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Quantization of Virtual Grothendieck Rings and Their Structure Including Quantum Cluster Algebras

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Abstract: The quantum Grothendieck ring of a certain category of finite-dimensional modules over a quantum loop algebra associated with a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ has a quantum cluster algebra structure of skew-symmetric type. Partly motivated by a search of a ring corresponding to a quantum cluster algebra of *skew-symmetrizable* type, the quantum *virtual* Grothendieck ring, denoted by $\mathfrak{K}_q(\mathfrak{g})$, is recently introduced by Kashiwara and Oh (Math Z 303(2):42, 2023) as a subring of the quantum torus based on the (q,t)-Cartan matrix specialized at $q = 1$. In this paper, we prove that $\mathfrak{K}_q(\mathfrak{g})$ indeed has a quantum cluster algebra structure of skew-symmetrizable type. This task essentially involves constructing distinguished bases of $\mathfrak{K}_q(\mathfrak{g})$ that will be used to make cluster variables and generalizing the quantum T-system associated with Kirillov–Reshetikhin modules to establish a quantum exchange relation of cluster variables. Furthermore, these distinguished bases naturally fit into the paradigm of Kazhdan–Lusztig theory and our study of these bases leads to some conjectures on quantum positivity and q-commutativity.

Contents

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1. Introduction

1.1. Background. Let $\mathsf{C} = (\mathsf{c}_{i,j})_{i,j \in I}$ be a Cartan matrix of finite type, and let $\mathfrak{g}$ be the finite-dimensional simple Lie algebra over $\mathbb{C}$ associated with C, where I is the index set of the simple roots of $\mathfrak{g}$. Since its inception as trigonometric solutions to the quantum Yang–Baxter equation [11,34], the quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ of $\mathfrak{g}$ has been one of the central objects in representation theory and mathematical physics, and various algebraic and geometric approaches have been taken to study the finite-dimensional modules over $U_q(\mathcal{L}\mathfrak{g})$. Moreover, for the last 15 years or so, as categorification became one of the major trends in representation theory and cluster algebra structures were discovered ubiquitously, the category $\mathscr{C}_\mathfrak{g}$ of finite-dimensional $U_q(\mathcal{L}\mathfrak{g})$-modules became a focal point of research where these new ideas and methods could be applied fruitfully, since the quantum Grothendieck ring of $\mathscr{C}_\mathfrak{g}$ provides a categorification of a cluster algebra and generalizes the Kazhdan–Lusztig(KL) theory.

To be more precise, the quantum cluster algebra $\mathcal{A}$, introduced by Berenstein–Fomin–Zelevinsky (BFZ) in [4,12], is a non-commutative $\mathbb{Z}[q^{\pm 1/2}]$-algebra contained in the quantum torus $\mathbb{Z}[\widetilde{X}_k^{\pm 1} | k \in \mathsf{K}]$ which is equipped with a distinguished set of generators (*quantum cluster variables*) grouped into subsets (*quantum clusters*), where K is an index set. Each cluster is defined inductively by a sequence of certain combinatorial algebraic operations (*mutations*) from an initial cluster. Since then, numerous connections and applications have been discovered in various fields of mathematics.

It is well-known that the quantum cluster algebra was introduced in an attempt to create an algebraic framework for the dual-canonical/upper-global basis $\mathbf{B}^*$ [40,41,54] of the quantum group $U_q(\mathfrak{g})$. Indeed, it is shown in [22,23] that the unipotent quantum coordinate algebra $A_q(\mathfrak{n})$ of $U_q(\mathfrak{g})$, which is the graded dual of the half of $U_q(\mathfrak{g})$, has a quantum cluster algebra structure, and intensive research has been performed to understand the structure in relation with $\mathbf{B}^*$ (see [42] for a survey). In these efforts, it turned out that categorification provides powerful methods [39,51,52,64].

When $\mathfrak{g}$ is of simply-laced type with its set of positive roots denoted by $\Phi^+_\mathfrak{g}$, we can consider the path algebra $\mathbb{C}Q$ of the Dynkin quiver Q associated with $\mathfrak{g}$ and obtain the Auslander–Reiten (AR) quiver Γ_Q of $\mathbb{C}Q$. In turn, Γ_Q can be understood as a heart of the AR-quiver $\widehat{\triangle}$ of the derived category $D^b(\mathrm{Rep}(\mathbb{C}Q))$, called the repetition quiver. In [29], which culminates preceding works [25,28,54,60,63,66,67], Hernandez and Leclerc defined the heart subcategory $\mathscr{C}^Q_\mathfrak{g}$ of $\mathscr{C}_\mathfrak{g}$ by using Γ_Q, and proved that the *quantum Grothendieck ring* $\mathcal{K}_\mathfrak{t}(\mathscr{C}^Q_\mathfrak{g})$ of $\mathscr{C}^Q_\mathfrak{g}$ is isomorphic to the integral form $A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathfrak{n})$ of $A_q(\mathfrak{n})$ and that the isomorphism sends the basis of $\mathcal{K}_\mathfrak{t}(\mathscr{C}^Q_\mathfrak{g})$ consisting of the elements corresponding to simple objects in $\mathscr{C}^Q_\mathfrak{g}$ to $\mathbf{B}^*$ of $A_q(\mathfrak{n})$ (cf. [61]).

To extend the results of [29,30] to non-simply-laced types, the Q-datum $\mathscr{Q}$ is introduced in [21] as a generalization of the Dynkin quivers of types ADE. Through the Q-datum for any finite type, the (combinatorial) AR-quiver $\Gamma_\mathscr{Q}$, the repetition quiver $\widehat{\triangle}^\sigma$, and the heart subcategory $\mathscr{C}^\mathscr{Q}_\mathfrak{g}$ of $\mathscr{C}_\mathfrak{g}$ are naturally defined, where σ is the Dynkin diagram automorphism of simply-laced $\mathbf{g}$ whose orbits produce the Dynkin diagram of $\mathfrak{g}$. One could possibly expect that $\mathcal{K}_\mathfrak{t}(\mathscr{C}^\mathscr{Q}_\mathfrak{g})$ would be isomorphic to $A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathfrak{n})$ of $U_q(\mathfrak{g})$ when $\mathfrak{g}$ is of non-simply-laced type, generalizing the result in types ADE to all types. However, further studies [18,31,46,62] show that the quantum Grothendieck ring $\mathcal{K}_\mathfrak{t}(\mathscr{C}^\mathscr{Q}_\mathfrak{g})$ is actually isomorphic to $A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathbf{n})$ of $U_q(\mathbf{g})$ associated with $\mathbf{g}$ of simply-laced type. Hence the structure of $\mathcal{K}_\mathfrak{t}(\mathscr{C}^\mathscr{Q}_\mathfrak{g})$ is intrinsically relevant to the counterpart of simply-laced type, and the quantum cluster algebra structure associated with $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^{\mathscr{Q}})$ is still of skew-symmetric type.

1.2. Overview of this paper. Since there are quantum cluster algebras of skew-symmetrizable type, a natural question arises:

Can we extend $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^{Q})$ (or $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}})$)in such a way to have a quantum cluster algebra structure of skew-symmetrizable type?

Partly motivated by this question, Kashiwara and Oh introduced the *quantum virtual Grothendieck ring* $\mathfrak{K}_q(\mathfrak{g})$ inside the quantum torus $\mathcal{X}_q(\mathfrak{g}):=\mathbb{Z}[X_{i,p}^{\pm 1} \mid (i,p) \in \widehat{\triangle}_0^{\mathfrak{g}}]$, where $\widehat{\triangle}_0^{\mathfrak{g}}$ is the set of vertices (i, p) $(i \in I,\ p \in \mathbb{Z})$ of the repetition quiver $\widehat{\triangle}^{\mathfrak{g}}$ with valued arrows induced from the (q,t)-Cartan matrix specialized at $q = 1$ ([47], see also Sect. 2.4). Pursuing the direction further, in this paper, we prove that $\mathfrak{K}_q(\mathfrak{g})$ indeed has a quantum cluster algebra structure of skew-symmetrizable type. In a subsequent paper, our result will be utilized to fully answer the above question and to genuinely extend the results of [29] in the sense that $A_q(\mathfrak{n})$ is involved even for $\mathfrak{g}$ of non-simply-laced type. We remark that the evaluation of $\mathfrak{K}_q(\mathfrak{g})$ at $q = 1$ coincides with the *folded t-character ring* (Remark 4.12), denoted by $\overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathfrak{g})$ (see (1.2) below), which is introduced by Frenkel–Hernandez–Reshetikhin in [14] to explore a (conjectural) quantum integrable model corresponding to what is called the folded Bethe Ansatz equation (see Remark 3.13).

Though we do not yet have an actual category that will replace $\mathscr{C}_{\mathfrak{g}}$ for our purpose (cf. [14, Remark 3.2, Remark 5.1]), we can still utilize an algebraic characterization of $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^{0})$ as the intersection of the kernels of screening operators in $\mathcal{Y}_{\mathtt{t}}(\mathfrak{g})$, where $\mathscr{C}_{\mathfrak{g}}^{0}$ is the skeleton subcategory of $\mathscr{C}_{\mathfrak{g}}$ and $\mathcal{Y}_{\mathtt{t}}(\mathfrak{g})$ is the quantum torus with respect to the (q,t)-Cartan matrix specialized at $t = 1$.

In order to give a quantum cluster algebra structure on $\mathfrak{K}_q(\mathfrak{g})$ in this paper, we need to construct quantum cluster variables and exchange relations for mutations. The former requires constructing distinguished bases for $\mathfrak{K}_q(\mathfrak{g})$ and the latter amounts to generalizing the quantum T-system associated with Kirillov–Reshetikhin (KR) modules as explained briefly below.

We establish three bases of $\mathfrak{K}_q(\mathfrak{g})$, denoted by F_q, E_q, and L_q respectively. The basis F_q is constructed by a generalization of Frenkel–Mukhin (FM) algorithm [15], which plays a crucial role in studying $\mathfrak{K}_q(\mathfrak{g})$. Furthermore, it induces two other important bases E_q and L_q of $\mathfrak{K}_q(\mathfrak{g})$. For $i \in I$, let $m^{(i)}[p,s] := X_{i,p}X_{i,p+2}\cdots X_{i,s}$ (see (3.2) for the notation). Then we denote by $F_q(m^{(i)}[p,s])$ the element in F_q corresponding to $m^{(i)}[p,s]$, and call it the *KR-polynomial*. Taking a q-commuting family consisting of these KR-polynomials as the quantum cluster of initial seed, we develop a *quantum folded T-system* to serve as the set of quantum exchange relations. After making compatible pairs available for our use (cf. [47]), we establish a quantum cluster algebra structure on a subalgebra and extend it to $\mathfrak{K}_q(\mathfrak{g})$.

It is worthwhile to remark that when $\mathfrak{g}$ is simply-laced, the basis L_q (resp. E_q) comes from simple (resp. standard) modules in $\mathscr{C}_{\mathfrak{g}}^{0}$, and the entries of the transition matrix between L_q and E_q are understood as analogues of the KL-polynomials. Thus our construction of L_q and E_q for all the finite types extends the KL-theory for $\mathscr{C}_{\mathfrak{g}}^{0}$. Moreover, we have conjectures related to positivity on KR-polynomials in F_q and *real* elements in L_q, and to BFZ-expectation that every quantum cluster monomial is an element in the canonical basis (see Conjecture I below).

Throughout this paper, the interplay between $\mathfrak{g}$ and its simply-laced type counterpart $\mathbf{g}$ and the Dynkin diagram automorphism σ (cf. (2.3) and (2.4)) provides important viewpoints leading to natural definitions. However, we emphasize that none of our main constructions, including bases F_q, E_q, and L_q, is obtained merely from combining objects in each orbit of σ. That is, none of our results is a consequence of simple folding. Rather, there seem to exist quite intriguing features of non-simply-laced type objects at the quantum level.

In the following subsections, we will review known results in Sects. 1.3 and 1.4 with some details, and present our results more rigorously in Sect. 1.5, and mention our future work in Sect. 1.6.

1.3. Quantum Grothendieck ring and quantum loop analogue of KL-theory. From the study for q-deformation of $\mathcal{W}$-algebras, the q-character[1] theory for $\mathscr{C}^0_{\mathfrak{g}}$ was invented by Frenkel–Reshetikhin [17] and further developed by Frenkel–Mukhin [15], which says that the (non-quantum) Grothendieck ring $K(\mathscr{C}^0_{\mathfrak{g}})$ of $\mathscr{C}^0_{\mathfrak{g}}$ is isomorphic to the commutative ring generated by the q-characters of fundamental modules $L(Y_{i,p})$ under the Chari–Pressley's classification [7,8]. For simply-laced type $\mathbf{g}$, Nakajima [60] and Varagnolo–Vasserot [67] constructed a non-commutative t-deformation of $K(\mathscr{C}^0_{\mathbf{g}})$ in a quantum torus $\mathcal{Y}_{\mathtt{t}}(\mathbf{g})$, denoted by $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$, based on a geometrical point of view. Since the specialization of $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$ at $\mathtt{t}=1$ recovers $K(\mathscr{C}^0_{\mathbf{g}})$, we call $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$ the *quantum Grothendieck ring* associated with $\mathscr{C}^0_{\mathbf{g}}$.

In particular, Nakajima established a KL-type algorithm to describe the composition multiplicity $P_{m,m'}$ of a simple module $L(m')$ inside a standard module $E(m)$ through equations in $K(\mathscr{C}^0_{\mathbf{g}})$: Denoting by $\mathcal{M}_+$ the parameterizing set of simple modules in $\mathscr{C}^0_{\mathbf{g}}$, we have

$$[E(m)] = [L(m)] + \sum_{m' \in \mathcal{M}_+;\ m' \prec_{\mathrm{N}} m} P_{m,m'}\,[L(m')].$$

It is proved by Nakajima [59,60] that the multiplicity $P_{m,m'}$ is equal to the specialization at $t=1$ of a polynomial $P_{m,m'}(t)$ with non-negative coefficients, which can be understood as a quantum loop analogue of KL-polynomial.

One step further, each q-character of simple module $L(m)$ (resp. standard module $E(m)$) allows a t-deformation in $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$, denoted by $L_t(m)$ (resp. $E_t(m)$), whose coefficients in $\mathbb{Z}[t^{\pm 1/2}]$ are non-negative. Its specialization at $t=1$ recovers the q-character of $L(m)$ (resp. $E(m)$) and the transition map between $\mathbf{L}_t=\{L_t(m)\}$ and $\mathbf{E}_t=\{E_t(m)\}$ in $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$ satisfies the following equation:

$$E_t(m) = L_t(m) + \sum_{m' \in \mathcal{M}_+;\, m' \prec_{\mathrm{N}} m} P_{m,m'}(t)\,L_t(m') \quad \text{where } P_{m,m'}(t) \in t\mathbb{Z}_{\geqslant 0}[t]. \tag{1.1}$$

We call $\mathbf{L}_t$ the *canonical basis* and $\mathbf{E}_t$ the *standard basis* of $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$, respectively (see Remark 5.10 also). In what follows, *positivity* generally means that polynomials of interest have non-negative coefficients as is the case with $P_{m,m'}(t) \in t\mathbb{Z}_{\geqslant 0}[t]$. We remark that, in these developments, the geometry of quiver varieties plays an essential role.

[1] In the main body of this paper, we sometimes call it *t-character* by replacing the role of q by t.

Despite the absence of fully developed theory of quiver varieties for general type $\mathfrak{g}$, Hernandez [24,25] constructed a conjectural KL-theory for $\mathscr{C}_{\mathfrak{g}}^0$ in a purely algebraic way. Let us explain this more precisely. Using the *quantum Cartan matrix* $\mathsf{C}(q)$, Hernandez constructed the quantum torus $\mathcal{Y}_{\mathtt{t}}(\mathfrak{g})$ and defined $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$ to be the intersection of the kernels of the t-deformed screening operators $S_{i,\mathtt{t}}$'s on $\mathcal{Y}_{\mathtt{t}}(\mathfrak{g})$. Then he constructed a basis $\mathbf{F}_t = \{F_t(m)\}$ by deforming the FM-algorithm and proved the positivity of $F_t(Y_{i,p}) = L_t(Y_{i,p})$. Then the basis $\mathbf{F}_t$ induces two other bases $\mathbf{E}_t = \{E_t(m)\}$ and $\mathbf{L}_t = \{L_t(m)\}$ satisfying (1.1) that enable us to establish a conjectural KL-theory, expecting the positivity of analogues of KL-polynomials and $L_t(m)$'s.

Recently, large parts of the conjectures for non-simply-laced g are proved by Fujita–Hernandez–Oh–Oya through so-called *propagation of positivity*. Let $\mathbf{g}$ be an unfolding of g as follows:

$$(g, \mathbf{g}) = (B_n, A_{2n-1}),\ \ (C_n, D_{n+1}),\ \ (F_4, E_6),\ \ (G_2, D_4).$$

Then it is proved in [18,29] that

$$\mathbb{K}_{\mathtt{t}}(\mathscr{C}_{\mathbf{g}}^0) \text{ and } \mathbb{K}_{\mathtt{t}}(\mathscr{C}_{g}^0) \text{ have the same presentation,}$$

where $\mathbb{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$. Hence the ring $\mathbb{K}_{\mathtt{t}}(\mathscr{C}_{g}^0)$ can be interpreted as the *boson-extension* of $A_q(\mathbf{n})$ of the simply-laced $\mathbf{g}$. Then the KL-theory and positivity are established for type B_n using the quantum Schur–Weyl duality functor [37,43] between $\mathscr{C}_{A_{2n-1}}^0$ and $\mathscr{C}_{B_n}^0$, and similar conjectures for CFG-types are mostly resolved in [18,19] using the quantum Schur–Weyl duality functor [36,46,62] for these types and the *degrees* (also called *g-vectors*) of (quantum) cluster algebra theory. As indicated above, the presentation of $\mathbb{K}_{\mathtt{t}}(\mathscr{C}_{g}^0)$ is of simply-laced type even for non-simply-laced g.

1.4. Quantum cluster algebra structure of skew-symmetric type on $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$. In the seminal paper [30], Hernandez–Leclerc proved that $K(\mathscr{C}_{\mathfrak{g}}^-)$ for a subcategory $\mathscr{C}_{\mathfrak{g}}^-$ of $\mathscr{C}_{\mathfrak{g}}^0$ has a cluster algebra structure of skew-symmetric type for any $\mathfrak{g}$ of finite type. To show the cluster algebra structure, they employed the T-system among Kirillov–Reshetikhin (KR) modules proved by Nakajima [59] for simply-laced types and by Hernandez [27] for non-simply-laced types. Then the result of [30] is extended to $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$ in [5,18,19,31,44,45] to obtain quantum cluster algebras of skew-symmetric type. Some important features of these works can be summarized as follows:

(a) The extension to whole category $\mathscr{C}_{\mathfrak{g}}^0$ in [44,45] involves a categorical language.
(b) The main idea of the extension to quantum cluster algebra in [5,18,31] is the quantization of T-system among KR modules.
(c) The monoidal categorification result in [45] tells us that every quantum cluster monomial of $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$ corresponds to an element of $\mathbf{L}_t$. This gives an affirmative answer to the BFZ-conjecture [12] on $\mathbf{B}^*$ and the quantum cluster monomials.
(d) As every KR-polynomial $F_t(m)$ appears as a quantum cluster variable of $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$, it is proved in [19,45] that $F_t(m) = L_t(m)$ for any KR-module $L(m)$.

Here we remark that the result of [45] is for $K(\mathscr{C}_{\mathfrak{g}}^0)$ and extended to $\mathcal{K}_{\mathtt{t}}(\mathscr{C}_{\mathfrak{g}}^0)$ in [19].

1.5. Main results of this paper. In this paper, we initiate a study of $\mathfrak{K}_q(\mathfrak{g})$ in the perspective of Sects. 1.3 and 1.4. Due to lack of a representation theory corresponding to $\mathfrak{K}_q(\mathfrak{g})$, we approach the ring $\mathfrak{K}_q(\mathfrak{g})$ by analyzing its construction in [47] and by exploiting (I) and (II), where

$$\begin{aligned}&\text{(I) } \mathfrak{K}_q(\mathfrak{g}) \text{ is a } q\text{-deformation of the commutative ring } \overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathfrak{g}), \text{ which is the specialization}\\ &\quad \text{of the refined ring } \overline{\mathcal{K}}_{\mathtt{q},\mathtt{t},\alpha}(\mathfrak{g}) \text{ of interpolating } (\mathtt{q},\mathtt{t})\text{-characters in [14] at } (\mathtt{q},\alpha)=(1,d),\\ &\text{(II) } \overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathbf{g}) \simeq K(\mathscr{C}^0_{\mathbf{g}}) \text{ if } \mathbf{g} \text{ is of simply-laced type,}\end{aligned} \tag{1.2}$$

(see Sect. 3.4 and [47, Introduction]). Here α is a factor to interpolate several characters (see [14, Remark 6.2(1)]) and d is the lacing number of $\mathfrak{g}$. In particular, if g is of non-simply-laced type, there exist a simply-laced $\mathbf{g}$ containing g as a non-trivial Lie subalgebra (e.g. see [35, Proposition 7.9] with (2.4)) and a surjective homomorphism

$$\overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathbf{g}) \twoheadrightarrow \overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathsf{g}) \simeq \mathfrak{K}(\mathsf{g}), \tag{1.3}$$

which is induced from the folding of generators of $\overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathbf{g}) \simeq K(\mathscr{C}^0_{\mathbf{g}})$.

The main results of this paper can be summarized into two statements:

(A) we construct bases F_q, E_q, and L_q of $\mathfrak{K}_q(\mathfrak{g})$, which play similar roles of $\mathbf{F}_t$, $\mathbf{E}_t$, and $\mathbf{L}_t$,

(B) we establish *skew-symmetrizable* quantum cluster algebra structures on subrings of $\mathfrak{K}_q(\mathfrak{g})$ (including itself) using the bases in (A).

Here we emphasize that our results can *not* be obtained from the folding in (1.3), as we do *not* have a surjective homomorphism $A_q(\mathbf{n}) \twoheadrightarrow A_q(\mathsf{n})$ from the canonical surjection $\mathbb{C}[\mathbf{N}] \twoheadrightarrow \mathbb{C}[\mathsf{N}]$, where $\mathbb{C}[N]$ denotes the unipotent coordinate ring of N of $\mathfrak{g}$.

1.5.1. Construction of bases and KL-paradigm for $\mathfrak{K}_q(\mathfrak{g})$ Let $\underline{\mathsf{C}}(t)$ be the (q,t)-Cartan matrix specialized at $q=1$, which is called *t-quantized Cartan matrix*. To construct the basis F_q of $\mathfrak{K}_q(\mathfrak{g})$, we apply a q-deformed version of FM-algorithm with respect to $\underline{\mathsf{C}}(t)$. However, there is no guarantee that the algorithm terminates in finite steps. To avoid this problem, we prove that the monomials (not including coefficients) of $F_q(X_{i,p})$ $((i,p) \in \widehat{\triangle}^{\mathsf{g}}_0)$ in F_q are obtained from those of the q-character of $L(Y_{\imath,p})$ of type $\mathbf{g}$ via (1.3) for $(\imath,p) \in \widehat{\triangle}^{\mathbf{g}}_0$. Furthermore, we prove that a similar phenomenon occurs for a KR-polynomial $F_q(m^{(i)}[p,s])$ (Proposition 5.20). This result implies that the outputs of the algorithm are indeed contained in $\mathfrak{K}_q(\mathfrak{g})$ and form a basis F_q. The basis F_q nicely characterizes an element in $\mathfrak{K}_q(\mathfrak{g})$ since each element in F_q has a unique dominant monomial (Theorem 5.27). Here we emphasize once more that general elements in F_q are not susceptible of similar manipulations based on (1.3) even in the specialization at $q=1$ (Example 3.11), and determining the $\mathbb{Z}[q^{\pm 1/2}]$-coefficients of $F_q(m^{(i)}[p,s])$ is a completely different problem even for a KR-polynomial $F_q(m^{(i)}[p,s])$.

We investigate properties of the KR-polynomials in F_q in detail, since they will be used as the quantum cluster variables of $\mathfrak{K}_q(\mathfrak{g})$ (Propositions 5.23 and 5.29). By applying the framework in [25], we construct the standard basis $\mathsf{E}_q = \{E_q(m)\}$ and the canonical basis $\mathsf{L}_q = \{L_q(m)\}$ fitting into the paradigm of Kazhdan–Lusztig theory:

$$E_q(m) = L_q(m) + \sum_{m' \in \mathcal{M};\, m' \prec_{\mathrm{N}} m} P_{m,m'}(q)\, L_q(m') \quad \text{where } P_{m,m'}(q) \in q\mathbb{Z}[q].$$

1.5.2. Quantum cluster algebra structure of skew-symmetrizable type on $\mathfrak{K}_q(\mathfrak{g})$ Based on the construction of bases for $\mathfrak{K}_q(\mathfrak{g})$, we show quantum cluster algebra structures on subrings of $\mathfrak{K}_q(\mathfrak{g})$ as the first task in the second part of this paper.

In [47], Kashiwara and Oh constructed a compatible pair $(\Lambda, \widetilde{B})$ arising from the isomorphism between the subtorus $\mathcal{X}_{q,Q}(\mathfrak{g})$ of $\mathcal{X}_q(\mathfrak{g})$ and the torus containing $A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathfrak{n})$, in which the exchange matrix $\widetilde{B}$ is skew-symmetrizable. Here $Q = (\triangle, \xi)$ is a Dynkin quiver of type $\mathfrak{g}$. Interpreting entries in Λ as pairing of KR-monomials (Theorem 8.1), we form an initial quantum cluster consisting of certain KR-polynomials $F_q(m)$ for each Dynkin quiver $Q = (\triangle, \xi)$ and its corresponding subring $\mathfrak{K}_{q,\xi}(\mathfrak{g})$.

As a quantum cluster should consist of mutually q-commutative elements, we prove that the family of $F_q(m)$ in the initial cluster are mutually q-commutative, using the truncation homomorphism (Proposition 6.3) and the properties of KR-polynomials. By investigating q-commuting conditions (Lemmas 6.6, 6.7, and 6.8) and multiplicative structure among KR-polynomials $F_q(m)$, we obtain *the quantum folded T-systems* among KR-polynomials $F_q(m)$ (Theorem 6.9):

$$F_q\big(\underline{m}^{(i)}[p,s)\big) * F_q\big(\underline{m}^{(i)}(p,s]\big) = q^{\alpha(i,k)} F_q\big(\underline{m}^{(i)}(p,s)\big) * F_q\big(\underline{m}^{(i)}[p,s]\big) + q^{\gamma(i,k)} \prod_{j;\ d(i,j)=1} F_q\big(\underline{m}^{(j)}(p,s)\big)^{-\mathsf{c}_{j,i}}.$$

Then we prove that $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ has a quantum cluster algebra structure of skew-symmetrizable type (Theorem 8.9) by using the quantum folded T-systems as mutation relations and applying special sequences of mutations. In the proof, we adopt the setup of [5,30] and use the *valued quivers* (Sect. 2.4) (equivalent to exchange matrices) for the sequences of mutations. As applications, we obtain a quantum cluster algorithm to compute KR-polynomials $F_q(m)$ (Proposition 8.6) and a sufficient condition for q-commutativity of certain pairs of KR-polynomials $F_q(m)$ (Theorem 8.10).

As the second task, we extend the result on $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ to the whole ring $\mathfrak{K}_q(\mathfrak{g})$. For this purpose, we construct a new quantum seed, whose valued quiver is a “sink-source" quiver reflecting features of $\mathfrak{g}$ and whose initial quantum cluster consists of certain KR-polynomials $F_q(m)$. Here the q-commutativity of the initial quantum cluster follows from Theorem 8.10. Finally, we prove that $\mathfrak{K}_q(\mathfrak{g})$ has a quantum cluster algebra structure of skew-symmetrizable type by establishing (a) a mutation equivalence between the valued quiver of $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ and that of $\mathfrak{K}_q(\mathfrak{g})$, and finding out (b) special sequences of mutations that yield every KR-polynomial $F_q(m)$ as a cluster variable.

Since every KR-polynomial $F_q(m)$ appears as a cluster variable and every quantum cluster monomial is expected to be a canonical basis element and *real*, we have the following conjecture:

Conjecture I. (a) *Every quantum cluster monomial of $\mathfrak{K}_q(\mathfrak{g})$ is contained in L_q.*

(b) *For every KR-polynomial $F_q(m)$, we have $F_q(m) = L_q(m)$ and $F_q(m)$ has non-negative coefficients.*

(c) *If $L_q(m)$ is real, that is, for any $k \in \mathbb{Z}_{\geqslant 1}$, there exists $t \in \mathbb{Z}$ such that $L_q(m)^k = q^t L_q(m^k)$, then it has non-negative coefficients.*

Also, we have two more conjectures on the q-commutativity of KR-polynomials $F_q(m)$ in Conjectures 4 and 5, which can be understood as natural generalizations of the results in [62] and [19,45], respectively.

1.6. Future work. In a forthcoming paper [33], we study the heart subring $\mathfrak{K}_{q,Q}(\mathfrak{g})$ of $\mathfrak{K}_q(\mathfrak{g})$ in terms of a generalization Q of the Dynkin quiver to non-simply-laced type, where the AR-quiver Γ_Q and the repetition quiver $\widehat{\triangle}$ are defined for $\mathfrak{g}$ of any finite type including BCFG. Since it is shown in this paper that $\mathfrak{K}_q(\mathfrak{g})$ has a quantum cluster algebra structure (of skew-symmetrizable type), as it is with $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathfrak{g}})$ in [18,29], it will be shown that each heart subring $\mathfrak{K}_{q,Q}(\mathfrak{g})$ is isomorphic to $A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathfrak{n})$ via a certain isomorphism ψ_Q and that the *normalized* dual-canonical/upper-global basis of $A_{\mathbb{Z}[q^{\pm 1/2}]}(\mathfrak{n})$ corresponds to the subset $\mathsf{L}_{q,Q} := \mathsf{L}_q \cap \mathfrak{K}_{q,Q}(\mathfrak{g})$ under ψ_Q. This justifies the name of L_q, the *canonical basis*. Here we would like to make an emphasis on the difference between the known result and our new result when g is non-simply-laced: in the previous $\mathcal{K}_{\mathtt{t}}(\mathscr{C}^Q_g)$-case, the corresponding $A_q(\mathbf{n})$ is of simply-laced type $\mathbf{g}$, while in the new $\mathfrak{K}_{q,Q}(g)$-case, the type of $A_q(n)$ is the same as that of g. Based on some investigation of the heart subrings, we will also clarify the presentation of

$$\mathbb{K}_q(\mathfrak{g}) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathfrak{K}_q(\mathfrak{g}),$$

which says that $\mathbb{K}_q(\mathfrak{g})$ can be understood as a boson-extension of $A_q(\mathfrak{n})$, as $\mathbb{K}_{\mathtt{t}}(\mathscr{C}^0_{\mathbf{g}})$ is for $A_q(\mathbf{n})$ of simply-laced type $\mathbf{g}$. Then we will show that the automorphisms of $\mathfrak{K}_q(\mathfrak{g})$, arising from the reflections on Dynkin quivers Q and the isomorphisms ψ_Q, preserve the canonical basis L_q of $\mathfrak{K}_q(\mathfrak{g})$ and induce the *braid group* action on $\mathfrak{K}_q(\mathfrak{g})$.

Convention. Throughout this paper, we use the following convention.

- For a statement $\mathtt{P}$, we set $\delta(\mathtt{P})$ to be 1 or 0 depending on whether $\mathtt{P}$ is true or not. As a special case, we use the notation $\delta_{i,j} := \delta(i = j)$ (Kronecker's delta).
- For $k, l \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 1}$, we write $k \equiv_s l$ if s divides $k - l$ and $k \not\equiv_s l$, otherwise.
- For a monoidal abelian category $\mathcal{C}$, we denote its Grothendieck ring by $K(\mathcal{C})$. The class of an object $X \in \mathcal{C}$ is denoted by $[X] \in K(\mathcal{C})$.
- A monomial in a Laurent polynomial ring $\mathbb{Z}[x_j^{\pm 1} \mid j \in J]$ is said to be *dominant* (resp. *anti-dominant*) if it is a product of non-negative (resp. non-positive) powers of x_i's.
- For elements $\{r_j\}_{j \in J}$ in a ring $(R, \star)$, parameterized by a totally ordered set $J = \{\cdots < j_{-1} < j_0 < j_1 < \cdots\}$, we write

$$\overset{\to}{\underset{j \in J}{\star}} r_j := \cdots \star r_{j_{-1}} \star r_{j_0} \star r_{j_1} \star \cdots .$$

- For integers $a, b \in \mathbb{Z}$, we set

$$[a, b] := \{x \in \mathbb{Z} \mid a \leqslant x \leqslant b\} \qquad (a, b] := \{x \in \mathbb{Z} \mid a < x \leqslant b\}$$
$$[a, b) := \{x \in \mathbb{Z} \mid a \leqslant x < b\} \qquad (a, b) := \{x \in \mathbb{Z} \mid a < x < b\}$$

  We refer to subsets of these forms as *intervals*.
- Let $X = \{x_j \mid j \in J\}$ be a parameterized by an index set J. Then for $j \in J$ and a subset $\mathcal{J} \subset J$, we set

$$(X)_j := x_j \quad \text{and} \quad (X)_{\mathcal{J}} := \{x_j \mid j \in \mathcal{J}\}.$$

2. Preliminaries

2.1. Cartan datum. Let $\mathfrak{g}$ be a Kac–Moody algebra of a symmetrizable type. We denote its Cartan matrix by $\mathsf{C} = (\mathsf{c}_{i,j})_{i,j \in I}$, Dynkin diagram[2] by $\triangle$, weight lattice by P, set of simple roots by $\Pi = \{\alpha_i \mid i \in I\}$ and set of simple coroots by $\Pi^\vee = \{h_i \mid i \in I\}$.

Let $\mathsf{D} = \mathrm{diag}(d_i \in \mathbb{Z}_{\geqslant 1} \mid i \in I)$ denote a diagonal matrix such that

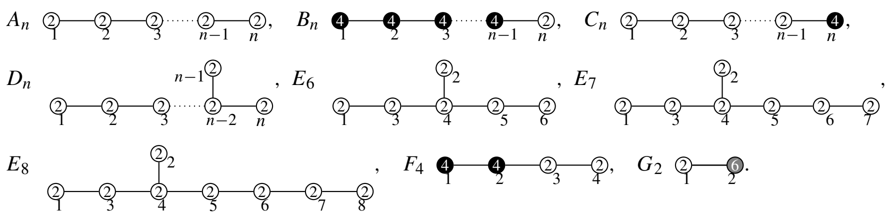
$$\overline{\mathsf{R}} = \mathsf{DC} \text{ and } \underline{\mathsf{R}} = \mathsf{CD}^{-1} \text{ become symmetric.}$$

We take D and the scalar product $(\cdot, \cdot)$ on P such that

$$(\alpha_i, \alpha_j) = d_i \mathsf{c}_{i,j} = d_j \mathsf{c}_{j,i} \in \mathbb{Z} \;\text{ and }\; (\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geqslant 1} \quad \text{for all } i \in I. \tag{2.1}$$

We also denote by $\Phi_\pm$ the set of positive (resp. negative) roots of $\mathfrak{g}$. For each $i \in I$, we choose $\varpi_i \in \mathsf{P}$ such that $\langle h_i, \varpi_j \rangle = \delta_{i,j}$ $(j \in I)$. The free abelian group $\mathsf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called *the root lattice*.

Throughout this paper, we use the following convention of finite Dynkin diagrams:



Here $\textcircled{t}_k$ means that $(\alpha_k, \alpha_k) = t$. For $i, j \in I$, we denote by $d(i, j)$ the smallest number of edges (i.e. the distance) between i and j in $\triangle$. For example, in the finite B_n-case, $d(n, n-1) = d(n-1, n) = 1$ and $d(n, n-2) = d(n-2, n) = 2$, and in the finite D_n-case, $d(n, n-1) = d(n-1, n) = 2$ and $d(n, n-2) = d(n-2, n) = 1$.

We denote by $\triangle_0$ the set of vertices and $\triangle_1$ the set of edges of $\triangle$, respectively. *Throughout this paper, we consider only connected Dynkin diagrams.* We sometimes use $\blacktriangle$ for non-simply-laced types to distinguish them from those of simply-laced types, and use $\triangle\!\!\!\!\cdot\;$ for finite types and, when an emphasis is needed, $\blacktriangle\!\!\!\!\cdot\;$ for finite non-simply-laced types. For each $\triangle\!\!\!\!\cdot\;$, our convention amounts to taking

$$\mathsf{D} := \mathrm{diag}((\alpha_i, \alpha_i)/2 \mid i \in \triangle\!\!\!\!\cdot\;_0) \text{ such that } \min((\alpha_i, \alpha_i)/2) = 1.$$

The Weyl group W of $\mathfrak{g}$ is generated by the reflections s_i $(i \in I)$ acting on P by

$$s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i \qquad (\lambda \in \mathsf{P},\ i \in I).$$

A *Coxeter element* of W is a product of the form $s_{i_1} \cdots s_{i_{|I|}}$ such that $\{i_k \mid 1 \leqslant k \leqslant |I|\} = I$. All Coxeter elements are conjugate in W when $\triangle$ is a tree [9,32], and their common order in W is finite when W is finite [65], in which case the order is called the *Coxeter number* and denoted by h.

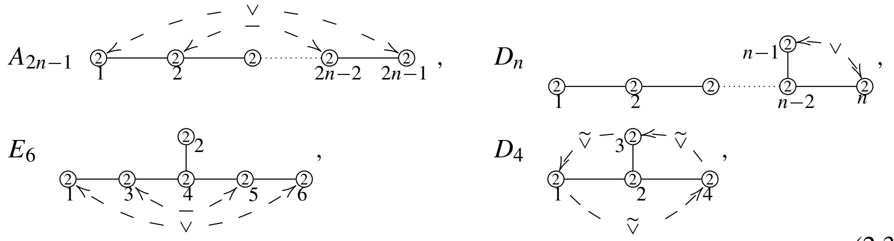
[2] Our convention is a variation of the Coxeter–Dynkin diagram in the sense that we connect vertices with single edges only. See the examples for the finite types. We will call them *Dynkin diagrams* for simplicity.

A bijection σ from $\triangle_0$ to itself is said to be a *Dynkin diagram automorphism* if $\langle h_i, \alpha_j \rangle = \langle h_{\sigma(i)}, \alpha_{\sigma(j)} \rangle$ for all $i, j \in \triangle_0$. *Throughout this paper, we assume that Dynkin diagram automorphisms σ satisfy the following condition:*

$$\textit{there is no } i \in \triangle_0 \textit{ such that } d(i, \sigma(i)) = 1. \tag{2.2}$$

The condition in (2.2) is referred to as an *admissibility* (see [55, §12.1.1]).

For each Dynkin diagram $\mathbb{\Delta}$ of finite type A_{2n-1}, D_n or E_6, there exists a unique non-identity Dynkin diagram automorphism $\vee$ of order 2 (except D_4-type, in which case, there are three automorphism of order 2 and two non-identity automorphisms $\widetilde{\vee}$ and $\widetilde{\vee}^2$ of order 3) satisfying the condition in (2.2).



(2.3)

For a Lie algebra $\mathbf{g}$ of simply-laced finite type associated to $\mathbb{\Delta}$ and a Dynkin diagram automorphism $\sigma (\neq \text{id})$ on $\mathbb{\Delta}$, we denote by g the Lie subalgebra of $\mathbf{g}$ such that it is non-simply-laced type [35, Proposition 7.9] and obtained via σ:

$$(\mathsf{g} \mid (\mathbf{g}, \sigma)) : \ (C_n \mid (A_{2n-1}, \vee)), \ (B_n \mid (D_{n+1}, \vee)), \ (F_4 \mid (E_6, \vee)), \ (G_2 \mid (D_4, \widetilde{\vee})). \tag{2.4}$$

Note that there exists a natural surjective map from $I^{\mathbf{g}}$ to I^{g} sending $I^{\mathbf{g}} \ni \imath \mapsto \bar{\imath} \in I^{\mathsf{g}}$, where $\bar{\imath}$ is an index in I^{g} which can be also understood as the orbit of i under σ.

2.2. Dynkin quiver. A *Dynkin quiver* $Q = (\triangle, \xi)$ of $\triangle$ is an oriented graph, whose underlying graph is $\triangle$, together with a function $\xi : \triangle_0 \to \mathbb{Z}$, called a height function of Q, which satisfies the following condition:

$$\xi_i = \xi_j + 1 \qquad \text{if } d(i, j) = 1 \text{ and } i \to j \text{ in } Q. \tag{2.5}$$

Remark 2.1. We emphasize here that *not* every Dynkin diagram $\triangle$ has a Dynkin quiver. For instance, if $\triangle$ is of affine type $A^{(1)}_{2n}$, there is no Dynkin quiver associated with $\triangle$. Thus, when we mention a Dynkin quiver $Q = (\triangle, \xi)$, it implies that $\triangle$ has one (see also [55, §14.1]).

Note that, since $\triangle$ is connected,

$$\text{height functions of } Q \text{ differ by integers.} \tag{2.6}$$

Conversely, to a Dynkin diagram $\triangle$ and a function $\xi : \triangle \to \mathbb{Z}$ satisfying $|\xi_i - \xi_j| = 1$ for $i, j \in I$ with $d(i, j) = 1$, we can define an orientation on $\triangle$ to obtain a Dynkin quiver in an obvious way. Thus it is enough to specify a pair $(\triangle, \xi)$ of a Dynkin diagram and a height function to present a Dynkin quiver.

For a Dynkin quiver $Q = (\triangle, \xi)$, we call $i \in \triangle_0$ a *source* (resp. *sink*) of Q (or ξ) if $\xi_i > \xi_j$ (resp. $\xi_i < \xi_j$) for all $j \in \triangle_0$ with $d(i, j) = 1$. For a Dynkin quiver $Q = (\triangle, \xi)$

and its source i, we denote by s_iQ the Dynkin quiver $(\triangle, s_i\xi)$, where $s_i\xi$ is the height function defined as follows:

$$(s_i\xi)_j = \xi_j - 2 \times \delta_{i,j}. \tag{2.7}$$

We call the operation from Q to s_iQ the *reflection of Q at a source i of Q*. Note that for Dynkin quivers $Q = (\triangle, \xi)$ and $Q' = (\triangle, \xi')$ with $\xi_i \equiv_2 \xi'_i$ for all $i \in \triangle_0$, there exists a sequence $i_1, \dots, i_r$ and an even integer $u \in 2\mathbb{Z}$ such that i_k is a source of $s_{i_{k-1}} \dots s_{i_1}Q$ $(1 \leqslant k \leqslant r)$ and $s_{i_r} \dots s_{i_1}Q - Q = u$ in the sense that $(s_{i_r} \cdots s_{i_1}\xi)_j = \xi'_j + u$ for all $j \in \triangle_0$.

