}}.$$

Then $f_{4k}(z) = h_0(4z) + h_1(4z)$. Since $f_{4k} \in \mathcal{M}^+_{\frac{1}{2}}(\Gamma_0(4))$, we have

$$f_{4k}\left(\frac{z}{4z+1}\right) = (4z+1)^{\frac{1}{2}} f(z).$$

By replacing $4z+1$ by z, and noting that $h_0(z \pm 1) = h_0(z)$, $h_1(z+1) = ih_1(z)$, and $h_1(z-1) = -ih_1(z)$, we have

$$h_0(-\tfrac{1}{z}) + ih_1(-\tfrac{1}{z}) = z^{\frac{1}{2}}(h_0(z) - ih_1(z)).$$

Now let $z = iy$ and note that $h_0(iy)$ and $h_1(iy)$ are real. Hence

$$h_0\left(\frac{i}{y}\right) = \frac{\sqrt{y}}{\sqrt{2}}(h_0(iy) + h_1(iy)), \quad h_1\left(\frac{i}{y}\right) = \frac{\sqrt{y}}{\sqrt{2}}(h_0(iy) - h_1(iy)).$$

Since h_0 and h_1 are meromorphic functions, the above equalities are true by replacing iy by z with $Im(z) > 0$. Hence

$$h_0(-\tfrac{1}{z}) = \frac{1-i}{2}\sqrt{z}(h_0(z) + h_1(z)), \quad h_1(-\tfrac{1}{z}) = \frac{1-i}{2}\sqrt{z}(h_0(z) - h_1(z)).$$

Therefore,

$$f_{4k}(-\tfrac{1}{z}) = h_0(-\tfrac{4}{z}) + h_1(-\tfrac{4}{z}) = \frac{1-i}{2}\sqrt{z}h_0(\tfrac{z}{4}).$$

For $f_{4k}(-\frac{1}{z} + \frac{1}{2})$, note that $h_1(z+2) = -h_1(z)$. Then

$$f_{4k}(-\tfrac{1}{z} + \tfrac{1}{2}) = h_0(-\tfrac{4}{z}) - h_1(-\tfrac{4}{z}) = \frac{1-i}{2}\sqrt{z}h_1(\tfrac{z}{4}).$$

□

So $\mu_1 = k, \mu_2 = 0$, and $a(-\nu)^{(1)} = 0$ for $\nu = 1, ..., k-1$. Therefore, the main term of (2.2) is

$$(2.7)\quad 2\pi i^{-\frac{1}{2}} \left(\frac{1+i}{4} \left(\frac{4k}{n}\right)^{\frac{1}{4}} I_{-\frac{1}{2}}(\pi\sqrt{4kn}) + \frac{1+i}{2} \left(\frac{k}{4n}\right)^{\frac{1}{4}} I_{-\frac{1}{2}}\left(4\pi\sqrt{\frac{nk}{4}}\right)\right) = \frac{2\cosh(\pi\sqrt{4kn})}{\sqrt{n}}.$$

We consider the other terms:

$$(2.8)\quad 2\pi i^{-\frac{1}{2}} \left(\sum_{\substack{c\in C_{00} \\ 4<c<\sqrt{n},\, 4|c}} c^{-1} A(c,n,4k) M(c,n,4k,\tfrac{1}{2}) + \frac{1+i}{2} \sum_{\substack{c\in C_{10} \\ 1<c<\sqrt{n}}} c^{-1} A(c,n,\tfrac{k}{4}) M(c,n,\tfrac{k}{4},\tfrac{1}{2}) \right)$$

We will prove that the above sum is smaller than the main term. By the trivial estimate, we obtain $|A(c,n,\nu_j)| \le c$ for any $j = 0,1,2$. Hence

$$\begin{aligned}
|(2.8)| &\le 2\pi \left(\sum_{\substack{c\in C_{00} \\ 4<c<\sqrt{n},\, 4|c}} M(c,n,4k,\tfrac{1}{2}) + \frac{1}{\sqrt{2}} \sum_{\substack{c\in C_{10} \\ 1<c<\sqrt{n}}} M(c,n,\tfrac{k}{4},\tfrac{1}{2}) \right) \\
&\le 2\pi \left(\sum_{\substack{c\in C_{00} \\ 4<c<\sqrt{n},\, 4|c}} \left(\frac{n}{4k}\right)^{-\frac{1}{4}} I_{-\frac{1}{2}}\left(\frac{8\pi\sqrt{nk}}{c}\right) + \frac{1}{\sqrt{2}} \sum_{\substack{c\in C_{10} \\ 1<c<\sqrt{n}}} \left(\frac{4n}{k}\right)^{-\frac{1}{4}} I_{-\frac{1}{2}}\left(\frac{2\pi\sqrt{nk}}{c}\right) \right) \\
&\le 2\pi\sqrt{n} \left(\frac{k}{n}\right)^{\frac{1}{4}} I_{-\frac{1}{2}}(\pi\sqrt{nk}) = 2\sqrt{2}\cosh(\pi\sqrt{nk}).
\end{aligned}$$

It is clearly smaller than (2.7).