For a reduced expression $\underline{w} = s_{i_1} \cdots s_{i_l}$ of $w \in \mathsf{W}$ or a sequence $\widetilde{w} = (i_1, \dots, i_l)_{i_1,\dots,i_l \in \triangle_0}$ of indices, we say that $\underline{w}$ (or $\widetilde{w}$) is *adapted to* $Q = (\triangle, \xi)$ if

$$i_k \text{ is a source of } s_{i_{k-1}}s_{i_{k-2}} \cdots s_{i_1}Q \text{ for all } 1 \leqslant k \leqslant l.$$

For a Dynkin quiver $Q = (\triangle, \xi)$, let $s_{i_1} \cdots s_{i_{|\triangle_0|}}$ be a Q-adapted reduced expression of a Coxeter element. Then the height function ξ' of the Dynkin quiver $s_{i_{|\triangle_0|}} \cdots s_{i_1}Q$ is given by

$$\xi'_i = \xi_i - 2 \quad \text{for any } i \in \triangle_0. \tag{2.8}$$

Note that, for $\mathfrak{g}$ of finite type, we can obtain a Dynkin quiver $Q = (\triangle\!\!\!\!\cdot\,, \xi)$ of the same type by assigning orientations to edges in $\triangle\!\!\!\!\cdot\,$. For each Dynkin quiver Q of a finite type, there exists a unique Coxeter element $\tau_Q \in \mathsf{W}$ whose reduced expressions are all adapted to Q. Note that, in finite type, there exists a unique element w_0 in W whose length is the largest. Also the element w_0 induces an involution $^* : I \to I$ given by $w_0(\alpha_i) = -\alpha_{i^*}$.

Convention 1. *Throughout this paper, we take a height function ξ on a finite Dynkin quiver $\triangle\!\!\!\!\cdot\,$ such that $\xi_1 \equiv_2 0$.*

Let $Q = (\triangle, \xi)$ be a Dynkin quiver and σ be a non-trivial Dynkin diagram automorphism of $\triangle$ satisfying (2.2). We call a Dynkin quiver Q *σ-fixed* if $\xi_i = \xi_{\sigma^k(i)}$ for all $i \in I$ and $0 \leqslant k < |\sigma|$. For a σ-fixed Dynkin quiver $Q = (\triangle\!\!\!\!\cdot\,^{\mathbf{g}}, \xi)$ of finite simply-laced type $\mathbf{g}$ and the pair $(\mathbf{g}, \mathsf{g})$ obtained via σ in (2.4), we obtain a Dynkin quiver $\overline{Q} = (\triangle\!\!\!\!\cdot\,^{\mathsf{g}}, \overline{\xi})$ of non-simply-laced type g by defining $\overline{\xi}_{\bar{\imath}} = \xi_\imath$ for all $\imath \in I^{\mathbf{g}}$.

2.3. t-quantized Cartan matrix. For an indeterminate x and integers $k \geqslant l \geqslant 0$, we set

$$[k]_x := \frac{x^k - x^{-k}}{x - x^{-1}}, \quad [k]_x! := \prod_{u=1}^{k} [u]_x \quad \text{and} \quad \begin{bmatrix} k \\ l \end{bmatrix}_x := \frac{[k]_x!}{[k-l]_x![l]_x!}.$$

For an indeterminate q and $i \in I$, we set $q_i = q^{d_i}$ where $\mathsf{D} = \mathrm{diag}(d_i \in \mathbb{Z}_{\geqslant 1} \mid i \in I)$ satisfies (2.1). For a given Cartan matrix C, we set $\mathcal{I} = (\mathcal{I}_{i,j})_{i,j \in I}$ the *adjacent matrix* of C by $\mathcal{I}_{i,j} = -\delta(i \neq j)\mathsf{c}_{i,j}$.

In [16], the (q,t)-deformation $\mathsf{C}(q,t) = (\mathsf{c}_{i,j}(q,t))_{i,j \in I}$ of finite Cartan matrix C is introduced, where

$$\mathsf{c}_{i,j}(q,t) := (q_i t^{-1} + q_i^{-1} t)\delta_{i,j} - [\mathcal{I}_{i,j}]_q.$$

Then we have two kinds of specializations of $\mathsf{C}(q,t)$. One is $\mathsf{C}(q) := \mathsf{C}(q,1)$, called the *quantum Cartan matrix*, and the other is $\underline{\mathsf{C}}(t) := \mathsf{C}(1,t)$, called the *t-quantized Cartan matrix*.

Throughout this paper, we mainly consider the following symmetric matrix

$$\underline{\mathsf{R}}(t) := \underline{\mathsf{C}}(t)\mathsf{D}^{-1}. \tag{2.9}$$

Note that $\underline{\mathsf{R}}(t)|_{t=1} = \underline{\mathsf{R}} \in \mathrm{GL}_{|I|}(\mathbb{Q})$. We regard $\underline{\mathsf{R}}(t)$ as an element of $\mathrm{GL}_{|I|}(\mathbb{Q}(t))$ and denote its inverse by $\widetilde{\underline{\mathsf{R}}}(t) = (\widetilde{\underline{\mathsf{R}}}_{i,j}(t))_{i,j\in I}$ provided it exists. Let

$$\widetilde{\underline{\mathsf{R}}}_{i,j}(t) = \sum_{u\in\mathbb{Z}} \widetilde{\mathsf{r}}_{i,j}(u)t^u \tag{2.10}$$

be the Laurent expansion of $\widetilde{\underline{\mathsf{R}}}_{i,j}(t)$ at $t = 0$. Note that $\widetilde{\underline{\mathsf{R}}}_{i,j}(t) = \widetilde{\underline{\mathsf{R}}}_{j,i}(t)$ for all $i, j \in I$. The closed formulae of $\underline{\mathsf{R}}(t)$ and $\widetilde{\underline{\mathsf{R}}}_{i,j}(t)$ for all finite types can be found in [47,48] (see also references therein)[3].

Lemma 2.2 ([20,29,47]). *Let $\widetilde{\underline{\mathsf{R}}}(t)$ be associated with C of finite type. Then, for any $i, j \in I$ and $u \in \mathbb{Z}$, we have*

(1) $\widetilde{\mathsf{r}}_{i,j}(u) = 0$ *if* $u \leqslant d(i,j)$ *or* $d(i,j) \equiv_2 u$,
(2) $\widetilde{\mathsf{r}}_{i,j}(d(i,j)+1) = \max(d_i, d_j)$.

For a Dynkin quiver Q, we choose a subset $\widetilde{\triangle}_0$ of $\triangle_0 \times \mathbb{Z}$ as follows:

$$\widetilde{\triangle}_0 := \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$

By Convention 1, $\widetilde{\triangle}_0$ does not depend on the choice of Q. For $i, j \in \widetilde{\triangle}_0$, we define an *even function* $\widetilde{\eta}_{i,j} : \mathbb{Z} \to \mathbb{Z}$ as follows:

$$\widetilde{\eta}_{i,j}(u) = \widetilde{\mathsf{r}}_{i,j}(u) + \widetilde{\mathsf{r}}_{i,j}(-u) \qquad \text{for } u \in \mathbb{Z}. \tag{2.11}$$

Lemma 2.3 ([5,47]). *We have*

$$\widetilde{\eta}_{i,j}(u-1) + \widetilde{\eta}_{i,j}(u+1) + \sum_{k;\ d(k,j)=1} \langle h_k, \alpha_j\rangle \widetilde{\eta}_{i,k}(u) = \delta_{u,1}\delta_{i,j} \times 2d_i.$$

2.4. Valued quiver. Let K be a (possibly infinite) countable index set with a decomposition $\mathsf{K} = \mathsf{K}_{\mathrm{ex}} \sqcup \mathsf{K}_{\mathrm{fr}}$. We call K_{ex} the set of *exchangeable indices* and K_{fr} the set of *frozen indices*.

We call an integer-valued $\mathsf{K} \times \mathsf{K}_{\mathrm{ex}}$ matrix $\widetilde{B} = (b_{i,j})_{i\in\mathsf{K}, j\in\mathsf{K}_{\mathrm{ex}}}$ an *exchange matrix* if it satisfies the following properties:

(a) For each $j \in \mathsf{K}_{\mathrm{ex}}$, there exist finitely many $i \in \mathsf{K}$ such that $b_{i,j} \neq 0$.
(b) Its principal part $B := (b_{i,j})_{i,j\in\mathsf{K}_{\mathrm{ex}}}$ is *skew-symmetrizable*; i.e., there exists a sequence $S = (\mathsf{t}_i \mid i \in \mathsf{K}_{\mathrm{ex}}, \mathsf{t}_i \in \mathbb{Z}_{\geqslant 1})$ such that $\mathsf{t}_i b_{i,j} = -\mathsf{t}_j b_{j,i}$ for all $i, j \in \mathsf{K}_{\mathrm{ex}}$. (2.12)

[3] In [47,48], $\widetilde{\underline{\mathsf{B}}}$ and $\widetilde{\mathsf{b}}$ are used instead of $\widetilde{\underline{\mathsf{R}}}$ and $\widetilde{\mathsf{r}}$, respectively.

For an exchange matrix $\widetilde{B}$, we associate a *valued quiver* $\mathcal{Q}_{\widetilde{B}}$ whose set of vertices is K and arrows between vertices are assigned by the following rules:

$$\begin{cases} \underset{k}{\bullet} \xrightarrow{\ulcorner a,b \lrcorner} \underset{l}{\bullet} & \text{if } l,k \in \mathsf{K}_{\mathrm{ex}},\ l \neq k,\ b_{kl} = a \geqslant 0 \text{ and } b_{lk} = b \leqslant 0, \\ \underset{k}{\circ} \xrightarrow{\ulcorner a,0 \lrcorner} \underset{l}{\bullet} \ \left(\text{resp. } \underset{k}{\circ} \xleftarrow{\ulcorner 0,b \lrcorner} \underset{l}{\bullet}\right) & \text{if } l \in \mathsf{K}_{\mathrm{ex}},\ k \in \mathsf{K}_{\mathrm{fr}} \text{ and } b_{kl} = a \geqslant 0 \text{ (resp. } b_{kl} = b \leqslant 0). \end{cases} \tag{2.13}$$

Here we do not draw an arrow between k and l if $b_{kl} = 0$ (and $b_{lk} = 0$ when $l, k \in \mathsf{K}_{\mathrm{ex}}$). Note that $\circ$ denotes a vertex in K_{fr}, and We call $\ulcorner a, b \lrcorner$ the *value* of an arrow.

Convention 2. *For some special values* $\ulcorner a, b \lrcorner$*, we will use the following scheme to draw a valued quiver for convenience: For* $l, k \in \mathsf{K}_{\mathrm{ex}}$ $l \neq k$*,*

(1) *if* $b_{kl} = 1$ *and* $b_{lk} = -b < 0$*, use* $\underset{k}{\bullet} \overset{<b}{\longrightarrow} \underset{l}{\bullet}$*,*

(2) *if* $b_{kl} = 2$ *and* $b_{lk} = -b < 0$*, use* $\underset{k}{\bullet} \overset{<b}{\Longrightarrow} \underset{l}{\bullet}$*,*

(3) *if* $b_{kl} = 3$ *and* $b_{lk} = -b < 0$*, use* $\underset{k}{\bullet} \overset{<b}{\Rrightarrow} \underset{l}{\bullet}$*,*

(4) *we usually skip* <1 *in an arrow when* ($\ulcorner a, -1 \lrcorner$ *and* $1 \leqslant a \leqslant 3$) *for notational simplicity,*

and for $l \in \mathsf{K}_{\mathrm{ex}}$ *and* $k \in \mathsf{K}_{\mathrm{fr}}$*,*

(5) *if* $b_{kl} = 1$ (*resp.* $b_{kl} = -1$)*, use* $\underset{k}{\circ} \longrightarrow \underset{l}{\bullet}$ (*resp.* $\underset{k}{\circ} \longleftarrow \underset{l}{\bullet}$)*,*

(6) *if* $b_{kl} = 2$ (*resp.* $b_{kl} = -2$)*, use* $\underset{k}{\circ} \Longrightarrow \underset{l}{\bullet}$ (*resp.* $\underset{k}{\circ} \Longleftarrow \underset{l}{\bullet}$)*,*

(7) *if* $b_{kl} = 3$ (*resp.* $b_{kl} = -3$)*, use* $\underset{k}{\circ} \Rrightarrow \underset{l}{\bullet}$ (*resp.* $\underset{k}{\circ} \Lleftarrow \underset{l}{\bullet}$).

Throughout this paper, we always apply Convention 2.

Definition 2.4. Let $\triangle$ be a Dynkin diagram. We set $\widetilde{\triangle}_0 \times \widetilde{\triangle}_0$-matrix $\widetilde{B}_{\widetilde{\triangle}_0}$ whose entries $b_{(i,p),(j,s)}$ are defined as follows:

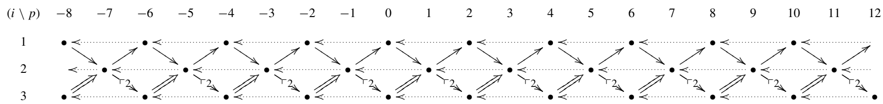
$$b_{(i,p),(j,s)} = \begin{cases} (-1)^{\delta(s>p)} \mathsf{c}_{i,j} & \text{if } |p-s| = 1 \text{ and } i \neq j, \\ (-1)^{\delta(s>p)} & \text{if } |p-s| = 2 \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

Note that $\widetilde{B}_{\widetilde{\triangle}_0}$ satisfies (2.12) with a sequence $S := (s_{i,p} \mid s_{i,p} = d_i)$ and without frozen vertices. We denote by $\widetilde{\triangle}$ the valued quiver associated to $\widetilde{B}_{\widetilde{\triangle}_0}$.

We call the arrows $(i, p) \leftarrow (i, p+2)$ in $\widetilde{\triangle}$ the *horizontal arrows* and the arrows between (i, p) and $(j, p+1)$ for $d(i, j) = 1$ the *vertical*[4] *arrows*.

Convention 3. *We use dashed arrows* $\leftarrow\!\cdots$ *for horizontal arrows in* $\widetilde{\triangle}$ *to distinguish them with vertical arrows in* $\widetilde{\triangle}$*.*

Example 2.5. Under Conventions 2 and 3, when $\blacktriangle$ is of finite type B_3, the valued quiver $\widetilde{\blacktriangle}$ is depicted as



(2.15)

[4] Visually, they are slant.

Remark 2.6. The valued quivers for simply-laced finite types coincide with the infinite quivers in [30, Section 2.1.3] where the infinite quivers are denoted by Γ.

Definition 2.7 (cf. [18, Definition 5.5]).

(1) We denote by $\overline{\triangle}$ the quiver obtained from $\widetilde{\triangle}$ by removing all horizontal arrows. We call $\overline{\triangle}$ the *valued repetition quiver* of $\triangle$.[5]
(2) A subset $\mathcal{R} \subset \widetilde{\triangle}_0 = \overline{\triangle}_0$ is said to be *convex* if it satisfies the following condition: For any oriented path $(x_1 \to x_2 \to \cdots \to x_l)$ consisting of (vertical) arrows in $\overline{\triangle}$, we have $\{x_1, x_2, \dots, x_l\} \subset \mathcal{R}$ if and only if $\{x_1, x_l\} \subset \mathcal{R}$.
(3) We say that a convex subset $\mathcal{R} \subset \widetilde{\triangle}_0$ has a *upper bound* if there exists $\max(p \mid (i,p) \in \mathcal{R})$ for each $i \in \triangle_0$.
(4) For a convex subset $\mathcal{R} \subset \widetilde{\triangle}_0$, we set $\mathcal{R}_{\mathrm{fr}} := \{(i,p) \mid p = \min(k \in \mathbb{Z} \mid (i,k) \in \mathcal{R})\}$ and $\mathcal{R}_{\mathrm{ex}} := \mathcal{R} \backslash \mathcal{R}_{\mathrm{fr}}$. We denote by ${}^{\mathcal{R}}\widetilde{\triangle}$ the valued quiver associated to ${}^{\mathcal{R}}\widetilde{B} := (b_{(i,p),(j,s)})_{(i,p)\in\mathcal{R},(j,s)\in\mathcal{R}_{\mathrm{ex}}}$.
(5) For a height function ξ on $\triangle$, let ${}^{\xi}\widetilde{B} := (b_{(i,p),(j,s)})_{(i,p),(j,s)\in{}^{\xi}\widetilde{\triangle}_0}$ and denote by ${}^{\xi}\widetilde{\triangle}$ the valued quiver associated to ${}^{\xi}\widetilde{B}$, where

$$
{}^{\xi}\widetilde{\triangle}_0 := \{(i,p) \in \widetilde{\triangle}_0 \mid p \leqslant \xi_i\}.
$$

Note that ${}^{\xi}\widetilde{\mathbb{\Delta}}_0$ is a convex subset of $\widetilde{\mathbb{\Delta}}$ for any height function ξ on $\mathbb{\Delta}$.

3. tCharacters of Quantum Loop Algebra and Virtual Grothendieck Rings

In this section, we first review the important properties of t-characters of finite-dimensional representations over quantum loop algebra briefly (see [15,17,25,27,59] for more details). Then we recall the virtual Grothendieck ring $\mathfrak{K}(\mathfrak{g})$ for any finite type $\mathfrak{g}$ (see [14,47] for non-simply-laced types).

3.1. Quantum loop algebras. Let t be an indeterminate. We denote by $\Bbbk := \overline{\mathbb{Q}(t)}$ the algebraic closure of the field $\mathbb{Q}(t)$ inside $\bigcup_{m\in\mathbb{Z}_{\geqslant 0}} \overline{\mathbb{Q}}((t^{1/m}))$. Let $\mathbf{g}$ be a complex finite-dimensional simple Lie algebra of simply-laced type. Note that, in this case, we can identify $\mathsf{C}(q)$ with $\underline{\mathsf{C}}(t)$ by exchanging q with t.

Convention 4. *Throughout this paper, we often use* ***bold symbols*** *to emphasize that those symbols are of simply-laced finite types. We also use $\imath$, $\jmath$ for indices in $I^{\mathbf{g}}$ for the same purpose.*

We denote by $U_t(\mathcal{L}\mathbf{g})$ the quantum loop algebra associated to $\mathbf{g}$, which is the $\Bbbk$-algebra given by the set of infinite generators, called the *Drinfeld generators*, subject to certain relations [1,10]. The quantum loop algebra $U_t(\mathcal{L}\mathbf{g})$ is a quotient of the corresponding (untwisted) quantum affine algebra $U_t'(\widehat{\mathbf{g}})$ and hence has a Hopf algebra structure.

[5] When we replace valued arrows with usual arrows, it is the usual repetition quiver $\widehat{\triangle}$ (see [47] for non-simply-laced types).

3.2. Finite dimensional modules and their t-characters. We denote by $\mathscr{C}_{\mathbf{g}}$ the category of finite-dimensional $U_t(\mathcal{L}\mathbf{g})$-modules of type $\mathbf{1}$. The category $\mathscr{C}_{\mathbf{g}}$ is a $\mathbb{k}$-linear rigid non-braided monoidal category. We say that V and W *commute* if $V \otimes W \simeq W \otimes V$ as $U_t(\mathcal{L}\mathbf{g})$-modules. We denote by $K(\mathscr{C})$ the Grothendieck ring of $\mathscr{C}_{\mathbf{g}}$. Note that the set of simple objects in $K(\mathscr{C}_{\mathbf{g}})$ are parameterized by the set $(1 + z\mathbb{k}[z])^{I^{\mathbf{g}}}$ of $I^{\mathbf{g}}$-tuples of monic polynomials, which is called *Drinfeld polynomials.*

In this paper, we usually consider the *skeleton* subcategory $\mathscr{C}^0_{\mathbf{g}}$ of $\mathscr{C}_{\mathbf{g}}$. The subcategory $\mathscr{C}^0_{\mathbf{g}}$ contains every *prime* simple module in $\mathscr{C}_{\mathbf{g}}$ up to *parameter shifts*. To explain $\mathscr{C}^0_{\mathbf{g}}$, we need to consider the Laurent polynomial $\mathcal{Y}$ generated by the set of variables $\{Y^{\pm 1}_{\imath,p}\}_{(\imath,p)\in\widetilde{\triangle}_0}$. Let us denote by $\mathcal{M}$ (resp. $\mathcal{M}_+$ and $\mathcal{M}_-$) the set of all monomials (resp. dominant monomials and anti-dominant monomials) of $\mathcal{Y}$. For a monomial $\mathbf{m}$ in $\mathcal{Y}$, we write

$$\mathbf{m} = \prod_{(\imath,p)\in\widetilde{\triangle}_0} Y_{\imath,p}^{u_{\imath,p}(\mathbf{m})} \quad \text{and} \quad \mathbf{m}_- = \prod_{(\imath,p)\in\widetilde{\triangle}_0} Y_{\imath,p}^{-u_{\imath,p}(\mathbf{m})} \tag{3.1}$$

with $u_{\imath,p}(\mathbf{m}) \in \mathbb{Z}$. For each $\mathbf{m} \in \mathcal{M}_+$, we denote by $L(\mathbf{m})$ the simple module in $\mathscr{C}$ whose Drinfeld polynomial is $\left(\prod_p (1 - q^p)^{u_{\imath,p}(\mathbf{m})}\right)_{\imath\in I^{\mathbf{g}}}$. Then the subcategory $\mathscr{C}^0_{\mathbf{g}}$ can be characterized by the Serre subcategory of $\mathscr{C}_{\mathbf{g}}$ generated by $\{L(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}_+\}$. Note that $\mathscr{C}^0_{\mathbf{g}}$ is a monoidal rigid subcategory of $\mathscr{C}_{\mathbf{g}}$. In [17], Frenkel-Reshetikhin proved that there exists an injective ring homomorphism

$$\chi_t : K(\mathscr{C}^0_{\mathbf{g}}) \to \mathcal{Y},$$

called *the t-character homomorphism.*[6] Note that each monomial of $\chi_t(L(\mathbf{m}))$ is often called a *ℓ-weight* of $L(\mathbf{m})$, since each monomial in $Y^{\pm 1}_{\imath,p}$ encodes the generalized eigenvalues of the commuting family consisting of certain Drinfeld generators as endomorphisms of $L(\mathbf{m})$ (see [17] for more detail). The existence of χ_t tells us that the Grothendieck ring $K(\mathscr{C}_{\mathbf{g}})$ is commutative, even though $\mathscr{C}_{\mathbf{g}}$ is not braided.

For an interval $[a, b] \subset \mathbb{Z}$, $\imath \in I^{\mathbf{g}}$, $k \in \mathbb{Z}_{\geqslant 1}$ and $(\imath, p) \in \widetilde{\triangle}_0$, we set dominant monomials

$$\mathbf{m}^{(\imath)}[a, b] := \prod_{(\imath,s)\in\widetilde{\triangle}_0;\ s\in[a,b]} Y_{\imath,s} \quad \text{and} \quad \mathbf{m}^{(\imath)}_{k,p} := \prod_{s=0}^{k-1} Y_{\imath,p+2s}, \tag{3.2}$$

and $\mathbf{m}^{(\imath)}(a, b]$, $\mathbf{m}^{(\imath)}[a, b)$, and $\mathbf{m}^{(\imath)}(a, b)$ are defined similarly.

The simple module $L(\mathbf{m}^{(\imath)}[p, s])$ ($p \leqslant s$) is called a *Kirillov–Reshetikhin* (*KR*) *module*. When $p = s$ and $(\imath, p) \in \widetilde{\triangle}_0$, we call $L(Y_{\imath,p})$ a *fundamental module*. Note that the Grothendieck ring $K(\mathscr{C}^0_{\mathbf{g}})$ is a polynomial ring in the isomorphism classes of the fundamental modules $L(Y_{\imath,p})$ [17].

For $(\imath, p) \in I^{\mathbf{g}} \times \mathbb{Z}$ with $(\imath, p \pm 1) \in \widetilde{\triangle}^{\mathbf{g}}_0$, we set

$$A_{\imath,p} := Y_{\imath,p-1} Y_{\imath,p+1} \prod_{\jmath:\ d(\imath,\jmath)=1} Y^{-1}_{\jmath,p} = Y_{\imath,p-1} Y_{\imath,p+1} \prod_{\jmath\neq\imath} Y^{\mathbf{c}_{\jmath,\imath}}_{\jmath,p}. \tag{3.3}$$

[6] It is usually called the q-character homomorphism in the literature.

Note that there is a *partial* ordering $\preccurlyeq_{\mathrm{N}}$ on the set of monomials $\mathcal{M}$, called the *Nakajima order*, defined as follows:

$$\mathbf{m} \preccurlyeq_{\mathrm{N}} \mathbf{m}' \text{ if and only if } \mathbf{m}^{-1}\mathbf{m}' \text{ is a product of elements in } \{A_{\imath,p+1} \mid (\imath, p) \in \widetilde{\triangle}_0^{\mathbf{g}}\}. \tag{3.4}$$

Theorem 3.1 ([15,16]). *For each dominant monomial* $\mathbf{m} \in \mathcal{M}_+$*, the monomials appearing in* $\chi_t(L(\mathbf{m})) - \mathbf{m}$ *are strictly less that* $\mathbf{m}$ *with respect to* $\preccurlyeq_{\mathrm{N}}$.

The t-characters of KR-modules satisfies a system of functional equations called T-systems:

Theorem 3.2 ([59, Theorem 1.1]). (*See also [27, Theorem 3.4].*) *For each* $(\imath, p), (\imath, s) \in \widetilde{\triangle}_0^{\mathbf{g}}$ *with* $p \leqslant s$*, we have*

$$\begin{aligned} \chi_t\big(L(\mathbf{m}^{(\imath)}[p, s))\big)\chi_t\big(L(\mathbf{m}^{(\imath)}(p, s])\big) &= \chi_t\big(L(\mathbf{m}^{(\imath)}[p, s])\big)\chi_t\big(L(\mathbf{m}^{(\imath)}(p, s))\big) \\ &+ \prod_{\jmath:\, d(\imath,\jmath)=1} \chi_t\big(L(\mathbf{m}^{(\jmath)}(p, s)\big). \end{aligned} \tag{3.5}$$

Let ξ be a height function on $\triangle^{\mathbf{g}}$. We denote by ${}^{\xi}\mathcal{M}_+$ the set of all dominant monomials in the variables $Y_{\imath,p}$'s for $(\imath, p) \in {}^{\xi}\widetilde{\triangle}_0$.

Definition 3.3. We define the subcategory $\mathscr{C}_{\mathbf{g}}^{\xi}$ as the Serre subcategory of $\mathscr{C}_{\mathbf{g}}$ such that $\operatorname{Irr} \mathscr{C}_{\mathbf{g}}^{\xi} = \{L(\mathbf{m}) \mid \mathbf{m} \in {}^{\xi}\mathcal{M}_+\}$.

Since ${}^{\xi}\widetilde{\triangle}_0$ is a convex subset of $\widetilde{\triangle}_0$, we have the following proposition:

Proposition 3.4. *The category* $\mathscr{C}_{\mathbf{g}}^{\xi}$ *is a monoidal subcategory of* $\mathscr{C}_{\mathbf{g}}$.

Proof. This assertion follows from the same argument of the proof of [30, Proposition 3.10]. □

3.3. Truncation. We denote by $\mathcal{Y}^{\xi}$ the Laurent polynomial ring generated by $Y_{\imath,p}$'s for $(\imath, p) \in {}^{\xi}\widetilde{\triangle}_0$. We define a linear map $(\cdot)_{\leqslant\xi} : \mathcal{Y} \to \mathcal{Y}^{\xi}$ by sending the monomials which contain some $Y_{\imath,p}$ with $(\imath, p) \notin {}^{\xi}\widetilde{\triangle}$ to zero and by keeping all the other terms.

Proposition 3.5. *For a height function* ξ*, the* $\mathbb{Z}$*-linear map* $(\cdot)_{\leqslant\xi} : K(\mathscr{C}_{\mathbf{g}}^{\xi}) \to \mathcal{Y}^{\xi}$ *given by*

$$[V] \mapsto {}^{\xi}\chi_t(V) := ((\cdot)_{\leqslant\xi} \circ \chi_t)(V)$$

gives an injective ring homomorphism $K(\mathscr{C}_{\mathbf{g}}^{\xi}) \hookrightarrow \mathcal{Y}^{\xi}$.

Proof. We can prove the assertion in the same way as in the proof of [28, Proposition 6.1]. □

3.4. (Virtual) Grothendieck rings. Recall that when $\mathbf{g}$ is of simply-laced finite type, the t-character homomorphism χ_t is an injection from $K(\mathscr{C}^0_{\mathbf{g}})$ into $\mathcal{Y}^{\mathbf{g}}$. Thus we can identify $K(\mathscr{C}^0_{\mathbf{g}})$ with

$$\mathfrak{K}(\mathbf{g}) := \chi_t\big(K(\mathscr{C}^0_{\mathbf{g}})\big).$$

We call $\mathfrak{K}(\mathbf{g})$ the Grothendieck ring of type $\mathbf{g}$ as well.

Proposition 3.6 ([15, Corollary 5.7]). *When* $\mathbf{g}$ *is of simply-laced type, we have*

$$\mathfrak{K}(\mathbf{g}) = \bigcap_{\imath \in I^{\mathbf{g}}} \Big(\mathbb{Z}[Y^{\pm 1}_{\jmath,l} \mid (\jmath, l) \in \widetilde{\vartriangle}^{\mathbf{g}}_0, \jmath \neq \imath] \otimes \mathbb{Z}[Y_{\imath,l}(1 + A^{-1}_{\imath,l+1}) \mid (\imath, l) \in \widetilde{\vartriangle}^{\mathbf{g}}_0]\Big) \subsetneq \mathcal{Y}^{\mathbf{g}}.$$

Now we move on to non-simply-laced finite types. For g associated with $(\mathbf{g}, \sigma)$ in (2.4), we consider the Laurent polynomial ring defined as follows: We first set

$$\mathcal{Y}^{\mathsf{g}} := \mathbb{Z}[X^{\pm 1}_{i,p} \mid (i, p) \in \widetilde{\vartriangle}^{\mathsf{g}}_0].$$

Then there exists a surjective ring homomorphism

$$\overline{\sigma} : \mathcal{Y}^{\mathbf{g}} \longrightarrow \mathcal{Y}^{\mathsf{g}} \qquad \text{sending} \qquad Y_{\sigma^k(\imath),p} \longmapsto X_{\bar{\imath},p} \tag{3.6}$$

for any $(\imath, p) \in \widetilde{\vartriangle}^{\mathbf{g}}_0$ and $0 \leqslant k < |\sigma|$. Finally, we set

$$\mathfrak{K}(\mathsf{g}) := \overline{\sigma}(\mathfrak{K}(\mathbf{g}))$$

and call it the *virtual Grothendieck ring* of type g. We call $\overline{\sigma}(L(\mathbf{m}))$ the *folded t-character* of $L(\mathbf{m})$.

Now we would like to unify the expression for $\mathfrak{K}(\mathfrak{g})$ for *any* finite type $\mathfrak{g}$ by replacing variables $Y_{i,p}$'s with $X_{i,p}$'s. Let $\mathcal{X}^{\mathfrak{g}}$ be the Laurent polynomial ring $\mathbb{Z}[X^{\pm 1}_{i,p} \mid (i, p) \in \widetilde{\vartriangle}^{\mathfrak{g}}_0]$. For $(i, p+1) \in \widetilde{\vartriangle}^{\mathfrak{g}}_0$, we set

$$B_{i,p} := X_{i,p-1} X_{i,p+1} \prod_{j : d(i,j)=1} X^{\mathsf{c}_{j,i}}_{j,p}. \tag{3.7}$$

Definition 3.7 [14, §3.4]. We define the commutative ring

$$\mathfrak{K}(\mathfrak{g}) = \bigcap_{i \in I^{\mathfrak{g}}} \Big(\mathbb{Z}[X^{\pm 1}_{j,l} \mid (j, l) \in \widetilde{\vartriangle}^{\mathfrak{g}}_0, j \neq i] \otimes \mathbb{Z}[X_{i,l}(1 + B^{-1}_{i,l+1}) \mid (i, l) \in \widetilde{\vartriangle}^{\mathfrak{g}}_0]\Big) \subsetneq \mathcal{X}^{\mathfrak{g}}. \tag{3.8}$$

Remark 3.8. Even though, we unify the expression for $\mathfrak{K}(\mathfrak{g})$ by using $X_{i,p}$, $\mathcal{X}$ and $B_{i,p}$, we sometimes use $Y_{\imath,p}$, $\mathcal{Y}$ and $A_{\imath,p}$ to emphasize that they are associated with $\mathbf{g}$ of simply-laced finite type.

Theorem 3.9 ([14, Proposition 3.3, Theorem 4.3]).

(1) *Every element of $\mathfrak{K}(\mathfrak{g})$ is characterized by the multiplicities of the dominant monomials contained in it.*
(2) *For each $m \in \mathcal{M}_+$, there is a unique element $F(m)$ of $\mathfrak{K}(\mathfrak{g})$ such that m is the unique dominant monomial of $F(m)$ with its coefficient* 1. *Therefore we have a basis $\{F(m) \mid m \in \mathcal{M}^{\mathfrak{g}}_+\}$ of $\mathfrak{K}(\mathfrak{g})$ parameterized by dominant monomials m.*

(3) *For each pair* $(\mathbf{g}, \mathsf{g})$ *obtained via* σ*, the map* $\overline{\sigma}$ *induces a surjective ring homomorphism from* $\mathfrak{K}(\mathbf{g})$ *to* $\mathfrak{K}(\mathsf{g})$.

An $\mathcal{X}$-monomial $\mathbf{m}$ is said to be *right-negative* if the factors $X_{j,l}$ appearing in m, for which l is maximal, have negative powers.

Corollary 3.10. *For each pair* $(\mathbf{g}, \mathsf{g})$ *obtained via* σ *and* $\mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}$*, assume that*

$$\text{every monomial in } F(\mathbf{m}) - \mathbf{m} \text{ is right-negative.} \tag{3.9}$$

Then $\overline{\sigma}(F(\mathbf{m})) = F\big(\overline{\sigma}(\mathbf{m})\big) \in \mathfrak{K}(\mathsf{g})$.

Proof. By Theorem 3.9 (3) and (3.9), $\overline{\sigma}(F(\mathbf{m}))$ is an element in $\mathfrak{K}(\mathsf{g})$ containing the unique dominant monomial $\overline{\sigma}(\mathbf{m})$. Thus our assertion follows. □

Example 3.11. For type X_n with $X = A$ or C, we write the polynomial $F(m)$ in Theorem 3.9 (2) by $F_{X_n}(m)$ to emphasize the type.

One can check (see the formulas of $F_{A_5}(Y_{4,-2})$ and $F_{A_5}(Y_{2,0})$ below)

(1) $F_{A_5}(Y_{4,-2}Y_{2,0})$ is equal to $F_{A_5}(Y_{4,-2})F_{A_5}(Y_{2,0})$, since $F_{A_5}(Y_{4,-2})F_{A_5}(Y_{2,0})$ has a unique dominant monomial $Y_{4,-2}Y_{2,0}$,
(2) $F_{A_5}(Y_{4,-2})F_{A_5}(Y_{2,0})$ contains a monomial $Y_{3,-1}Y_{5,-1}Y_{2,0}Y_{4,0}^{-1}$,
(3) $F_{A_5}(Y_{2,-2}Y_{2,0})$ is different from $F_{A_5}(Y_{4,-2})F_{A_5}(Y_{2,0})$, since $F_{A_5}(Y_{4,-2})F_{A_5}(Y_{2,0})$ contains $Y_{4,-2}Y_{2,0}$.

In particular, (2) tells us that that $\overline{\sigma}(F_{A_5}(Y_{4,-2})F_{A_5}(Y_{2,0}))$ contains a monomial

$$\overline{\sigma}(Y_{3,-1}Y_{5,-1}Y_{2,0}Y_{4,0}^{-1}) = X_{3,-1}X_{1,-1},$$

which is dominant but not equal to $X_{2,-2}X_{2,0}$. Hence $\overline{\sigma}(F_{A_5}(Y_{4,-2}Y_{2,0}))$ can *not* be $F_{C_3}(X_{2,-2}X_{2,0})$. Note that $F_{A_5}(Y_{4,-2}Y_{2,0})$ does not satisfy (3.9), while $F_{A_5}(Y_{2,-2}Y_{2,0})$ satisfies that property. Therefore, $\overline{\sigma}(F_{A_5}(Y_{2,-2}Y_{2,0})) = F_{C_3}(X_{2,-2}X_{2,0})$ by Corollary 3.10.

Here we present $F_{A_5}(Y_{4,-2})$ and $F_{A_5}(Y_{2,0})$ explicitly for reader's convenience:

$$\begin{aligned}
F_{A_5}(Y_{4,-2}) = {} & Y_{4,-2} + \boxed{Y_{3,-1}Y_{4,0}^{-1}Y_{5,-1}} + Y_{2,0}Y_{3,1}^{-1}Y_{5,-1} + Y_{3,-1}Y_{5,1}^{-1} + Y_{1,1}Y_{2,2}^{-1}Y_{5,-1} \\
& + Y_{2,0}Y_{3,1}^{-1}Y_{4,0}Y_{5,1}^{-1} + Y_{1,3}^{-1}Y_{5,-1} + Y_{1,1}Y_{2,2}^{-1}Y_{4,0}Y_{5,1}^{-1} + Y_{2,0}Y_{4,2}^{-1} + Y_{1,3}^{-1}Y_{4,0}Y_{5,1}^{-1} \\
& + Y_{1,1}Y_{2,2}^{-1}Y_{3,1}Y_{4,2}^{-1} + Y_{1,3}^{-1}Y_{3,1}Y_{4,2}^{-1} + Y_{1,1}Y_{3,3}^{-1} + Y_{1,3}^{-1}Y_{2,2}Y_{3,3}^{-1} + Y_{2,4}^{-1}, \\
F_{A_5}(Y_{2,0}) = {} & \boxed{Y_{2,0}} + Y_{1,1}Y_{2,2}^{-1}Y_{3,1} + Y_{1,3}^{-1}Y_{3,1} + Y_{1,1}Y_{3,3}^{-1}Y_{4,2} + Y_{1,3}^{-1}Y_{2,2}Y_{3,3}^{-1}Y_{4,2} + Y_{1,1}Y_{4,4}^{-1}Y_{5,3} \\
& + Y_{2,4}^{-1}Y_{4,2} + Y_{1,3}^{-1}Y_{2,2}Y_{4,4}^{-1}Y_{5,3} + Y_{1,1}Y_{5,5}^{-1} + Y_{2,4}^{-1}Y_{3,3}Y_{4,4}^{-1}Y_{5,3} + Y_{1,3}^{-1}Y_{2,2}Y_{5,5}^{-1} \\
& + Y_{3,5}^{-1}Y_{5,3} + Y_{2,4}^{-1}Y_{3,3}Y_{5,5}^{-1} + Y_{3,5}^{-1}Y_{4,4}Y_{5,5}^{-1} + Y_{4,6}^{-1},
\end{aligned}$$

where the product of boxed monomials yields the non-right-native monomial $Y_{3,-1}Y_{5,-1}Y_{2,0}Y_{4,0}^{-1}$.

Note that if $\mathbf{m}, \mathbf{m}' \in \mathcal{M}^{\mathbf{g}}$ with $\mathbf{m} \preccurlyeq_{\mathrm{N}} \mathbf{m}'$, then we have

$$\overline{\sigma}(\mathbf{m}) \preccurlyeq_{\mathrm{N}} \overline{\sigma}(\mathbf{m}') \in \mathcal{M}^{\mathsf{g}}. \tag{3.10}$$

It is proved in [15,27] that, for $\mathbf{m}^{(\imath)}[p,s] \in \mathcal{M}_+^{\mathbf{g}}$, $F(\mathbf{m}^{(\imath)}[p,s])$ satisfies the condition in (3.9) and

$$F(\mathbf{m}^{(\imath)}[p,s]) = \chi_t(L(\mathbf{m}^{(\imath)}[p,s])).$$

Thus we have

$$\overline{\sigma}(F(\mathbf{m}^{(\imath)}[p,s])) = F(m^{(i)}[p,s]) \tag{3.11}$$

and (3.5) is changed into the following form: For any finite type $\mathfrak{g}$ and $(i,p),(i,s) \in \widetilde{\triangle}_0$ with $p \leqslant s$, we have

$$F\big(m^{(i)}[p,s)\big)F\big(m^{(i)}(p,s]\big) = F\big(m^{(i)}[p,s]\big)F\big(m^{(i)}(p,s)\big) + \prod_{j;\ d(i,j)=1} F\big(m^{(j)}(p,s)\big)^{-\mathsf{c}_{j,i}}. \tag{3.12}$$

We call (3.12) the *folded T-systems*.

Definition 3.12. (1) For a height function ξ on $\triangle^{\mathbf{g}}$ of simply-laced finite type, we set

$${}^{\xi}\mathfrak{K}(\mathbf{g}) := {}^{\xi}\chi_t(K(\mathscr{C}^{\xi})).$$

(2) For a height function ξ on $\triangle^{\mathsf{g}}$ of non-simply-laced finite type, we set

$${}^{\xi}\mathfrak{K}(\mathsf{g}) := \overline{\sigma}\big({}^{\underline{\xi}}\mathfrak{K}(\mathbf{g})\big),$$

where $\underline{\xi}$ is the σ-fixed height function on $\triangle^{\mathbf{g}}$ such that

$$\underline{\xi}_{\sigma^k(\imath)} = \xi_{\bar{\imath}} \qquad \text{for any } 0 \leqslant k < |\sigma| \text{ and } \imath \in \sigma^{-1}(\bar{\imath}).$$

We call ${}^{\xi}\mathfrak{K}(\mathsf{g})$ the *truncated virtual Grothendieck ring* and ${}^{\xi}\overline{\chi}_t(\mathbf{m})$ the *folded truncated t-character* of $L(\mathbf{m})$ with respect to ξ, defined as below:

$$\begin{tikzcd}
K(\mathscr{C}^{\underline{\xi}\,\mathbf{g}}) \arrow[r, "{}^{\underline{\xi}}\chi_t"'] \arrow[rr, bend left, "{}^{\xi}\overline{\chi}_t"] & {}^{\underline{\xi}}\mathfrak{K}(\mathbf{g}) \arrow[r, "\overline{\sigma}"'] & {}^{\xi}\mathfrak{K}(\mathsf{g})
\end{tikzcd}$$

Remark 3.13. Let G be a simply-connected complex Lie group associated with g of *non-simply-laced type*. In [14], the authors formulate (conjectural) *folded integrable models* of g corresponding to *folded Bethe Ansatz equations*. Then $\mathfrak{K}(\mathsf{g})$, denoted by $\mathcal{K}_t^-(\mathsf{g})$ in [14],[7] plays the role of describing the spectra of the transfer-matrix $t_V(z,u)$ with a finite-dimensional $U_t(\mathcal{L}\mathbf{g})$-module V in the folded integrable model, as in the role of $\mathfrak{K}(\mathbf{g}) \simeq K(\mathscr{C}_{\mathbf{g}}^0)$ in the integrable models for simply-laced types (cf. [13,14] for more details). We remark that our main interest is to study the structure of the *quantization of* $\mathfrak{K}(\mathsf{g})$ introduced independently in [47] with other motivations related to *canonical basis* and *quantum cluster algebra structure*. In contrast, the authors of [14] mainly focus on a study of the folded integrable models associated with g. It would be interesting to find connections between our results and those in [14].

[7] In our introduction, we use $\overline{\mathcal{K}}_{1,\mathtt{t},d}(\mathfrak{g})$ instead.

4. Quantization

In this section, we quantize the Laurent polynomial ring $\mathcal{X}$ with the resulting ring denoted by $\mathcal{X}_q$, via the inverse matrix $\widetilde{\underline{\mathsf{R}}}(t)$ of (2.9) associated with $\underline{\mathsf{C}}(t)$ following [47] (see also [14]), and define its subalgebra $\mathfrak{K}_q(\mathfrak{g})$ that is regarded as a quantization of $\mathfrak{K}(\mathfrak{g})$.

4.1. Quantum torus. Let q be an indeterminate. Let us recall that $\widetilde{\mathsf{r}}_{i,j}(u)$ ($u \in \mathbb{Z}$) in (2.10) and the even function $\widetilde{\eta}_{i,j} : \mathbb{Z} \to \mathbb{Z}$ defined in (2.11).

Definition 4.1 ([25,47,60,67]). Let $(\mathcal{X}_q, *)$ be the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra with the generators $\{\widetilde{X}_{i,p}^{\pm 1} \mid (i,p) \in \widetilde{\triangle}_0\}$ with the defining relations

$$\widetilde{X}_{i,p} * \widetilde{X}_{i,p}^{-1} = \widetilde{X}_{i,p}^{-1} * \widetilde{X}_{i,p} = 1 \quad \text{and} \quad \widetilde{X}_{i,p} * \widetilde{X}_{j,s} = q^{\underline{\mathcal{N}}(i,p;j,s)} \widetilde{X}_{j,s} * \widetilde{X}_{i,p},$$

where $(i,p), (j,s) \in \widetilde{\triangle}_0$ and

$$\underline{\mathcal{N}}(i,p;j,s) := \widetilde{\mathsf{r}}_{i,j}(p-s-1) - \widetilde{\mathsf{r}}_{i,j}(s-p-1) - \widetilde{\mathsf{r}}_{i,j}(p-s+1) + \widetilde{\mathsf{r}}_{i,j}(s-p+1). \tag{4.1}$$

We call $\mathcal{X}_q$ the *quantum torus associated with* $\underline{\mathsf{C}}(t)$ (see Definition 7.1 below).

Remark 4.2. For simply-laced finite types, the quantum torus $\mathcal{X}_q$ was already defined in [25,60,67], whereas for non-simply-laced finite types, it is introduced in [47] very recently.

Note that since $\widetilde{\underline{\mathsf{R}}}(t)$ is symmetric,

$$\underline{\mathcal{N}}(i,p;j,s) = \underline{\mathcal{N}}(j,p;i,s) = -\underline{\mathcal{N}}(i,s;j,p) = -\underline{\mathcal{N}}(j,s;i,p),$$

and it follows from Lemma 2.2 that

$$\underline{\mathcal{N}}(i,p;j,s) = \widetilde{\mathsf{r}}_{i,j}(p-s-1) - \widetilde{\mathsf{r}}_{i,j}(p-s+1) \qquad \text{if } p > s. \tag{4.2}$$

Moreover, for $p \in \mathbb{Z}$ and $i, j \in \triangle_0$ such that $(i,p), (j,p) \in \widetilde{\triangle}_0$, Lemma 2.2 tells that

$$\widetilde{X}_{i,p} * \widetilde{X}_{j,p} = \widetilde{X}_{j,p} * \widetilde{X}_{i,p}. \tag{4.3}$$

By specializing q at 1, the quantum torus $\mathcal{X}_q$ recovers the commutative Laurent polynomial ring $\mathcal{X}$, while $\mathcal{X}_q$ is non-commutative; i.e., there exists a $\mathbb{Z}$-algebra homomorphism $\mathrm{ev}_{q=1} : \mathcal{X}_q \to \mathcal{X}$ given by $q^{\frac{1}{2}} \mapsto 1$ and $\widetilde{X}_{i,p} \mapsto X_{i,p}$.

We say that $\widetilde{m} \in \mathcal{X}_q$ is a $\mathcal{X}_q$*-monomial* if it is a product of the generators $\widetilde{X}_{i,p}^{\pm 1}$ and $q^{\pm\frac{1}{2}}$. For a $\mathcal{X}_q$-monomial $\widetilde{m} \in \mathcal{X}_q$, we set $u_{i,p}(\widetilde{m}) := u_{i,p}\big(\mathrm{ev}_{q=1}(\widetilde{m})\big)$ (see (3.1)). An $\mathcal{X}_q$-monomial $\widetilde{m}$ is said to be *right-negative* if $\mathrm{ev}_{q=1}(\widetilde{m})$ is right-negative. Note that a product of right negative $\mathcal{X}$-monomials (resp. $\mathcal{X}_q$-monomials) is right negative. A $\mathcal{X}_q$-monomial $\widetilde{m}$ is called *dominant* if $\mathrm{ev}_{q=1}(\widetilde{m})$ is dominant. Moreover, for $\mathcal{X}_q$-monomials $\widetilde{m}, \widetilde{m}'$ in $\mathcal{X}_q$, we define

$$\widetilde{m} \preccurlyeq_{\mathsf{N}} \widetilde{m}' \quad \text{if and only if} \quad \mathrm{ev}_{q=1}(\widetilde{m}) \preccurlyeq_{\mathsf{N}} \mathrm{ev}_{q=1}(\widetilde{m}').$$

For $i \in \triangle_0$, we call $\mathcal{X}$-monomial m (resp. $\mathcal{X}_q$-monomial $\widetilde{m}$) *i-dominant* if $u_{i,p}(m) \geqslant 0$ (resp. $u_{i,p}(\widetilde{m}) \geqslant 0$) for all p such that $(i,p) \in \widetilde{\triangle}_0$. For $J \subset \triangle_0$, we call $\mathcal{X}$-monomial

m (resp. $\mathcal{X}_q$-monomial $\widetilde{m}$) *J-dominant* if m (resp. $\widetilde{m}$) is j-dominant for all $j \in J$. For monomials $\widetilde{m}, \widetilde{m}'$ in $\mathcal{X}_q$, we define

$$\underline{\mathcal{N}}(\widetilde{m}, \widetilde{m}') := \sum_{(i,p),(j,s)\in\widetilde{\triangle}_0} u_{i,p}(\widetilde{m})u_{j,s}(\widetilde{m}')\underline{\mathcal{N}}(i,p;j,s). \tag{4.4}$$

There exists the $\mathbb{Z}$-algebra anti-involution $\overline{(\cdot)}$ on $\mathcal{X}_q$ ([25,47]) given by

$$q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}, \qquad \widetilde{X}_{i,p} \mapsto q_i \widetilde{X}_{i,p}. \tag{4.5}$$

Thus, for any $\mathcal{X}_q$-monomial $\widetilde{m} \in \mathcal{X}_q$, there exists a unique $r \in \frac{1}{2}\mathbb{Z}$ such that $q^r\widetilde{m}$ is $\overline{(\cdot)}$-invariant. A monomial of this form is called *bar-invariant* and denoted by $\underline{\widetilde{m}}$. For an example,

$$\mathsf{X}_{i,p} := q^{\frac{d_i}{2}}\widetilde{X}_{i,p} \text{ is bar-invariant.}$$

More generally, for a family $\left(u_{i,p} \mid (i,p) \in \widetilde{\triangle}_0\right)$ of integers with finitely many non-zero components, the expression

$$q^{\frac{1}{2}\sum_{(i,p)<(j,s)} u_{i,p}u_{j,s}\underline{\mathcal{N}}(j,s;i,p)} \overset{\rightarrow}{\underset{(i,p)\in\widetilde{\triangle}_0}{*}} \mathsf{X}_{i,p}^{u_{i,p}} \tag{4.6}$$

does not depend on the choice of an ordering on $\widetilde{\triangle}_0$ and is bar-invariant.

Remark 4.3. Note that the relations in Definition 4.1 do not change when we replace $\widetilde{X}_{i,p}$ with $\mathsf{X}_{i,p}$, and $\underline{\widetilde{m}}$ depends only on $\mathrm{ev}_{q=1}(\widetilde{m})$. Therefore, for every monomial m in $\mathcal{X}$, we denote by $\underline{m}$ the bar-invariant monomial in $\mathcal{X}_q$ corresponding to m. Also the notation $Y_{i,p}$ of $(\mathbf{Y}_t, *)$ in [29, Section 3] corresponds to $\mathsf{X}_{i,p}$, the bar-invariant monomial, in this paper.

For $(i,p) \in \widetilde{\triangle}_0$, we set

$$\widetilde{B}_{i,p} := \underline{B_{i,p}} \in \mathcal{X}_q. \tag{4.7}$$

Definition 4.4. Let $\mathbf{B}_q^-$ be the $\mathbb{Z}[q^{\pm1/2}]$-subalgebra of $\mathcal{X}_q$ generated by $\widetilde{B}_{i,p}^{-1}$'s for $(i,p) \in I \times \mathbb{Z}$. For $k \in \mathbb{Z}_{\geqslant 1}$, we denote by $\mathbf{B}_q^{-k}$ the $\mathbb{Z}[q^{\pm1/2}]$-span of the monomials $\overset{\rightarrow}{\underset{1\leqslant s\leqslant k}{*}} \widetilde{B}_{i_s,p_s}^{-1}$.

For bar-invariant $\mathcal{X}_q$-monomials $\underline{m_1}$ and $\underline{m_2}$, we set $\underline{m_1} \cdot \underline{m_2} := \underline{m_1 m_2}$, and for $\underline{m_k}$ ($k \in \mathbb{Z}_{\geqslant 1}$), we set

$$\prod_k \underline{m_k} := \underline{\prod_k m_k}. \tag{4.8}$$

Definition 4.5. (cf. [18, Definition 5.5]) For a subset $\mathsf{S} \subset \widetilde{\triangle}_0$, we denote by ${}^{\mathsf{S}}\mathcal{X}_q$ the quantum subtorus of $\mathcal{X}_q$ generated by $\widetilde{X}_{i,p}^{\pm1}$ for $(i,p) \in \mathsf{S} \subset \widetilde{\triangle}_0$. In particular, for a height function ξ on $\triangle$, we denote by ${}^{\xi}\mathcal{X}_q$ the quantum subtorus generated by $\widetilde{X}_{i,p}^{\pm1}$ for $(i,p) \in {}^{\xi}\widetilde{\triangle}_0$.

Proposition 4.6. ([47, Proposition 5.7]) *For* $i, j \in I$ *and* $p, s, t, u \in \mathbb{Z}$ *with* $(i, p), (j, s+1), (i, t+1), (j, u+1) \in \widetilde{\triangle}_0$, *we have*

$$\widetilde{X}_{i,p} * \widetilde{B}^{-1}_{j,s} = q^{\beta(i,p;j,s)}\, \widetilde{B}^{-1}_{j,s} * \widetilde{X}_{i,p} \quad \text{and} \quad \widetilde{B}^{-1}_{i,t} * \widetilde{B}^{-1}_{j,u} = q^{\alpha(i,t;j,u)}\, \widetilde{B}^{-1}_{j,u} * \widetilde{B}^{-1}_{i,t}.$$

Here,

$$\beta(i, p; j, s) = \delta_{i,j}(-\delta_{p-s,1} + \delta_{p-s,-1})(\alpha_i, \alpha_i), \tag{4.9}$$

$$\alpha(i, t; j, u) = \begin{cases} \pm(\alpha_i, \alpha_i) & \text{if } (i, t) = (j, u \pm 2), \\ \pm 2(\alpha_i, \alpha_j) & \text{if } d(i, j) = 1 \text{ and } t = u \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

4.2. Quantization $\mathfrak{K}_q(\mathfrak{g})$ *of* $\mathfrak{K}(\mathfrak{g})$. We briefly recall the construction of $\mathfrak{K}_q(\mathfrak{g})$, defined in [25,47,60,67], by mainly following the argument in [24,25]. For each $i \in I$, we define the free $\mathcal{X}_q$-left module

$${}^L\mathcal{X}_{i,q} := \bigoplus_{r:\ (i,r)\in\widetilde{\triangle}_0} \mathcal{X}_q \cdot \widetilde{s}_{i,r} \tag{4.11}$$

whose basis elements are denoted by $\widetilde{s}_{i,r}$. We also regard ${}^L\mathcal{X}_{i,q}$ as a $\mathcal{X}_q$-bimodule by defining right $\mathcal{X}_q$-module action $\cdot$ as follows:

$$\widetilde{s}_{i,r} \cdot \widetilde{m} = q_i^{-2u_{i,r}(\widetilde{m})}\, \widetilde{m} \cdot \widetilde{s}_{i,r}, \tag{4.12}$$

where $\widetilde{m}$ is an $\mathcal{X}_q$-monomial (see Remark 4.11, cf. [25, Lemma 4.6]). Let $\mathcal{X}_{i,q}$ be the quotient of ${}^L\mathcal{X}_{i,q}$ by the $\mathcal{X}_q$-submodule generated by the elements

$$\widetilde{B}_{i,r+1}\, \widetilde{s}_{i,r} - q_i \widetilde{s}_{i,r+2} \quad \text{for } (i, r) \in \widetilde{\triangle}_0. \tag{4.13}$$

By following arguments in [25, Proposotion 4.8] and [5, Lemma 4.3.1], we have the following lemma:

Lemma 4.7. *For each* l *with* $(i, l) \in \widetilde{\triangle}_0$, *the* $\mathcal{X}_q$*-left module* $\mathcal{X}_{i,q}$ *is free over any* $\{\widetilde{s}_{i,r_0}\}$, *where* $(i, r_0) \in \widetilde{\triangle}_0$.

For all $i \in I$, we define

$$\begin{tikzcd} S_{i,q} :\ \mathcal{X}_q \arrow[r, "\widetilde{S}_{i,q}"'] \arrow[rr, bend left, two heads] & {}^L\mathcal{X}_{i,q} \arrow[r, two heads] & \mathcal{X}_{i,q}\ , \end{tikzcd} \tag{4.14}$$

where each map is defined as follows (recall (4.11) for definition of ${}^L\mathcal{X}_{i,q}$):

(a) The map $\widetilde{S}_{i,q}$ is defined by

$$\widetilde{S}_{i,q}(\widetilde{m}) = \frac{1}{q_i^{-2} - 1} \sum_{r:\ (i,r)\in\widetilde{\triangle}_0} \boldsymbol{\mathcal{T}} \widetilde{s}_{i,r},\ \widetilde{m} \boldsymbol{\mathcal{U}}$$

for an $\mathcal{X}_q$-monomial $\widetilde{m}$, where ${}^L\mathcal{X}_{i,q}$ is regarded as the $\mathcal{X}_q$-bimodule. Here $\boldsymbol{\mathcal{T}}\ \boldsymbol{\mathcal{U}}$ denotes the commutator.

(b) The map from ${}^L\mathcal{X}_{i,q}$ to $\mathcal{X}_{i,q}$, denoted by an double-headed arrow, is the surjective map sending an element of ${}^L\mathcal{X}_{i,q}$ to its image in $\mathcal{X}_{i,q}$ (recall (4.13)).

By direct computation, we have the following:

Proposition 4.8. *The map $S_{i,q}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linear map and derivation with respect to $*$, that is,*

$$S_{i,q}(\widetilde{m}_1 * \widetilde{m}_2) = \widetilde{m}_1 \cdot S_{i,q}(\widetilde{m}_2) + S_{i,q}(\widetilde{m}_1) \cdot \widetilde{m}_2, \tag{4.15}$$

where the $\cdot$ indicates the $\mathcal{X}_q$-bimodule actions of $\mathcal{X}_{i,q}$ induced from ${}^L\mathcal{X}_{i,q}$.

Definition 4.9. For $i \in \triangle_0$, we denote by $\mathfrak{K}_{i,q}(\mathfrak{g})$ the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{X}_q$ generated by

$$\widetilde{X}_{i,l} * (1 + q_i^{-1}\widetilde{B}_{i,l+1}^{-1}) \quad \text{and} \quad \widetilde{X}_{j,s}^{\pm 1} \quad \text{for } j \in \triangle_0 \setminus \{i\} \quad \text{and} \quad (i,l), (j,s) \in \widetilde{\triangle}_0.$$

By using the same arguments as in [15,24,25], we have

$$\mathfrak{K}_{i,q}(\mathfrak{g}) = \mathrm{Ker}(S_{i,q}). \tag{4.16}$$

Therefore, we call $S_{i,q}$ *the i-th q-screening operator* with respect to $\mathfrak{K}_{i,q}(\mathfrak{g})$.

Definition 4.10. [47] We set

$$\mathfrak{K}_q(\mathfrak{g}) := \bigcap_{i \in I} \mathfrak{K}_{i,q}(\mathfrak{g})$$

and call it the *quantum virtual Grothendieck ring associated to* $\underline{\mathsf{C}}(t)$.

Remark 4.11. Using the fact that $S_{i,q}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linear derivation (or by its definition with (4.12)), one can check that Then it follows from the definition of $S_{i,q}$, (4.12) and (4.13) that

$$S_{i,q}(\widetilde{X}_{i,l}^{-1} + q_i^{-1}\widetilde{X}_{i,l}^{-1} * \widetilde{B}_{i,l-1}) = (-\widetilde{X}_{i,l}^{-1})\widetilde{s}_{i,l} + (q_i^{-1}\widetilde{X}_{i,l}^{-1} * \widetilde{B}_{i,l-1})\widetilde{s}_{i,l-2} = 0.$$

In fact, $\mathfrak{K}_{i,q}(\mathfrak{g})$ is realized as the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathcal{X}_q$ generated by $\widetilde{X}_{i,l}^{-1} + q_i^{-1}\widetilde{X}_{i,l}^{-1} * \widetilde{B}_{i,l-1}$ and $\widetilde{X}_{j,s}^{\pm 1}$ for $j \in \triangle_0 \setminus \{i\}$ and $(i,l), (j,s) \in \widetilde{\triangle}_0$ (cf. (4.16)).

Remark 4.12. Since the following diagram commutes (cf. [25])

$$\begin{tikzcd} \mathcal{X}_q \arrow[r, "S_{i,q}"] \arrow[d, "\mathrm{ev}_{q=1}"'] & \mathcal{X}_{i,q} \arrow[d, "\mathrm{ev}_{q=1}"] \\ \mathcal{X} \arrow[r, "S_i"'] & \mathcal{X}_i \end{tikzcd} \tag{4.17}$$

where S_i is the i-th screening operator with respect to $\underline{\mathsf{C}}(t)$, we have $\mathrm{ev}_{q=1}\big(\mathfrak{K}_q(\mathfrak{g})\big) \subset \mathfrak{K}(\mathfrak{g})$. However, the opposite inclusion $\mathrm{ev}_{q=1}\big(\mathfrak{K}_q(\mathfrak{g})\big) \supset \mathfrak{K}(\mathfrak{g})$ is not trivial (for non-simply-laced types). We resolve this issue in the next section.

5. Bases of $\mathfrak{K}_q(\mathbf{g})$ and Kazhdan–Lusztig Analogues

Let $(\mathbf{g}, \mathsf{g})$ be a pair in (2.4). It is known in [57,58] (see also [25]) that the basis $\mathbf{F}_q$ of $\mathfrak{K}_q(\mathbf{g})$ with properties (5.1) below can be constructed algorithmically by using a deformed Frenkel–Mukhin (FM for short) algorithm (cf. [15]) with respect to $\mathsf{C}(q)$ (so-called t-algorithm [25]). This basis enables us to construct other important bases of $\mathfrak{K}_q(\mathbf{g})$ (see (5.5), Theorem 5.9). In the second part of this section, we will construct a basis F_q of $\mathfrak{K}_q(\mathsf{g})$ by a deformed FM-algorithm with respect to $\underline{\mathsf{C}}(t)$, and verify that it has similar properties to (5.1) by following the framework in [25]. Moreover, we construct other bases E_q and L_q of $\mathfrak{K}_q(\mathsf{g})$ from the basis F_q in the spirit of [25,58], where analogues of Kazhdan–Lusztig polynomials [49] were studied (see Theorem 5.31, Remarks 5.10 and 5.32).

5.1. *Bases of* $\mathfrak{K}_q(\mathbf{g})$.

Note that $\mathsf{C}(q)$ coincides with $\underline{\mathsf{C}}(t)$ for simply-laced finite types, when we replace q with t. Thus,

throughout this subsection, we switch the roles of q and t.

This makes our notations more compatible with the literature where only simply-laced types are considered, and we presents previously known results in this subsection.

Remark 5.1. When $\mathbf{g}$ is of simply-laced type, the variable $A_{\imath,p}$ in (3.3) coincides with $B_{\imath,p}$ in (3.7) by replacing Y with X. Since this subsection presents previously known results for $\mathbf{g}$, we do not introduce a new notation for $\mathbf{B}_q^-$ in this case, and just denote it by $\mathbf{B}_t^-$ following the above convention. Namely, $\mathbf{B}_t^-$ is the $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $\mathcal{Y}_t$ generated by $\widetilde{A}_{\imath,p}^{-1}$'s for $(\imath, p) \in I^{\mathbf{g}} \times \mathbb{Z}$. In a similar way, we write $\mathbf{B}_t^{-k}$ instead of $\mathbf{B}_q^{-k}$ for $\mathbf{g}$ (see Definition 4.4).

Remark 5.2. Inevitably, we have used several notations for monomials. We recall those notations for convenience of the reader. We say that $m \in \mathcal{X}$ is an *$\mathcal{X}$-monomial* (or just *monomial* if there is no confusion) if m is a product of $X_{i,p}^{\pm 1}$'s for $(i, p) \in I \times \mathbb{Z}$, while $\widetilde{m} \in \mathcal{X}_q$ is said to be an *$\mathcal{X}_q$-monomial* if $\widetilde{m}$ is a product of $\widetilde{X}_{i,p}^{\pm 1}$'s and $q^{\pm 1/2}$ so that $\mathrm{ev}_{q=1}(\widetilde{m})$ becomes an $\mathcal{X}$-monomial. With regard to the $\mathbb{Z}$-algebra anti-involution (4.5), we frequently consider the *bar-invariant monomial* $\widetilde{m}$ (4.6) corresponding to an $\mathcal{X}$-monomial m such that $\mathrm{ev}_{q=1}(\widetilde{m}) = m$ and $\overline{\widetilde{m}} = \widetilde{m}$, which is denoted by $\underline{m}$ for simplicity (see Remark 4.3). Under Convention 4, we denote by $\mathbf{m} \in \mathcal{Y}$ and $\widetilde{\mathbf{m}}, \underline{\mathbf{m}} \in \mathcal{Y}_t$ those monomials for simply-lace types, replacing $X_{i,p}^{\pm 1}, \widetilde{X}_{i,p}^{\pm 1}, q$ with $Y_{\imath,p}^{\pm 1}, \widetilde{Y}_{\imath,p}^{\pm 1}, t$, respectively.

In [25] (cf. [58,60]), the algorithm for constructing basis $\mathbf{F}_t := \{F_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ was proposed, so called *t-algorithm*. The structure and properties of the algorithm can be summarized as follows:

(a) For each dominant $\mathcal{Y}_t$-monomial $\widetilde{\mathbf{m}}$, we construct an element $F_t(\widetilde{\mathbf{m}})$ by adding monomials $\widetilde{\mathbf{m}}' \in \widetilde{\mathbf{m}}\mathbf{B}_t^{-k}$ in an inductive way as k increases from 0. In the process, the coefficient for each monomial is also determined in an inductive way.

(b) If there appears a unique $\widetilde{\mathbf{m}}'$ with the smallest $k \in \mathbb{Z}_{\geqslant 1}$ satisfying

(i) $\widetilde{\mathbf{m}}'$ is anti-dominant and $\widetilde{\mathbf{m}}' \in \widetilde{\mathbf{m}}\mathbf{B}_t^{-k}$ is generated in the performing step,

(ii) any monomial generated in the previous step is contained in $\widetilde{\mathbf{m}}\mathbf{B}_t^{-s}$ $(0 \leqslant s < k)$, not anti-dominant, and strictly larger than $\widetilde{\mathbf{m}}'$ with respect to $\prec_{\mathrm{N}}$,

then, the coefficient of $\widetilde{\mathbf{m}}'$ is contained in $t^{\frac{1}{2}\mathbb{Z}}$. Furthermore, the sum of all monomials with coefficients obtained from the steps so far, denoted by $F_t(\widetilde{\mathbf{m}})$, is contained in the kernel of $S_{\imath,t}$ for all $\imath$. Hence $F_t(\widetilde{\mathbf{m}})$ is an element of $\mathfrak{K}_t(\mathbf{g})$ and the t-algorithm terminates. (5.1)

Furthermore, each $F_t(\widetilde{\mathbf{m}})$ satisfies the following properties:

(1) $F_t(\widetilde{\mathbf{m}}) \in \mathfrak{K}_t(\mathbf{g}) \cap \widetilde{\mathbf{m}}\mathbf{B}_t^{-}$.
(2) $F_t(\widetilde{\mathbf{m}})$ is bar-invariant if $\widetilde{\mathbf{m}}$ is bar-invariant.
(3) Every monomial of $F_t(\widetilde{\mathbf{m}}) - \widetilde{\mathbf{m}}$ is strictly less than $\widetilde{\mathbf{m}}$ with respect to $\prec_{\mathrm{N}}$.

Remark 5.3. Another characterization of $\mathfrak{K}_{\imath,t}(\mathbf{g})$ in Remark 4.11 allows us to consider the lowest ℓ-weight version of the t-algorithm, that is, a t-deformation of *reversed* Frenkel–Mukhin algorithm which is an algorithm starting from the lowest ℓ-weight monomial. For instance, the formulas in [25, Lemma 4.13] can be reformulated in terms of anti-dominant monomial with $\widetilde{A}_{\imath,k}$'s. The reversed algorithm seems to be already known to experts in the theory of q-characters (e.g. see [15], [56]).

Let $\widetilde{\mathbf{m}}_-$ be an anti-dominant (bar-invariant) $\mathcal{Y}_t$-monomial. We denote by $F_t(\widetilde{\mathbf{m}}_-)$ the unique element of $\mathfrak{K}_t(\mathbf{g})$ generated by the *reversed* t-algorithm (referred above) with respect to $\widetilde{\mathbf{m}}_-$. Then one can verify that $F_t(\widetilde{\mathbf{m}}_-)$ satisfies similar properties to (5.1) after modifying notations and terminologies associated with $\widetilde{\mathbf{m}}_-$. For example, the property (3) in (5.1) associated with $\widetilde{\mathbf{m}}_-$ is restated as every monomial appearing in $F_t(\widetilde{\mathbf{m}}_-) - \widetilde{\mathbf{m}}_-$ is strictly *greater* than $\widetilde{\mathbf{m}}_-$ with respect to $\prec_{\mathrm{N}}$. Throughout this section, we often refer to these properties.

Theorem 5.4. [59, Theorem 3.1] [27, Theorem 4.1, Lemma 4.4] *For $(\imath, p), (\imath, s) \in \widetilde{\triangle}_0$ with $p < s$, the element $F_t(\underline{\mathbf{m}}^{(\imath)}[p, s]) \in \mathfrak{K}_t(\mathbf{g})$ is of the form*

$$F_t(\underline{\mathbf{m}}^{(\imath)}[p, s]) = \underline{\mathbf{m}}^{(\imath)}[p, s] * (1 + \widetilde{A}^{-1}_{\imath,s+1} * \chi),$$

where $\underline{\mathbf{m}}^{(\imath)}[p, s] := \underline{\mathbf{m}^{(\imath)}[p, s]}$ and χ is a (non-commutative) $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-polynomial in $\widetilde{A}^{-1}_{\jmath,k+1}$ $(\jmath, k) \in \widetilde{\triangle}_0$. In particular, we have

$$F_t(\underline{\mathbf{m}}^{(\imath)}[p, s]) = F_t(\underline{\mathbf{m}}_-^{(\imath^*)}[p + \mathsf{h}, s + \mathsf{h}]),$$

where $\underline{\mathbf{m}}_-^{(\imath^*)}[p + \mathsf{h}, s + \mathsf{h}] := (\underline{\mathbf{m}}^{(\imath^*)}[p + \mathsf{h}, s + \mathsf{h}])_-$ and

(1) *$F_t(\underline{\mathbf{m}}^{(\imath)}[p, s])$ has the unique dominant (resp. anti-dominant) monomial $\underline{\mathbf{m}}^{(\imath)}[p, s]$ (resp. $\underline{\mathbf{m}}_-^{(\imath^*)}[p + \mathsf{h}, s + \mathsf{h}]$),*

(2) *all* $\mathcal{Y}_t$*-monomials of* $F_t(\underline{\mathbf{m}}^{(\imath)}[p,s]) - \underline{\mathbf{m}}^{(\imath)}[p,s] - \underline{\mathbf{m}}_{-}^{(\imath^*)}[p+\mathsf{h}, s+\mathsf{h}]$ *are products of* $\widetilde{Y}_{\jmath,u}^{\pm1}$ *with* $p < u < s+\mathsf{h}$ *and right-negative.*
(3) *for* $((\imath,p),(\jmath,p) \in \widetilde{\triangle}_0,\ \jmath \neq \imath)$, $F_t(\underline{\mathbf{m}}^{(\imath)}[p,s])$ *and* $F_t(\underline{\mathbf{m}}^{(\jmath)}[p,s])$ *commute; i.e.,* $F_t(\underline{\mathbf{m}}^{(\imath)}[p,s]) * F_t(\underline{\mathbf{m}}^{(\jmath)}[p,s]) = F_t(\underline{\mathbf{m}}^{(\jmath)}[p,s]) * F_t(\underline{\mathbf{m}}^{(\imath)}[p,s])$.

It is well known that, for $r \in 2\mathbb{Z}$ and $\imath \in \triangle_0$,

$$\mathsf{T}_r(F_t(\underline{\mathbf{m}}^{(\imath)}[p,s])) = F_t(\underline{\mathbf{m}}^{(\imath)}[p+r, s+r]), \tag{5.2}$$

where T_r is the $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-algebra automorphism of $\mathcal{Y}_t$ sending $\widetilde{Y}_{\imath,p}$ to $\widetilde{Y}_{\imath,p+r}$.

Theorem 5.5. [25, Theorem 5.11]

(a) *For every dominant* (*resp. anti-dominant*) *monomial* $\widetilde{\mathbf{m}} \in \mathcal{Y}_t$, $F_t(\widetilde{\mathbf{m}})$ *is the unique element in* $\mathfrak{K}_t(\mathbf{g})$ *such that* $\widetilde{\mathbf{m}}$ *is the unique dominant* (*resp. anti-dominant*) *monomial of* $F_t(\widetilde{\mathbf{m}})$.
(b) *Every monomial appearing in* $F_t(\widetilde{\mathbf{m}}) - \widetilde{\mathbf{m}}$ *is strictly less* (*resp. strictly greater*) *than* $\widetilde{\mathbf{m}}$ *with respect to* $\prec_{\mathrm{N}}$.
(c) *The set* $\mathbf{F}_t := \{F_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ *forms a bar-invariant* $\mathbb{Z}[t^{\pm\frac{1}{2}}]$*-basis of* $\mathfrak{K}_t(\mathbf{g})$.

Remark 5.6. We remark that an element in $\mathfrak{K}_t(\mathbf{g})$ is characterized by the multiplicities of its dominant (resp. anti-dominant) monomials by Theorem 5.5. Then it yields that $\mathrm{ev}_{t=1}(F_t(\widetilde{\mathbf{m}})) = F(\mathrm{ev}_{t=1}(\widetilde{\mathbf{m}}))$.

Example 5.7. We present $F_t(\widetilde{Y}_{2,0})$ of type D_4 (cf. [58, Example 5.3.2]) by organizing the monomials appearing in $F_t(\widetilde{Y}_{2,0})$ as a directed graph $\Gamma(\widetilde{Y}_{2,0})$ such that $F_t(\widetilde{Y}_{2,0})$ is the sum of the monomials on the vertices of the directed graph, see (5.4). Note that in this example, we write the $\mathcal{Y}_t$-monomials according to the order given by

$$(\imath,p) < (\jmath,s) \iff (\,p < s\,) \text{ or } (\,p = s \text{ and } \imath < \jmath\,). \tag{5.3}$$

We use the convention of [17,58] for the directed oriented graph $\Gamma(\widetilde{Y}_{2,0})$: For monomials $\widetilde{\mathbf{m}}_1$ and $\widetilde{\mathbf{m}}_2$, we use an colored directed edge $f(t)\ \widetilde{\mathbf{m}}_1 \xrightarrow{\imath,k} g(t)\ \widetilde{\mathbf{m}}_2$ if $\mathrm{ev}_{t=1}(\widetilde{\mathbf{m}}_2) = \mathrm{ev}_{t=1}(\widetilde{\mathbf{m}}_1\widetilde{A}_{\imath,k}^{-1})$, where $f(t), g(t) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$. Then the directed colored graphs $\Gamma(\widetilde{Y}_{2,0})$ of $F_t(\widetilde{Y}_{2,0})$ is given as below:

$$
\begin{tikzcd}[column sep=tiny]
 & & & \widetilde{Y}_{2,0} \arrow[d, "2,1"] & & & \\
 & & & t^2\,\widetilde{Y}_{1,1}*\widetilde{Y}_{3,1}*\widetilde{Y}_{4,1}*\widetilde{Y}^{-1}_{2,2} \arrow[dl, "1,2"'] \arrow[d, "3,2"] \arrow[dr, "4,2"] & & & \\
 & & t\,\widetilde{Y}_{3,1}*\widetilde{Y}_{4,1}*\widetilde{Y}^{-1}_{1,3} \arrow[dl, "3,1"'] \arrow[drr, "4,2"] & t\,\widetilde{Y}_{1,1}*\widetilde{Y}_{4,1}*\widetilde{Y}^{-1}_{3,3} \arrow[dll, "1,2"] \arrow[drr, "4,2"] & t\,\widetilde{Y}_{1,1}*\widetilde{Y}_{3,1}*\widetilde{Y}^{-1}_{4,3} \arrow[dl, "1,2"'] \arrow[dr, "3,2"] & & \\
 & t\,\widetilde{Y}_{4,1}*\widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{1,3}*\widetilde{Y}^{-1}_{3,3} \arrow[dl, "2,3"'] \arrow[drrrrr, "4,2"] & & t\,\widetilde{Y}_{3,1}*\widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{1,3}*\widetilde{Y}^{-1}_{4,3} \arrow[dl, "2,3"'] \arrow[drrr, "3,2"] & & t\,\widetilde{Y}_{1,1}*\widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{3,3}*\widetilde{Y}^{-1}_{4,3} \arrow[dl, "2,3"'] \arrow[dr, "1,2"] & \\
t\,\widetilde{Y}_{4,1}*\widetilde{Y}_{4,3}*\widetilde{Y}^{-1}_{2,4} \arrow[d, "4,4"'] & & t\,\widetilde{Y}_{3,1}*\widetilde{Y}_{3,3}*\widetilde{Y}^{-1}_{2,4} \arrow[d, "3,4"'] & & t\,\widetilde{Y}_{1,1}*\widetilde{Y}_{1,3}*\widetilde{Y}^{-1}_{2,4} \arrow[d, "1,4"'] & & t^2\,\widetilde{Y}^{*2}_{2,2}*\widetilde{Y}^{-1}_{1,3}*\widetilde{Y}^{-1}_{3,3}*\widetilde{Y}^{-1}_{4,3} \arrow[d, "2,3"'] \\
\widetilde{Y}_{4,1}*\widetilde{Y}^{-1}_{4,5} \arrow[d, "4,2"'] & & \widetilde{Y}_{3,1}*\widetilde{Y}^{-1}_{3,5} \arrow[d, "3,2"'] & & \widetilde{Y}_{1,1}*\widetilde{Y}^{-1}_{1,5} \arrow[d, "1,2"'] & & (t^{-1}+t)\,\widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{2,4} \arrow[d, "2,3"'] \\
\widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{4,3}*\widetilde{Y}^{-1}_{4,5} \arrow[dr, "2,3"] & & \widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{3,3}*\widetilde{Y}^{-1}_{3,5} \arrow[dr, "2,3"] & & \widetilde{Y}_{2,2}*\widetilde{Y}^{-1}_{1,3}*\widetilde{Y}^{-1}_{1,5} \arrow[dr, "2,3"] & & t^3\,\widetilde{Y}_{1,3}*\widetilde{Y}_{3,3}*\widetilde{Y}_{4,3}*\widetilde{Y}^{*-2}_{2,2} \arrow[dlllll, "4,4"'] \arrow[dlll, "3,4"'] \arrow[dl, "1,4"] \\
 & t\,\widetilde{Y}_{1,3}*\widetilde{Y}_{3,3}*\widetilde{Y}^{-1}_{2,4}*\widetilde{Y}^{-1}_{4,5} \arrow[dr, "3,4"'] \arrow[drrr, "1,4"] & & t\,\widetilde{Y}_{1,3}*\widetilde{Y}_{4,3}*\widetilde{Y}^{-1}_{2,4}*\widetilde{Y}^{-1}_{3,4} \arrow[dl, "4,4"] \arrow[dr, "1,4"] & & t\,\widetilde{Y}_{3,3}*\widetilde{Y}_{4,3}*\widetilde{Y}^{-1}_{1,5}*\widetilde{Y}^{-1}_{2,4} \arrow[dlll, "4,4"'] \arrow[dl, "3,4"] & \\
 & & \widetilde{Y}_{1,3}*\widetilde{Y}^{-1}_{3,5}*\widetilde{Y}^{-1}_{4,5} \arrow[dr, "1,4"] & \widetilde{Y}_{3,3}*\widetilde{Y}^{-1}_{1,5}*\widetilde{Y}^{-1}_{4,5} \arrow[d, "3,4"] & \widetilde{Y}_{4,3}*\widetilde{Y}^{-1}_{1,5}*\widetilde{Y}^{-1}_{3,5} \arrow[dl, "4,4"'] & & \\
 & & & \widetilde{Y}_{2,4}\widetilde{Y}^{-1}_{1,5}*\widetilde{Y}^{-1}_{3,5}*\widetilde{Y}^{-1}_{4,5} \arrow[d, "2,5"] & & & \\
 & & & t^{-1}\,\widetilde{Y}^{-1}_{2,6} & & &
\end{tikzcd}
\tag{5.4}
$$

For a dominant monomial $\mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}$, we set

$$E_t(\underline{\mathbf{m}}) := t^a \left(\mathop{\overset{\rightarrow}{*}}_{p\in\mathbb{Z}} \left(\mathop{*}_{\imath\in I^{\mathbf{g}};\,(\imath,p)\in\widetilde{\triangle}_0} F_t(\widetilde{Y}_{\imath,p})^{u_{\imath,p}(\mathbf{m})} \right) \right), \tag{5.5}$$

where a is an element in $\frac{1}{2}\mathbb{Z}$ such that $\underline{\mathbf{m}}$ appears in $E_t(\underline{\mathbf{m}})$ with the coefficient 1. Here $\mathop{*}_{\imath} F_t(\widetilde{Y}_{\imath,p})^{u_{\imath,p}(\mathbf{m})}$ is well-defined by Theorem 5.4 (3). Note that $E_t(\underline{\mathbf{m}})$ contains $\underline{\mathbf{m}}$ as its maximal monomial with respect to $\prec_{\mathrm{N}}$. In particular, by Theorem 5.5, we have

$$E_t(\underline{\mathbf{m}}) = F_t(\underline{\mathbf{m}}) + \sum_{\mathbf{m}'\prec_{\mathrm{N}}\mathbf{m}} C_{\mathbf{m},\mathbf{m}'} F_t(\underline{\mathbf{m}}') \tag{5.6}$$

with $C_{\mathbf{m},\mathbf{m}'} \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$. Note that the set $\mathbf{E}_t := \{E_t(\underline{\mathbf{m}}) \mid \underline{\mathbf{m}} \in \mathcal{M}_+^{\mathbf{g}}\}$ also forms a $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-basis since

$$\sharp\{\mathbf{m}' \in \mathcal{M}_+ \mid \mathbf{m}' \prec_{\mathrm{N}} \mathbf{m}\} < \infty \quad \text{for each } \mathbf{m} \in \mathcal{M}_+. \tag{5.7}$$

We call $\mathbf{E}_t$ the *standard basis* of $\mathfrak{K}_t(\mathbf{g})$.

Remark 5.8. We should point out that the t-algorithm (explained in the beginning of Sect. 5.1) might progress infinitely many times. In fact, $F_t(\widetilde{\mathbf{m}})$ was constructed in a completion of $\mathfrak{K}_t(\mathbf{g})$ at first. Interestingly, the property (1) in (5.1) is guaranteed once we prove

$$F_t(\widetilde{Y}_{\imath,p}) \in \mathfrak{K}_t(\mathbf{g}). \tag{5.8}$$

More precisely, (5.8) implies $\mathbf{E}_t \subset \mathfrak{K}_t(\mathbf{g})$. Then it is known (e.g. see the proof of [25, Proposition 6.3] for more detail) that $\mathbf{E}_t$ has the unit-triangular property with $\mathbf{F}_t$, that is, $F_t(\underline{\mathbf{m}})$ can be written as a linear combination of elements in $\mathbf{E}_t \subset \mathfrak{K}_t(\mathbf{g})$, so the proof for (1) in (5.1) is reduced to prove (5.8). Then (5.8) is deduced from [58,60].

Note that $Y_{\iota,p}$ is a minimal element in $\mathcal{M}_+$ with respect to the partial order $\preccurlyeq_{\mathrm{N}}$. Thus (5.6) tells that

$$E_t(\underline{Y_{\iota,p}}) = F_t(\underline{Y_{\iota,p}}).$$

Using the bases $\mathbf{F}_t$ and $\mathbf{E}_t$, the third basis $\mathbf{L}_t := \{L_t(\underline{\mathbf{m}})\}$ of $\mathfrak{K}_t(\mathbf{g})$ has been constructed in an inductive way using $\preccurlyeq_{\mathrm{N}}$ such that

$$E_t(\underline{Y_{\iota,p}}) = F_t(\underline{Y_{\iota,p}}) = L_t(\underline{Y_{\iota,p}}) \tag{5.9}$$

and $L_t(\underline{\mathbf{m}})$ for general $\mathbf{m} \in \mathcal{M}_+$ is characterized as in the following theorem.