Let $f_{4k-1}(z) = q^{-4k+1} + \sum_{n=1}^{\infty} a(n)q^n$. In the same way as in Lemma 2.6, let

$$h_0(z) = \sum_{n=1}^{\infty} a(4n)q^n, \quad h_1(z) = q^{-k+\frac{1}{4}} + \sum_{n=0}^{\infty} a(4n+1)q^{n+\frac{1}{4}}.$$

We can prove

$$\begin{aligned}
z^{-\frac{1}{2}} f_{4k-1}(-\tfrac{1}{z}) &= \frac{1-i}{2} \sum_{n=1}^{\infty} a(4n) q^{\frac{n}{4}}, \\
z^{-\frac{1}{2}} f_{4k-1}(-\tfrac{1}{z} + \tfrac{1}{2}) &= \frac{1-i}{2} q^{\frac{1}{16}} (q^{-\frac{k}{4}} + \sum_{n=0}^{\infty} a(4n+1) q^{\frac{n}{4}}).
\end{aligned}$$

Therefore, the main term of (2.2) is

$$2\pi i^{-\frac{1}{2}}\left((-1)^n\frac{1+i}{4}\left(\frac{4k-1}{n}\right)^{\frac{1}{4}}I_{-\frac{1}{2}}(\pi\sqrt{(4k-1)n})+(-1)^n\frac{1+i}{2}\left(\frac{k-\frac{1}{4}}{4n}\right)^{\frac{1}{4}}I_{-\frac{1}{2}}\left(2\pi\sqrt{n(k-\frac{1}{4})}\right)\right)$$
$$=(-1)^n\frac{2\cosh(\pi\sqrt{(4k-1)n})}{\sqrt{n}}.$$

Now we are left to deal with the error term (2.3). Unfortunately, we cannot make the implied constant explicit. It is expected since $f_{4k}(z)$ and $f_{4k}(z)+Nf_0(z)$ have the same main term in (2.2) for any N. So we can only conclude that the main term is bigger than the error term for $n \geq n_0$ for some n_0. This concludes the proof of Theorem 0.2.

Remark 2.9. We can apply the same technique to $j_m(z)$ in [8], page 23. It is defined as $j_0(z)=1, \quad j_1(z)=j(z)-744$, and for $m\geq 2$,

$$j_m(z)=j_1(z)|T_0(m)=\sum_{\substack{d|m\\ ad=m}}\sum_{b=0}^{d-1}j_1\left(\frac{az+b}{d}\right).$$

It has the q-expansion

$$j_m(z)=q^{-m}+\sum_{n=1}^{\infty}c_m(n)q^n.$$

From the definition, it is clear that $c_m(n)$ are all positive integers. We have the following series expression for $c_m(n)$ ([6], page 314):

$$c_m(n)=2\pi\sum_{k=1}^{\infty}\frac{A(k,n,m)}{k}\left(\frac{m}{n}\right)^{\frac{1}{2}}I_1\left(\frac{4\pi\sqrt{mn}}{k}\right), \quad A(k,n,m)=\sum_{\substack{h(\text{mod } k),\,(h,k)=1\\ hh'\equiv -1(\text{mod } k)}}e\left(-\frac{nh+mh'}{k}\right).$$

Hence we obtain $c_m(n)\sim\dfrac{m^{\frac{1}{4}}e^{4\pi\sqrt{mn}}}{\sqrt{2}n^{\frac{3}{4}}}$.

Remark 2.10. For each positive integer $D\equiv 0,1(\text{mod } 4)$, let $g_D(z)\in\mathcal{M}^{+}_{\frac{3}{2}}(\Gamma_0(4))$ be the unique modular form with a Fourier expansion of the form ([8], page 72)

$$g_D(z)=q^{-D}+\sum_{\substack{d\geq 0\\ d\equiv 0,3\,(\text{mod } 4)}}B(D,d)q^d.$$

Zagier [11] proved that $B(D,d)=-A(D,d)$, where $f_d(z)=q^{-d}+\sum_{D>0}A(D,d)q^D$. Using our method, we can prove that if $4|D$, the coefficient $B(D,d)$ is a negative integer for all sufficiently large d. Since $B(D,d)=-A(D,d)$, this shows that given D with $4|D$, $A(D,d)$ is positive for all sufficiently large d.