Theorem 5.9. [60] (see also [25]) *For a dominant monomial $\mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}$, there exists a unique element $L_t(\underline{\mathbf{m}})$ in $\mathfrak{K}_t(\mathbf{g})$ such that $\overline{L_t(\underline{\mathbf{m}})} = L_t(\underline{\mathbf{m}})$ and*

$$E_t(\underline{\mathbf{m}}) = L_t(\underline{\mathbf{m}}) + \sum_{\mathbf{m}' \prec_{\mathrm{N}} \mathbf{m}} P_{\mathbf{m},\mathbf{m}'}(t) L_t(\underline{\mathbf{m}'}) \quad \text{with } P_{\mathbf{m},\mathbf{m}'}(t) \in t\mathbb{Z}[t]. \tag{5.10}$$

We call $\mathbf{L}_t$ the *canonical basis* of $\mathfrak{K}_t(\mathbf{g})$.

Remark 5.10. In a highly influential paper [49], Kazhdan and Lusztig conjectured a realization of the composition multiplicities of Verma modules for $\mathfrak{g}$ in terms of a certain class of polynomials defined by Iwahori–Hecke algebras, so-called *Kazhdan–Lusztig polynomials* (KL polynomials, for short). The *Kazhdan–Lusztig conjecture* states that the specialization of the KL polynomials at 1 coincides with the composition multiplicities of Verma modules. This is proved independently by Beilinson–Bernstein [2,3] and Brylinski–Kashiwara [6]. Moreover, it is shown in [50] that the KL polynomials can be interpreted as the Poincaré polynomials for local intersection cohomology of Schubert varieties. This geometric interpretation gives the positivity of the KL polynomials.

A similar story has been developed in the representation theory of quantum loop algebras. In [57,58,60], it is proved by Nakajima that the specialization of $P_{\mathbf{m},\mathbf{m}'}(t)$ at $t = 1$ gives the composition multiplicity of $L(\underline{\mathbf{m}'})$ in the standard module $E(\underline{\mathbf{m}})$. Furthermore, $P_{\mathbf{m},\mathbf{m}'}(t)$ coincides with the Poincaré polynomial of intersection cohomology of graded quiver varieties, which implies the positivity of $P_{\mathbf{m},\mathbf{m}'}(t)$. Consequently, the polynomials $P_{\mathbf{m},\mathbf{m}'}(t)$ may be viewed as analogs of KL polynomials. We also remark that there have been recent developments ([18,19]) associated with $P_{\mathbf{m},\mathbf{m}'}(t)$ for the quantum loop algebras beyond ADE-types.

Theorem 5.11. [60]

(a) *For a dominant monomial $\mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}$, every monomial in $L_t(\underline{\mathbf{m}})$ has a quantum positive coefficient; that means, each coefficient of a monomial in $L_t(\underline{\mathbf{m}})$ contained in $\mathbb{Z}_{\geqslant 0}[t^{\pm\frac{1}{2}}]$. In particular, we have* $\mathrm{ev}_{t=1}(L_t(\underline{\mathbf{m}})) = \chi_q(L(\underline{\mathbf{m}}))$.
(b) *For each monomial $\mathbf{m}^{(\imath)}[p,s]$, we have $F_t(\underline{\mathbf{m}^{(\imath)}[p,s]}) = L_t(\underline{\mathbf{m}^{(\imath)}[p,s]})$.*
(c) *The coefficient $P_{\mathbf{m},\mathbf{m}'}(t)$ in (5.10) is actually contained in $t\mathbb{Z}_{\geqslant 0}[t]$.*

Remark 5.12. Let recapitulate the main points in this subsection. From the t-algorithm, we obtain a basis $\{F_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ of $\mathfrak{K}_t(\mathbf{g})$. One crucial step is to prove that $F_t(\widetilde{Y}_{\imath,p})$ is contained in $\mathfrak{K}_t(\mathbf{g})$. Then it is proved in [25,60] that there are frameworks for constructing two other bases $\{E_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ and $\{L_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ of $\mathfrak{K}_t(\mathbf{g})$. In particular, the basis $\{L_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ is constructed using the two other bases through the induction on $\mathcal{M}_+$ via $\preccurlyeq_{\mathrm{N}}$, and there are uni-triangular transition maps (5.6) and (5.10) between the three bases.

As $L_t(\underline{\mathbf{m}})$ can be understood as a t-quantization of $L(\mathbf{m})$ by Theorem 5.11 (a), the T-system among KR modules is also t-quantized as follows:

Theorem 5.13. [29, Proposition 5.6] (see also [59, Section 4]) *For $(\imath, p), (\imath, s) \in \widetilde{\triangle}_0$ with $p < s$, there exists an equation in $\mathfrak{K}_t(\mathbf{g})$:*

$$\begin{aligned} L_t\big(\underline{\mathbf{m}}^{(\imath)}[p,s)\big) * L_t\big(\underline{\mathbf{m}}^{(\imath)}(p,s]\big) &= t^x L_t\big(\underline{\mathbf{m}}^{(\imath)}[p,s]\big) * L_t\big(\underline{\mathbf{m}}^{(\imath)}(p,s)\big) \\ &\quad + t^y \prod_{\jmath;\, d(\imath,\jmath)=1} L_t\big(\underline{\mathbf{m}}^{(\jmath)}(p,s)\big), \end{aligned} \tag{5.11}$$

where $L_t\big(\underline{\mathbf{m}}^{(\jmath)}(p,s)\big)$ and $L_t\big(\underline{\mathbf{m}}^{(\jmath')}(p,s)\big)$ $(\jmath, \jmath' \in I)$ are pairwise commute and

$$y = \frac{\widetilde{\mathsf{r}}_{\imath,\imath}(2(s-p)+1) + \widetilde{\mathsf{r}}_{\imath,\imath}(2(s-p)-1)}{2} \quad \text{and} \quad x = y-1.$$

5.2. Bases of $\mathfrak{K}_q(\mathsf{g})$. Assume that g is of *non-simply-laced finite type*. Since $\mathsf{C}(q)$ can not be identified with $\underline{\mathsf{C}}(t)$ anymore,

> *we come back to the convention of the previous sections (not the previous subsection).*

Let $\mathfrak{K}_{i,q}^{\infty}(\mathsf{g})$ is the completion of $\mathfrak{K}_{i,q}(\mathsf{g})$ satisfying $\mathfrak{K}_{i,q}^{\infty}(\mathsf{g}) \cap \mathcal{X}_q = \mathfrak{K}_{i,q}(\mathsf{g}) = \mathrm{Ker}(S_{i,q})$ (see Lemma 5.36). Then we define

$$\mathfrak{K}_q^{\infty}(\mathsf{g}) = \bigcap_{i \in I} \mathfrak{K}_{i,q}^{\infty}(\mathsf{g}),$$

which can be viewed as a completion of $\mathfrak{K}_q(\mathsf{g})$ as in [25, Section 5.2]. By following the construction of $\{F_t(\underline{\mathbf{m}}) \mid \mathbf{m} \in \mathcal{M}_+^{\mathbf{g}}\}$ in [24,25], we can establish an analog of the t-algorithm in [25, Definition 5.19] on $\mathfrak{K}_q^{\infty}(\mathsf{g})$, called *$q$-algorithm* in the setting of Sect. 4.2.

Roughly speaking, the algorithm is given inductively by computing all possible quantized i-expansions while determining "correct" coefficients in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ of resulting monomials, so that the resulting element is contained in $\mathfrak{K}_q^{\infty}(\mathsf{g})$ (consequently, $\mathfrak{K}_q(\mathsf{g})$) (cf. [15, Section 5.5], [25, Definition 5.19]).

Let us summarize the q-algorithm. For $\mathcal{X}$-monomials m_1 and m_2, we use an colored directed edge $m_1 \xrightarrow{i,p} m_2$ if $m_2 = m_1 B_{i,p}^{-1}$. For $\mathcal{X}$-monomials m and m', we say that m' is generated from m if there exists a finite sequence $\{(i_1,p_1),(i_2,p_2),\cdots,(i_\ell,p_\ell)\} \subset I \times \mathbb{Z}$ such that

$$m = m_0 \xrightarrow{i_1,p_1} m_1 \xrightarrow{i_2,p_2} \cdots \xrightarrow{i_{\ell-1},p_{\ell-1}} m_\ell = m',$$

where m_k is an $\mathcal{X}$-monomial of $E_J(m_{k-1})$ defined in (5.34) for $1 \leqslant k \leqslant \ell$ for some $J \subset I$.

Let $\widetilde{m}$ be a dominant $\mathcal{X}_q$-monomial. Then we collect all possible $\mathcal{X}$-monomials generated from $\widetilde{m}$, and then enumerate them by

$$\cdots < m_v < \cdots < m_0 = \widetilde{m}, \tag{5.12}$$

where $<$ is a total order compatible with $\prec_{\mathrm{N}}$ at $q = 1$. Let $\widetilde{m}_v$ be an $\mathcal{X}_q$-monomial determined *inductively* from assuming the existence of

$$F_{J,q}(\widetilde{m}_u) \in \bigcap_{i\in J} \mathfrak{K}^{\infty}_{i,q}(\mathsf{g})$$

for some $u < v$ and $J \subset I$, where $F_{J,q}(\widetilde{m}_u)$ contains $\widetilde{m}_u$ as a unique J-dominant monomial. Note that $\widetilde{m}_v$ is uniquely determined up to a coefficient in $q^{\frac{1}{2}\mathbb{Z}}$. For this reason, we fix an order defined as in (5.3) on spectral parameters to write them uniquely.

For $J \subsetneq I$, we denote by $(s(m_v)(q))_{v\in\mathbb{Z}_{\geqslant 0}}$ and $(s_J(m_v)(q))_{v\in\mathbb{Z}_{\geqslant 0}}$ the sequences in $\mathbb{Z}[q^{\pm\frac{1}{2}}]^{\mathbb{Z}_{\geqslant 0}}$ defined inductively as follows:

$$\begin{aligned} s_J(m_v)(q) &= \sum_{u<v} (s(m_u)(q) - s_J(m_u))\, c_J(q)(m_v), \\ s(m_v)(q) &= \begin{cases} s_J(m_v)(q) & \text{if } m_v \text{ is not } J\text{-dominant}, \\ 0 & \text{if } m_v \text{ is dominant}, \end{cases} \end{aligned} \tag{5.13}$$

where $s(m_0)(q) = 1$, $s_J(m_0)(q) = 0$ and $c_J(q)(m_v)$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficient of $\widetilde{m}_v$ in $F_{J,q}(\widetilde{m}_u)$. Here we assume that $F_{J,q}(\widetilde{m}_u) = 0$ if m_u is not J-dominant, so $c_J(q)(m_u) = 0$ in this case. Note that the sequences $(s(m_v)(q))_{v\in\mathbb{Z}_{\geqslant 0}}$ and $(s_J(m_v)(q))_{v\in\mathbb{Z}_{\geqslant 0}}$ are well-defined, and $s(m_v)(q)$ does not depend on the choice of $J \subsetneq I$. This can be proved as in [25, Lemma 5.20]. Finally, we define

$$F_q(\widetilde{m}) := \sum_{v\geqslant 0} s(\widetilde{m}_v)(q)\, \widetilde{m}_v. \tag{5.14}$$

Remark 5.14. We need to make some remarks on the q-algorithm:

(1) One can prove the validity of the q-algorithm in our setting using the arguments in [25]. More precisely, define

$$\overline{\mathsf{R}}(t) := \mathsf{D}\underline{\mathsf{C}}(t) = \left(\overline{\mathsf{R}}_{ij}(t)\right)_{i,j\in I^{\mathsf{g}}}.$$

Then we consider the $\mathbb{C}$-algebra $\mathscr{H}$ generated by $b_i[m]$ for $i \in I$, $m \in \mathbb{Z}\setminus\{0\}$ and central elements c_r for $r > 0$, with defining relations

$$\mathcal{T} b_i[m], b_j[r] \mathcal{U} = \delta_{m,-r}(t^m - t^{-m})\overline{\mathsf{R}}_{ij}(t^m)c_{|m|},$$

where $i, j \in I$ and $m, r \in \mathbb{Z}\setminus\{0\}$. Put $x_j[m] := \sum_{i\in I} \widetilde{\underline{\mathsf{C}}}_{ij}(t^m) b_i[m] \in \mathscr{H}$ for $j \in I$ and $m \in \mathbb{Z}$. Note that

$$\mathcal{T} x_i[m], x_j[r] \mathcal{U} = \delta_{m,-r}\underline{\mathsf{R}}_{ji}(t^m)(t^m - t^{-m})c_{|m|}.$$

With these definitions, one can check that the formulations in [26] recover precisely the quantum torus $\mathcal{X}_q$ (Definition 4.1), the q-deformed screening operators $S_{i,q}$ (in the sense of Remark 4.12), the quantum virtual Grothendieck ring $\mathfrak{K}_q(\mathfrak{g})$ (Definition 4.10), and so on. Now the q-algorithm (in $\mathfrak{K}_q^\infty(\mathfrak{g})$) follows verbatim from [25] without any complications.

(2) In the computational view point, the elements $\widetilde{X}_{i,p}$ and $q_i\widetilde{B}_{i,p}$ in the q-algorithm play the roles of $\widetilde{Y}_{i,p}$ and $t\widetilde{A}_{i,p}$ in the t-algorithm. However, for a dominant $\mathcal{X}_q$-monomial $\widetilde{m}$ such that $\mathrm{ev}_{q=1}(\widetilde{m}) = \overline{\sigma}(\mathrm{ev}_{t=1}(\widetilde{\mathbf{m}}))$, it does not mean that $F_q(\widetilde{m})$ is obtained from $F_t(\widetilde{\mathbf{m}})$ by the above replacement for non-simply-laced types (see also Remark 5.17).

(3) We emphasize that the set of monomials in (5.12) might be infinitely countable, but that the number of nonzero $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients in (5.14) should be finite. Thus the formula on the right-hand side of (5.14) makes sense, and it is actually a finite sum. The proof is non-trivial. See Proposition 5.20 and (5.25) below (cf. [26]).

We say that

(a) the q-algorithm is *well-defined for step r* if there exist $s_J(m_v)(q)$ and $s(m_v)(q)$ defined in (5.13) for $0 \leqslant v \leqslant r$, and
(b) the q-algorithm *never fails* if it is well-defined for all steps.

When the q-algorithm never fails, it yields, for each dominant monomial $\widetilde{m}$ in $\mathcal{X}_q$,

$$F_q(\widetilde{m}) \in \mathfrak{K}_q^\infty(\mathfrak{g}) \tag{5.15}$$

containing $\widetilde{m}$ as a unique dominant monomial. The q-algorithm is well-defined and never fails by following the framework of [25, Section 5.3]. Since the proof is quite parallel to [25] as indicated in Remark 5.14 (1), we do not provide a proof here. Instead, we will detail the algorithm and its consequences in Example 5.16. Before presenting the example, we record the following consequence.

Proposition 5.15. *Let $\mathfrak{K}_q^{\infty,f}(\mathfrak{g})$ be the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-submodule of $\mathfrak{K}_q^\infty(\mathfrak{g})$ generated by elements in $\mathfrak{K}_q^\infty(\mathfrak{g})$ with finitely many dominant monomials. Then the set*

$$\left\{ F_q(\widetilde{m}) \mid \widetilde{m} \text{ is a dominant monomial in } \mathcal{X}_q \right\}$$

is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis of $\mathfrak{K}_q^{\infty,f}(\mathfrak{g})$. Indeed, $\mathfrak{K}_q^{\infty,f}(\mathfrak{g})$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathfrak{K}_q^\infty(\mathfrak{g})$

Example 5.16. We illustrate the q-algorithm by computing $F_q(\widetilde{X}_{2,5})$ for type G_2. For $n \in \mathbb{Z} \setminus \{0\}$, we use $\widetilde{X}_{i,p}^n$ to denote $\widetilde{X}_{i,p}^{*n}$ for simplicity. We compute all $\mathcal{X}_q$-monomials with non-zero $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients starting from $\widetilde{X}_{2,5}$ as follows:

<u>**Step 1**</u>. For $J = \{2\}$, since $E_J(X_{2,5}) = F_J(X_{2,5}) = X_{2,5}(1 + B_{2,6}^{-1}) = X_{2,5} + X_{1,6}^3X_{2,7}^{-1}$, we will determine the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficient of $X_{1,6}^3X_{2,7}^{-1}$. For this, we compute $F_{2,q}(\widetilde{X}_{2,5})$. Note that $\widetilde{X}_{2,5} * (1 + q^{-3}\widetilde{B}_{2,6}^{-1}) \in \ker(S_{2,q})$ (see (4.16) and Remark 4.11). We compute

$$\widetilde{B}_{2,6} = \underline{X_{2,5}X_{2,7}X_{1,6}^{-3}} = q^3\widetilde{X}_{1,6}^{-3} * \widetilde{X}_{2,7} * \widetilde{X}_{2,5}.$$

It follows from Proposition 5.35 and Lemma 5.36 that $F_{2,q}(\widetilde{X}_{2,5}) = \widetilde{X}_{2,5} * (1+q^{-3}\widetilde{B}_{2,6}^{-1})$. Then we have

$$s_J(X_{1,6}^3X_{2,7}^{-1})(q) = q^3.$$

Since $X_{1,6}^3 X_{2,7}^{-1}$ is not 2-dominant, we see that $s(X_{1,6}^3 X_{2,7}^{-1})(q) = q^3$. Hence, we obtain the new term $q^3 \widetilde{X}_{1,6}^3 * \widetilde{X}_{2,7}^{-1}$ in this step.

Step 2. Let us consider $\widetilde{X}_{1,6}^3 * \widetilde{X}_{2,7}^{-1}$. In this step, set $J = \{1\}$. Since

$$E_J(X_{1,6}^3 X_{2,7}^{-1}) = F_J(X_{1,6})^3 X_{2,7}^{-1} = \left(X_{1,6}(1 + B_{1,7}^{-1})\right)^3 X_{2,7}^{-1} = (X_{1,6} + X_{1,8}^{-1} X_{2,7})^3 X_{2,7}^{-1},$$

we will determine the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients of $X_{1,6}^2 X_{1,8}^{-1}$, $X_{1,6} X_{1,8}^{-2} X_{2,7}$ and $X_{1,8}^{-3} X_{2,7}^2$. Let us only do the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficient of $X_{1,6}^2 X_{1,8}^{-1}$ explicitly because the computation for the other monomials is almost identical. By **Step 1**, we know that

$$s_J(X_{1,6}^3 X_{2,7}^{-1})(q) = 0, \quad s(X_{1,6}^3 X_{2,7}^{-1})(q) = q^3. \tag{5.16}$$

On the other hand, since $S_{i,q}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-derivation (see Proposition 4.8 and Remark 4.11) and (4.16) holds, we have

$$(\widetilde{X}_{1,6} + q^{-1} \widetilde{X}_{1,6} * \widetilde{B}_{1,7}^{-1})^3 * X_{2,7}^{-1} \in \ker(S_{1,q}).$$

Here $\widetilde{B}_{1,7} = \widetilde{X}_{2,7}^{-1} * \widetilde{X}_{1,8} * \widetilde{X}_{1,6}$. Thus, it follows from Proposition 5.35 and Lemma 5.36 that

$$F_{1,q}(\widetilde{X}_{1,6}^3 * \widetilde{X}_{2,7}^{-1}) = (\widetilde{X}_{1,6} + q^{-1} \widetilde{X}_{1,6} * \widetilde{B}_{1,7}^{-1})^3 * \widetilde{X}_{2,7}^{-1}.$$

The expansion of $(\widetilde{X}_{1,6} + q^{-1} \widetilde{X}_{1,6} * \widetilde{B}_{1,7}^{-1})^3$ is

$$\begin{aligned} &\widetilde{X}_{1,6}^3 + (q^{-2}+1+q^2) \widetilde{X}_{1,6}^2 * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} \\ &\quad + (q^3+q^5+q^7) \widetilde{X}_{1,6} * \widetilde{X}_{2,7}^2 * \widetilde{X}_{1,8}^{-2} + q^{15} \widetilde{X}_{2,7}^3 * \widetilde{X}_{1,8}^{-3}, \end{aligned} \tag{5.17}$$

where $\widetilde{X}_{1,6}^2 * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} * \widetilde{X}_{2,7}^{-1} = q^{-3} \widetilde{X}_{1,6}^2 * \widetilde{X}_{1,8}^{-1}$ due to Definition 4.1. By (5.16) and (5.17), we have

$$s_J(X_{1,6}^2 X_{1,8}^{-1})(q) = q^3(q^{-5} + q^{-3} + q^{-1}) = q^{-2} + 1 + q^2.$$

Since $X_{1,6}^2 X_{1,8}^{-1}$ is not 1-dominant, we set $s(X_{1,6}^2 X_{1,8}^{-1})(q) = s_J(X_{1,6}^2 X_{1,8}^{-1})(q)$. Hence, we have a new term $(q^{-2} + 1 + q^2) \widetilde{X}_{1,6}^2 * \widetilde{X}_{1,8}^{-1}$. Similarly, one can compute the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients of $X_{1,6} X_{1,8}^{-2} X_{2,7}$ and $X_{1,8}^{-3} X_{2,7}^2$. As a result, we have the following terms in this step:

$$(q^{-2} + 1 + q^2) \widetilde{X}_{1,6}^2 * \widetilde{X}_{1,8}^{-1}, \quad (1 + q^2 + q^4) \widetilde{X}_{1,6} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2}, \quad q^9 \widetilde{X}_{2,7}^2 * \widetilde{X}_{1,8}^{-3}.$$

Step 3. Let us consider the monomials $X_{1,6} X_{2,7} X_{1,8}^{-2}$ and $X_{2,7}^2 X_{1,8}^{-3}$. In this step, set $J = \{2\}$. Then we observe

$$\begin{aligned} E_J(X_{1,6} X_{2,7} X_{1,8}^{-2}) &= X_{1,6} F_J(X_{2,7}) X_{1,8}^{-2} = X_{1,6} X_{2,7} X_{1,8}^{-2} + X_{1,6} X_{1,8} X_{2,9}^{-1}, \\ E_J(X_{2,7}^2 X_{1,8}^{-3}) &= \left(F_J(X_{2,7})\right)^2 X_{1,8}^{-3} = X_{2,7}^2 X_{1,8}^{-3} + 2 X_{2,7} X_{2,9}^{-1} + X_{1,8}^3 X_{2,7}^{-2}. \end{aligned}$$

As in **Step 1** and **Step 2**, we compute new terms obtained from them, respectively:

$$(1+q^2+q^4)\widetilde{X}_{1,6}*\widetilde{X}_{2,7}*\widetilde{X}_{1,8}^{-2}\xrightarrow{2,8}(q^{-3}+q^{-1}+q)\widetilde{X}_{1,6}*\widetilde{X}_{1,8}*\widetilde{X}_{2,9}^{-1},$$
$$q^9\widetilde{X}_{2,7}^2*\widetilde{X}_{1,8}^{-3}\xrightarrow{2,8}(q^{-3}+q^3)\widetilde{X}_{2,7}*\widetilde{X}_{2,9}^{-1}\xrightarrow{2,8}q^6\widetilde{X}_{1,8}^3*\widetilde{X}_{2,9}^{-2}.$$

Step 4. Set $J=\{1\}$. Let us consider the monomial $X_{1,6}X_{1,8}X_{2,9}^{-1}$ which is 1-dominant. As in **Step 2**, one can check

$$\begin{aligned}F_{1,q}(\widetilde{X}_{1,6}*\widetilde{X}_{1,8}*\widetilde{X}_{2,9}^{-1}) &= \widetilde{X}_{1,6}*\widetilde{X}_{1,8}*(1+q^{-1}\widetilde{B}_{1,9}^{-1}+q^{-3}\widetilde{B}_{1,9}^{-1}*\widetilde{B}_{1,7}^{-1})*\widetilde{X}_{2,9}^{-1}\ \big(\in \ker(S_{1,q})\big)\\ &= \widetilde{X}_{1,6}*\widetilde{X}_{1,8}*\widetilde{X}_{2,9}^{-1}+q^{-1}\widetilde{X}_{1,6}*\widetilde{X}_{1,10}^{-1}+q\widetilde{X}_{2,7}*\widetilde{X}_{1,8}^{-1}*\widetilde{X}_{1,10}^{-1}.\end{aligned}$$

On the other hand, we have seen

$$s(X_{1,6}X_{1,8}X_{2,9}^{-1})(q)=q^{-3}+q^{-1}+q \quad \text{and} \quad s_J(X_{1,6}X_{1,8}X_{2,9}^{-1})(q)=0.$$

Hence, we obtain new terms $(q^{-4}+q^{-2}+1)\widetilde{X}_{1,6}*\widetilde{X}_{1,10}^{-1}$ and $(q^{-2}+1+q^2)\widetilde{X}_{2,7}*\widetilde{X}_{1,8}^{-1}*\widetilde{X}_{1,10}^{-1}$.

Step 5. In the case of $X_{1,8}^3X_{2,9}^{-2}$ which is 1-dominant, the computation in this step is similar to **Step 2** (up to shift of spectral parameter). As a result, we have

$$q^6\widetilde{X}_{1,8}^3*\widetilde{X}_{2,9}^{-2}\xrightarrow{1,9}(q^{-2}+1+q^2)\widetilde{X}_{1,8}^2*\widetilde{X}_{2,9}^{-1}*\widetilde{X}_{1,10}^{-1}\xrightarrow{1,9}(q^{-3}+q^{-1}+1)$$
$$\widetilde{X}_{1,8}*\widetilde{X}_{2,10}^{-2}\xrightarrow{1,9}q^3\widetilde{X}_{2,9}*\widetilde{X}_{1,10}^{-3}$$

Step 6. We consider $X_{2,9}X_{1,10}^{-3}$ which is 2-dominant. One can check that we have new term $q^{-3}\widetilde{X}_{2,11}^{-1}$ from $X_{2,9}X_{1,10}^{-3}$ by similar computations as in **Step 1**.

Now the sum of all $\mathcal{X}_q$-monomials obtained from the steps so far, denoted by $F_q(\widetilde{X}_{2,5})$, can be read in (5.19) below. Then it follows from **Step 1**–**Step 6** that $F_q(\widetilde{X}_{2,5})\in\ker(S_{i,q})$ for $i=1,2$. For example, $F_q(\widetilde{X}_{2,5})$ is written as

$$\begin{aligned}F_q(\widetilde{X}_{2,5}) &= F_{1,q}(\widetilde{X}_{2,5})+q^3F_{1,q}(\widetilde{X}_{1,6}^3*\widetilde{X}_{2,7}^{-1})+(q^{-3}+q^{-1}+q)F_{1,q}(\widetilde{X}_{1,6}*\widetilde{X}_{1,8}*\widetilde{X}_{2,9}^{-1})\\ &\quad+(q^{-3}+q^3)F_{1,q}(\widetilde{X}_{2,7}*\widetilde{X}_{2,9}^{-1})+q^6F_{1,q}(\widetilde{X}_{1,8}^3*\widetilde{X}_{2,9}^{-2})+q^{-3}F_{1,q}(\widetilde{X}_{2,11}^{-1})\end{aligned} \tag{5.18}$$

which is clearly in $\ker(S_{1,q})$ (recall Proposition 5.35). The case of $i=2$ is similar.

The $\mathcal{X}_q$-monomial $q^{-3}\widetilde{X}_{2,11}^{-1}$ satisfies the obvious counterpart of (b) in (5.1) with respect to $F_q(\widetilde{X}_{2,5})$, that is, the q-algorithm terminates at this step and the Laurent polynomial $F_q(\widetilde{X}_{2,5})$ is in $\mathfrak{K}_q(\mathfrak{g})$. Indeed, for a dominant $\mathcal{X}_q$-monomial $\widetilde{m}$, the q-algorithm allows us to write $F_q(\widetilde{m})$ as a linear combination of $F_{i,q}(\,\cdot\,)$'s over $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ as in (5.18). This is a general fact that plays a key role in proving Proposition 5.20 (see Sect. 5.3 for more details).

The directed colored graphs $\Gamma(X_{2,5})$ and $\Gamma(\widetilde{X}_{2,5})$ of $F(X_{2,5})\in\mathfrak{K}(\mathfrak{g})$ and $F_q(\widetilde{X}_{2,5})\in\mathfrak{K}_q(\mathfrak{g})$, respectively, are given as follows:

$$
\begin{tikzcd}[column sep=tiny]
 & X_{2,5} \arrow[d, "2,6"] & \\
 & X_{1,6}^3X_{2,7}^{-1} \arrow[d, "1,7"] & \\
 & 3X_{1,6}^2X_{1,8}^{-1} \arrow[d, "1,7"] & \\
 & 3X_{1,6}X_{2,7}X_{1,8}^{-2} \arrow[dl, "1,7"'] \arrow[dr, "2,8"] & \\
X_{2,7}^2X_{1,8}^{-3} \arrow[d, "2,8"] & & 3X_{1,6}X_{1,8}X_{2,9}^{-1} \arrow[d, "1,9"] \\
2X_{2,7}X_{2,9}^{-1} \arrow[d, "2,8"] & & 3X_{1,6}X_{1,10}^{-1} \arrow[d, "1,7"] \\
X_{1,8}^3X_{2,9}^{-2} \arrow[dr, "1,9"'] & & 3X_{2,7}X_{1,8}^{-1}X_{1,10}^{-1} \arrow[dl, "2,8"] \\
 & 3X_{1,8}^2X_{2,9}^{-1}X_{1,10}^{-1} \arrow[d, "1,9"] & \\
 & 3X_{1,8}X_{1,10}^{-2} \arrow[d, "1,9"] & \\
 & X_{2,9}X_{1,10}^{-3} \arrow[d, "2,10"] & \\
 & X_{2,11}^{-1} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & \widetilde{X}_{2,5} \arrow[d, "2,6"] & \\
 & q^3\widetilde{X}_{1,6}^3 * \widetilde{X}_{2,7}^{-1} \arrow[d, "1,7"] & \\
 & (q^{-2}+1+q^2)\widetilde{X}_{1,6}^2 * \widetilde{X}_{1,8}^{-1} \arrow[d, "1,7"] & \\
 & (1+q^2+q^4)\widetilde{X}_{1,6} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2} \arrow[dl, "1,7"'] \arrow[dr, "2,8"] & \\
q^9\widetilde{X}_{2,7}^2 * \widetilde{X}_{1,8}^{-3} \arrow[d, "2,8"] & & (q^{-3}+q^{-1}+q)\widetilde{X}_{1,6} * \widetilde{X}_{1,8} * \widetilde{X}_{2,9}^{-1} \arrow[d, "1,9"] \\
(q^{-3}+q^3)\widetilde{X}_{2,7} * \widetilde{X}_{2,9}^{-1} \arrow[d, "2,8"] & & (q^{-4}+q^{-2}+1)\widetilde{X}_{1,6} * \widetilde{X}_{1,10}^{-1} \arrow[d, "1,7"] \\
q^6\widetilde{X}_{1,8}^3 * \widetilde{X}_{2,9}^{-2} \arrow[dr, "1,9"'] & & (q^{-2}+1+q^2)\widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} * \widetilde{X}_{1,10}^{-1} \arrow[dl, "2,8"] \\
 & (q^{-2}+1+q^2)\widetilde{X}_{1,8}^2 * \widetilde{X}_{2,9}^{-1} * \widetilde{X}_{1,10}^{-1} \arrow[d, "1,9"] & \\
 & (q^{-3}+q^{-1}+1)\widetilde{X}_{1,8} * \widetilde{X}_{1,10}^{-2} \arrow[d, "1,9"] & \\
 & q^3\widetilde{X}_{2,9} * \widetilde{X}_{1,10}^{-3} \arrow[d, "2,10"] & \\
 & q^{-3}\widetilde{X}_{2,11}^{-1} &
\end{tikzcd}
\tag{5.19}
$$

Here $F(X_{2,5})$ is obtained from $\mathrm{ev}_{t=1}(\mathsf{T}_5(F_t(\widetilde{Y}_{2,0})))$ (see Example 5.7 for $F_t(\widetilde{Y}_{2,0})$) by folding the $\mathcal{Y}$-monomials (recall Remark 5.6).

Remark 5.17. Let us recall that $F(X_{\bar{\imath},p})$ is obtained from $F(Y_{\imath,p})$ by folding the monomials of $F(Y_{\imath,p})$ via (3.6) (see Corollary 3.10). However, we would like to emphasize that we do not know yet whether $F_q(\widetilde{X}_{\bar{\imath},p})$ could be obtained directly from $F_t(\widetilde{Y}_{\imath,p})$ by folding $\mathcal{Y}_t$-monomials with some modification of coefficients in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$.

Definition 5.18. For $f \in \mathcal{X}_q$, we set

$$
\begin{aligned}
\mathcal{M}(f) &:= \{\mathrm{ev}_{q=1}(\widetilde{m}) \mid \widetilde{m} \text{ is a monomial in } f\}, & \underline{\mathcal{M}}(f) &:= \{\underline{m} \mid m \in \mathcal{M}(f)\},\\
\mathcal{M}_+(f) &:= \{\mathrm{ev}_{q=1}(\widetilde{m}) \mid \widetilde{m} \text{ is a dominant monomial in } f\}, & \underline{\mathcal{M}}_+(f) &:= \{\underline{m} \mid m \in \underline{\mathcal{M}}_+(f)\}.
\end{aligned}
$$

For $P \in \mathfrak{K}_q(\mathsf{g})$, a monomial m in P is called *maximal monomial* (resp. *minimal monomial*) if its ℓ-weight is not lower (resp. not higher) than any other monomial in P with respect to $\preccurlyeq_{\mathrm{N}}$.

Lemma 5.19. (cf. [15, Lemma 5.6]) *For $P \in \mathfrak{K}_q(\mathsf{g})$, any maximal* (*resp. minimal*) *monomial in P is dominant* (*resp. anti-dominant*).

Proof. Let us first consider a maximal monomial in P, denoted by $\widetilde{m}$. Take $i \in I$. By Definition 4.9 and Proposition 4.6, we have

$$
P \in \mathfrak{K}_{i,q}(\mathsf{g}) = \mathbb{Z}[q^{\pm\frac{1}{2}}][\widetilde{X}_{j,l}^{\pm1}]_{(j,l)\in\widetilde{\triangle}_0,\, j\neq i} \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}][\widetilde{X}_{i,l} + q_i^{-1}\widetilde{X}_{i,l} * \widetilde{B}_{i,l+1}^{-1}]_{(i,l)\in\widetilde{\triangle}_0}.
$$

Hence, the element P can be written in the following form:

$$P = \sum \widetilde{m}_{(1)} * \widetilde{p}_{(2)},$$

where $\widetilde{m}_{(1)} \in \mathbb{Z}[q^{\pm\frac{1}{2}}][\widetilde{X}_{j,l}]_{(j,l)\in\widetilde{\triangle}_0,\, j\neq i}$ are monomials and, $\widetilde{p}_{(2)} \in \mathbb{Z}[q^{\pm\frac{1}{2}}][\widetilde{X}_{i,l} + q_i^{-1}\widetilde{X}_{i,l} * \widetilde{B}_{i,l+1}^{-1}]_{(i,l)\in\widetilde{\triangle}_0}$ are of the form

$$\widetilde{p}_{(2)} = n\, c(q) \overrightarrow{\underset{\substack{(i,l)\in\widetilde{\triangle}_0,\\ \text{finite}}}{*}} (\widetilde{X}_{i,l} + q_i^{-1}\widetilde{X}_{i,l} * \widetilde{B}_{i,l+1}^{-1})^{n_{i,l}}$$

for some $n_{i,l} \in \mathbb{Z}_{\geqslant 1}$, $n \in \mathbb{Z}$ and $c(q) \in q^{\frac{1}{2}\mathbb{Z}}$. In particular, the maximal monomial $\widetilde{m}$ is a monomial in $\widetilde{m}_{(1)} * \widetilde{p}_{(2)}$. Since $X_{i,l}B_{i,l+1}^{-1} \prec_{\mathrm{N}} X_{i,l}$ the monomial $\widetilde{m}$ should be obtained from $\widetilde{m}_{(1)}$ and $\widetilde{X}_{i,l}$'s,. Otherwise, it contradicts the assumption that m is a maximal monomial. Since $i \in I$ is arbitrary and $P \in \mathfrak{K}_q(\mathsf{g})$, the maximal monomial m should be dominant. In the case of minimal monomials, the proof is almost identical because of another characterization of $\mathfrak{K}_{i,q}(\mathsf{g})$ in Remark (4.11). □

The following proposition plays a crucial role in proving fundamental results established on $\mathfrak{K}_q(\mathsf{g})$.

Proposition 5.20. *For $p < s$, let $m^{(i)}[p,s]$ be given such that $\overline{\sigma}(\mathbf{m}^{(\imath)}[p,s]) = m^{(i)}[p,s]$ (i.e. $\bar{\imath} = i$).*

(1) *For each $m' \in \mathcal{M}(F_q(\underline{m}^{(i)}[p,s]))$, there exists $\mathbf{m}' \in \mathcal{M}(F_t(\underline{\mathbf{m}}^{(\imath)}[p,s]))$ such that $\overline{\sigma}(\mathbf{m}') = m'$.*
(2) *We have $F_q(\underline{m}^{(i)}[p,s]) \in \mathfrak{K}_q(\mathsf{g})$.*

Proof. We will give a proof of Proposition 5.20 in Sect. 5.3. □

Definition 5.21. We call an element of the form $F_q(\underline{m}^{(\imath)}[p,s])$ a *KR-polynomial*. In particular, we call $F_q(X_{i,p})$ a *fundamental polynomial*. We also call a monomial of the form $m^{(\imath)}[p,s]$ a *KR-monomial*.

Corollary 5.22. *For $p < s$, let $m^{(i)}[p,s]$ be such that $\overline{\sigma}(\mathbf{m}^{(\imath)}[p,s]) = m^{(i)}[p,s]$ (i.e. $\bar{\imath} = i$). Then we have*

$$\overline{\sigma}\left(\mathcal{M}(F_t(\underline{\mathbf{m}}^{(\imath)}[p,s]))\right) = \mathcal{M}(F_q(\underline{m}^{(i)}[p,s])).$$

Proof. The inclusion $\supset$ follows from Proposition 5.20 (1). Let us prove the opposite inclusion $\subset$. Let $m \in \overline{\sigma}\left(\mathcal{M}(F_t(\underline{\mathbf{m}}^{(\imath)}[p,s]))\right)$ be an $\mathcal{X}$-monomial, where we write $m = \overline{\sigma}(\mathbf{m})$ for some $\mathcal{Y}$-monomial $\mathbf{m} \in \mathcal{M}(F_t(\underline{\mathbf{m}}^{(\imath)}[p,s]))$. We have seen

$$\mathrm{ev}_{t=1}\left(F_t(\underline{\mathbf{m}}^{(\imath)}[p,s])\right) = F(\mathbf{m}^{(\imath)}[p,s]) \tag{5.20}$$

(see Remark 5.6), and then the quantum positivity for $F_t(\underline{\mathbf{m}}^{(\imath)}[p,s])$ in Theorem 5.11 with (5.20) implies that all the coefficients of $F(\mathbf{m}^{(\imath)}[p,s])$ should be positive. In particular, the coefficient of $\mathbf{m}$ in $F(\mathbf{m}^{(\imath)}[p,s])$ is positive. Since it follows from Corollary 3.10 and Theorem 5.4 that

$$\overline{\sigma}\left(F(\mathbf{m}^{(i)}[p,s])\right) = F(m^{(i)}[p,s]), \tag{5.21}$$

the $\mathcal{X}$-monomial m appears in $F(m^{(i)}[p,s])$ with a positive coefficient. But, we have

$$\mathrm{ev}_{q=1}\left(F_q(\underline{m}^{(i)}[p,s])\right) = F(m^{(i)}[p,s]) \tag{5.22}$$

(see Corollary 5.28), which implies that there exists a term $f(q)\underline{m}$ in $F_q(\underline{m}^{(i)}[p,s])$ such that $\mathrm{ev}_{q=1}(f(q)\underline{m}) = f(1)m$ is a term in $F(m^{(i)}[p,s])$ with $f(1) > 0$. □

Proposition 5.23. *For each* $(i,p) \in \widetilde{\blacktriangle}_0$, *we have*

(a) $F_q(\mathsf{X}_{i,p}) = F_q(\mathsf{X}^{-1}_{i^*,p+\mathsf{h}})$ *contains only one anti-dominant monomial* $\mathsf{X}^{-1}_{i^*,p+\mathsf{h}}$.

(b) *All* $\mathcal{X}_q$*-monomials of* $F_q(\mathsf{X}_{i,p}) - \mathsf{X}_{i,p} - \mathsf{X}^{-1}_{i^*,p+\mathsf{h}}$ *are products of* $\widetilde{X}^{\pm 1}_{j,u}$ *with* $p < u < p+\mathsf{h}$.

(c) $F_q(\widetilde{X}_{i,p})$ *and* $F_q(\widetilde{X}_{j,p})$ $((i,p),(j,p) \in \widetilde{\blacktriangle}_0,\ j \neq i)$ *commute.*

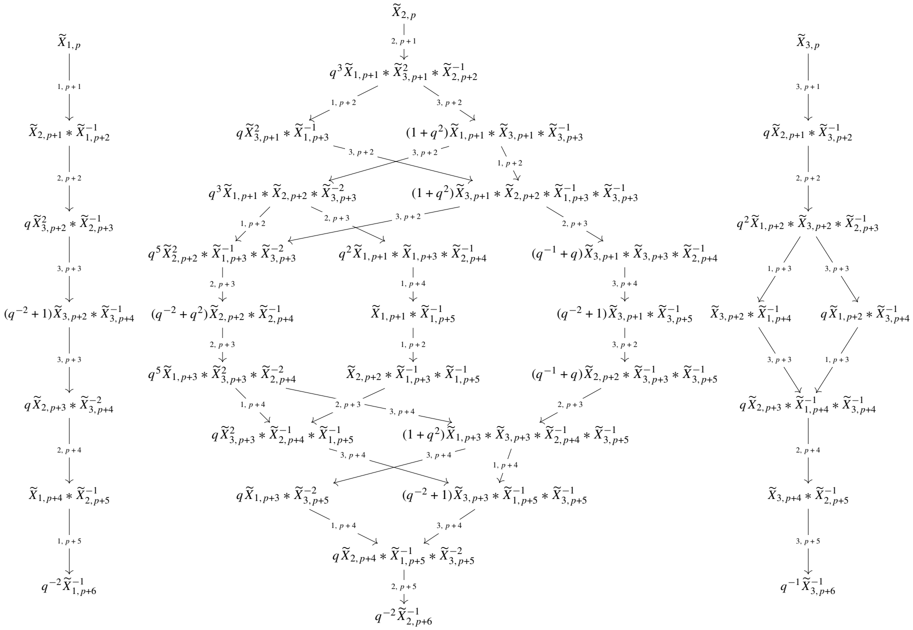
Proof. Since $F_q(\mathsf{X}_{i,p})$ is an element in $\mathfrak{K}_q(\mathfrak{g})$, it contains an anti-dominant monomial by Lemma 5.19. Then Theorem 5.4 and Proposition 5.20 tell that $F_q(\mathsf{X}_{i,p})$ has the unique antidominant monomial $\mathsf{X}^{-1}_{i^*,p+\mathsf{h}}$. Thus (a) follows. By (3.10), (b) follows from (a) and Proposition 5.20. Finally, (c) follows from the same argument as in [25, Lemma 5.12 (iv)]. □

Example 5.24. As in Example 5.16 for type G_2, one may compute the formula of $F_q(\widetilde{X}_{1,10})$ to obtain

$$\widetilde{X}_{1,10} + q^2\widetilde{X}_{2,11}\widetilde{X}^{-1}_{1,12} + q^2\widetilde{X}^2_{1,12}\widetilde{X}^{-1}_{2,13} + (q^{-1}+q)\widetilde{X}_{1,12}\widetilde{X}^{-1}_{1,14} + q^3\widetilde{X}_{2,13}\widetilde{X}^{-2}_{1,14} + \widetilde{X}_{1,14}\widetilde{X}^{-1}_{2,15} + q^{-1}\widetilde{X}^{-1}_{1,16}.$$

Then $F_q(\mathsf{X}_{1,10}) = (\mathsf{X}_{1,10} * \widetilde{X}^{-1}_{1,10})F_q(\widetilde{X}_{1,10}) = q^{\frac{1}{2}}F_q(\widetilde{X}_{1,10}) \in \mathfrak{K}_q(\mathfrak{g})$ is bar-invariant. Note that $\widetilde{X}_{1,10} * \widetilde{X}_{2,10} = \widetilde{X}_{2,10} * \widetilde{X}_{1,10}$ and there is no dominant $\mathcal{X}_q$-monomial in $\mathcal{M}_+(F_q(\widetilde{X}_{1,10}) * F_q(\widetilde{X}_{2,10}))$ except for $\widetilde{X}_{1,10} * \widetilde{X}_{2,10}$ (cf. Example 5.16). Hence we have $F_q(\widetilde{X}_{1,10}) * F_q(\widetilde{X}_{2,10}) = F_q(\widetilde{X}_{2,10}) * F_q(\widetilde{X}_{1,10})$.

Example 5.25. By the q-algorithm starting from $\widetilde{X}_{i,p}$ as in Example 5.16, one can compute the explicit formulas of $F_q(\widetilde{X}_{i,p})$ for $1 \leqslant i \leqslant 3$ of the finite type B_3 as follows:



For a dominant monomial $m \in \mathcal{M}_+^{\mathfrak{g}}$, we set

$$E_q(\underline{m}) := q^b \left(\mathop{\overset{\rightarrow}{*}}_{p \in \mathbb{Z}} \left(\mathop{*}_{i \in I;\, (i,p) \in \triangle_0} F_q(\mathsf{X}_{i,p})^{u_{i,p}(m)} \right) \right) \in \mathfrak{K}_q(\mathfrak{g}), \tag{5.23}$$

where b is an element in $\frac{1}{2}\mathbb{Z}$ such that $\underline{m}$ appears in $E_q(\underline{m})$ with the coefficient 1. By Proposition 5.20, we have

$$E_q(\underline{m}) \in \mathfrak{K}_q(\mathfrak{g}) \tag{5.24}$$

and there are finitely many dominant monomials in $E_q(\underline{m})$. As we regard $E_q(\underline{m})$ as an element of $\mathfrak{K}_q^{\infty,f}(\mathfrak{g})$ (recall Proposition 5.15), we obtain a uni-triangular transition map as in (5.6) between $\{E_q(\underline{m})\}$ and $\{F_q(\underline{m})\}$ in $\mathfrak{K}_q^{\infty,f}(\mathfrak{g})$ by Proposition 5.15:

$$E_q(\underline{m}) = F_q(\underline{m}) + \sum_{m' \prec_{\mathrm{N}} m} C_{m,m'} F_q(\underline{m'}) \quad \text{in } \mathfrak{K}_q^{\infty,f}(\mathfrak{g}), \tag{5.25}$$

where $C_{m,m'} \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Note that the summation in (5.25) is finite since $E_q(\underline{m})$ has finitely many dominant monomials. Hence, (5.25) implies that $F_q(\underline{m})$ can be written as a linear combination of $E_q(\underline{m'})$ for $\underline{m'} \preccurlyeq_{\mathrm{N}} \underline{m}$, so $F_q(\underline{m}) \in \mathfrak{K}_q(\mathfrak{g})$ by (5.23) and (5.24). Until now, we have proved the following.

Proposition 5.26. *The sets*

$$\mathsf{E}_q := \{E_q(\underline{m}) \mid m \in \mathcal{M}_+^{\mathfrak{g}}\} \quad \text{and} \quad \mathsf{F}_q := \{F_q(\underline{m}) \mid m \in \mathcal{M}_+^{\mathfrak{g}}\}$$

are $\mathbb{Z}[q^{\pm\frac{1}{2}}]$*-bases of* $\mathfrak{K}_q(\mathfrak{g})$*, respectively.*

In particular, we call E_q the *standard basis* of $\mathfrak{K}_q(\mathfrak{g})$. Now let us further investigate the basis F_q of $\mathfrak{K}_q(\mathfrak{g})$, which is characterized as follows:

Theorem 5.27. *Let $\widetilde{m} \in \mathcal{X}_q$ be a dominant (resp. anti-dominant) monomial.*

(a) *The Laurent (non-commutative) polynomial $F_q(\widetilde{m})$ is the unique element in $\mathfrak{K}_q(\mathfrak{g})$ such that $\widetilde{m}$ is the unique dominant (resp. anti-dominant) monomial occurring in $F_q(\widetilde{m})$.*
(b) *Every monomial in $F_q(\widetilde{m}) - \widetilde{m}$ is strictly less (resp. greater) than $\widetilde{m}$ with respect to $\prec_{\mathrm{N}}$.*
(c) *The set F_q forms a bar-invariant $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis of $\mathfrak{K}_q(\mathfrak{g})$.*

Proof. We prove only the dominant case because the proof for the anti-dominant case is almost identical.

Let us first prove (a). Thanks to (5.25), $F_q(\widetilde{m})$ can be written as a linear combination of $E_q(\widetilde{m}')$ for $\widetilde{m}' \preccurlyeq_{\mathrm{N}} \widetilde{m}$, where the sum is finite due to Proposition 5.20(2). Hence, $F_q(\widetilde{m}) \in \mathfrak{K}_q(\mathfrak{g})$. Note that $F_q(\widetilde{m})$ has the unique dominant $\mathcal{X}_q$-monomial $\widetilde{m}$ by its construction through the q-algorithm (see (5.13)).

Let $G_q(\widetilde{m})$ be another element in $\mathfrak{K}_q(\mathfrak{g})$ such that $\widetilde{m}$ is the unique dominant $\mathcal{X}_q$-monomial occurring in $G_q(\widetilde{m})$. Suppose that $F_q(\widetilde{m})-G_q(\widetilde{m}) \neq 0$. Then $F_q(\widetilde{m})-G_q(\widetilde{m})$ contains a maximal $\mathcal{X}_q$-monomial $\widetilde{m}'$ different from $\widetilde{m}$. Since $F_q(\widetilde{m}) - G_q(\widetilde{m}) \in \mathfrak{K}_q(\mathfrak{g})$, it follows from Lemma 5.19 that the $\mathcal{X}_q$-monomial $\widetilde{m}'$ is dominant. This implies that $G_q(\widetilde{m})$ has a dominant $\mathcal{X}_q$-monomial not equal to $\widetilde{m}$, which contradicts the assumption on $G_q(\widetilde{m})$.

Second, (b) is a direct consequence of the q-algorithm. Finally, let us prove (c). The linear independence follows from the uniqueness of the dominant $\mathcal{X}_q$-monomial of $F_q(\underline{m})$. Take an element $\chi \in \mathfrak{K}_q(\mathfrak{g})$. We enumerate $\mathcal{M}_+(\chi)$ by $m_0,\ m_1,\ \ldots,\ m_L$. Let us write $\lambda_k \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ be the coefficients of $\underline{m}_k$ in χ for $k = 0, 1, \ldots, L$. Then, the element $\chi - \sum_{k=0}^{L} \lambda_k F_q(\underline{m}_k) \in \mathfrak{K}_q(\mathfrak{g})$ has no dominant $\mathcal{X}_q$-monomial. If it is non-zero, then it has at least one dominant $\mathcal{X}_q$-monomials by Lemma 5.19, which yields a contradiction. Hence, we conclude that the set $\{F_q(\underline{m}) \mid m \in \mathcal{M}_+^{\mathfrak{g}}\}$ generates $\mathfrak{K}_q(\mathfrak{g})$. □

Corollary 5.28. *Let $\widetilde{m} \in \mathcal{X}_q$ be a dominant monomial. Then we have*

$$\mathrm{ev}_{q=1}(F_q(\widetilde{m})) = F(\mathrm{ev}_{q=1}(\widetilde{m})).$$

Proof. It follows from (4.17) that $\mathrm{ev}_{q=1}(F_q(\widetilde{m})) \in \mathfrak{K}(\mathfrak{g})$, where $\mathrm{ev}_{q=1}(F_q(\widetilde{m}))$ has the unique dominant monomial $\mathrm{ev}_{q=1}(\widetilde{m}) \in \mathcal{X}$ by Theorem 5.27 (a). Thus our assertion is proved from Theorem 3.9(2). □

For an interval $[a,b]$, $i \in I$, $(i,t) \in \widetilde{\triangle}_0$ and $k \in \mathbb{Z}_{\geqslant 1}$, we define

$$m^{(i)}[a,b] := \prod_{\substack{(i,p)\in\widetilde{\triangle}_0 \\ p\in[a,b]}} X_{i,p} \quad \text{and} \quad m^{(i)}_{k,t} := \prod_{s=0}^{k-1} X_{i,t+2s}. \tag{5.26}$$

We define $m^{(i)}(a,b]$, $m^{(i)}[a,b)$, and $m^{(i)}(a,b)$ in a similar way. As in the simply-laced cases (5.2), we have

$$\mathsf{T}_r(F_q(\underline{m}^{(i)}[p,s])) = F_q(\underline{m}^{(i)}[p+r,s+r]) \quad \text{for any } r \in 2\mathbb{Z}, \tag{5.27}$$

where $r \in 2\mathbb{Z}$ and T_r is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra automorphism of $\mathcal{X}_q$ sending $\widetilde{X}_{i,p}$ to $\widetilde{X}_{i,p+r}$.

Proposition 5.29. *For* $(i,p),(i,s) \in \widetilde{\Delta}_0$ *with* $p < s$, *the element* $F_q(\underline{m}^{(i)}[p,s])$ *is of the form*

$$F_q(\underline{m}^{(i)}[p,s]) = \underline{m}^{(i)}[p,s] * (1 + \widetilde{B}^{-1}_{i,s+1} * \chi) \tag{5.28}$$

where χ *is a (non-commutative)* $\mathbb{Z}[q^{\pm\frac{1}{2}}]$*-polynomial in* $\widetilde{B}^{-1}_{j,k+1}$, $(j,k) \in \widetilde{\Delta}_0$. *In particular, we have*

$$F_q(\underline{m}^{(i)}[p,s]) = F_q(\underline{m}_{-}^{(i^*)}[p+\mathsf{h}, s+\mathsf{h}]) \tag{5.29}$$

and

(1) $F_q(\underline{m}^{(i)}[p,s])$ *contains the unique dominant monomial* $\underline{m}^{(i)}[p,s]$,
(2) $F_q(\underline{m}^{(i)}[p,s])$ *contains the unique anti-dominant monomial* $\underline{m}_{-}^{(i^*)}[p+\mathsf{h}, s+\mathsf{h}]$,
(3) *all* $\mathcal{X}_q$*-monomials of* $F_q(\underline{m}^{(i)}[p,s]) - \underline{m}^{(i)}[p,s] - \underline{m}_{-}^{(i^*)}[p+\mathsf{h}, s+\mathsf{h}]$ *are product of* $\widetilde{X}^{\pm 1}_{j,u}$ *with* $p < u < s+\mathsf{h}$ *and right-negative.*

Proof. (1) follows from Theorem 5.27 (a). (2) and (5.29) follow from the reversed version of the q-algorithm (see Remark 5.3) and (1). Finally, (5.28) and (3) are the direct consequences of Theorem 5.4 and Proposition 5.20. □

Conjecture 1. *For* $(i,p),(i,s) \in \widetilde{\Delta}_0$ *with* $p < s$, *every monomial in* $F_q(\underline{m}^{(i)}[p,s])$ *has a quantum positive coefficient; that means, each coefficient of a monomial in* $F_q(\underline{m}^{(i)}[p,s])$ *is contained in* $\mathbb{Z}_{\geqslant 0}[q^{\pm\frac{1}{2}}]$.

Remark 5.30. In the proof of Corollary 5.22, we have seen that the coefficients of monomials in $F(\underline{m}^{(i)}[p,s])$ are positive. In Sect. 8, we will provide a quantum cluster algebra theoretic algorithm for computing $F_q(\underline{m}^{(i)}[p,s])$, which starts from an initial quantum cluster variable $\underline{m}^{(i)}[p,s]$ (see Proposition 8.6 below). This may be viewed as an evidence of Conjecture 1, which is compatible with the quantum positivity conjecture of quantum cluster algebras ([4]).

By the following theorem, we have the third basis, denoted by

$$\mathsf{L}_q := \{L_q(\underline{m}) \mid m \in \mathcal{M}_+^{\mathfrak{g}}\},$$

and called the *canonical basis* of $\mathfrak{K}_q(\mathfrak{g})$. We remark that the reason why we call it the canonical basis is further explained in [33].

Theorem 5.31. *For* $m \in \mathcal{M}_+^{\mathfrak{g}}$, *there exists a unique element* $L_q(\underline{m})$ *in* $\mathfrak{K}_q(\mathfrak{g})$ *such that*

(a) $\overline{L_q(\underline{m})} = L_q(\underline{m})$,
(b) $E_q(\underline{m}) = L_q(\underline{m}) + \sum_{m' \prec_{\mathrm{N}} m} P_{m,m'}(q) L_q(\underline{m}')$ *with* $P_{m,m'}(q) \in q\mathbb{Z}[q]$.

Proof. For $\underline{m} \in \mathcal{M}_+^{\mathfrak{g}}$, we will construct $L_q(\underline{m})$ inductively using some dominant $\mathcal{X}_q$-monomials below, which are all less than $\underline{m}$ with respect to the Nakajima order $\preccurlyeq_{\mathrm{N}}$.
Step 1. Let us first collect all dominant $\mathcal{X}_q$-monomials obtained from $\underline{m}$ in an inductive way. Let $\underline{\mathcal{M}}_1 := \underline{\mathcal{M}}_+\big(E_q(\underline{m})\big) = \big\{\underline{m}_{1,1}, \underline{m}_{1,2}, \ldots, \underline{m}_{1,\ell_1} = \underline{m}\big\}$. Then we define

$$\underline{\mathcal{M}}_n := \bigcup_{1 \leqslant k \leqslant \ell_{n-1}} \underline{\mathcal{M}}_+\big(E_q(\underline{m}_{n-1,k})\big),$$

where $\underline{\mathcal{M}}_{n-1} = \left\{ \underline{m}_{n-1,1}, \underline{m}_{n-1,2}, \ldots, \underline{m}_{n-1,\ell_{n-1}} \right\}$ for $n \geqslant 2$. Note that

$$\underline{\mathcal{M}}_{+}\big(E_q(\underline{m})\big) = \underline{\mathcal{M}}_1 \subset \underline{\mathcal{M}}_2 \subset \underline{\mathcal{M}}_3 \subset \cdots$$

The above chain has finite length, that is, there exists N such that $\underline{\mathcal{M}}_n = \underline{\mathcal{M}}_{n+1}$ for $n \geqslant N$ because we can apply the same argument as in the proof of [25, Lemma 3.13 and Lemma 3.14]. For simplicity, let us relabel the dominant $\mathcal{X}_q$-monomials in $\underline{\mathcal{M}}_N$ as follows:

$$\underline{\mathsf{m}}_1 < \underline{\mathsf{m}}_2 < \cdots < \underline{\mathsf{m}}_M = \underline{m}. \tag{5.30}$$

where $<$ is also a total order compatible with $\preccurlyeq_{\mathrm{N}}$. In particular, $E_q(\underline{\mathsf{m}}_1)$ has no dominant $\mathcal{X}_q$-monomial other than $\underline{\mathsf{m}}_1$ by construction.

Step 2. We construct $L_q(\underline{m})$ by inductive argument on (5.30) as follows. Since $E_q(\underline{\mathsf{m}}_1)$ has the unique dominant $\mathcal{X}_q$-monomial $\underline{\mathsf{m}}_1$ by construction, we have $E_q(\underline{\mathsf{m}}_1) = F_q(\underline{\mathsf{m}}_1)$. If we set $L_q(\underline{\mathsf{m}}_1) = E_q(\underline{\mathsf{m}}_1)$, then the initial step is done because $\overline{E_q(\underline{\mathsf{m}}_1)} = \overline{F_q(\underline{\mathsf{m}}_1)} = F_q(\underline{\mathsf{m}}_1) = E_q(\underline{\mathsf{m}}_1)$.

Suppose that $L_q(\underline{\mathsf{m}}_k)$ is well-defined and uniquely determined for $1 \leqslant k \leqslant M-1$. By the property (b), one can write

$$L_q(\underline{\mathsf{m}}_k) = E_q(\underline{\mathsf{m}}_k) + \sum_{\underline{\mathsf{m}}_l \prec_{\mathrm{N}} \underline{\mathsf{m}}_k} Q_{\underline{\mathsf{m}}_l,\underline{\mathsf{m}}_k}(q) E_q(\underline{\mathsf{m}}_l).$$

By (5.25), $L_q(\underline{\mathsf{m}}_k)$ can be written as a linear combination of $F_q(\underline{\mathsf{m}}_l)$ for $1 \leqslant l \leqslant k$. In particular, the coefficient of $F_q(\underline{\mathsf{m}}_k)$ is 1 due to the property (a). Hence, the finiteness described in (5.30) implies that

$$F_q(\underline{\mathsf{m}}_k) \text{ can be written as a linear combination of } L_q(\underline{\mathsf{m}}_l) \text{ for } 1 \leqslant l \leqslant k. \tag{5.31}$$

By replacing $F_q(\underline{m}')$ in (5.25) with (5.31), we have

$$E_q(\underline{m}) = F_q(\underline{m}) + \sum_{1 \leqslant l \leqslant M-1} \alpha_l(q) L_q(\underline{\mathsf{m}}_l). \tag{5.32}$$

Let us take $\beta_l(q) \in \mathbb{Z}[q^{\pm 1}]$ such that $\beta_l(q)$ is symmetric in q and q^{-1}, and $\alpha_l(q) - \beta_l(q) \in q\mathbb{Z}[q]$ for all $1 \leqslant l \leqslant L-1$. This is possible by the following way. Let us write $\alpha_l(q)$ by $\alpha_l^+(q) + \alpha_l^0(q) + \alpha_l^-(q)$, where $\alpha_l^{\pm}(q) \in q^{\pm 1}\mathbb{Z}[q^{\pm 1}]$ and $\alpha_l^0(q) \in \mathbb{Z}$. Then we define $\beta_l(q) = \beta_l^+(q) + \beta_l^0(q) + \beta_l^-(q)$ by setting $\beta_l^+(q) = \alpha_l^-(q^{-1})$, $\beta_l^-(q) = \alpha_l^-(q)$ and $\beta_l^0(q) = \alpha_l^0(q)$. Now, we define

$$L_q(\underline{m}) = F_q(\underline{m}) + \sum_{1 \leqslant l \leqslant M-1} \beta_l(q) L_q(\underline{\mathsf{m}}_l) \in \mathfrak{K}_q(\mathsf{g}).$$

Then, $L_q(\underline{m})$ satisfies the properties (a) and (b) due to the our choice of $\beta_l(q)$, which is the desired element of $\mathfrak{K}_q(\mathsf{g})$. Note that it follows from Proposition 5.26 and (b) that L_q is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-basis of $\mathfrak{K}_q(\mathsf{g})$.

Step 3. Let us prove the uniqueness of $L_q(\underline{m})$. Assume that $L'_q(\underline{m}) \in \mathfrak{K}_q(\mathsf{g})$ satisfies (a) and (b). By (5.30) and (b), we have

$$L'_q(\underline{\mathsf{m}}_1) = E_q(\underline{\mathsf{m}}_1) = L_q(\underline{\mathsf{m}}_1).$$

By induction on (5.30), we suppose that $L_q(\underline{\mathsf{m}_k}) = L'_q(\underline{\mathsf{m}_k})$ for $1 \leqslant k \leqslant M-1$. By (b) and induction hypothesis, $E_q(\underline{m})$ is written as

$$E_q(\underline{m}) = L_q(\underline{m}) + \sum_{1 \leqslant k \leqslant M-1} P_{m,\mathsf{m}_k}(q) L_q(\underline{\mathsf{m}_k}) = L'_q(\underline{m}) + \sum_{1 \leqslant k \leqslant M-1} P'_{m,\mathsf{m}_k}(q) L_q(\underline{\mathsf{m}_k}),$$

where $P_{m,\mathsf{m}_k}(q)$, $P'_{m,\mathsf{m}_k}(q) \in q\mathbb{Z}[q]$. Hence we have

$$L_q(\underline{m}) = L'_q(\underline{m}) + \sum_{1 \leqslant k \leqslant M-1} \left(P'_{m,\mathsf{m}_k}(q) - P_{m,\mathsf{m}_k}(q)\right) L_q(\underline{\mathsf{m}_k}). \tag{5.33}$$

By taking the bar involution on both sides of (5.33), it follows from (a) that for $1 \leqslant k \leqslant M-1$,

$$\overline{P'_{m,\mathsf{m}_k}(q) - P_{m,\mathsf{m}_k}(q)} = P_{m,\mathsf{m}_k}(q) - P'_{m,\mathsf{m}_k}(q) \in q\mathbb{Z}[q] \cap q^{-1}\mathbb{Z}[q^{-1}] = \{0\}.$$

This implies that $L'_q(\underline{m}) = L_q(\underline{m})$ by (5.33). □

Remark 5.32. In the viewpoint of Kazhdan–Lusztig theory (explained briefly in Remark 5.10), we regard the polynomials $P_{m,m'}(q)$'s as new KL-type polynomials, which generalize Nakajima's KL-type polynomials, since the t-quantized Cartan matrices for types ADE are equal to the quantum Cartan matrices and the basis in Theorem 5.31 essentially coincides with Nakajima's as explained in [25,26]. It would be very interesting to find a geometric or representation theoretic interpretation behind $P_{m,m'}(q)$ in the spirit of Kazhdan–Lusztig theory.

Remark 5.33. We emphasize that the basis $\mathsf{L}_q = \{L_q(\underline{m}) \mid m \in \mathcal{M}_+^{\mathsf{g}}\}$ of $\mathfrak{K}_q(\mathsf{g})$ is quite different from the $\mathbf{L}_t$ of $\mathfrak{K}_t(\mathbf{g}) \simeq \mathcal{K}_{\mathfrak{t}}(\mathscr{C}_{\mathbf{g}}^0)$, that is, $L_q(\underline{m})$ cannot be obtained from $L_t(\underline{\mathbf{m}})$ by folding $\mathcal{Y}_t$-monomials with some modification of coefficients in $\mathbb{Z}[t^{\pm\frac{1}{2}}]$, where $m = \overline{\sigma}(\mathbf{m})$. We give an example to illustrate this phenomenon. Let us consider $L_t(\widetilde{Y}_{1,1})$ and $L_t(\widetilde{Y}_{4,-2})$ of the finite type A_5. One may observe that $L_t(\widetilde{Y}_{1,1})$ q-commutes with $L_t(\widetilde{Y}_{4,-2})$, which implies that $L_t(\widetilde{Y}_{1,1} * \widetilde{Y}_{4,-2})$ coincides with $L_t(\widetilde{Y}_{1,1}) * L_t(\widetilde{Y}_{4,-2})$ up to $q^{\mathbb{Z}}$ [29, Corollary 5.5]. On the other hand, for type C_3, $L_q(\widetilde{X}_{1,1})$ does not q-commute with $L_q(\widetilde{X}_{2,-2})$. This implies that $L_q(\widetilde{X}_{1,1} * \widetilde{X}_{2,-2})$ is not equal to $L_q(\widetilde{X}_{1,1}) * L_q(\widetilde{X}_{2,-2})$ up to $q^{\mathbb{Z}}$. In fact, $L_q(\widetilde{X}_{1,1} * \widetilde{X}_{2,-2})$ has two dominant $\mathcal{X}_q$-monomials, while $L_t(\widetilde{Y}_{1,1} * \widetilde{Y}_{4,-2})$ has only one dominant $\mathcal{Y}_t$-monomial.

Conjecture 2. *For $(i, p), (i, s) \in \widetilde{\mathbf{\Delta}}_0$ with $p < s$, we have*

$$L_q(\underline{m^{(i)}}[p, s]) = F_q(\underline{m^{(i)}}[p, s]),$$

where $\underline{m^{(i)}}[p, s] := \underline{m^{(i)}[p, s]}$ denotes the bar-invariant $\mathcal{X}_q$-monomial corresponding to $m^{(i)}[p, s]$ (5.26) as in Remark 4.3.

Example 5.34. Let us illustrate Theorem 5.31 in the case of $L_q(\underline{X_{2,5}X_{1,10}})$ for type G_2. *Step 1*. By (5.23), we have

$$E_q(\underline{X_{2,5}X_{1,10}}) = q^{\frac{3}{2}} F_q(\mathsf{X}_{2,5}) * F_q(\mathsf{X}_{1,10}).$$

Let us recall the formulas of $F_q(\mathsf{X}_{2,5})$ and $F_q(\mathsf{X}_{1,10})$ in Examples 5.16 and 5.24, respectively. Then we observe that there exist two bar-invariant dominant $\mathcal{X}_q$-monomials with $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients in $E_q(\underline{X_{2,5}X_{1,10}})$, namely, $\underline{X_{2,5}X_{1,10}}$ and $\left(q^{-1} + q + q^3\right) \underline{X_{1,6}}$.

Step 2. By *Step 1*, we have

$$E_q(\underline{X_{2,5}X_{1,10}}) = F_q(\underline{X_{2,5}X_{1,10}}) + \left(q^{-1} + q + q^3\right) F_q(\underline{X_{1,6}}),$$

which corresponds to (5.32) in this case, that is, $M = 2$ and $\alpha_1(q) = q^{-1} + q + q^3$. Set $\beta_1(q) = q + q^{-1}$ by construction in the proof of Theorem 5.31. Then we have

$$L_q(\underline{X_{2,5}X_{1,10}}) = F_q(\underline{X_{2,5}X_{1,10}}) + \left(q^{-1} + q\right) F_q(\underline{X_{1,6}}),$$

which is bar-invariant. Note that $L_q(\underline{X_{2,5}X_{1,10}})$ has two dominant $\mathcal{X}_q$-monomials $\underline{X_{2,5}X_{1,10}}$ and $\underline{X_{1,6}}$. Moreover, we verify

$$E_q(\underline{X_{2,5}X_{1,10}}) = L_q(\underline{X_{2,5}X_{1,10}}) + P_{X_{2,5}X_{1,10},X_{1,6}}(q)L_q(\underline{X_{1,6}}),$$

where $P_{X_{2,5}X_{1,10},X_{1,6}}(q) = q^3 \in q\mathbb{Z}_{\geqslant 0}[q]$, that is, $L_q(\underline{X_{2,5}X_{1,10}})$ is the unique element in $\mathfrak{K}_q(\mathsf{g})$ satisfying the properties (a) and (b) in Theorem 5.31.

5.3. Proof of Proposition 5.20. To prove Proposition 5.20, we utilize some analogues of the results in [25], where we will skip some proof of them when they can be obtained from the corresponding arguments in [25].

For $J^{\mathsf{g}} \subset I^{\mathsf{g}}$, we set $\overline{J^{\mathsf{g}}} = \{\bar{\imath} \mid \imath \in J^{\mathsf{g}}\} \subset I^{\mathfrak{g}}$. Let $J \subset I$ be given such that $J = \overline{J^{\mathsf{g}}}$ for some $J^{\mathsf{g}} \subset I^{\mathsf{g}}$. Let us define $\mathfrak{K}_J(\mathfrak{g}) \subset \mathcal{X}$ as follows:

$$\mathfrak{K}_J(\mathfrak{g}) = \bigcap_{j\in J}\left(\mathbb{Z}[X_{k,l}^{\pm 1} \mid (k,l) \in \widetilde{\triangle}_0^{\mathfrak{g}},\ j \neq k \in I] \otimes \mathbb{Z}[X_{j,l}(1 + B_{j,l+1}^{-1}) \mid (j,l) \in \widetilde{\triangle}_0^{\mathfrak{g}}]\right).$$

Note that $\mathfrak{K}_I(\mathfrak{g}) = \mathfrak{K}(\mathfrak{g})$. We also define $\mathfrak{K}_{J,q}(\mathfrak{g}) \subset \mathcal{X}_q$ as above by replacing the letters X and B with $\widetilde{X}$ and $\widetilde{B}$, respectively.

Proposition 5.35. *Let $J \subset I$ with $|J| \leqslant 2$. For a J-dominant monomial m, there exists a unique $F_{J,q}(\underline{m}) \in \mathfrak{K}_{J,q}(\mathfrak{g})$ such that $\underline{m}$ is the unique J-dominant $\mathcal{X}_q$-monomial of $F_{J,q}(\underline{m})$. Moreover,*

$$\{F_{J,q}(\underline{m}) \mid m \textit{ is } J\textit{-dominant}\} \textit{ is a } \mathbb{Z}[q^{\pm\frac{1}{2}}]\textit{-basis of } \mathfrak{K}_{J,q}(\mathfrak{g}).$$

Proof. If $|J| = 1$, then our assertion is a folded version of [25, Proposition 4.12], where its proof is parallel with the replacement in Remark 5.14 (2). If $|J| = 2$, one may construct $F_{J,q}(\mathsf{X}_{i,p}) \in \mathfrak{K}_{J,q}(\mathfrak{g})$ for $i \in J$ by explicit computation in rank 2. Note that the computation in this case is done by [25, Appendix] for types $A_1 \times A_1$ and A_2, Examples 5.16, 5.24 for type G_2, Example A.1 for type C_2, and Example A.2 for type B_2. Hence our assertion is proved by a similar argument to the proof of Proposition 5.26 with the q-deformation of (5.34) defined similarly as in (5.23). □

For $m \in \mathcal{M}_+^J$, we define

$$E_J(m) = \prod_{j\in J;(j,p)\in\triangle_0} F_J(X_{j,p})^{u_{j,p}(m)} \in \mathfrak{K}_J(\mathfrak{g}), \tag{5.34}$$

where $F_J(X_{j,p}) := \mathrm{ev}_{q=1}(F_{J,q}(\mathsf{X}_{i,p}))$ is a unique element in $\mathfrak{K}_J(\mathfrak{g})$ such that $X_{j,p}$ is the unique dominant monomial of $F_J(X_{j,p})$ (cf. Remark 4.12 and Remark 5.6). Let $\mathfrak{K}_{i,q}^{\infty}(\mathfrak{g})$ be the completion of $\mathfrak{K}_{i,q}(\mathfrak{g})$ given by the method in [25, Section 5.2.2]. Put $\mathfrak{K}_{J,q}^{\infty}(\mathfrak{g}) = \bigcap_{j\in J} \mathfrak{K}_{j,q}^{\infty}(\mathfrak{g})$.

Lemma 5.36. (1) *A non-zero element of $\mathfrak{K}^{\infty}_{J,q}$ has at least one J-dominant $\mathcal{X}_q$-monomial.*
(2) *We have*

$$\mathfrak{K}_{J,q}(\mathsf{g}) = \mathfrak{K}^{\infty}_{J,q}(\mathsf{g}) \bigcap \mathcal{X}_q.$$

Proof. Part (1) follows from an analog of the proof of Lemma 5.19 (cf. [15, Lemma 5.6]), and part (2) follows from the same argument as in the proof of [25, Lemma 5.7] by using Proposition 5.35. □

For $i \in I^{\mathsf{g}}$, take $\imath \in I^{\mathbf{g}}$ such that $\bar{\imath} = i$ and put

- $D^{\mathbf{g}}_{\mathbf{m}^{(\imath)}[p,s]} = (\mathbf{m}^{(k)})_{k \geqslant 0}$: the countable set as in [25, Section 5.2.3] associated with $\mathbf{m}^{(\imath)}[p, s]$,
- $D^{\mathsf{g}}_{m^{(i)}[p,s]} = (m^{(k)})_{k \geqslant 0}$: the analogue of the above one for $m^{(i)}[p, s]$ in terms of (5.34).

Remark 5.37. The set $D^{\mathbf{g}}_{\mathbf{m}^{(\imath)}[p,s]}$ may be an infinitely countable set. If we enumerate the monomials in the countable set as follows:

$$\cdots < \mathbf{m}^{(2)} < \mathbf{m}^{(1)} < \mathbf{m}^{(0)} = \mathbf{m}^{(\imath)}[p, s].$$

Then the t-algorithm determines $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-coefficients of the monomials $\underline{\mathbf{m}}^{(k)}$'s. Let $(\mathsf{c}^{\mathbf{g}}(\mathbf{m}^{(r)}))_{r \geqslant 0}$ be the sequence of $\mathbb{Z}[t^{\pm\frac{1}{2}}]$-coefficients for $\underline{\mathbf{m}}^{(r)}$'s determined by the t-algorithm starting from $\underline{\mathbf{m}}^{(\imath)}[p, s]$. It was known in [26] that the sequence $(\mathsf{c}^{\mathbf{g}}(\mathbf{m}_k))_{k \geqslant 0}$ should have finitely many non-zero coefficients, that is, $F_t(\underline{\mathbf{m}}^{(\imath)}[p, s]) \in \mathfrak{K}_t(\mathbf{g})$. Note that $\underline{\mathcal{M}}(F_t(\underline{\mathbf{m}}^{(\imath)}[p, s])) \subset \{\underline{\mathbf{m}}^{(k)} \mid k \geqslant 0\}$.

Let us enumerate the finite set $\mathcal{M}(F_t(\underline{\mathbf{m}}^{(\imath)}[p, s]))$ as follows:

$$\mathbf{m}_N < \cdots < \mathbf{m}_2 < \mathbf{m}_1 < \mathbf{m}_0 = \mathbf{m}^{(\imath)}[p, s],$$

where $<$ is a total order compatible with $\prec_{\mathrm{N}}$. In particular, $\mathbf{m}_N$ is an anti-dominant $\mathcal{Y}$-monomial, i.e. $\mathbf{m}_N = \mathbf{m}_{-}^{(\imath^*)}[p+\mathsf{h}, s+\mathsf{h}]$ by Theorem 5.4. It follows from Corollary 3.10 and Theorem 5.4 that

$$\mathsf{M} := \{\overline{\sigma}(\mathbf{m}_k) \mid 1 \leqslant k \leqslant N\} \subset D^{\mathsf{g}}_{m^{(i)}[p,s]}.$$

Then we enumerate the $\mathcal{X}$-monomials in M by

$$m_{-}^{(i)}[p+\mathsf{h}, s+\mathsf{h}] = \mathsf{m}_{N'} <' \cdots <' \mathsf{m}_1 <' \mathsf{m}_0 = m^{(i)}[p, s], \tag{5.35}$$

where $<'$ is a total order compatible with $\prec_{\mathrm{N}}$.

Definition 5.38. Set $\widetilde{\mathsf{c}}^{\mathsf{g}}(m^{(i)}[p, s]) = 1$ and $\widetilde{\mathsf{c}}^{\mathsf{g}}_J(m^{(i)}[p, s]) = 0$. For $J \subset I$ with $|J| \leqslant 2$ and $\mathsf{m} \in \mathsf{M}$ such that $\mathsf{m} \neq m^{(i)}[p, s]$, we define

$$\widetilde{\mathsf{c}}^{\mathsf{g}}_J(\mathsf{m}) = \sum_{\substack{\mathsf{m} \in \mathsf{M} \\ \mathsf{m} <' \mathsf{m}'}} \left(\widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}') - \widetilde{\mathsf{c}}^{\mathsf{g}}_J(\mathsf{m}')\right) \left[F_{J,q}(\underline{\mathsf{m}'})\right]_{\underline{\mathsf{m}}},$$

$$\widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}) = \begin{cases} \widetilde{\mathsf{c}}_J(\mathsf{m}) & \text{if } \mathsf{m} \text{ is not } J\text{-dominant}, \\ 0 & \text{if } \mathsf{m} \text{ is dominant}, \end{cases}$$

where $\left[F_{J,q}(\underline{\mathsf{m}'})\right]_{\underline{\mathsf{m}}}$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficient of $\underline{\mathsf{m}}$ in $F_{J,q}(\underline{\mathsf{m}'})$. Here $F_{J,q}(\underline{\mathsf{m}'})$ is assumed to be 0 when m' is not J-dominant.

Lemma 5.39. *The sequences* $(\widetilde{\mathsf{c}}^{\mathfrak{g}}_J(\mathsf{m}))_{\mathsf{m}\in\mathsf{M}}$ *and* $(\widetilde{\mathsf{c}}^{\mathfrak{g}}(\mathsf{m}))_{\mathsf{m}\in\mathsf{M}}$ *are well-defined, and* $(\widetilde{\mathsf{c}}^{\mathfrak{g}}(\mathsf{m}))_{\mathsf{m}\in\mathsf{M}}$ *is not depend on the choice of* J *with* $|J| \leqslant 2$.

Proof. We prove our assertion by induction on (5.35). Suppose that our assertion holds for the $\mathcal{X}$-monomials m_k with $0 < k < N'$. The well-definedness of $\widetilde{\mathsf{c}}^{\mathfrak{g}}_J(\mathsf{m}_{k+1})$ follows from its definition.

For $J_1, J_2 \subset I$ such that $J_1 \neq J_2$ and $\max\{|J_1|, |J_2|\} \leqslant 2$, if m_{k+1} is not both J_1-dominant and J_2-dominant, then we should verify

$$\widetilde{\mathsf{c}}^{\mathfrak{g}}_{J_1}(\mathsf{m}_{k+1}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}_{J_2}(\mathsf{m}_{k+1}).$$

For $J \subset I$ with $|J| \leqslant 2$, we set

$$\chi^k_J = \sum_{l \leqslant k} \left(\widetilde{\mathsf{c}}^{\mathfrak{g}}(\mathsf{m}_l) - \widetilde{\mathsf{c}}^{\mathfrak{g}}_J(\mathsf{m}_k)\right) F_{J,q}(\underline{\mathsf{m}}_l).$$

For simplicity, put

$$\chi^k_i := \chi^k_J \quad \text{and} \quad \widetilde{\mathsf{c}}^{\mathfrak{g}}_i(\mathsf{m}) := \widetilde{\mathsf{c}}^{\mathfrak{g}}_J(\mathsf{m})$$

when $J = \{i\}$ for $i \in I$. Note that χ^k_J is well-defined by induction hypothesis. In particular, $\chi^k_J \in \mathfrak{K}_{J,q}(\mathfrak{g})$ by Proposition 5.35.

Let us take $j_1 \in J_1$ and $j_2 \in J_2$ such that m_{k+1} is not both j_1-dominant and j_2-dominant. Set $J = \{j_1, j_2\}$. Since m_{k+1} is not J-dominant, we have

$$\chi^k_J - \chi^k_{j_1} \in \sum_{l > k+1} \mathbb{Z}[q^{\pm\frac{1}{2}}] F_{j_1,q}(\underline{\mathsf{m}}_l) \quad \text{and} \quad \widetilde{\mathsf{c}}^{\mathfrak{g}}_J(\mathsf{m}_{k+1}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}_{j_1}(\mathsf{m}_{k+1})$$

by similar computations in the proof of [25, Lemma 5.21] under the current setting. Similarly, we also have

$$\widetilde{\mathsf{c}}^{\mathfrak{g}}_J(\mathsf{m}_{k+1}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}_{j_2}(\mathsf{m}_{k+1}), \quad \widetilde{\mathsf{c}}^{\mathfrak{g}}_{J_1}(\mathsf{m}_{k+1}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}_{j_1}(\mathsf{m}_{k+1}), \quad \widetilde{\mathsf{c}}^{\mathfrak{g}}_{J_2}(\mathsf{m}_{k+1}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}_{j_2}(\mathsf{m}_{k+1}).$$

Hence, we conclude that $\widetilde{\mathsf{c}}^{\mathfrak{g}}_{J_1}(\mathsf{m}_{k+1}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}_{J_2}(\mathsf{m}_{k+1})$. This completes the proof. □

By Proposition 5.35 and Lemma 5.39, we set

$$\chi := \sum_{\mathsf{m}\in\mathsf{M}} \widetilde{\mathsf{c}}^{\mathfrak{g}}(\mathsf{m})\underline{\mathsf{m}} \in \mathcal{X}_q \quad \text{and} \quad \chi_i := \sum_{\mathsf{m}\in\mathsf{M}} \mu_i(\mathsf{m}) F_{i,q}(\underline{\mathsf{m}}) \in \mathfrak{K}_{i,q}(\mathfrak{g}),$$

where $\mu_i(\mathsf{m}) = \widetilde{\mathsf{c}}^{\mathfrak{g}}(\mathsf{m}) - \widetilde{\mathsf{c}}^{\mathfrak{g}}_i(\mathsf{m})$.

Remark 5.40. It follows from Corollary 3.10 and Theorem 5.4 that

$$\overline{\sigma}(F(\mathbf{m}^{(\imath)}[p,s])) = F(m^{(i)}[p,s]) \in \mathfrak{K}(\mathfrak{g}). \tag{5.36}$$

On the other hand, it follows from Corollary 5.28 and Proposition 5.35 that $F(m^{(i)}[p,s])$ should be written as a finite linear combination of $\mathrm{ev}_{q=1}(F_{J,q}(\underline{m}))$ for $J \subset I$ with $|J| \leqslant 2$. Combining this fact with (5.36), we conclude that each $\underline{\mathsf{m}}$ ($\mathsf{m} \in \mathsf{M}$) appears in both χ and χ_i for all $i \in I$, so it makes sense to compare their coefficients (e.g. see Example 5.7 and Examples 5.16).

Now, we are ready to prove Proposition 5.20.

Proof of Proposition 5.20. Let us compute the coefficient of $\underline{\mathsf{m}'}$ in $\chi - \chi_i$ for $\mathsf{m}' \in \mathsf{M}$.

Case 1. m' is not i-dominant. By definition of $\widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}')$, we have

$$\begin{aligned}(\text{coefficient of } \underline{\mathsf{m}'} \text{ in } \chi - \chi_i) &= \widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}') - \sum_{\substack{\mathsf{m} \in \mathsf{M} \\ \mathsf{m}' \preccurlyeq' \mathsf{m}}} \mu_i(\mathsf{m}) \left[F_{i,q}(\underline{\mathsf{m}})\right]_{\underline{\mathsf{m}'}} \\ &= (\widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}') - \widetilde{\mathsf{c}}_i^{\mathsf{g}}(\mathsf{m}')) \left[F_{i,q}(\underline{\mathsf{m}'})\right]_{\underline{\mathsf{m}'}} = 0,\end{aligned}$$

where $F_{i,q}(\underline{\mathsf{m}'}) = 0$ since m' is not i-dominant.