More precisely, we can get an asymptotic expression for $B(D,d)$. For simplicity, let $g_{4k}(z)=q^{-4k}+\sum_{n=0}^{\infty}a(n)q^n$. In this case, $\kappa_1=0, \kappa_2=\frac{3}{4}$. So it has a Fourier expansion at the other cusps:

$$g_{4k}(-\tfrac{1}{z})=iz^{\frac{3}{2}}\sum_{n=-\mu_1}^{\infty}a(n)^{(1)}q^{\frac{n}{4}},\quad g_{4k}(-\tfrac{1}{z}+\tfrac{1}{2})=iz^{\frac{3}{2}}q^{\frac{3}{16}}\sum_{n=-\mu_2}^{\infty}a(n)^{(2)}q^{\frac{n}{4}}.$$

Let

$$h_0(z)=q^{-k}+\sum_{n=0}^{\infty}a(4n)q^n,\quad h_1(z)=\sum_{n=0}^{\infty}a(4n+3)q^{n+\frac{3}{4}}.$$

Then $g_{4k}(z)=h_0(4z)+h_1(4z)$. In the same way as for f_{4k}, we can show

$$g_{4k}(-\tfrac{1}{z})=\frac{1+i}{8}z^{\frac{3}{2}}h_0(\tfrac{z}{4}),\quad g_{4k}(-\tfrac{1}{z}+\tfrac{1}{2})=\frac{1+i}{8}z^{\frac{3}{2}}h_1(\tfrac{z}{4}).$$

Here in (2.2), $\mu_0=4k$, and $A(4,n,\mu_0)=1-i$. Hence

$$a(n)\sim 2\pi i^{-\frac{3}{2}}\left(\frac{1-i}{4\sqrt{2}}\left(\frac{n}{k}\right)^{\frac{1}{4}}I_{\frac{1}{2}}(2\pi\sqrt{nk})+\frac{1-i}{8}\sqrt{2}\left(\frac{n}{k}\right)^{\frac{1}{4}}I_{\frac{1}{2}}(2\pi\sqrt{nk})\right)=-\pi\left(\frac{n}{k}\right)^{\frac{1}{4}}I_{\frac{1}{2}}(2\pi\sqrt{nk}).$$

Since $I_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi}}\frac{\sinh(z)}{\sqrt{z}}$, we obtain

$$a(n)\sim-\frac{2\sinh(\pi\sqrt{4kn})}{\sqrt{4k}}.$$

In the same way, we can show that if $g_{4k+1}(z)=q^{-4k-1}+\sum_{n=0}^{\infty}a(n)q^n$,

$$a(n)\sim(-1)^{n-1}\frac{2\sinh(\pi\sqrt{(4k+1)n})}{\sqrt{4k+1}}.$$

For example, when $k=2, n=7$, we have $a(n)\sim 22505067826.5\ldots$. The actual value is 22505066244.

Remark 2.11. For each positive integer $d\equiv 0,1 \pmod 4$, consider $v_d\in\mathcal{M}^{+}_{-\frac{1}{2}}(\Gamma_0(4))$ which is the unique modular form with a Fourier expansion of the form ([11], page 19)

$$v_d(z)=q^{-d}+\sum_{\substack{n\geq 0\\ n\equiv 0,3 \pmod 4}}a(n)q^n.$$

If $f\in\mathcal{M}^{+}_{-\frac{1}{2}}(\Gamma_0(4))$, $f(\frac{az+b}{cz+d})=(\frac{c}{d})^{-1}\epsilon_d(cz+d)^{-\frac{1}{2}}f(z)$. In this case, $\kappa_1=0,\kappa_2=\frac{3}{4}$. So this case is similar to g_D in the above remark. Let $v_{4k}(z)=q^{-4k}+\sum_{n=0}^{\infty}a(n)q^n$. In this case, we have a series representation of $a(n)$ without the error term ([6], page 313):

$$a(n) = 2\pi i^{\frac{1}{2}} \sum_{j=0}^{2} \sum_{\nu=1}^{\mu_j} a(-\nu)^{(j)} \sum_{0<c\in C_{j0}} c^{-1} A(c,n,\nu_j) M(c,n,\nu_j,r).$$

In this case, we can show that $a(n)$ is positive for all n, and

$$a(n) \sim 4\pi \left(\frac{k}{n}\right)^{\frac{3}{4}} I_{-\frac{3}{2}}(2\pi\sqrt{nk}).$$

Since $I_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi}}\,\frac{z\sinh(z)-\cosh(z)}{z^{\frac{3}{2}}}$, we have

$$a(n) \sim \frac{2\pi\sqrt{4kn}\sinh(\pi\sqrt{4kn}) - 2\cosh(\pi\sqrt{4kn})}{\pi n^{\frac{3}{2}}}.$$

Similarly, if $v_{4k+1}(z) = q^{-4k-1} + \sum_{n=0}^{\infty} a(n)q^n$,

$$a(n) \sim (-1)^n \frac{2\pi\sqrt{(4k+1)n}\,\sinh(\pi\sqrt{(4k+1)n}) - 2\cosh(\pi\sqrt{(4k+1)n})}{\pi n^{\frac{3}{2}}}.$$

For example, when $k=2, n=7$, we obtain $a(n) \sim -27774695413.6\ldots$. The actual value is -27774693612.

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