Case 2. m' is i-dominant. By uniqueness of i-dominant $\mathcal{X}_q$-monomial for $F_{i,q}(\underline{\mathsf{m}})$ with $\mathsf{m}' \preccurlyeq' \mathsf{m}$, we have $\widetilde{\mathsf{c}}_i^{\mathsf{g}}(\mathsf{m}') = 0$, and the coefficient of $\underline{\mathsf{m}'}$ in χ_i is $\mu_i(\mathsf{m}') = \widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}') - \widetilde{\mathsf{c}}_i^{\mathsf{g}}(\mathsf{m}') = \widetilde{\mathsf{c}}^{\mathsf{g}}(\mathsf{m}')$. This implies that the coefficient of $\underline{\mathsf{m}'}$ in $\chi - \chi_i$ is 0 in this case.

By *Case 1* and *Case 2*, we have $\chi = \chi_i \in \mathfrak{K}_{i,q}(\mathsf{g})$ and then $\chi \in \mathfrak{K}_q(\mathsf{g})$. Note that χ has unique dominant $\mathcal{X}_q$-monomial $\underline{m}^{(i)}[p,s]$ by Definition 5.38 (or our choice of M). Since $F_q(\underline{m}^{(i)}[p,s]) - \chi \in \mathfrak{K}_q^{\infty}(\mathsf{g})$ has no dominant $\mathcal{X}_q$-monomial, we conclude $F_q(\underline{m}^{(i)}[p,s]) = \chi \in \mathfrak{K}_q(\mathsf{g})$ by Lemma 5.36. □

6. Subrings of $\mathfrak{K}_q(\mathfrak{g})$ nd the Quantum Folded T-Systems

In this section, we prove the quantum folded T-systems, which play a crucial role in this paper. To do this, we consider a subring $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ of $\mathfrak{K}_q(\mathfrak{g})$ for a height function ξ. We mainly employ the framework in [29,30] (see also [5]).

6.1. Subring. Let S be a convex set of $\widetilde{\triangle}_0$ (recall Definition 2.7 (2)). We denote by ${}^{\mathsf{S}}\mathcal{X}$ (resp. ${}^{\mathsf{S}}\mathcal{X}_q$) the subring of $\mathcal{X}$ (resp. $\mathcal{X}_q$) generated by $X_{i,p}^{\pm 1}$ (resp. $\widetilde{X}_{i,p}^{\pm 1}$) for $(i,p) \in \mathsf{S}$. Let ${}^{\mathsf{S}}\mathcal{M}_+$ be the set all dominant monomials in the variables $X_{i,p}$'s for $(i,p) \in \mathsf{S}$. We define the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module $\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g})$ as the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-submodule of $\mathfrak{K}_q(\mathfrak{g})$ given by

$$\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g}) := \bigoplus_{m \in {}^{\mathsf{S}}\mathcal{M}_+} \mathbb{Z}[q^{\pm\frac{1}{2}}] F_q(\underline{m}). \tag{6.1}$$

Lemma 6.1. (cf. [18, Lemma 5.6]) *The set ${}^{\mathsf{S}}\mathcal{M}_+$ is an ideal of the partially ordered set $(\mathcal{M}_+, \preccurlyeq_{\mathrm{N}})$; i.e., it is closed under taking smaller elements in $\mathcal{M}_+$ with respect to $\preccurlyeq_{\mathrm{N}}$.*

Proof. Let $m \in {}^{\mathsf{S}}\mathcal{M}_+$ and $mM \in \mathcal{M}_+$ where $M \in \mathbf{B}_q^{-k}$ for some $k \in \mathbb{Z}_{\geqslant 1}$. For a factor $B_{i,p}^{-1}$ of M, the monomial m should have factors $X_{i,p-1}$ and $X_{i,p+1}$ due to (3.7). Thus we have an oriented path from $(i, p+1)$ to a vertex in S and another oriented path from a vertex in S to $(i, p-1)$ (these paths are possibly of length zero) in $\widetilde{\triangle}_0$. Hence we have an oriented path whose end points are in S factoring through both $(i, p-1)$ and $(i, p+1)$. By convexity of S and the definition of $B_{i,p}$ (3.7), $M \in {}^{\mathsf{S}}\mathcal{X}$ and $mM \in {}^{\mathsf{S}}\mathcal{M}_+$ as we desired. □

Proposition 6.2. *For a convex subset S in $\widetilde{\triangle}_0$, the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module $\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g})$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathfrak{K}_q(\mathfrak{g})$. Moreover, we have*

$$\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g}) = \bigoplus_{m \in {}^{\mathsf{S}}\mathcal{M}_+} \mathbb{Z}[q^{\pm\frac{1}{2}}] E_q(\underline{m}) = \bigoplus_{m \in {}^{\mathsf{S}}\mathcal{M}_+} \mathbb{Z}[q^{\pm\frac{1}{2}}] L_q(\underline{m}). \tag{6.2}$$

Proof. Let $m_1, m_2 \in {}^{\mathsf{S}}\mathcal{M}_+$. By Theorem 5.27 and Proposition 5.29, $F_q(\underline{m}_1) * F_q(\underline{m}_2) \in \mathfrak{K}_q(\mathfrak{g})$ is written as shown below.

$$F_q(\underline{m}_1) * F_q(\underline{m}_2) = \sum_{\substack{m \in \mathcal{M}_+ \\ m \preccurlyeq_{\mathrm{N}} m_1 m_2}} c_{\underline{m}} F_q(\underline{m}), \tag{6.3}$$

where $c_{\underline{m}} \in \mathbb{Z}[q^{\pm\frac{1}{2}}] \setminus \{0\}$. Then it follows from Lemma 6.1 that $m \in {}^{\mathsf{S}}\mathcal{M}_+$ for a monomial $m \preccurlyeq_{\mathrm{N}} m_1 m_2$ above. Hence, we conclude that $\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g})$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathfrak{K}_q(\mathfrak{g})$ by definition (6.1) of $\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g})$.

Since $\mathfrak{K}_{q,\mathsf{S}}(\mathfrak{g})$ is given by (6.1), (6.2) follows from $\mathfrak{K}_q(\mathfrak{g})$-analogue of (5.6) and (b) in Theorem 5.31. $\square$

6.2. Truncation. Let ξ be a height function of $\triangle$ and set ${}^{\xi}\mathcal{X}_q := {}^{\xi\widetilde{\triangle}_0}\mathcal{X}_q$ for simplicity. For a (non-commutative) Laurent polynomial $x \in \mathcal{X}_q$, we denote by $x_{\leqslant\xi}$ the element of ${}^{\xi}\mathcal{X}_q$ obtained from x by discarding all the monomials containing $\widetilde{X}_{i,p}^{\pm 1}$ with $(i,p) \in \widetilde{\triangle}_0 \setminus {}^{\xi}\widetilde{\triangle}_0$.

The map

$$(\cdot)_{\leqslant\xi} : \mathcal{X}_q \longrightarrow {}^{\xi}\mathcal{X}_q \quad \text{given by} \quad x \longmapsto x_{\leqslant\xi}$$

is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linear map, which is not $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra homomorphism. For $m \in \mathcal{M}_+$, we denote by $F_q(\underline{m})_{\leqslant\xi}$ the image of $F_q(\underline{m})$ under the map $(\cdot)_{\leqslant\xi}$.

Let us recall Definition 2.7 and (6.1). We set

$$\mathfrak{K}_{q,\xi}(\mathfrak{g}) := \mathfrak{K}_{q,{}^{\xi}\widetilde{\triangle}_0}(\mathfrak{g}). \tag{6.4}$$

Proposition 6.3. *For a height function ξ on $\triangle$, the map $(\cdot)_{\leqslant\xi}$ restricts to the injective $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra homomorphism*

$$(\cdot)_{\leqslant\xi} : \mathfrak{K}_{q,\xi}(\mathfrak{g}) \hookrightarrow {}^{\xi}\mathcal{X}_q.$$

Proof. The injectivity follows from Theorem 5.27. Let us take $m_1, m_2 \in {}^{\xi}\mathcal{M}_+ := (\mathcal{M}_+)_{\leqslant\xi}$. We consider a linear expansion of $F_q(\underline{m}_1) * F_q(\underline{m}_2)$ as in (6.3). Then we claim that

$$F_q(\underline{m}_1)_{\leqslant\xi} * F_q(\underline{m}_2)_{\leqslant\xi} = \sum_{\substack{m \in {}^{\xi}\mathcal{M}_+ \\ m \preccurlyeq_{\mathrm{N}} m_1 m_2}} c_{\underline{m}} F_q(\underline{m})_{\leqslant\xi} \quad (c_{\underline{m}} \neq 0). \tag{6.5}$$

Take a $\mathcal{X}_q$-monomial $\widetilde{m}'$ (resp. $\widetilde{m}''$) appearing in $F_q(\underline{m}_1)_{\leqslant\xi}$ (resp. $F_q(\underline{m}_2)_{\leqslant\xi}$). If $\mathrm{ev}_{q=1}(\widetilde{m}'\widetilde{m}'') \in \mathcal{M}_+$, then $\mathrm{ev}_{q=1}(\widetilde{m}'\widetilde{m}')' \in {}^{\xi}\mathcal{M}_+$ by Lemma 6.1. Furthermore, by Theorem 5.27 and definition of ${}^{\xi}\mathcal{X}_q$, $F_q(\underline{m}_1)_{\leqslant\xi} * F_q(\underline{m}_2)_{\leqslant\xi}$ is written as a linear combination of $\{F_q(\underline{m})_{\leqslant\xi} \mid m \in {}^{\xi}\mathcal{M}_+\}$. Thus, $F_q(\widetilde{m}'\widetilde{m}'')_{\leqslant\xi}$ appears in the right-hand side of (6.5) up to $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. This proves the above claim.

Finally, we have

$$(\cdot)_{\leqslant\xi}(F_q(\underline{m}_1) * F_q(\underline{m}_2)) = \sum_{\substack{m \in {}^{\xi}\mathcal{M}_+ \\ m \preccurlyeq_{\mathrm{N}} m_1 m_2}} c_{\underline{m}} F_q(\underline{m})_{\leqslant\xi} = F_q(\underline{m}_1)_{\leqslant\xi} * F_q(\underline{m}_2)_{\leqslant\xi}.$$

by Proposition 6.2 and (6.5), which completes the proof. $\square$

Definition 6.4. For $m \in \mathcal{M}_+$, we say $L_q(m)$ (resp. $F_q(m)$) *real* if, for any $k \in \mathbb{Z}_{\geqslant 1}$, we have $(L_q(m))^k = q^t L_q(m^k)$ (resp. $(F_t(m))^k = q^t F_q(m^k)$) for some $t \in \mathbb{Z}$.

Corollary 6.5. *For each KR-monomial $\underline{m}^{(i)}[p,s]$, $F_q(\underline{m}^{(i)}[p,s])$ is real.*

Proof. Let ξ be a height function with $\xi_i = s$. Then we have

$$(F_q(\underline{m}^{(i)}[p,s]))_{\leqslant\xi} = \underline{m}^{(i)}[p,s],$$

by (5.28) in Proposition 5.29. Since

$$\mathrm{ev}_{q=1}\left(\left(F_q(\underline{m}^{(i)}[p,s])^{*n}\right)_{\leqslant\xi}\right) = (m^{(i)}[p,s])^n = \mathrm{ev}_{q=1}\left(\left(F_q(\underline{m}^{(i)}[p,s]^{*n})\right)_{\leqslant\xi}\right),$$

our assertion follows from Proposition 6.3. □

In accordance with Theorem 5.11 for simply-laced type $\mathbf{g}$, one can also expect that $L_q(m)$ has quantum positivity. Moreover, it is proved in [19,45] that every cluster monomial in $\mathfrak{K}_q(\mathbf{g})$ corresponds to a real element in $\mathbf{L}_t$. Based on some computational evidence, we suggest the following conjecture:

Conjecture 3. *For $m \in \mathcal{M}_+$, if $L_q(m)$ is real, then $L_q(m)$ has a quantum positive coefficient.*

6.3. Quantum folded T-system. For $f, g \in \mathcal{X}_q$, we say that f and g *q-commute or are q-commutative* if $fg = q^k gf$ for some $k \in \frac{1}{2}\mathbb{Z}$. In this subsection, we shall prove the functional equations among KR-polynomials $F_q(m^{(i)}[p,s])$'s, called the *quantum folded T-system*. For simply-laced finite type, the quantum folded T-system is nothing but the quantum T-system, investigated in [29] (see also [18,31]).

Lemma 6.6. *For $(i,p), (i,s) \in \widetilde{\triangle}_0$ with $p < s$, let $j, j' \in \triangle_0$ such that $d(i,j) = d(i,j') = 1$. Then we have*

$$F_q\big(\underline{m}^{(j)}(p,s)\big) * F_q\big(\underline{m}^{(j')}(p,s)\big) = F_q\big(\underline{m}^{(j')}(p,s)\big) * F_q\big(\underline{m}^{(j)}(p,s)\big).$$

Proof. Note that one can take a height function ξ of $\triangle$ satisfying (a) $\xi_j = \max\{\xi_i \mid i \in I\} = s$ and (b) $\xi_j = \xi_{j'} = s-1$. Then, by (5.28), we have

$$F_q\big(\underline{m}^{(j)}(p,s)\big)_{\leqslant\xi} = \underline{m}^{(j)}(p,s) \quad \text{and} \quad F_q\big(\underline{m}^{(j')}(p,s)\big)_{\leqslant\xi} = \underline{m}^{(j')}(p,s).$$

By Proposition 6.3, we have

$$F_q\big(\underline{m}^{(j)}(p,s)\big) * F_q\big(\underline{m}^{(j')}(p,s)\big) = q^{\beta} F_q\big(\underline{m}^{(j')}(p,s)\big) * F_q\big(\underline{m}^{(j)}(p,s)\big)$$

for some $\beta \in \frac{1}{2}\mathbb{Z}$.

Now, let us prove that $\beta = 0$ by induction on $k = (p-s)/2$. When $k = 1$, we have $\underline{m}^{(j)}(p,s) = \widetilde{X}_{j,p+1}$. In this case, $\beta = 0$ by (4.3). Suppose that $k > 1$. By the induction hypothesis, we have

$$\underline{m}^{(j)}(p,s-2) * \underline{m}^{(j')}(p,s-2) = \underline{m}^{(j')}(p,s-2) * \underline{m}^{(j)}(p,s-2).$$

Then we have

$$\underline{m}^{(j)}(p,s) * \underline{m}^{(j')}(p,s) = q^{\underline{\mathcal{N}}(\widetilde{X}_{j,s-1}, m^{(j')}(p,s)) + \underline{\mathcal{N}}(m^{(j)}(p,s), \widetilde{X}_{j',s-1})} \underline{m}^{(j)}(p,s) * \underline{m}^{(j')}(p,s).$$

Since

$$\begin{aligned}
\underline{\mathcal{N}}(\widetilde{X}_{j,s-1}, m^{(j')}(p,s)) &= \sum_{i=0}^{(p-s)/2-1} \underline{\mathcal{N}}(j, s-1; j', p+1+2i) \\
&= \sum_{i=0}^{(p-s)/2-1} \widetilde{\mathsf{r}}_{j,j'}(s-p-2i-3) - \widetilde{\mathsf{r}}_{j,j'}(s-p-2i-1), \\
\underline{\mathcal{N}}(m^{(j)}(p,s), \widetilde{X}_{j',s-1}) &= -\underline{\mathcal{N}}(\widetilde{X}_{j',s-1}, m^{(j)}(p,s)) = -\sum_{i=0}^{(p-s)/2-1} \underline{\mathcal{N}}(j', s-1; j, p+1+2i) \\
&= \sum_{i=0}^{(p-s)/2-1} -\widetilde{\mathsf{r}}_{j',j}(s-p-2i-3) + \widetilde{\mathsf{r}}_{j',j}(s-p-2i-1),
\end{aligned}$$

our assertion follows from the fact that $\widetilde{\mathsf{r}}_{j,j'}(u) = \widetilde{\mathsf{r}}_{j',j}(u)$ for all $u \in \mathbb{Z}$ (cf. [47, Section 4]). □

Lemma 6.7. *For $(i,p), (i,s) \in \widetilde{\triangle}_0$ with $p < s$, we have*

$$F_q\big(\underline{m}^{(i)}[p,s]\big) * F_q\big(\underline{m}^{(i)}(p,s)\big) = F_q\big(\underline{m}^{(i)}(p,s)\big) * F_q\big(\underline{m}^{(i)}[p,s]\big).$$

Proof. Let $\imath \in I^{\mathrm{g}}$ such that $\bar{\imath} = i$. By Theorem 3.9 and (3.11), we have

$$\overline{\sigma}\left(F(\mathbf{m}^{(\imath)}[p,s])F(\mathbf{m}^{(\imath)}(p,s))\right) = F(m^{(i)}[p,s])F(m^{(i)}(p,s)). \tag{6.6}$$

Put $M = m^{(i)}[p,s]m^{(i)}(p,s)$. It follows from [27, Lemma 5.6 (2)] and (6.6) that the set of dominant $\mathcal{X}$-monomials in $F(m^{(i)}[p,s])F(m^{(i)}(p,s))$ is given by

$$M,\ MB_{i,s-1}^{-1},\ MB_{i,s-1}^{-1}B_{i,s-3}^{-1},\ \dots,\ MB_{i,s-1}^{-1}B_{i,s-3}^{-1}\cdots B_{i,p+3}^{-1} \tag{6.7}$$

with multiplicity 1. By (5.22) (see also Remark 4.12), we have the set of dominant $\mathcal{X}_q$-monomials in $F_q(\underline{m}^{(i)}[p,s]) * F_q(\underline{m}^{(i)}(p,s))$ and the set of those in $F_q(\underline{m}^{(i)}(p,s)) * F_q(\underline{m}^{(i)}[p,s])$; namely, their specialization at $q=1$ is (6.7).

Now, it follows from Theorem 5.27 that it is enough to compare the dominant $\mathcal{X}_q$-monomials of $F_q(\underline{m}^{(i)}[p,s]) * F_q(\underline{m}^{(i)}(p,s))$ with those of $F_q(\underline{m}^{(i)}(p,s)) * F_q(\underline{m}^{(i)}[p,s])$ to prove our assertion. At this point, we can apply the $\mathfrak{sl}_2$-reduction argument as in [31, Remark 9.10] based on Proposition 4.6 (see also the proof of [18, Proposition 6.10]). Consequently, we conclude that

$$F_q(\underline{m}^{(i)}[p,s]) * F_q(\underline{m}^{(i)}(p,s)) = q^a F_q(\underline{m}^{(i)}(p,s)) * F_q(\underline{m}^{(i)}[p,s]) \quad \text{for some } a \in \mathbb{Z}/2. \tag{6.8}$$

Finally, to complete our assertion, it suffices to show that a in (6.8) vanishes, that is,

$$\underline{m}^{(i)}[p,s] * \underline{m}^{(i)}(p,s) = \underline{m}^{(i)}(p,s) * \underline{m}^{(i)}[p,s].$$

By an induction on $(p-s)/2$, we have

$$\begin{aligned}\underline{\mathcal{N}}(\underline{m}^{(i)}[p,s],\underline{m}^{(i)}(p,s)) &= \underline{\mathcal{N}}(\widetilde{X}_{i,s},\underline{m}^{(i)}(p,s-2]) + \underline{\mathcal{N}}(\underline{m}^{(i)}[p,s-2),\widetilde{X}_{i,s-2}) \\ &\overset{\star}{=} \underline{\mathcal{N}}(\widetilde{X}_{i,s},\underline{m}^{(i)}(p,s]) + \underline{\mathcal{N}}(\underline{m}^{(i)}[p,s),\widetilde{X}_{i,s-2}) \\ &\overset{\dagger}{=} \underline{\mathcal{N}}(\widetilde{X}_{i,s},\underline{m}^{(i)}(p,s]) + \underline{\mathcal{N}}(\underline{m}^{(i)}(p,s],\widetilde{X}_{i,s}) = 0,\end{aligned}$$

where $\overset{\star}{=}$ follows from $\underline{\mathcal{N}}(\widetilde{X}_{i,t},\widetilde{X}_{i,t}) = 0$ and $\overset{\dagger}{=}$ follows from $\underline{\mathcal{N}}(\widetilde{X}_{i,t},\widetilde{X}_{i,t'}) = \underline{\mathcal{N}}(\widetilde{X}_{i,t\pm 2},\widetilde{X}_{i,t'\pm 2})$. □

For $(i,p),(i,s) \in \widetilde{\triangle}_0$ with $p<s$, we set $m(i;p,s) := \prod_{j;\, d(i,j)=1} m^{(j)}(p,s)^{-\mathsf{c}_{j,i}}$, where $m^{(j)}(p,s)$ is given as in (5.26).

Lemma 6.8. *For $(i,p),(i,s) \in \widetilde{\triangle}_0$ with $p<s$, we have*

$$F_q(\underline{m}(i;p,s)) = \prod_{j;\, d(i,j)=1} F_q(\underline{m}^{(j)}(p,s))^{-\mathsf{c}_{j,i}},$$

where the order of the product does not matter.

Proof. By Lemma 6.6, $\prod_{j;\, d(i,j)=1} F_q(\underline{m}^{(j)}(p,s))^{-\mathsf{c}_{j,i}}$ is well-defined. Let ξ be a height function on $\triangle\!\!\!\!\triangle$ such that $\xi_i = s$ and $\xi_j = s-1$ for $j \in \triangle\!\!\!\!\triangle_0$ with $d(i,j)=1$. Then we have

$$\left(\prod_{j;\, d(i,j)=1} F_q(\underline{m}^{(j)}(p,s))^{-\mathsf{c}_{j,i}}\right)_{\leqslant \xi} = \underline{m}(i;p,s),$$

which implies the assertion. □

Now, we are in a position to state and prove the quantum folded T-system (cf. Theorem 5.13).

Theorem 6.9 (Quantum folded T-system). *For $(i,p),(i,s) \in \widetilde{\triangle\!\!\!\!\triangle}_0$ with $p<s$ and $k=(s-p)/2 \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\begin{aligned}F_q\big(\underline{m}^{(i)}[p,s)\big) * F_q\big(\underline{m}^{(i)}(p,s]\big) &= q^{\alpha(i,k)} F_q\big(\underline{m}^{(i)}(p,s)\big) * F_q\big(\underline{m}^{(i)}[p,s]\big) \\ &\quad + q^{\gamma(i,k)} \prod_{j;\, d(i,j)=1} F_q\big(\underline{m}^{(j)}(p,s)\big)^{-\mathsf{c}_{j,i}},\end{aligned}$$

where $\gamma(i,k) = \dfrac{1}{2}\big(\widetilde{\mathsf{r}}_{i,i}(2k-1) + \widetilde{\mathsf{r}}_{i,i}(2k+1)\big)$ and $\alpha(i,k) = \gamma(i,k) - d_i$.

Proof. First, we claim that

$$F_q\big(\underline{m}^{(i)}[p,s)\big) * F_q\big(\underline{m}^{(i)}(p,s]\big) = q^{\alpha} F_q\big(\underline{m}^{(i)}[p,s]\big) \cdot F_q\big(\underline{m}^{(i)}(p,s)\big) + q^{\gamma} F_q\big(\underline{m}(i;p,s)\big)$$

for some $\alpha,\gamma \in \frac{1}{2}\mathbb{Z}$. By using the q-algorithm and the argument in [27, Lemma 5.6] (or [31, Theorem 9.6, Lemma 9.9]), the product of $F_q\big(\underline{m}^{(i)}[p,s)\big)$ and $F_q\big(\underline{m}^{(i)}(p,s]\big)$ has exactly distinct k dominant monomials

$$M_1,\ M_2,\ \ldots,\ M_k,$$

where $\mathrm{ev}_{q=1}(M_1) = m^{(i)}[p,s)m^{(i)}(p,s]$. Moreover, $M_1, \dots, M_{k-1}$ exhaust the dominant monomials occurring in $F_q(\underline{m}^{(i)}[p,s])F_q(\underline{m}^{(i)}(p,s))$ and

$$\mathrm{ev}_{q=1}(M_k) = \left(m^{(i)}[p,s)B_{i,s-1}^{-1}B_{i,s-3}^{-1}\cdots B_{i,p+1}^{-1}\right)m^{(i)}(p,s] = m(i;p,s).$$

Hence, our claim follows from Theorem 5.27 and Lemma 6.8.

Second, we compute $\alpha = \alpha(i,k)$ and $\gamma = \gamma(i,k)$ explicitly. By Theorem 5.27, Lemma 6.6 implies that

$$F_q\big(\underline{m}(i;p,s)\big) = \prod_{j;\ d(i,j)=1} F_q\big(\underline{m}^{(j)}(p,s)\big)^{-\mathsf{c}_{j,i}}.$$

Also, by Lemma 6.7, we also have

$$F_q\big(\underline{m}^{(i)}[p,s]\big) * F_q\big(\underline{m}^{(i)}(p,s)\big) = F_q\big(\underline{m}^{(i)}(p,s)\big) * F_q\big(\underline{m}^{(i)}[p,s]\big).$$

Thus it suffices to compute α, γ such that

$$\underline{m}^{(i)}[p,s) * \underline{m}^{(i)}(p,s] = q^{\alpha}\underline{m}^{(i)}[p,s] * \underline{m}^{(i)}(p,s) = q^{\alpha}\underline{m}^{(i)}(p,s) * \underline{m}^{(i)}[p,s]$$

and

$$\left(\underline{m}^{(i)}[p,s)\cdot \widetilde{B}_{i,s-1}^{-1}\cdot \widetilde{B}_{i,s-3}^{-1}\cdots \widetilde{B}_{i,p+1}^{-1}\right) * \underline{m}^{(i)}(p,s] = q^{\gamma}\ \underline{m}(i;p,s).$$

The coefficient α can be computed as follows:

$$\begin{aligned}
\alpha &= \sum_{a=1}^{k-1}\underline{\mathcal{N}}(i,p;i,p+2a) + \frac{1}{2}\underline{\mathcal{N}}(i,p;i,p+2k)\\
&= \sum_{a=1}^{k-1}\left(\widetilde{\mathsf{r}}_{i,i}(2a+1) - \widetilde{\mathsf{r}}_{i,i}(2a-1)\right) + \frac{1}{2}\left(\widetilde{\mathsf{r}}_{i,i}(2k+1) - \widetilde{\mathsf{r}}_{i,i}(2k-1)\right)\\
&= -\widetilde{\mathsf{r}}_{i,i}(1) + \frac{1}{2}\left(\widetilde{\mathsf{r}}_{i,i}(2k+1) + \widetilde{\mathsf{r}}_{i,i}(2k-1)\right) = -d_i + \frac{1}{2}\left(\widetilde{\eta}_{i,i}(2k+1) + \widetilde{\eta}_{i,i}(2k-1)\right).
\end{aligned}$$

Note that $\underline{m} := \left(\underline{m}^{(i)}[p,s)\cdot \widetilde{B}_{i,s-1}^{-1}\cdot \widetilde{B}_{i,s-3}^{-1}\cdots \widetilde{B}_{i,p+1}^{-1}\right)$ is contained in $F_q(\underline{m}^{(i)}[p,s))$ with coefficient 1, and $\underline{m}\cdot\underline{m}^{(i)}(p,s] = \prod_{j;\ d(i,j)=1}\underline{m}^{(j)}(p,s)^{-\mathsf{c}_{j,i}}$. Thus we have

$$\begin{aligned}
\underline{m} * \underline{m}^{(i)}(p,s] &= \left(\left(\underline{m}^{(i)}(p,s]\right)^{-1}\cdot \prod_{j;\ d(i,j)=1}\underline{m}^{(j)}(p,s)^{-\mathsf{c}_{j,i}}\right) * \underline{m}^{(i)}(p,s]\\
&= q^{\gamma}\prod_{j;\ d(i,j)=1}\underline{m}^{(j)}(p,s)^{-\mathsf{c}_{j,i}},
\end{aligned}$$

where

$$\gamma = \frac{1}{2}\sum_{j;\ d(i,j)=1} -\mathsf{c}_{j,i}\sum_{a=1}^{k}\sum_{b=1}^{k}\underline{\mathcal{N}}(j,p+2a-1;i,p+2b)$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{j;\, d(i,j)=1} -\mathsf{c}_{j,i} \sum_{a=1}^{k} \sum_{b=1}^{k} \big(\widetilde{\mathsf{r}}_{j,i}(2(a-b)-2) - \widetilde{\mathsf{r}}_{j,i}(2(a-b)) - \widetilde{\mathsf{r}}_{j,i}(2(b-a)) \\
&\quad + \widetilde{\mathsf{r}}_{j,i}(2(b-a)+2)\big) \\
&= \frac{1}{2} \sum_{j;\, d(i,j)=1} -\mathsf{c}_{j,i} \sum_{a=1}^{k} \big(\widetilde{\mathsf{r}}_{j,i}(2(a-k)-2) - \widetilde{\mathsf{r}}_{j,i}(2(a-1)) - \widetilde{\mathsf{r}}_{j,i}(2(1-a)) \\
&\quad + \widetilde{\mathsf{r}}_{j,i}(2(k-a)+2)\big) \\
&= \frac{1}{2} \sum_{j;\, d(i,j)=1} -\mathsf{c}_{j,i} \sum_{a=1}^{k} \big(-\widetilde{\mathsf{r}}_{j,i}(2(a-1)) + \widetilde{\mathsf{r}}_{j,i}(2(k-a)+2)\big) = \frac{1}{2} \sum_{j;\, d(i,j)=1} -\mathsf{c}_{j,i}\widetilde{\mathsf{r}}_{j,i}(2k) \\
&= \frac{1}{2} \sum_{j;\, d(i,j)=1} -\mathsf{c}_{j,i}\widetilde{\eta}_{j,i}(2k) = \frac{1}{2} \sum_{j;\, d(i,j)=1} -\mathsf{c}_{j,i}\widetilde{\eta}_{i,j}(2k).
\end{aligned}
$$

Then our proof is completed by Lemma 2.3. □

Example 6.10. Let us recall the formula of $F_q(\widetilde{X}_{2,5})$ in (5.19). Also, $F_q(\mathsf{X}_{2,5}) = q^{\frac{3}{2}} F_q(\widetilde{X}_{2,5}) \in \mathfrak{K}_q(\mathsf{g})$ and it is bar-invariant with respect to (4.5). Note that $F_q(\widetilde{X}_{2,7}) = \mathsf{T}_2(F_q(\widetilde{X}_{2,5}))$ and $F_q(\mathsf{X}_{2,7}) = q^{\frac{3}{2}} F_q(\widetilde{X}_{2,7})$. Clearly, these computations implies that $F_q(\mathsf{X}_{2,5}) * F_q(\mathsf{X}_{2,7})$ has two dominant $\mathcal{X}_q$-monomials, namely, $\underline{X_{2,5}X_{2,7}}$ and $\underline{X_{1,6}^3}$. By Theorem 5.27, we should have

$$
F_q(\mathsf{X}_{2,5}) * F_q(\mathsf{X}_{2,7}) = q^{\frac{3}{2}} F_q(\underline{X_{2,5}X_{2,7}}) + q^{\frac{9}{2}} F_q(\underline{X_{1,6}})^3. \tag{6.9}
$$

On the other hand, we obtain

$$
\begin{aligned}
&d_2 = 3, \quad \gamma(2,1) = \frac{1}{2}\big(\widetilde{\mathsf{r}}_{2,2}(1) + \widetilde{\mathsf{r}}_{2,2}(3)\big) = \frac{9}{2}, \quad \alpha(2,1) = \gamma(2,1) - d_2 = \frac{3}{2}, \\
&-\mathsf{c}_{1,2} = 3,
\end{aligned}
$$

where $\widetilde{\mathsf{r}}_{2,2}(1) = 3$ and $\widetilde{\mathsf{r}}_{2,2}(3) = 6$ from (2.10). Hence (6.9) illustrates Theorem 6.9.

7. Quantum Cluster Algebra

In this section we recall the definition of skew-symmetrizable quantum cluster algebras of infinite rank, following [4], [22, §8], [30] and [45].

7.1. Quantum seed. Let K be an index set described in Sect. 2.4. Let $L = (\lambda_{i,j})_{i,j\in\mathsf{K}}$ be a skew symmetric integer-valued $\mathsf{K} \times \mathsf{K}$-matrix. Let q be an indeterminate.

Definition 7.1. We define $(\mathscr{P}(L), \star)$ as the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra, called the *quantum torus associated to* L, generated by a family of elements $\{Z_i\}_{i\in\mathsf{K}}$ with the defining relations

$$
Z_i \star Z_j = q^{\lambda_{i,j}} Z_j \star Z_i \qquad (i, j \in \mathsf{K}).
$$

We denote by $\mathfrak{F}(L)$ the skew field of fractions of $\mathscr{P}(L)$.

For $\mathbf{a} = (a_i)_{i\in\mathsf{K}} \in \mathbb{Z}^{\oplus\mathsf{K}}$, we define the element $Z^{\mathbf{a}}$ of $\mathscr{F}(L)$ as

$$Z^{\mathbf{a}} := q^{\frac{1}{2}\sum_{i>j} a_i a_j \lambda_{i,j}} \overset{\rightarrow}{\underset{i\in\mathsf{K}}{\star}} Z_i^{a_i} \tag{7.1}$$

(cf. (4.6)). Here we take a total order $<$ on the set K. Note that $Z^{\mathbf{a}}$ does not depend on the choice of a total order on K. We have

$$Z^{\mathbf{a}} \star Z^{\mathbf{b}} = q^{\frac{1}{2}\sum_{i,j\in\mathsf{K}} a_i b_j \lambda_{i,j}} Z^{\mathbf{a}+\mathbf{b}}.$$

Let $(\mathcal{A}, \star)$ be a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra. We say that a family $\{z_i\}_{i\in\mathsf{K}}$ of elements of $\mathcal{A}$ is *L-commuting* if it satisfies $z_i \star z_j = q^{\lambda_{i,j}} z_j \star z_i$ for any $i, j \in \mathsf{K}$. In that case we can define $z^{\mathbf{a}}$ for any $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus\mathsf{K}}$ as in (7.1). We say that an L-commuting family $\{z_i\}_{i\in\mathsf{K}}$ is *algebraically independent* if the algebra map $\mathscr{P}(L) \to \mathcal{A}$ given by $Z_i \mapsto z_i$ is injective.

Let $\widetilde{B} = (b_{i,j})_{i\in\mathsf{K}, j\in\mathsf{K}_{\mathrm{ex}}}$ be an integer-valued $\mathsf{K} \times \mathsf{K}_{\mathrm{ex}}$-exchange matrix satisfying (2.12). We say that the pair $(L, \widetilde{B})$ is *compatible with a diagonal matrix* $\mathrm{diag}(\mathsf{d}_i \in \mathbb{Z}_{\geqslant 1} \mid i \in \mathsf{K})$, if we have

$$\sum_{k\in\mathsf{K}} b_{ki}\lambda_{kj} = \delta_{i,j}\mathsf{d}_i, \quad \text{equivalently,} \quad (L\widetilde{B})_{ji} = -\delta_{i,j}\mathsf{d}_i, \tag{7.2}$$

for any $i \in \mathsf{K}_{\mathrm{ex}}$ and $j \in \mathsf{K}$. We also call the pair $(L, \widetilde{B})$ a *compatible pair* for short.

Let $(L, \widetilde{B})$ be a compatible pair and $\mathcal{A}$ a $\mathbb{Z}[q^{\pm 1/2}]$-algebra. We say that $\mathscr{S} = (\{z_i\}_{i\in\mathsf{K}}, L, \widetilde{B})$ is a *quantum seed* in $\mathcal{A}$ if $\{z_i\}_{i\in\mathsf{K}}$ is an algebraically independent L-commuting family of elements of $\mathcal{A}$. The set $\{z_i\}_{i\in\mathsf{K}}$ is called the *quantum cluster* of $\mathscr{S}$ and its elements the *quantum cluster variables*. The quantum cluster variables z_i ($i \in \mathsf{K}_{\mathrm{fr}}$) are called the *frozen variables*. The elements $z^{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus\mathsf{K}}$) are called the *quantum cluster monomials*.

7.2. Mutation. For $k \in \mathsf{K}_{\mathrm{ex}}$, we define a $\mathsf{K} \times \mathsf{K}$-matrix $E = (e_{i,j})_{i,j\in\mathsf{K}}$ and a $\mathsf{K}_{\mathrm{ex}} \times \mathsf{K}_{\mathrm{ex}}$-matrix $F = (f_{i,j})_{i,j\in\mathsf{K}_{\mathrm{ex}}}$ as follows:

$$e_{i,j} = \begin{cases} \delta_{i,j} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, -b_{i,k}) & \text{if } i \neq j = k, \end{cases} \qquad f_{i,j} = \begin{cases} \delta_{i,j} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, b_{k,j}) & \text{if } i = k \neq j. \end{cases}$$

The *mutation* $\mu_k(L, \widetilde{B}) := (\mu_k(L), \mu_k(\widetilde{B}))$*of a compatible pair* $(L, \widetilde{B})$ *in direction* k is given by

$$\mu_k(L) := (E^T)\, L\, E, \quad \mu_k(\widetilde{B}) := E\, \widetilde{B}\, F.$$

We define

$$a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{i,k}) & \text{if } i \neq k, \end{cases} \qquad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{i,k}) & \text{if } i \neq k. \end{cases} \tag{7.3}$$

and set $\mathbf{a}' := (a'_i)$ and $\mathbf{a}'' := (a''_i) \in \mathbb{Z}^{\oplus\mathsf{K}}$.

Let $\mathcal{A}$ be a $\mathbb{Z}[q^{\pm 1/2}]$-algebra contained in a skew field K. Let $\mathscr{S} = (\{z_i\}_{i \in \mathsf{K}}, L, \widetilde{B})$ be a quantum seed in $\mathcal{A}$. Define the elements $\mu_k(z)_i$ of K by

$$\mu_k(z)_i := \begin{cases} z^{\mathbf{a}'} + z^{\mathbf{a}''} & \text{if } i = k, \\ z_i & \text{if } i \neq k. \end{cases} \tag{7.4}$$

Then $\{\mu_k(z)_i\}$ is an algebraically independent $\mu_k(L)$-commuting family in K. We call

$$\mu_k(\mathscr{S}) := \big(\{\mu_k(z)_i\}_{i \in \mathsf{K}}, \mu_k(L), \mu_k(\widetilde{B})\big)$$

the *mutation of $\mathscr{S}$ in direction k*. It becomes a new quantum seed in K; that means,

(1) $\big(\mu_k(L), \mu_k(\widetilde{B})\big)$ is compatible with the diagonal matrix of $(L, \widetilde{B})$,
(2) $\{\mu_k(z)_i\}_{i \in \mathsf{K}}$ is $\mu_k(L)$-commuting.

Definition 7.2. Let $\mathscr{S} = (\{z_i\}_{i \in \mathsf{K}}, L, \widetilde{B})$ and $\mathscr{S}' = (\{z'_i\}_{i \in \mathsf{K}'}, L', \widetilde{B}')$ be quantum seeds in a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathcal{A}$.

(i) We say that *$\mathscr{S}'$ is mutated from $\mathscr{S}$* if the following condition is satisfied: For any finite subset J of K', there exist
  (a) a finite sequence $(k_1, k_2, \ldots, k_r)$ in K_{ex},
  (b) an injective map $\sigma \colon \mathsf{J} \to \mathsf{K}$, depending on the choice of J, such that
    (1) $\sigma(\mathsf{J}_{\mathrm{ex}}) \subset \mathsf{K}_{\mathrm{ex}}$, where $\mathsf{J}_{\mathrm{ex}} := \mathsf{J} \cap (\mathsf{K}')_{\mathrm{ex}}$,
    (2) $z'_j = \upmu(z)_{\sigma(j)}$ for all $j \in \mathsf{J}$,
    (3) $(\widetilde{B}')_{(i,j)} = \upmu(\widetilde{B})_{\sigma(i),\sigma(j)}$ for any $(i, j) \in \mathsf{J} \times \mathsf{J}^{\mathrm{ex}}$,

where $\upmu := \mu_{k_r} \circ \cdots \circ \mu_{k_1}$.

(ii) We say that the quantum seeds $\mathscr{S}$ and $\mathscr{S}'$ are *mutation equivalent* if $\mathscr{S}'$ is mutated from $\mathscr{S}$ and $\mathscr{S}$ is also mutated from $\mathscr{S}'$. In this case, we write $\mathscr{S} \simeq \mathscr{S}'$.

7.3. Mutation of valued quiver. Recall that we can associate the valued quiver $\mathcal{Q}_{\widetilde{B}}$ to an exchange matrix $\widetilde{B}$. Here we describe the algorithm transforming a valued quiver $\mathcal{Q}$ into a new valued quiver $\mu_k(\mathcal{Q})$ ($k \in \mathsf{K}_{\mathrm{ex}}$), which corresponds to $\mu_k(\widetilde{B})$.

Algorithm 7.3. For $k \in \mathsf{K}_{\mathrm{ex}}$, the *valued quiver mutation* μ_k transforms $\mathcal{Q}$ into a new valued quiver $\mu_k(\mathcal{Q})$ via the following rules, where we assume (i) $ac > 0$ or $bd > 0$, and (ii) we do not perform $(\mathcal{NC})$ and $(\mathcal{C})$ below, if i and j are frozen at the same time:

$(\mathcal{NC})$ For each full-subquiver $\begin{tikzcd} i \arrow[r, "\ulcorner a,b \lrcorner"'] \arrow[rr, bend left, "\ulcorner e,f \lrcorner"] & k \arrow[r, "\ulcorner c,d \lrcorner"'] & j \end{tikzcd}$ in $\mathcal{Q}$, we change the value of the arrow from i to j into $\ulcorner e + ac, f - bd \lrcorner$:

$$i \xrightarrow[\ulcorner e+ac, f-bd \lrcorner]{} j \ .$$

$(\mathcal{C})$ For each full-subquiver $\begin{tikzcd} i \arrow[r, "\ulcorner a,b \lrcorner"'] & k \arrow[r, "\ulcorner c,d \lrcorner"'] & j \arrow[ll, bend right, "\ulcorner e,f \lrcorner"'] \end{tikzcd}$ with $(e, f) \neq (0, 0)$ in $\mathcal{Q}$, we change the valued arrow between i and j as follows:

$$\begin{cases} i \xleftarrow[\ulcorner e-bd, f+ac \lrcorner]{} j & \text{if } f + ac \leqslant 0 \leqslant e - bd, \\ i \xrightarrow[\ulcorner f+ac, e-bd \lrcorner]{} j & \text{if } f + ac \geqslant 0 \geqslant e - bd. \end{cases}$$

($\mathcal{R}$) Reverse the direction of each arrow incident to the vertex k and change the value $\ulcorner a, b \lrcorner$ of each arrow into $\ulcorner -b, -a \lrcorner$.

Here if there is no arrow between i and j in ($\mathcal{NC}$) and ($\mathcal{C}$), then put $e = f = 0$ and follow the same rule.

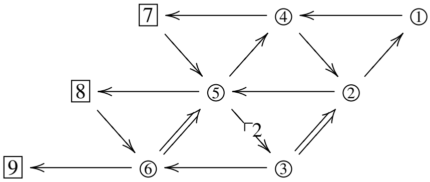
Example 7.4. Consider the following 9×6 integer-valued matrix:

$$\widetilde{B} = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 & -2 & 1 \\ -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{7.5}$$

By taking $\mathsf{K}_{\mathrm{ex}} = \{1, 2, 3, 4, 5, 6\}$ and $\mathsf{K}_{\mathrm{fr}} = \{7, 8, 9\}$, one can see that its principal part is skew-symmetrizable with $\mathsf{S} = \mathrm{diag}(2, 2, 1, 2, 2, 1)$.

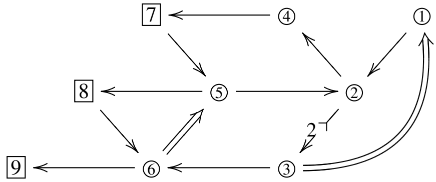
Using Convention 2, the valued quiver $\mathcal{Q}$ associated to $\widetilde{B}$ in (7.5) can be drawn as

$\mathcal{Q} =$

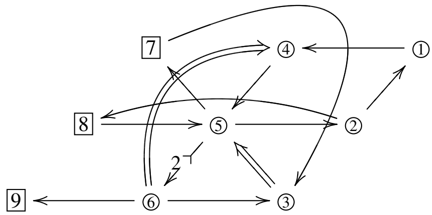


Here $\boxed{k}$ denotes $k \in \mathsf{K}_{\mathrm{fr}}$. Then $\mu_2(\mathcal{Q})$ and $\mu_5(\mathcal{Q})$ are depicted as follows:

$\mu_2(\mathcal{Q}) =$



$\mu_5(\mathcal{Q}) =$



7.4. Quantum cluster algebra. Let $\mathscr{S} = (\{z_i\}_{i \in \mathsf{K}}, L, \widetilde{B})$ be a quantum seed in a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathcal{A}$. The *quantum cluster algebra* $\mathscr{A}_{q^{1/2}}(\mathscr{S})$ associated to the quantum seed $\mathscr{S}$ is the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of the skew field K generated by all the quantum cluster variables in the quantum seeds obtained from $\mathscr{S}$ by any *finite* sequence of mutations. Here we call $\mathscr{S}$ the *initial quantum seed* of the quantum cluster algebra $\mathscr{A}_{q^{1/2}}(\mathscr{S})$.

Lemma 7.5. *Let $\mathscr{S}$ and $\mathscr{S}'$ be quantum seeds in $\mathcal{A}$. If $\mathscr{S}'$ is mutated from $\mathscr{S}$, then $\mathscr{A}_{q^{1/2}}(\mathscr{S}')$ is isomorphic to $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra of $\mathscr{A}_{q^{1/2}}(\mathscr{S})$. Furthermore, if $\mathscr{S}$ and $\mathscr{S}'$ are mutation equivalent to each other, then we have*

$$\mathscr{A}_{q^{1/2}}(\mathscr{S}') \simeq \mathscr{A}_{q^{1/2}}(\mathscr{S}).$$

Proof. This assertion follows from Definition 7.2. □

Definition 7.6. A *quantum cluster algebra structure* associated with a quantum seed $\mathscr{S}$ in a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathcal{A}$, contained in a skew field K, is a family $\mathscr{F}$ of quantum seeds in $\mathcal{A}$ satisfying the following conditions:

(a) For any quantum seed $\mathscr{S}$ in $\mathscr{F}$, the quantum cluster algebra $\mathscr{A}_{q^{1/2}}(\mathscr{S})$ is isomorphic to $\mathcal{A}$ as a $\mathbb{Z}[q^{\pm 1/2}]$-algebra.
(b) Any mutation of a quantum seed in $\mathscr{F}$ is in $\mathscr{F}$.
(c) For any pair $\mathscr{S}$, $\mathscr{S}'$ of quantum seeds in $\mathscr{F}$, we have $\mathscr{S}' \simeq \mathscr{S}$.

8. Quantum Cluster Algebra Structure on $\mathfrak{K}_{q,\xi}(\mathfrak{g})$

In this section, we will prove that the ring $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ has a quantum cluster algebra structure based on the recent work [47] by Kashiwara–Oh. As applications, we obtain

- a quantum cluster algebra algorithm to compute the KR-polynomials $F_q(\underline{m}^{(i)}[a,b])$ for KR-monomials $m^{(i)}[a,b]$,
- a q-commutativity for KR-polynomials $F_q(\underline{m}^{(i)}_{k,r})$ and $F_q(\underline{m}^{(j)}_{l,t})$ satisfying certain conditions on the pair of their KR-monomials $(m^{(i)}_{k,r},\, m^{(j)}_{l,t})$.

Note that, in this section, we shall employ the framework in [5,30] for our goal.

8.1. Compatible pair. Let S be a convex subset of $\widetilde{\triangle}_0$ with an upper bound (recall Definition 2.7). For each $j \in \triangle_0$, we set

$$\xi_j := \max(s \mid (j,s) \in \mathsf{S}).$$

Recall the exchange matrices $\widetilde{B}_{\widetilde{\triangle}_0}$ and ${}^{\xi}\widetilde{B}$ in Definition 2.4 and Definition 2.7.

Theorem 8.1. [47, Theorem 7.1] (see also [19]) *Define*

$$\Lambda_{(i,p),(j,s)} = \underline{\mathcal{N}}(m^{(i)}[p,\xi_i], m^{(j)}[s,\xi_j]) \qquad (i,p),(j,s) \in \mathsf{S}.$$

Then the pair $((\Lambda_{(i,p),(j,s)})_{(i,p),(j,s)\in\mathsf{S}}, {}^{\mathsf{S}}\widetilde{B})$ *is compatible with* $\mathrm{diag}(2d_{i,p} := 2d_i \mid (i,p) \in \mathsf{S})$.

Recall that the subset ${}^{\xi}\widetilde{\triangle}_0$ is convex without frozen indices. Thus the pair $({}^{\xi}L, {}^{\xi}\widetilde{B})$ is compatible with $\mathrm{diag}(2d_{i,p} := 2d_i \mid (i,p) \in {}^{\xi}\widetilde{\triangle}_0)$, where

$${}^{\xi}L = \big(\Lambda_{(i,p),(j,s)}\big)_{(i,p),(j,s)\in{}^{\xi}\widetilde{\triangle}_0} \quad \text{and} \quad \Lambda_{(i,p),(j,s)} = \underline{\mathcal{N}}\big(m^{(i)}[p,\xi_i], m^{(j)}[s,\xi_j]\big). \tag{8.1}$$

8.2. Sequence of mutations. Let us consider the valued quiver ${}^{\xi}\widetilde{\triangle}$ associated to the height function ξ of Q. Note that, for a source i of Q,

$$\begin{aligned}
&\text{(i) the vertex } (i,\xi_i) \text{ is located at the boundary of } {}^{\xi}\widetilde{\triangle} \text{ determined by } \xi, \text{ and} \\
&\quad \text{vertically sink and horizontally source,} \\
&\text{(ii) } s_i\xi \text{ is a height function defined as in (2.7).}
\end{aligned} \tag{8.2}$$

For a source i of Q, we set an *infinite* sequence of mutations

$$\overset{\boldsymbol{i}}{\underset{\xi}{}}\upmu := \cdots \circ \mu_{(i,\xi_i-4)} \circ \mu_{(i,\xi_i-2)} \circ \mu_{(i,\xi_i)} \tag{8.3}$$

and call it the *forward shift* at i (see [30] for $\mathcal{K}_{\mathfrak{t}}(\mathscr{C}^0_{\mathfrak{g}})$-cases).

Proposition 8.2. *For a Dynkin quiver $Q = (\triangle, \xi)$ and a source i, we have*

$$\overset{\boldsymbol{i}}{\underset{\xi}{}}\upmu({}^{\xi}\widetilde{\triangle}) \simeq {}^{s_i\xi}\widetilde{\triangle}.$$

Proof. We shall prove our assertion by an inductive argument on the sequence $\overset{\boldsymbol{i}}{\underset{\xi}{}}\upmu$. For this, we observe first two steps $\mu_{(i,\xi_i)}$ and $\mu_{(i,\xi_i-2)} \circ \mu_{(i,\xi_i)}$.

Step 1. Let us consider $\mu_{(i,\xi_i)}({}^{\xi}\widetilde{\triangle})$. In this case, the vertex (i,ξ) in ${}^{\xi}\widetilde{\triangle}$ (marked with $*$ below) is vertically sink and horizontally source in ${}^{\xi}\widetilde{\triangle}$ by (2.14) and (8.2) (i) as follows:

$$\begin{tikzcd}[column sep=small]
& (j,\xi_j-4) \arrow[dr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"'] & & (j,\xi_j-2) \arrow[ll] \arrow[dr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"'] & & (j,\xi_j) \arrow[ll] \arrow[dr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"'] & \\
{} & & (i,\xi_i-4) \arrow[ll] \arrow[ur, "\ulcorner -c_{i,j},c_{j,i}\lrcorner"] \arrow[dr, "\ulcorner -c_{ij'},c_{j'i}\lrcorner"] & & (i,\xi_i-2) \arrow[ll] \arrow[ur, "\ulcorner -c_{i,j},c_{j,i}\lrcorner"] \arrow[dr, "\ulcorner -c_{ij'},c_{j'i}\lrcorner"] & & (i,\xi_i)^* \arrow[ll] \\
& (j',\xi_{j'}-4) \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] & & (j',\xi_{j'}-2) \arrow[ll] \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] & & (j',\xi_{j'}) \arrow[ll] \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] &
\end{tikzcd}$$

Here j and j' are indices in $\triangle_0$ such that $d(i,j) = d(i,j') = 1$. Note that, in order to observe the behavior with respect to $\mu_{(i,\xi_i)}$, it suffices to consider the full-subquiver described as above.

Applying Algorithm 7.3, $\mu_{(i,\xi_i)}({}^{\xi}\widetilde{\triangle})$ can be depicted as follows:

$$\begin{tikzcd}[column sep=small]
& (j,\xi_j-4) \arrow[dr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"'] & & (j,\xi_j-2) \arrow[ll] \arrow[dr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"'] & & (j,\xi_j) \arrow[ll] & \\
{} & & (i,\xi_i-4) \arrow[ll] \arrow[ur, "\ulcorner -c_{i,j},c_{j,i}\lrcorner"] \arrow[dr, "\ulcorner -c_{ij'},c_{j'i}\lrcorner"] & & (i,\xi_i-2)^* \arrow[ll] \arrow[rr] & & (i,\xi_i) \arrow[ul, "\ulcorner -c_{i,j},c_{j,i}\lrcorner"'] \arrow[dl, "\ulcorner -c_{ij'},c_{j'i}\lrcorner"] \\
& (j',\xi_{j'}-4) \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] & & (j',\xi_{j'}-2) \arrow[ll] \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] & & (j',\xi_{j'}) \arrow[ll] &
\end{tikzcd}$$

in which the vertex (i,ξ_i-2) (marked with $*$ above) becomes vertically sink and horizontally source.

Step 2. Let us consider $(\mu_{(i,\xi_i-2)} \circ \mu_{(i,\xi_i)})({}^{\xi}\widetilde{\triangle})$. Applying Algorithm 7.3 again, $(\mu_{(i,\xi_i-2)} \circ \mu_{(i,\xi_i)})({}^{\xi}\widetilde{\triangle})$ becomes

$$\begin{tikzcd}[column sep=small]
& (j,\xi_j-4) \arrow[dr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"'] & & (j,\xi_j-2) \arrow[ll] \arrow[drrr, "\ulcorner -c_{j,i},c_{i,j}\lrcorner"] & & (j,\xi_j) \arrow[ll] & \\
{} & & (i,\xi_i-4)^* \arrow[ll] \arrow[rr] \arrow[dr, phantom] & & (i,\xi_i-2) \arrow[ul, "\ulcorner -c_{i,j},c_{j,i}\lrcorner"'] & & (i,\xi_i) \arrow[ll] \arrow[ul, "\ulcorner -c_{i,j},c_{j,i}\lrcorner"'] \arrow[dl, "\ulcorner -c_{ij'},c_{j'i}\lrcorner"] \\
& (j',\xi_{j'}-4) \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] & & (j',\xi_{j'}-2) \arrow[ll] \arrow[ur, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"] \arrow[urrr, "\ulcorner -c_{j'i},c_{ij'}\lrcorner"'] & & (j',\xi_{j'}) \arrow[ll] &
\end{tikzcd}$$

which is isomorphic to

$$\begin{tikzcd}[column sep=small]
 & (j,\xi_j-4) \arrow[l] \arrow[dr, "\ulcorner -\mathsf{c}_{j,i},\mathsf{c}_{i,j} \lrcorner"] & & & (j,\xi_j-2) \arrow[lll] \arrow[dr, "\ulcorner -\mathsf{c}_{j,i},\mathsf{c}_{i,j} \lrcorner"] & & (j,\xi_j) \arrow[ll] \\
(i,\xi_i-6) \arrow[ur, "\ulcorner -\mathsf{c}_{i,j},\mathsf{c}_{j,i} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{ij'},\mathsf{c}_{j'i} \lrcorner"'] & & (i,\xi_i-4)^{*} \arrow[ll] \arrow[r] & (i,\xi_i-2) \arrow[ur, "\ulcorner -\mathsf{c}_{i,j},\mathsf{c}_{j,i} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{ij'},\mathsf{c}_{j'i} \lrcorner"'] & & (i,\xi_i) \arrow[ll] \arrow[ur, "\ulcorner -\mathsf{c}_{i,j},\mathsf{c}_{j,i} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{ij'},\mathsf{c}_{j'i} \lrcorner"'] & \\
 & (j',\xi_{j'}-4) \arrow[l] \arrow[ur, "\ulcorner -\mathsf{c}_{j'i},\mathsf{c}_{ij'} \lrcorner"'] & & & (j',\xi_{j'}-2) \arrow[lll] \arrow[ur, "\ulcorner -\mathsf{c}_{j'i},\mathsf{c}_{ij'} \lrcorner"'] & & (j',\xi_{j'}) \arrow[ll]
\end{tikzcd} \tag{8.4}$$

Here the vertex $(i, \xi_i - 4)$ (marked with $*$ in (8.4)) becomes also vertically sink and horizontally source.

By *Step 1* and *Step 2*, we observe that the full-subquiver consisting of the rightmost 6-vertices in (8.4) are isomorphic to the rightmost 6-vertices of ${}^{s_i\xi}\widetilde{\triangle}$. Furthermore, since the local circumstance of $(i, \xi_i - 4)$ in $(\mu_{(i,\xi_i-2)} \circ \mu_{(i,\xi_i)})({}^{\xi}\widetilde{\triangle})$ is the same as the one of $(i, \xi_i - 2)$ in $\mu_{(i,\xi_i)}({}^{\xi}\widetilde{\triangle})$, we can apply an induction on k for the valued quiver

$$(\mu_{(i,\xi_i-2k)} \cdots \circ \mu_{(i,\xi_i-4)} \circ \mu_{(i,\xi_i-2)} \circ \mu_{(i,\xi_i)})({}^{\xi}\widetilde{\triangle}) \qquad \text{for all } k \in \mathbb{Z}_{\geqslant 1}.$$

Finally, our assertion comes from the definition of ${}^{s_i\xi}\widetilde{\triangle}$. □

The following proposition is a direct consequence of Proposition 8.2 and the definition of ${}^{\xi}\widetilde{\triangle}$.

Proposition 8.3. *Let i, j be sources of $Q = (\triangle, \xi)$. Then we have*

$$({}^{\boldsymbol{j}}_{s_i\xi}\upmu \circ {}^{\boldsymbol{i}}_{\xi}\upmu)({}^{\xi}\widetilde{\triangle}) \simeq ({}^{\boldsymbol{i}}_{s_j\xi}\upmu \circ {}^{\boldsymbol{j}}_{\xi}\upmu)({}^{\xi}\widetilde{\triangle}).$$

Thus, for any Q-adapted reduced expression $s_{i_1} \cdots s_{i_n}$ of the Coxeter element τ_Q,

$$^{\boldsymbol{Q}}\upmu := {}^{\boldsymbol{i}_n}_{s_{i_n}\cdots s_{i_1}\xi}\upmu \circ \cdots \circ {}^{\boldsymbol{i}_2}_{s_{i_1}\xi}\upmu \circ {}^{\boldsymbol{i}_1}_{\xi}\upmu \text{ is well-defined.} \tag{8.5}$$

Theorem 8.4. *For Dynkin quivers $Q = (\triangle, \xi)$ and $Q = (\triangle, \xi')$, there exists a sequence of mutations $\upmu$ such that*

$$\upmu({}^{\xi}\widetilde{\triangle}) \simeq {}^{\xi'}\widetilde{\triangle} \qquad \text{as valued quivers.}$$

In particular, we have

$$^{\boldsymbol{Q}}\upmu({}^{\xi}\widetilde{\triangle}) \simeq {}^{\xi}\widetilde{\triangle} \qquad \text{as valued quivers.}$$

Proof. This assertion follows from (2.6), (2.8) and Proposition 8.2. □

8.3. Quantum cluster algebra structure on $\mathfrak{K}_{q,\xi}(\mathfrak{g})$. For each $s \in \mathbb{Z}$, we denote by ${}^{(s)}\xi$ the height function such that ${}^{(s)}\xi_i \in \{s, s-1\}$ for all $i \in \triangle_0$ and ${}^{(s)}Q = (\triangle, {}^{(s)}\xi)$. Note that ${}^{(s)}\xi$ is uniquely determined by Convention 1.

Example 8.5. For $\mathfrak{g}$ of type A_5 and $s = 3$, the height function ${}^{(3)}\xi$ of ${}^{(s)}Q$ is given as follows:

$${}^{(3)}\xi_1 = {}^{(3)}\xi_3 = {}^{(3)}\xi_5 = 2 \quad \text{and } {}^{(3)}\xi_2 = {}^{(3)}\xi_4 = 3.$$

Here we choose ${}^{(3)}\xi_1 = 2$ by Convention 1.

For a height function ξ, we set

$$^{\xi}\mathfrak{K}_q(\mathfrak{g}) := \bigoplus_{m \in {}^{\xi}\mathcal{M}_+} \mathbb{Z}[q^{\pm\frac{1}{2}}](F_q(\underline{m}))_{\leqslant \xi} \subset {}^{\xi}\mathcal{X}_q.$$

Note that ${}^{\xi}\mathfrak{K}_q(\mathfrak{g}) = (\cdot)_{\leqslant \xi}\left(\mathfrak{K}_{q,\xi}(\mathfrak{g})\right) \simeq \mathfrak{K}_{q,\xi}(\mathfrak{g})$.

For simplicity of notations, we set

(a) ${}^{s}\widetilde{\Delta} := {}^{(s)\xi}\widetilde{\Delta}$, ${}^{s}\mathcal{X}_q := {}^{(s)\xi}\mathcal{X}_q$, $(-)_{\leqslant s} := (-)_{\leqslant {}^{(s)}\xi}$,

(b) ${}^{s}L := {}^{(s)\xi}L$, ${}^{s}B := {}^{(s)\xi}\widetilde{B}$,

(c) ${}^{s}\mu := {}^{(s)}\mathcal{Q}\mu$, ${}^{s}\mathfrak{K}_q(\mathfrak{g}) := {}^{(s)\xi}\mathfrak{K}_q(\mathfrak{g})$ and $\mathfrak{K}_{q,s}(\mathfrak{g}) := \mathfrak{K}_{q,{}^{(s)}\xi}(\mathfrak{g})$.

From now on, we fix $s \in \mathbb{Z}$ and $\widetilde{\Delta}$. Let us denote by ${}^{s}\mathcal{A}_q$ the quantum cluster algebra whose initial seed is

$$^{s}\mathcal{S} := (\{v_{i,p} := \underline{m^{(i)}[p,s]}\}_{(i,p)\in {}^{s}\widetilde{\Delta}_0}, {}^{s}L, {}^{s}B). \tag{8.6}$$

For $n \geqslant 0$, let $v_{i,p}^{(n)}$ be the quantum cluster variable obtained at vertex (i,p) after applying the sequence of mutations ${}^{s}\mu$ n-times. Then we give a quantum cluster algebra algorithm to compute $F_q(\underline{m}^{(i)}[a,b])$ for KR-monomials $m^{(i)}[a,b]$. The following proposition establishes an analogue of [30, Theorem 3.1] and [5, Proposition 6.3.1].

Proposition 8.6. *For each $(i,p) \in {}^{s}\widetilde{\Delta}_0$ and $n \geqslant 0$,*

$$v_{i,p}^{(n)} = {}^{s}F_q(\underline{m}^{(i)}[p-2n, s-2n]) := (F_q(\underline{m}^{(i)}[p-2n, s-2n]))_{\leqslant s}. \tag{8.7}$$

In particular, if $2n \geqslant \mathsf{h}$, we have

$$v_{i,p}^{(n)} = F_q(\underline{m}^{(i)}[p-2n, s-2n]).$$

Proof. Let us apply induction on n for this assertion. For $n=0$, it follows from (5.28) in Proposition 5.29. Let $n \geqslant 0$ and $(i,p) \in {}^{s}\widetilde{\Delta}_0$. Suppose we have applied ${}^{s}\mu$ n-times on ${}^{s}\mathcal{S}$, and $(n+1)$-times on all vertices preceding (i,p) in the sequence ${}^{s}\mu$, and that all those previous vertices satisfy (8.7).

Thanks to Theorem 8.4, the corresponding valued quivers coincide up to a shift of spectral parameters in labeling of vertices. Then, the argument in the proof of Proposition 8.2 tells us that the vertex (i,p) is vertically sink or horizontally source, that is, one of the following configurations:

$$\begin{tikzcd}[column sep=small] & (j, p+{}^{(s)}\xi_{j,i}) \arrow[d, "{\ulcorner -\mathsf{c}_{j,i}, \mathsf{c}_{i,j} \lrcorner}"] & \\ (i,p-2) & (i,p) \arrow[l] \arrow[r] & (i,p+2) \\ & (j', p+{}^{(s)}\xi_{j'i}) \arrow[u, "{\ulcorner -\mathsf{c}_{j'i}, \mathsf{c}_{ij'} \lrcorner}"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & (j, p+{}^{(s)}\xi_{j,i}) \arrow[d, "{\ulcorner -\mathsf{c}_{j,i}, \mathsf{c}_{i,j} \lrcorner}"] & \\ (i,p-2) & (i,p) \arrow[l] \arrow[r] & (i,p+2) \end{tikzcd} \quad \begin{tikzcd}[column sep=small] (i,p-2) & (i,p) \arrow[l] \arrow[r] & (i,p+2) \\ & (j', p+{}^{(s)}\xi_{j'i}) \arrow[u, "{\ulcorner -\mathsf{c}_{j'i}, \mathsf{c}_{ij'} \lrcorner}"'] & \end{tikzcd} \tag{8.8}$$

where ${}^{(s)}\xi_{k,i} := (-1)^{\delta\left({}^{(s)}\xi_k < {}^{(s)}\xi_i\right)}$ for $k \in \Delta_0$ with $d(i,k) = 1$. In this proof, we only consider the first case in (8.8) since the computation below is almost identical for the other cases.

By the definition of ${}^{(s)}\xi$, we have ${}^{(s)}\xi_{j,i} = {}^{(s)}\xi_{j',i}$ for all $j, j' \in \triangle_0$ with $d(j,i) = d(j',i) = 1$. Now let us assume that i is a source of ${}^{(s)}\xi$ since the proof for the cases when i is a sink of ${}^{(s)}\xi$ is similar. Then the quantum exchange relation has the form

$$v^{(n+1)}_{i,p} * v^{(n)}_{i,p} = q^{\alpha} v^{(n+1)}_{i,p+2} \cdot v^{(n)}_{i,p-2} + q^{\beta} \prod_{j;\ d(j,i)=1} \left(v^{(n)}_{i,p-1}\right)^{-\mathsf{c}_{j,i}} \tag{8.9}$$

for some $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$, where

$$q^{\alpha}\left(v^{(n+1)}_{i,p+2} \cdot v^{(n)}_{i,p-2}\right) * \left(v^{(n)}_{i,p}\right)^{-1} \text{ and } q^{\beta}\left(\prod_{j;\ d(j,i)=1} \left(v^{(n)}_{i,p-1}\right)^{-\mathsf{c}_{j,i}}\right) * \left(v^{(n)}_{i,p}\right)^{-1} \text{ are bar-invariant.} \tag{8.10}$$

Here the dot product $\cdot$ is given in (4.8).

The rest of this proof is devoted to show that the above quantum exchange relation coincides with the truncated image of the quantum folded T-system in Theorem 6.9. For this, it suffices to assume that $s = 0$ and hence $p \in \mathbb{Z}_{\leqslant 0}$. For each $(i,p) \in {}^{0}\widetilde{\triangle}_0$, we set $k := \max(u \mid p + 2u \leqslant 0)$. By the induction hypothesis, we have

$$\begin{aligned} u^{(n+1)}_{i,p} * {}^{0}F_q(\underline{m}^{(i)}_{k,p-2n}) &= q^{\alpha}\left({}^{0}F_q(\underline{m}^{(i)}_{k-1,p-2n}) \cdot {}^{0}F_q(\underline{m}^{(i)}_{k+1,p-2n-2})\right) \\ &+ q^{\gamma} \prod_{j;\ d(j,i)=1} {}^{0}F_q(\underline{m}^{(j)}_{k,p-1-2n})^{-\mathsf{c}_{j,i}} \end{aligned}$$

On the other hand, the corresponding truncated image of the quantum folded T-system in Theorem 6.9 is

$$\begin{aligned} {}^{0}F_q(\underline{m}^{(i)}_{k,p-2n-2}) * {}^{0}F_q(m^{(i)}_{k,p-2n}) = q^{\alpha'}&\left({}^{0}F_q(\underline{m}^{(i)}_{k-1,p-2n}) \cdot {}^{0}F_q(\underline{m}^{(i)}_{k+1,p-2n-2})\right) \\ &+ q^{\gamma'} \prod_{j;\ d(j,i)=1} {}^{0}F_q(\underline{m}^{(j)}_{k,p-1-2n})^{-\mathsf{c}_{j,i}}, \end{aligned} \tag{8.11}$$

where

$$\gamma' = \frac{1}{2}\left(\widetilde{\mathsf{r}}_{i,i}(2k-1) + \widetilde{\mathsf{r}}_{i,i}(2k+1)\right) \quad \text{and} \quad \alpha' = \gamma' + d_i.$$

By using the dominant monomials in (8.11) and bar-invariance in (8.10),

$$q^{\alpha}\left(\underline{m}^{(i)}_{k-1,p-2n} \cdot \underline{m}^{(i)}_{k+1,p-2n-2}\right) * \left(\underline{m}^{(i)}_{k,p-2n}\right)^{-1} \text{ and } q^{\gamma}\left(\prod_{j;\ d(j,i)=1} \left(\underline{m}^{(j)}_{k,p-1-2n}\right)^{-\mathsf{c}_{j,i}}\right) * \left(\underline{m}^{(i)}_{k,p-2n}\right)^{-1}$$

are bar-invariant. Thus we have

$$\begin{aligned} \alpha = \frac{1}{2}\sum_{a=0}^{k-1}\Bigg(\sum_{b=0}^{k-2}&\left(\widetilde{\eta}_{i,i}(2(a-b)+1) - \widetilde{\eta}_{i,i}(2(a-b)-1)\right) + \sum_{b=0}^{k}\Big(\widetilde{\eta}_{i,i}(2(a-b)+3) \\ &-\widetilde{\eta}_{i,i}(2(a-b)+1)\Big)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{a=0}^{k-1}\big(\widetilde{\eta}_{i,i}(2a+1) - \widetilde{\eta}_{i,i}(2a-2k+3) + \widetilde{\eta}_{i,i}(2a+3) - \widetilde{\eta}_{i,i}(2a-2k+1)\big) \\
&= \frac{1}{2}\big(\widetilde{\eta}_{i,i}(2k+1) + \widetilde{\eta}_{i,i}(2k-1)\big) + \widetilde{\eta}_{i,i}(1) = \alpha'
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma &= \frac{1}{2}\sum_{j;d(i,j)=1} -\mathsf{c}_{j,i}\left(\sum_{a=0}^{k-1}\left(\sum_{b=0}^{k-1}\widetilde{\eta}_{i,i}(2(a-b)+2) - \widetilde{\eta}_{i,i}(2(a-b))\right)\right) \\
&= \frac{1}{2}\sum_{j;d(i,j)=1} -\mathsf{c}_{j,i}\left(\sum_{a=0}^{k-1}\big(\widetilde{\eta}_{i,i}(2a+2) - \widetilde{\eta}_{i,i}(2a-2k+2)\big)\right) = \frac{1}{2}\sum_{j;d(i,j)=1} -\mathsf{c}_{j,i}\widetilde{\eta}_{i,i}(2k) \\
&\overset{\dagger}{=} \frac{1}{2}\big(\widetilde{\eta}_{i,i}(2k+1) + \widetilde{\eta}_{i,i}(2k-1)\big) = \gamma'.
\end{aligned}
$$

Here $\overset{\dagger}{=}$ holds by Lemma 2.3.

Since ${}^0F_q(m^{(i)}_{k,p-2n})$ is invertible in the skew-field of fractions ${}^0\mathfrak{F}_q$ of the quantum torus ${}^0\mathcal{X}_q$, we conclude that

$$
v^{(n+1)}_{i,p} = {}^0F_q(\underline{m}^{(i)}_{k,p-2n-2}),
$$

as desired. The second assertion follows from Proposition 5.29. □

Let ${}^{(s)}\mathcal{T}_q$ be the quantum torus associated with ${}^sL$ generated by $v_{i,p}$ for $(i,p) \in {}^s\widetilde{\triangle}_0$. Then, ${}^{(s)}\mathcal{T}_q$ is isomorphic to ${}^s\mathcal{X}_q$. Thus, ${}^s\mathcal{A}_q$ can be understood as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-subalgebra in ${}^{(s)}\mathcal{T}_q$.

By following the argument in the proof of [5, Lemma 6.4.1], we have the following lemma:

Lemma 8.7. *The assignment*

$$
\Omega : v_{i,p} \mapsto F_q(\underline{m}^{(i)}[p,s])
$$

extends to a well-defined injective $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra homomorphism

$$
\Omega : {}^{(s)}\mathcal{T}_q \mapsto \mathcal{X}_q.
$$

Moreover, the restriction of Ω to the quantum cluster algebra ${}^s\mathcal{A}_q$ has its image in the quantum torus $\mathcal{X}_q$ and the $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra homomorphisms Ω and $(\cdot)_{\leqslant s}$ satisfy the following commutative diagram:

$$
\begin{tikzcd}
{}^s\mathcal{A}_q \arrow[r, "\Omega"] \arrow[dr, "{}^s\Omega"'] & \mathcal{X}_q \arrow[d, "(\cdot)_{\leqslant s}"] \\
& {}^s\mathcal{X}_q,
\end{tikzcd}
\tag{8.12}
$$

where ${}^s\Omega$ is the map induced from the assignment $v_{i,p} \to \underline{m}^{(i)}[p,s]$.

Let ${}^{s}R_q$ be the image of the quantum cluster algebra ${}^{s}\mathcal{A}_q$ under the map Ω:

$$R_{q,s} := \Omega({}^{s}\mathcal{A}_q).$$

We prove an analogue of [30, Theorem 5.1] and [5, Proposition 6.4.2] below.

Proposition 8.8. *We have*

$$R_{q,s} = \mathfrak{K}_{q,s}(\mathfrak{g}).$$

Proof. Let us recall $v_{i,p} := \underline{m^{(i)}[p,s]}$ and ${}^{(s)}\xi_i \in \{s-1, s\}$. By Proposition 8.6 and Lemma 8.7, we have

$$\Omega\left(v^{(n)}_{i,{}^{(s)}\xi_i}\right) = F_q\left(\mathsf{X}_{i,{}^{(s)}\xi_i-2n}\right) \quad \text{for } i \in \triangle_0 \text{ and } n \in \mathbb{Z}_{\geqslant 0}.$$

Since $\mathfrak{K}_{q,s}(\mathfrak{g})$ is generated by $F_q\left(\mathsf{X}_{i,p}\right)$ for all $(i,p) \in {}^{s}\widetilde{\triangle}$ as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra by Theorem 5.27 (see also (5.23)) below), we have the following inclusion:

$$\mathfrak{K}_{q,s}(\mathfrak{g}) \subset R_{q,s}.$$

Next, let us prove the reverse inclusion. As we see in Sect. 4.2, there exist $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-derivations $S_{i,q} : \mathcal{X}_q \to \mathcal{X}_{i,q}$ such that

$$\bigcap_{i \in \widetilde{\triangle}_0} \mathrm{Ker}(S_{i,q}) = \mathfrak{K}_q(\mathfrak{g}). \tag{8.13}$$

Let us prove by induction that all cluster variables Z in ${}^{s}\mathcal{A}_q$ satisfy $\Omega(Z) \in \mathfrak{K}_{q,s}(\mathfrak{g})$. Let Z be a quantum cluster variable in ${}^{s}\mathcal{A}_q$. If Z belongs to the initial cluster variables, it is done by definition of Ω. Let us assume that Z does not belong to the initial cluster variables. Then Z is obtained from a finite sequence of mutations. Then we have

$$ZZ_1 = q^{\alpha}M_1 + q^{\beta}M_2,$$

where Z_1, M_1 and M_2 are quantum cluster monomials of ${}^{s}\mathcal{A}_q$. By the induction hypothesis,

$$\Omega(Z_1), \quad \Omega(M_1), \quad \Omega(M_2) \in \mathfrak{K}_{q,s}(\mathfrak{g}). \tag{8.14}$$

Note that $\Omega(Z_1) \neq 0$. By Lemma 8.7, we have

$$\Omega(Z) * \Omega(Z_1) = q^{\alpha}\Omega(M_1) + q^{\beta}\Omega(M_2).$$

Since $S_{i,q}$ $(i \in \triangle_0)$ is a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-linear derivation (Proposition 4.8),

$$\begin{aligned} S_{i,q}(\Omega(Z) * \Omega(Z_1)) &= \Omega(Z) \cdot S_{i,q}(\Omega(Z_1)) + S_{i,q}(\Omega(Z)) \cdot \Omega(Z_1) \\ &= q^{\alpha} S_{i,q}(\Omega(M_1)) + q^{\beta} S_{i,q}(\Omega(M_2)). \end{aligned}$$

By the induction hypothesis and (8.13), we have

$$S_{i,q}(\Omega(Z_1)) = S_{i,q}(\Omega(M_1)) = S_{i,q}(\Omega(M_2)) = 0.$$

Then Lemma 4.7 tells us that $S_{i,q}(\Omega(Z)) = 0$, that is, $\Omega(Z) \in \mathfrak{K}_{i,q}(\mathfrak{g})$ for all $i \in \triangle_0$. Hence, $\Omega(Z) \in \mathfrak{K}_{q,s}(\mathfrak{g})$ due to (8.13) and (8.14), as we desired. $\square$

Now, we present the main result in this section.

Theorem 8.9. *For each height function ξ on $\triangle$, the ring $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ has a quantum cluster algebra structure whose initial quantum seed is*

$$\mathscr{S}_\xi = \left(\{F_q(\underline{m}^{(i)}[p, \xi_i])\}_{(i,p)\in {}^\xi\widetilde{\triangle}_0}, {}^\xi L, {}^\xi\widetilde{B}\right). \tag{8.15}$$

Proof. Our assertion for ${}^{(s)}\xi$ already holds by Proposition 8.8. Let $j \in {}^s\widetilde{\triangle}_0$ be a source of ${}^{(s)}\xi$. Then we have

$$\mu^{j}_{{}^{(s)}\xi}\left(\mathscr{S}_{{}^{(s)}\xi}\right) = \left(\{F_q(\underline{m}^{(i)}[p - 2\delta_{i,j}, {}^{(s)}\xi_i - 2\delta_{i,j}])\}_{(i,p)\in {}^s\widetilde{\triangle}_0}, {}^{s_j {}^{(s)}\xi}L, {}^{s_j {}^{(s)}\xi}\widetilde{B}\right) = \mathscr{S}_{s_j {}^{(s)}\xi}, \tag{8.16}$$

by Proposition 8.2 and Proposition 8.6. Let Q (resp. ${}^{(s)}Q$) be the Dynkin quiver of $\triangle$ corresponding to ξ (resp. ${}^{(s)}\xi$). Since any Dynkin quivers of $\triangle$ are connected by a finite sequence of reflections (up to constant on their height functions), so are Q and ${}^{(s)}Q$. Then the quantum seed $\mathscr{S}_{{}^{(s)}\xi}$ is mutation equivalent to $\mathscr{S}_\xi$ by (8.16) and T_r ($r \in 2\mathbb{Z}$) (see (5.27) for the definition of T_r). Hence, it follows from Lemma 7.5 and Proposition 8.8 that $\mathfrak{K}_{q,s}(\mathfrak{g}) \simeq \mathscr{A}_{q^{1/2}}(\mathscr{S}_{{}^{(s)}\xi}) \simeq \mathscr{A}_{q^{1/2}}(\mathscr{S}_\xi) \simeq \mathfrak{K}_{q,\xi}(\mathfrak{g})$, so $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ has a quantum cluster algebra structure. □

As an application of Theorem 8.9, we obtain q-commutativities of $F_q(\underline{m}^{(i)}_{k,r})$ and $F_q(\underline{m}^{(j)}_{l,t})$ satisfying certain conditions as follows.

Theorem 8.10. *For a pair $(m^{(i)}_{k,r}, m^{(j)}_{l,t})$, $\left(F_q(\underline{m}^{(i)}_{k,r}), F_q(\underline{m}^{(j)}_{l,t})\right)$ is a q-commuting pair if*

(a) $r - d(i,j) \leqslant t \leqslant t + 2(l-1) \leqslant r + 2(k-1) + d(i,j)$ *or*
(b) $t - d(i,j) \leqslant r \leqslant r + 2(k-1) \leqslant t + 2(l-1) + d(i,j)$.

In particular, $F_q(m^{(i)}_{k,r})$ q-commutes with $F_q(X_{j,p})$ if

$$r - d(i,j) \leqslant p \leqslant r + 2(k-1) + d(i,j).$$

Proof. Under the conditions (a) and (b), there exists a height function ξ on $\triangle$ such that $\xi_i = r + 2(k-1)$ and $\xi_j = t + 2(l-1)$. Then we have

$$F_q(\underline{m}^{(i)}_{k,r}) = F_q(\underline{m}^{(i)}[\xi_i - 2(k-1), \xi_i]) \ \text{ and } \ F_q(\underline{m}^{(j)}_{l,t}) = F_q(\underline{m}^{(i)}[\xi_j - 2(l-1), \xi_j])$$

which can be viewed as initial quantum cluster variables in $\mathscr{S}_\xi$. Thus our assertion follows from Theorem 8.9. □

The conjecture below is proved in [62] when $\mathfrak{g}$ is of finite AD-type.

Conjecture 4. *For a pair $(m^{(i)}_{k,r}, m^{(j)}_{l,t})$, $F_q(\underline{m}^{(i)}_{k,r})$ and $F_q(\underline{m}^{(j)}_{l,t})$ q-commute unless there exist $1 \leqslant u \leqslant \mathsf{h}$ and $0 \leqslant s \leqslant \min(k,l) - 1$ satisfying*

$$|k + r - l - t| = u + |k - l| + 2s \quad \text{and} \quad \widetilde{\mathsf{r}}_{i,j}(u-1) \neq 0. \tag{8.17}$$

9. Extension to $\mathfrak{K}_q(\mathfrak{g})$

In this section, we will extend Theorem 8.9 to $\mathfrak{K}_q(\mathfrak{g})$, that is, the quantum virtual Grothendieck ring $\mathfrak{K}_q(\mathfrak{g})$ has also a quantum cluster algebra structure (of skew-symmetrizable type) isomorphic to its subalgebra $\mathfrak{K}_{q,\xi}(\mathfrak{g})$ for each height function ξ on $\widetilde{\triangle}$.

9.1. Sink-source quiver. For an integer $s \in \mathbb{Z}$, recall the height function ${}^{(s)}\xi$ on $\triangle$. Now let us consider a new valued quiver ${}^{s}\overleftarrow{\varDelta}$ whose set of vertices is ${}^{s}\widetilde{\triangle}_0$ and the exchange matrix ${}^{s}\mathfrak{B}$ is given as follows:

$$\mathfrak{b}_{(i,p),(j,t)} = \begin{cases} -\mathsf{c}_{i,j} & \text{if either (a) } t-p=1,\ i \neq j \text{ and } p \equiv_4 \xi_i = s-1, \\ & \qquad \text{or (b) } p-t=1,\ i \neq j \text{ and } p \not\equiv_4 \xi_i = s, \\ \mathsf{c}_{i,j} & \text{if either (a}') \ p-t=1,\ i \neq j \text{ and } p \equiv_4 \xi_i = s, \\ & \qquad \text{or (b}') \ t-p=1,\ i \neq j \text{ and } p \not\equiv_4 \xi_i = s-1, \\ 1 & \text{if either (A) } |p-t|=2,\ i=j \text{ and } p \equiv_4 \xi_i = s, \\ & \qquad \text{or (B) } |p-t|=2,\ i=j \text{ and } p \not\equiv_4 \xi_i = s-1, \\ -1 & \text{if either (A}') \ |p-t|=2,\ i=j \text{ and } p \not\equiv_4 \xi_i = s, \\ & \qquad \text{or (B}') \ |p-t|=2,\ i=j \text{ and } p \equiv_4 \xi_i = s-1, \\ 0 & \text{otherwise.} \end{cases} \tag{9.1}$$

Note that ${}^{s}\mathfrak{B}$ satisfies (2.12) with the sequence $S = (s_{i,p} \mid s_{i,p} = d_i)$ and without frozen vertices.

Example 9.1. Here are a couple of examples of ${}^{s}\overleftarrow{\varDelta}$ for non-simply-laced types:

(1) Let us assume that $s = 0$ and $\mathfrak{g}$ is of type B_3. By Convention 1, we have

$$\xi_1 = \xi_3 = 0, \quad \xi_2 = -1.$$

Recall that ${}^{s}\widetilde{\triangle}_0 = \{(i,p) \in I^{\mathfrak{g}} \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z} \text{ and } p \leqslant \xi_i\}$. Let us compute $\mathfrak{b}_{(i,p),(j,t)}$ in (9.1) for $(i,p) = (2,-1)$. If $(j,t) = (1,0)$ or $(3,0)$, then this is the case (a), so we have $\mathfrak{b}_{(i,p),(j,t)} = -\mathsf{c}_{i,j}$. If $(j,t) = (1,-2)$ or $(3,-2)$, then the pair of (i,p) and (j,t) does not satisfies (b) and (a$'$). Thus, $\mathfrak{b}_{(i,p),(j,t)} = 0$ in this case. Finally, if $(j,t) = (2,-3)$, then this is the case (B$'$), so $\mathfrak{b}_{(i,p),(j,t)} = -1$. Consequently, the valued quiver ${}^{0}\overleftarrow{\varDelta}$ can be depicted as follows:

$$\begin{tikzcd}[column sep=small]
\cdots & & (1,-4) \arrow[ll] \arrow[rr] & & (1,-2) \arrow[dl] & & (1,0) \arrow[ll] \\
\cdots \arrow[r] & (2,-5) \arrow[ur] \arrow[dr, "2"] & & (2,-3) \arrow[ll] \arrow[rr] & & (2,-1) \arrow[ur] \arrow[dr, "2"] & \\
\cdots & & (3,-4) \arrow[ll] \arrow[rr] & & (3,-2) \arrow[ul, Rightarrow] & & (3,0) \arrow[ll]
\end{tikzcd} \tag{9.2}$$

(2) For $s = 0$ and $\mathfrak{g}$ of type G_2, ${}^{0}\overleftarrow{\varDelta}$ can be depicted as follows:

$$\begin{tikzcd}[column sep=small]
\cdots & & (1,-4) \arrow[ll] \arrow[rr] & & (1,-2) \arrow[dl, Rrightarrow] & & (1,0) \arrow[ll] \\
\cdots \arrow[r] & (2,-5) \arrow[ur, "3"] & & (2,-3) \arrow[ll] \arrow[rr] & & (2,-1) \arrow[ur, "3"] &
\end{tikzcd}$$

Remark 9.2. Note that every vertex (i,p) in ${}^{s}\overleftarrow{\varDelta}_0$ is either

$$\begin{aligned} &\text{(i) vertically sink and horizontally source, or} \\ &\text{(ii) vertically source and horizontally sink.} \end{aligned} \tag{9.3}$$

More precisely, when

(i)$'$ $\xi_i = s$ and $p \equiv_4 s$, or $\xi_i = (s-1)$ and $p \not\equiv_4 s-1$, (i,p) satisfies (i),
(ii)$'$ $\xi_i = s$ and $p \not\equiv_4 s$, or $\xi_i = (s-1)$ and $p \equiv_4 s-1$, (i,p) satisfies (ii).

Thus, comparing with the quiver (2.15) of type B_3 in Example 2.5, every vertex in (9.2) satisfies (9.3), while none in (2.15) does.

For each $(i, p) \in {}^{s}\overleftarrow{\varDelta}_0 = {}^{s}\widetilde{\Delta}_0$, we assign ${}^{s}\mathfrak{u}_{i,p} \in \mathfrak{K}_q(\mathfrak{g})$ at (i, p), which is defined by

$$ {}^{s}\mathfrak{u}_{i,p} := F_q\left(\underline{m}^{(i)}[{}^{s}o_{i,p}, {}^{s}o_{i,p} + 2\,{}^{s}l_{i,p}]\right), \tag{9.4} $$

where

$$ {}^{s}l_{i,p} := \left({}^{(s)}\xi_i - p\right)/2 \in \mathbb{Z}_{\geqslant 0} \quad \text{and} \quad {}^{s}o_{i,p} := {}^{(s)}\xi_i - 2 \times \left\lfloor \frac{{}^{s}l_{i,p} + \delta\left({}^{(s)}\xi_i = s\right)}{2} \right\rfloor. \tag{9.5} $$

Example 9.3. By replacing vertices (i, p)'s in ${}^{s}\overleftarrow{\varDelta}_0$ with $\mathfrak{u}_{i,p}$ in Example 9.1, we can obtain the following pictures:

(1) For $s = 0$ and $\mathfrak{g}$ of type B_3, we have

$$
\begin{tikzcd}
\cdots & & F_q(m^{(1)}[-2,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(1)}[-2,0]) \arrow[dl] & & F_q(m^{(1)}[0,0]) \arrow[ll] \\
\cdots \arrow[r] & F_q(m^{(2)}[-3,1]) \arrow[ur] \arrow[dr, "\ulcorner 2"] & & F_q(m^{(2)}[-1,1]) \arrow[ll] \arrow[rr] & & F_q(m^{(2)}[-1,-1]) \arrow[ur] \arrow[dr, "\ulcorner 2"] & \\
\cdots & & F_q(m^{(3)}[-2,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(3)}[-2,0]) \arrow[ul, Rightarrow] & & F_q(m^{(3)}[0,0]) \arrow[ll]
\end{tikzcd}
$$

(2) For $s = 0$ and $\mathfrak{g}$ of type G_2, we have

$$
\begin{tikzcd}
\cdots & & F_q(m^{(1)}[-2,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(1)}[-2,0]) \arrow[dl, Rrightarrow] & & F_q(m^{(1)}[0,0]) \arrow[ll] \\
\cdots \arrow[r] & F_q(m^{(2)}[-3,1]) \arrow[ur, "3\urcorner"] & & F_q(m^{(2)}[-1,1]) \arrow[ll] \arrow[rr] & & F_q(m^{(2)}[-1,-1]) \arrow[ur, "3\urcorner"] &
\end{tikzcd}
$$

Let us define a matrix ${}^{s}\Lambda = ({}^{s}\Lambda_{(i,p),(j,t)})_{(i,p),(j,t) \in {}^{s}\widetilde{\Delta}_0}$ such that

$$ {}^{s}\Lambda_{(i,p),(j,t)} = \underline{\mathcal{N}}(m^{(i)}[{}^{s}o_{i,p}, {}^{s}o_{i,p} + 2\,{}^{s}l_{i,p}], m^{(j)}[{}^{s}o_{j,t}, {}^{s}o_{j,s} + 2\,{}^{s}l_{j,t}]). $$

Theorem 9.4. *The pair* $({}^{s}\Lambda, {}^{s}\mathfrak{B})$ *is compatible with* $\mathrm{diag}(2d_{i,p} := 2d_i \mid (i, p) \in {}^{s}\overleftarrow{\varDelta}_0)$.

Proof. Let $(i, p), (j, t) \in {}^{s}\overleftarrow{\varDelta}_0$. In this proof, we only consider the case of $\xi_j = s$ and $t \equiv_4 \xi_j$, since the other cases are similar. Set $a_1 = {}^{s}o_{i,p}$, $a_2 = a_1 + 2{}^{s}l_{i,p}$, $b_1 = {}^{s}o_{j,t}$ and $b_2 = b_1 + 2{}^{s}l_{l,t}$. By (9.1), we have

$$ \begin{aligned} -({}^{s}\Lambda{}^{s}\mathfrak{B})_{(i,p),(j,t)} &= \delta(t \neq \xi_j)\ {}^{s}\Lambda_{(i,p),(j,t+2)} + {}^{s}\Lambda_{(i,p),(j,t-2)} + \sum_{k;\ d(j,k)=1} \mathsf{c}_{k,j}\,{}^{s}\Lambda_{(i,p),(k,t-1)} \\ &= \delta(t \neq \xi_j)^{s}\underline{\mathcal{N}}(m^{(i)}[a_1, a_2], m^{(j)}[b_1, b_2 - 2]) \\ &\quad + \underline{\mathcal{N}}(m^{(i)}[a_1, a_2], m^{(j)}[b_1 - 2, b_2]) \\ &\quad + \sum_{k;\ d(j,k)=1} \mathsf{c}_{k,j}\underline{\mathcal{N}}(m^{(i)}[a_1, a_2], m^{(k)}[b_1 - 1, b_2 - 1]) \\ &\overset{*}{=} \underline{\mathcal{N}}(m^{(i)}[a_1, a_2],\ B_{j,b_1-1}B_{j,b_1+3} \cdots B_{j,b_2-1}) \end{aligned} $$

where $\overset{*}{=}$ holds by (3.7) and (4.4). Then it follows from (4.9) in Proposition 4.6 that

$$ -({}^{s}\Lambda{}^{s}\mathfrak{B})_{(i,p),(j,t)} = \underline{\mathcal{N}}(m^{(i)}[a_1, a_2], B_{j,b_1-1}B_{j,b_1+3} \cdots B_{j,b_2-1}) $$

$$= \sum_{x=0}^{\frac{a_2-a_1}{2}} \sum_{y=0}^{\frac{b_2-b_1}{2}} \delta_{i,j}(-\delta(a_1+2x-b_1-2y=-2) \\ + \delta(a_1+2x-b_1-2y=0))2d_i \\ = \delta_{i,j} \sum_{x=0}^{\frac{a_2-a_1}{2}} (-\delta(a_1+2x-b_1=-2)+\delta(a_1+2x-b_2=0))2d_i.$$

If $i = j$, we have the following:

(1) $[a_1, a_2]$ and $[b_1, b_2]$ are inclusive, that is, either $[a_1, a_2] \subset [b_1, b_2]$ or $[b_1, b_2] \subset [a_1, a_2]$;
(2) if $a_k = b_k$, then $b_l - a_l = 2$ or 0 for $\{k, l\} = \{1, 2\}$.

Thus we can conclude that

$$-({}^s\Lambda^s\mathfrak{B})_{(i,p),(j,t)} = \delta((i, p) = (j, t))2d_i,$$

as we desired. □

Lemma 9.5. *The set* $\{{}^s\mathfrak{u}_{i,p}\}_{\in {}^s\widetilde{\triangle}_0}$ *forms a* q*-commuting family in* $\mathfrak{K}_q(\mathfrak{g})$.

Proof. From Theorem 8.10, our assertion easily follows. □

Theorem 9.6. *The family of quantum seeds*

$$\mathfrak{S}_s = (\{{}^s\mathfrak{u}_{i,p}\}_{(i,p)\in {}^s\widetilde{\triangle}_0}, {}^s\Lambda, {}^s\mathfrak{B}) \quad \textit{for } s \in \mathbb{Z}, \tag{9.6}$$

gives a quantum cluster algebra structure on $\mathfrak{K}_q(\mathfrak{g})$.

The rest of this paper will be devoted to proving Theorem 9.6. Let ${}^s\mathscr{A}_q(\mathfrak{g})$ be the quantum cluster algebra generated by the quantum seed $\mathfrak{S}_s$. To prove Theorem 9.6, we need to show that

$${}^s\mathscr{A}_q(\mathfrak{g}) = \mathfrak{K}_q(\mathfrak{g}). \tag{9.7}$$

Then the proof of (9.7) is separated into two steps as follows:

Step 1. For the inclusion ${}^s\mathscr{A}_q(\mathfrak{g}) \subset \mathfrak{K}_q(\mathfrak{g})$, we will prove the following proposition in Sect. 9.2.

Proposition 9.7. *For any finite sequence* μ *of mutations, a cluster variable in* $\mu(\mathfrak{S}_s)$ *is contained in* $\mathfrak{K}_q(\mathfrak{g})$.

The key observation for proving Proposition 9.7 is that the mutated variables from $\mathfrak{S}_s$ are understood as the ones from $\mathscr{S}_{s'}$ for some $s' \in \mathbb{Z}$, which implies ${}^s\mathscr{A}_q(\mathfrak{g}) \subset \mathfrak{K}_q(\mathfrak{g})$.

Step 2. The opposite inclusion will be proved as the following proposition is shown in Sect. 9.3.

Proposition 9.8. *For* $(i, p) \in \widetilde{\triangle}_0$*, there exists a finite sequence* μ *of mutations such that* $\mu(\mathfrak{S}_s)$ *contains* $F_q(\mathsf{X}_{i,p})$ *as its cluster variable.*

Since $\mathfrak{K}_q(\mathfrak{g})$ is generated by $F_q(\mathsf{X}_{i,p})$ for $(i, p) \in \widetilde{\triangle}_0$ by Theorem 5.27 (see also (5.23) above), the opposite inclusion for proving (9.7) follows from Proposition 9.8.

9.2. Proof of Theorem 9.6: Step 1: proof of Proposition 9.7. For $k \leqslant s$, we set

$$\langle k \rangle := \{(i,k) \in {}^s\widetilde{\triangle}_0\} \quad \text{and} \quad \langle k,s \rangle := \{(i,p) \in {}^s\widetilde{\triangle}_0 \mid k \leqslant p \leqslant s\}.$$

We understand $\langle k,s \rangle = \emptyset$ for $k > s$.

Lemma 9.9. *For the valued quiver ${}^s\widetilde{\triangle}$, we have*

$$\mu_{(i_1,s)} \circ \mu_{(i_2,s)} \circ \cdots \circ \mu_{(i_r,s)}({}^s\widetilde{\triangle}) \simeq \mu_{(j_1,s)} \circ \mu_{(j_2,s)} \circ \cdots \circ \mu_{(j_r,s)}({}^s\widetilde{\triangle}),$$

where $\{(i_t,s)\}_{1 \leqslant t \leqslant r} = \{(j_t,s)\}_{1 \leqslant t \leqslant r} = \langle s \rangle$. Thus, $\mu_{\langle s \rangle}$ is well-defined on ${}^s\widetilde{\triangle}$, that is, $\mu_{\langle s \rangle}({}^s\widetilde{\triangle})$ is uniquely determined.

Proof. Note that (a) each $(i_k,s) \in \langle s \rangle$ is vertically sink and horizontally source, (b) all the length 2 paths passing through (i_k,s) start from $(i',s-1)$ and end at $(i_k,s-2)$ where $d(i',i_k)=1$, and (c) there is no arrow between (i_k,s) and $(i_{k'},s)$ for $i_k \neq i_{k'}$.

$$ {}^s\widetilde{\triangle} = \begin{tikzcd}
\cdots & (i_k,s-4) \arrow[l] \arrow[dr, "{\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k} \lrcorner}"] & & (i_k,s-2) \arrow[ll] \arrow[dr, "{\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k} \lrcorner}"] & & (i_k,s) \arrow[ll] \\
\cdots & & (i',s-3) \arrow[ll] \arrow[ur, "{\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'} \lrcorner}"] \arrow[dr, "{\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'} \lrcorner}"] & & (i',s-1) \arrow[ll] \arrow[ur, "{\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'} \lrcorner}"] \arrow[dr, "{\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'} \lrcorner}"] & \\
\cdots & (i_{k'},s-4) \arrow[l] \arrow[ur, "{\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}} \lrcorner}"] & & (i_{k'},s-2) \arrow[ll] \arrow[ur, "{\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}} \lrcorner}"] & & (i_{k'},s) \arrow[ll]
\end{tikzcd}$$

Thus the mutation $\mu_{(i_k,s)}$ of ${}^s\widetilde{\triangle}$ at (i_k,s) does not affect the local circumstance of $(i_{k'},s)$ and the arrows between $(i_k,s-2)$ and $(i',s-1)$ for $d(i_k,i')=1$ are canceled out by the mutation $\mu_{(i_k,s)}$.

$$\begin{aligned}
&\xrightarrow{\mu_{(i_k,s)}} \begin{tikzcd}
\cdots & (i_k,s-4) \arrow[l] \arrow[dr, "{\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k} \lrcorner}"] & & (i_k,s-2) \arrow[ll] \arrow[rr] & & (i_k,s) \arrow[dl, "{\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k} \lrcorner}"] \\
\cdots & & (i',s-3) \arrow[ll] \arrow[ur, "{\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'} \lrcorner}"] \arrow[dr, "{\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'} \lrcorner}"] & & (i',s-1) \arrow[ll] \arrow[dr, "{\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'} \lrcorner}"] & \\
\cdots & (i_{k'},s-4) \arrow[l] \arrow[ur, "{\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}} \lrcorner}"] & & (i_{k'},s-2) \arrow[ll] \arrow[ur, "{\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}} \lrcorner}"] & & (i_{k'},s) \arrow[ll]
\end{tikzcd} \\
&\xrightarrow{\mu_{(i_{k'},s)}} \begin{tikzcd}
\cdots & (i_k,s-4) \arrow[l] \arrow[dr, "{\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k} \lrcorner}"] & & (i_k,s-2) \arrow[ll] \arrow[rr] & & (i_k,s) \arrow[dl, "{\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k} \lrcorner}"] \\
\cdots & & (i',s-3) \arrow[ll] \arrow[ur, "{\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'} \lrcorner}"] \arrow[dr, "{\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'} \lrcorner}"] & & (i',s-1) \arrow[ll] & \\
\cdots & (i_{k'},s-4) \arrow[l] \arrow[ur, "{\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}} \lrcorner}"] & & (i_{k'},s-2) \arrow[ll] \arrow[rr] & & (i_{k'},s) \arrow[ul, "{\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}} \lrcorner}"]
\end{tikzcd}
\end{aligned} \tag{9.8}$$

Hence the assertions follow. $\square$

Lemma 9.10. *For the valued quiver ${}^s\widetilde{\triangle}$ and $k \leqslant s$, the valued quiver*

$$\mu_{\langle k,s \rangle}({}^s\widetilde{\triangle}) := \mu_{\langle k \rangle} \circ \mu_{\langle k+1 \rangle} \circ \cdots \circ \mu_{\langle s \rangle}({}^s\widetilde{\triangle}) \text{ is uniquely determined.} \tag{9.9}$$

Thus $\mu_{\langle k \rangle}$ is well-defined on $\mu_{\langle k+1,s \rangle}({}^s\widetilde{\triangle})$ and hence $\mu_{\langle k,s \rangle}$ is well-defined on ${}^s\widetilde{\triangle}$.

Proof. The assertion for $k = s$ holds by the previous lemma. As we can observe in (9.8), (a) each $(i',s-1) \in \langle s-1 \rangle$ is vertically sink and horizontally source, (b) all the length 2 paths passing through $(i',s-1)$ start from (i,s) and end at $(i',s-3)$ where $d(i',i)=1$ and (c) there is no path between $(i',s-1)$ and $(i'',s-1)$. Thus $\mu_{(i',s-1)} \circ \mu_{(i'',s-1)} =$

$\mu_{(i'',s-1)} \circ \mu_{(i',s-1)}$ on $\mu_{\langle s\rangle}({}^s\widetilde{\triangle})$. Thus the assertion holds for $k = s-1$, and $\mu_{\langle s-1\rangle}$ yields arrows from (i, s) to $(i', s-3)$, and hence $\mu_{\langle s-1,s\rangle}({}^s\widetilde{\triangle})$ can be depicted as follows:

$$\begin{tikzcd}[column sep=small]
\cdots & (i_k, s-4) \arrow[l] \arrow[dr, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"] & & (i_k, s-2) \arrow[ll] \arrow[rr] & & (i_k, s) \arrow[dlll, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"'] \\
(i', s-5) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-3) \arrow[ll] \arrow[rr] \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-1) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
\cdots & (i_{k'}, s-4) \arrow[l] \arrow[ur, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"'] & & (i_{k'}, s-2) \arrow[ll] \arrow[rr] & & (i_{k'}, s) \arrow[ulll, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"]
\end{tikzcd} \tag{9.10}$$

By the same reasons for $\mu_{\langle s\rangle}$ and $\mu_{\langle s-1\rangle}$, the sequence of mutations $\mu_{\langle s-2\rangle}$ is well-defined. Furthermore, by the mutation rules, the arrows between $(i_k, (s-2) \pm 2)$ and $(i', s-3)$ for $d(i_k, i') = 1$ are canceled out by the mutation $\mu_{\langle s-1\rangle}$. Thus $\mu_{\langle s-2,s\rangle}({}^s\widetilde{\triangle})$ can be depicted as follows:

$$\begin{tikzcd}[column sep=tiny]
\cdots & (i_k, s-6) \arrow[l] \arrow[dr, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"] & & (i_k, s-4) \arrow[ll] \arrow[rr] & & (i_k, s-2) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"'] & & (i_k, s) \arrow[ll] \\
(i', s-7) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-5) \arrow[ll] \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-3) \arrow[ll] \arrow[rr] & & (i', s-1) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
\cdots & (i_{k'}, s-6) \arrow[l] \arrow[ur, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"'] & & (i_{k'}, s-4) \arrow[ll] \arrow[rr] & & (i_{k'}, s-2) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"] & & (i_{k'}, s) \arrow[ll]
\end{tikzcd} \tag{9.11}$$

As in the previous cases, $\mu_{\langle s-3\rangle}$ is well-defined, $\mu_{\langle s-3\rangle}$ yields arrows from $(i, s-2)$ to $(i', s-3 \pm 2)$ as $\mu_{\langle s-1\rangle}$ did, and hence $\mu_{\langle s-3,s\rangle}({}^s\widetilde{\triangle})$ can be depicted as follows:

$$\begin{tikzcd}[column sep=tiny]
\cdots & (i_k, s-6) \arrow[l] \arrow[dr, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"] & & (i_k, s-4) \arrow[ll] \arrow[rr] & & (i_k, s-2) \arrow[dlll, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"'] \arrow[dr, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"] & & (i_k, s) \arrow[ll] \\
\cdots & & (i', s-5) \arrow[ll] \arrow[rr] \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-3) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-1) \arrow[ll] \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
\cdots & (i_{k'}, s-6) \arrow[l] \arrow[ur, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"'] & & (i_{k'}, s-4) \arrow[ll] \arrow[rr] & & (i_{k'}, s-2) \arrow[ulll, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"] \arrow[ur, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"'] & & (i_{k'}, s) \arrow[ll]
\end{tikzcd} \tag{9.12}$$

Then one can see that

(i) the full-subquiver of $\mu_{\langle s-2,s\rangle}({}^s\widetilde{\triangle})$ obtained by excluding vertices in $\langle s\rangle$ is isomorphic to the valued quiver $\mu_{\langle s\rangle}({}^s\widetilde{\triangle})$ in (9.8),

(ii) the full-subquiver of $\mu_{\langle s-3,s\rangle}({}^s\widetilde{\triangle})$ obtained by excluding vertices in $\langle s-1, s\rangle$ is isomorphic to the valued quiver $\mu_{\langle s-1,s\rangle}({}^s\widetilde{\triangle})$ in (9.10).

Thus the induction works. □

Remark 9.11. In the previous lemmas, we observe the following:

(1) Each $\mu_{(i,p)}$ in $\mu_{\langle k,s\rangle}$ happens when (i, p) is vertically sink and horizontally source, and the arrows adjacent to (i, p) are given as follows: for any j with $d(i, j) = 1$,

$$\begin{cases}
\begin{tikzcd}[column sep=small]
(i, p-2) & & (i, p) \arrow[ll] \arrow[rr] & & (i, p+2) \\
& (j, p-1) \arrow[ur, "\ulcorner -\mathsf{c}_{j,i}, \mathsf{c}_{i,j} \lrcorner"] & & &
\end{tikzcd} & \text{if } p \equiv_2 s, \\
\begin{tikzcd}[column sep=small]
(i, p-2) & & (i, p) \arrow[ll] \arrow[rr] & & (i, p+2) \\
& & & (j, p+1) \arrow[ul, "\ulcorner -\mathsf{c}_{j,i}, \mathsf{c}_{i,j} \lrcorner"'] &
\end{tikzcd} & \text{if } p \not\equiv_2 s.
\end{cases}$$

(2) Each $\mu_{(i,p)}$ in $\mu_{\langle k,s\rangle}$ does not affect on the local circumstance of the vertex (j, s) for $|s-p| > 2$ in the valued quiver obtained by applying the preceding mutations on ${}^s\widetilde{\triangle}$.

Example 9.12. By applying $\mu_{\langle s-4\rangle}$ on the valued quiver $\mu_{\langle s-3,s\rangle}({}^{s}\widetilde{\triangle})$ in (9.12), we observe that the local circumstance of vertices in $\langle s-1, s\rangle$ are preserved as explained in Remark 9.11 (2):

$$\mu_{\langle s-4,s\rangle}({}^{s}\widetilde{\triangle}) = \begin{tikzcd}[column sep=small]
\cdots & (i_k,s-6) \arrow[l] \arrow[rr] & & (i_k,s-4) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k}\lrcorner"'] & & (i_k,s-2) \arrow[ll] \arrow[dr, "\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k}\lrcorner"] & & (i_k,s) \arrow[ll] \\
(i',s-7) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & & (i',s-5) \arrow[ll] \arrow[rr] & & (i',s-3) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & & (i',s-1) \arrow[ll] \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & \\
\cdots & (i_{k'},s-6) \arrow[l] \arrow[rr] & & (i_{k'},s-4) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}}\lrcorner"] & & (i_{k'},s-2) \arrow[ll] \arrow[ur, "\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}}\lrcorner"'] & & (i_{k'},s) \arrow[ll]
\end{tikzcd} \tag{9.13}$$

For notational simplicity, let us keep the following notations:

- $\Upsilon_s(\langle k,s\rangle) := \mu_{\langle k,s\rangle}({}^{s}\widetilde{\triangle})$ (in (9.9)), $\Upsilon_s := {}^{s}\widetilde{\triangle}$, $\Theta_s := {}^{s}\overleftarrow{\Box}$, $\Omega_s := {}^{s}\overrightarrow{\Box}$,
- for a valued quiver Γ, a quiver ${}^{X}\Gamma$ denotes the full-subquiver of Γ whose vertices are in $X \subseteq \Gamma_0$,

where ${}^{s}\overrightarrow{\Box}$ is the quiver obtained from ${}^{s}\overleftarrow{\Box}$ by reversing the orientation of arrows in ${}^{s}\overleftarrow{\Box}$. By Remark 9.11 (2), we have

$$ {}^{\langle -\infty,k-3\rangle}\Upsilon_s(\langle k,s\rangle) \simeq {}^{\langle -\infty,k-3\rangle}\Upsilon_s, \tag{9.14}$$

for any $k \leqslant s$. The lemma below concerns ${}^{\langle k-3,s\rangle}\Upsilon_s(\langle k,s\rangle)$.

Lemma 9.13. *For $r \in \mathbb{Z}_{\geqslant 0}$, as a finite quiver,*

(a) ${}^{\langle s-2r+1,s\rangle}\Upsilon_s(\langle s-2r+1,s\rangle) \simeq {}^{\langle s-2r+1,s\rangle}\Upsilon_s$.
(b) ${}^{\langle s-2r+1,s\rangle}\Upsilon_s(\langle s-2r,s\rangle) \simeq {}^{\langle s-2r+1,s\rangle}\Upsilon_s$ *and*

$$ {}^{\langle s-2r-3,\min(s-2r+2,s)\rangle}\Upsilon_s(\langle s-2r,s\rangle) \simeq \begin{cases} {}^{\langle s-3,s\rangle}\Omega_s & \text{if } r=0, \\ {}^{\langle s-5,s\rangle}\Theta_s & \text{otherwise.} \end{cases}$$

Proof. (a) Recall $\langle s-2r+1,s\rangle = \emptyset$ if $r=0$, so this case trivially holds. The cases of $r=1$ and $r=2$ are already verified in (9.10) and (9.12), respectively. One observes that in the general case (i.e. $r \geqslant 3$), the mutation patterns in the intermediate steps are identical with (9.8) and (9.11) up to the shift of the second parameters. This completes the proof of (a).
(b) Let us consider the cases of $0 \leqslant r \leqslant 2$ precisely as follows:
Case 1. $r=0$. By (9.8), ${}^{\langle s-3,s\rangle}\Upsilon_s(\langle s\rangle)$ and ${}^{\langle s+1,s\rangle}\Upsilon_s(\langle s\rangle)$ are

$$\begin{tikzcd}[column sep=small]
& (i_k,s-2) \arrow[rr] & & (i_k,s) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k}\lrcorner"'] \\
(i',s-3) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & & (i',s-1) \arrow[ll] & \\
& (i_{k'},s-2) \arrow[rr] & & (i_{k'},s) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}}\lrcorner"]
\end{tikzcd} \quad \text{and} \quad \emptyset, \tag{9.15}$$

Case 2. $r=1$. By (9.11), ${}^{\langle s-5,s\rangle}\Upsilon_s(\langle s-2,s\rangle)$ and ${}^{\langle s-1,s\rangle}\Upsilon_s(\langle s-2,s\rangle)$ are

$$\begin{tikzcd}[column sep=small]
& (i_k,s-4) \arrow[rr] & & (i_k,s-2) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'},\mathsf{c}_{i',i_k}\lrcorner"'] & & (i_k,s) \arrow[ll] \\
(i',s-5) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & & (i',s-3) \arrow[ll] \arrow[rr] & & (i',s-1) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & \\
& (i_{k'},s-4) \arrow[rr] & & (i_{k'},s-2) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'},\mathsf{c}_{i',i_{k'}}\lrcorner"] & & (i_{k'},s) \arrow[ll]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small]
& (i_k,s) \\
(i',s-1) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k},\mathsf{c}_{i_k,i'}\lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}},\mathsf{c}_{i_{k'},i'}\lrcorner"'] & \\
& (i_{k'},s)
\end{tikzcd}$$

Case 3. $r = 2$. By (9.13), ${}^{\langle s-7,s-2\rangle}\Upsilon_s(\langle s-4, s\rangle)$ and ${}^{\langle s-3,s\rangle}\Upsilon_s(\langle s-4, s\rangle)$ are

$$\begin{tikzcd}[column sep=small]
& (i_k, s-6) \arrow[rr] & & (i_k, s-4) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"] & & (i_k, s-2) \arrow[ll] \\
(i', s-7) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-5) \arrow[ll] \arrow[rr] & & (i', s-3) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
& (i_{k'}, s-6) \arrow[rr] & & (i_{k'}, s-4) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"'] & & (i_{k'}, s-2) \arrow[ll]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
& (i_k, s-2) \arrow[dr, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"] & & (i_k, s) \arrow[ll] \\
(i', s-3) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-1) \arrow[ll] \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
& (i_{k'}, s-2) \arrow[ur, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"'] & & (i_{k'}, s) \arrow[ll]
\end{tikzcd}
\tag{9.16}$$

One may further observe from *Case 1–Case 3* that

- ${}^{\langle s-2r+1,s\rangle}\Upsilon_s(\langle s-2r, s\rangle) \simeq {}^{\langle s-2r+1,s\rangle}\Upsilon_s$ for $r \geqslant 1$ (by a similar argument as in (a)),
- ${}^{\langle s-2r-3,s-2r+2\rangle}\Upsilon_s(\langle s-2r, s\rangle)$ for $r \geqslant 1$ is isomorphic to ${}^{\langle s-5,s\rangle}\Theta_s$ as finite quivers, where ${}^{\langle s-3,s\rangle}\Upsilon_s(\langle s\rangle) \simeq {}^{\langle s-3,s\rangle}\Omega_s$.

Hence we complete the proof of (b). □

For $r \in \mathbb{Z}_{\geqslant 1}$, we define

$$\mu_{\langle s-2r,s\rangle\!\rangle} := \begin{cases} \mu_{\langle s\rangle} \circ \mu_{\langle s-4,s\rangle} \circ \cdots \circ \mu_{\langle s-2r+4,s\rangle} \circ \mu_{\langle s-2r,s\rangle} & \text{if } r \equiv_2 0, \\ \mu_{\langle s-2,s\rangle} \circ \mu_{\langle s-6,s\rangle} \circ \cdots \circ \mu_{\langle s-2r+4,s\rangle} \circ \mu_{\langle s-2r,s\rangle} & \text{if } r \equiv_2 1. \end{cases}$$

By Lemma 9.13 (b), ${}^{\langle s-2r+1,s\rangle}\Upsilon_s(\langle s-2r, s\rangle) \simeq {}^{\langle s-2r+1,s\rangle}\Upsilon_s$. By Lemma 9.10 and Remark 9.11 (2), $\mu_{\langle s-2r+4,s\rangle}$ is well-defined on ${}^{\langle s-2r+1,s\rangle}\Upsilon_s(\langle s-2r, s\rangle)$. Thus it makes sense to define

$$\Upsilon_s(\langle s-2r, s\rangle\!\rangle) := \mu_{\langle s-2r,s\rangle\!\rangle}({}^{s}\widetilde{\triangle\!\!\!\!\triangle}).$$

Then we have a generalization of Lemma 9.13.

Proposition 9.14. *For $r \in \mathbb{Z}_{\geqslant 0}$, we have*

$${}^{\langle s-2r-3,s\rangle}\Upsilon_s(\langle s-2r, s\rangle\!\rangle) \simeq \begin{cases} {}^{\langle s-2r-3,s\rangle}\Omega_s & \textit{if } r \equiv_2 0, \\ {}^{\langle s-2r-3,s\rangle}\Theta_s & \textit{if } r \equiv_2 1. \end{cases}$$

Proof. We first consider the case $0 \leqslant r \leqslant 2$, and then the general case $r \geqslant 3$.
Case 1. $0 \leqslant r \leqslant 2$. The assertion for $r = 0$ and $r = 1$ is shown by (9.8) and (9.11), respectively. Let us consider the case $r = 2$. By (9.13), we may consider ${}^{\langle s-7,s\rangle}\Upsilon_s(\langle s-4, s\rangle)$ by separating it into two parts ${}^{\langle s-7,s-2\rangle}\Upsilon_s(\langle s-4, s\rangle)$ and ${}^{\langle s-2,s\rangle}\Upsilon_s(\langle s-4, s\rangle)$, where each one is shown in (9.16). Then ${}^{\langle s-7,s\rangle}\Upsilon_s(\langle s-4, s\rangle\!\rangle)$ is understood as a concatenation of ${}^{\langle s-7,s-2\rangle}\Upsilon_s(\langle s-4, s\rangle)$ and $\mu_{\langle s\rangle}\big({}^{\langle s-2,s\rangle}\Upsilon_s(\langle s-4, s\rangle)\big)$ due to Remark 9.11 (2), where the common vertices are overlapped. Since $\mu_{\langle s\rangle}\big({}^{\langle s-2,s\rangle}\Upsilon_s(\langle s-4, s\rangle)\big)$ is isomorphic to the valued quiver in (9.15), the assertion for $r = 2$ is proved.
Case 2. $r \geqslant 3$. The proof idea in this case is identical with *Case 1*, that is, by using the same argument as in *Case 1*, we observe that the finite valued quiver

$$\Gamma_1 := {}^{\langle s-2r-3,s-2r+6\rangle}\Big(\mu_{\langle s-2r+4,s\rangle} \circ \mu_{\langle s-2r,s\rangle}({}^{s}\widetilde{\triangle\!\!\!\!\triangle})\Big)$$

is a concatenation of

$$\begin{tikzcd}[column sep=small]
 & (i_k, s-2r-2) \arrow[rr] & & (i_k, s-2r) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"'] & & (i_k, s-2r+2) \arrow[ll] \\
(i', s-2r-3) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-2r-1) \arrow[ll] \arrow[rr] & & (i', s-2r+1) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
 & (i_{k'}, s-2r-2) \arrow[rr] & & (i_{k'}, s-2r) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"] & & (i_{k'}, s-2r+2) \arrow[ll]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
 & (i_k, s-2r+2) \arrow[rr] & & (i_k, s-2r+4) \arrow[dl, "\ulcorner -\mathsf{c}_{i_k,i'}, \mathsf{c}_{i',i_k} \lrcorner"'] & & (i_k, s-2r+6) \arrow[ll] \\
(i', s-2r+1) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & & (i', s-2r+3) \arrow[ll] \arrow[rr] & & (i', s-2r+5) \arrow[ur, "\ulcorner -\mathsf{c}_{i',i_k}, \mathsf{c}_{i_k,i'} \lrcorner"] \arrow[dr, "\ulcorner -\mathsf{c}_{i',i_{k'}}, \mathsf{c}_{i_{k'},i'} \lrcorner"'] & \\
 & (i_{k'}, s-2r+2) \arrow[rr] & & (i_{k'}, s-2r+4) \arrow[ul, "\ulcorner -\mathsf{c}_{i_{k'},i'}, \mathsf{c}_{i',i_{k'}} \lrcorner"] & & (i_{k'}, s-2r+6) \arrow[ll]
\end{tikzcd}$$

where we regard the common vertices to be overlapped in the concatenation. Since

$$^{\langle s-2r+5,s\rangle}\Big(\mu_{\langle s-2r+4,s\rangle} \circ \mu_{\langle s-2r,s\rangle}({}^{s}\widetilde{\mathbb{A}})\Big) \simeq {}^{\langle s-2r+5,s\rangle}\Upsilon_s \quad \text{by Lemma 9.13 (b),}$$

and $\mu_{\langle s-2r+8,s\rangle}$ does not contribute to Γ_1, we complete the proof by applying the same argument to ${}^{\langle s-2r+5,s\rangle}\Upsilon_s$ as in *Case 1.* □

Let us write μ in Proposition 9.7 as

$$\mu = \mu_{(i_l,p_l)} \circ \mu_{(i_{l-1},p_{l-1})} \circ \cdots \circ \mu_{(i_1,p_1)}. \tag{9.17}$$

Take $t \in \mathbb{Z}$ such that $t \ll \min(p_k \mid 1 \leqslant k \leqslant l)$ and $s - t \equiv_4 2$. By our choice of t, it follows from Proposition 9.14 that

$${}^{\langle t-3,s\rangle}\Upsilon_s(\langle t, s\rangle\!\rangle) \simeq {}^{\langle t-3,s\rangle}\Theta_s \quad \text{as a valued quiver,}$$

where

$$s - t = 4u + 2 \qquad \text{for some } u \in \mathbb{Z}_{\geqslant 0}. \tag{9.18}$$

Recall the quantum seeds

$$\begin{aligned}
\mathscr{S}_s &= \big(\{{}^{s}\mathfrak{v}_{i,p} := F_q(\underline{m}^{(i)}[p, {}^{(s)}\xi_i])\}_{(i,p)\in{}^{s}\widetilde{\mathbb{A}}_0}, {}^{s}L, {}^{s}\widetilde{B}\big) \text{ associated to } {}^{(s)}\xi \text{ in (8.15),} \\
\mathfrak{S}_s &= \left(\left\{{}^{s}\mathfrak{u}_{i,p} = F_q\big(\underline{m}^{(i)}[{}^{s}o_{i,p}, {}^{s}o_{i,p} + 2\,{}^{s}l_{i,p}]\big)\right\}_{(i,p)\in{}^{s}\widetilde{\mathbb{A}}_0}, {}^{s}\Lambda, {}^{s}\mathfrak{B}\right) \text{ in (9.6).}
\end{aligned} \tag{9.19}$$

Proposition 9.15. *Every mutation $\mu_{(i,p)}$ in $\mu_{\langle t,s\rangle\!\rangle}$ on the cluster $\{{}^{s}\mathfrak{v}_{i,p}\}$ corresponds to the quantum folded T-system in Theorem 6.9, and furthermore,*

$$\text{the mutation } \mu_{(i,p)} \text{ sends } F_q(\underline{m}^{(i)}[a,b]) \text{ to } \mathsf{T}_{-2}(F_q(\underline{m}^{(i)}[a,b])) = F_q(\underline{m}^{(i)}[a-2,b-2]),$$

when $F_q(\underline{m}^{(i)}[a,b])$ is the quantum cluster variable sitting at (i,p) and obtained from the subsequence of mutations previous to the $\mu_{(i,p)}$ in $\mu_{\langle t,s\rangle\!\rangle}$.

Proof. First, let us consider a mutation $\mu_{(i,p)}$ in $\mu_{\langle t,s\rangle}$. When $(i,p) = (i,s)$ (i.e. one of the vertices located in the right-most of ${}^{s}\widetilde{\mathbb{A}}_0$), the local circumstance of (i,s) described in Remark 9.11 (1) tells us that the quantum exchange relation is given by

$$\mu_{(i,s)}\big(F_q(\mathsf{X}_{i,s})\big) * F_q(\mathsf{X}_{i,s}) = q^{\alpha(i,1)} F_q(\underline{m}^{(i)}[s-2,s]) + q^{\gamma(i,1)} \prod_{j;\, d_{i,j}=1} F_q(\mathsf{X}_{j,s-1})^{-\mathsf{c}_{j,i}},$$

where $q^{\alpha(i,1)}$ and $q^{\gamma(i,1)}$ are determined to be bar-invariant as in the sense of (8.9). Consequently, it corresponds to the quantum folded T-system in Theorem 6.9 and hence $\mu_{(i,s)}(F_q(\mathsf{X}_{i,s})) = F_q(\mathsf{X}_{i,s-2})$ as we desired. Note that another mutation at (i', s) does not affect the mutation at (i,s) as shown in Lemma 9.9.

Second, let us consider a mutation at $(j, s-1)$, which appears later than any (i, s) in $\mu_{\langle t,s\rangle}$. Let us keep in mind that the cluster variable located at (i', s) is already mutated by former mutations, which is $F_q(\mathsf{X}_{i',s-2})$. Then the quantum exchange relation is given as follows (recall Remark 9.11 (1)):

$$\mu_{(j,s-1)}\big(F_q(\mathsf{X}_{j,s-1})\big) * F_q(\mathsf{X}_{j,s-1}) = q^{\alpha(j,1)} F_q(\underline{m}^{(j)}[s-3, s-1]) + q^{\gamma(j,1)} \prod_{i;\, d_{j,i}=1} F_q(\mathsf{X}_{i,s-2})^{-\mathsf{c}_{i,j}},$$

which coincides with the quantum folded T-system in Theorem 6.9. Hence $\mu_{(j,s-1)}(F_q(\mathsf{X}_{j,s-1})) = F_q(\mathsf{X}_{j,s-3})$, as we desired.

Finally, by using this argument and the local circumstance of (k, p) in the order for applying $\mu_{(k,p)}$, described in Remark 9.11 (1), one can conclude that each mutation $\mu_{(i,p)}$ in $\mu_{\langle t,s\rangle}$ corresponds to shifting the second parameters of cluster variables by -2. The assertion for mutations in $\mu_{\langle t+4r,s\rangle}$ $(r \geqslant 1)$ follows from Lemma 9.13 (b), Remark 9.11 (2) and the argument for mutations in $\mu_{\langle t,s\rangle}$. □

Recall $u \in \mathbb{Z}_{\geqslant 0}$ in (9.18) depending on $\langle t, s\rangle$. For $(j, a) \in {}^s\widetilde{\triangle}_0$ with $t \leqslant a \leqslant s$, we remark that

$$\begin{aligned}&\text{(A) there exists } 0 \leqslant e \leqslant u \text{ such that } s-4e-2 \leqslant a < \min(s+1, s-4e+2), \text{ equivalently}\\ &\qquad a \in \{s-4e-2, s-4e-1, s-4e, s-4e+1\},\\ &\text{(B) } {}^{(s)}\xi_j = s \text{ if } a = s-4e-2 \text{ or } s-4e, \text{ and } {}^{(s)}\xi_j = s-1, \text{ otherwise.}\end{aligned} \tag{9.20}$$

Proposition 9.16. *For $(j, a) \in {}^s\widetilde{\triangle}_0$ with $t \leqslant a \leqslant s$,*

$$\big(\mu_{\langle t,s\rangle\!\rangle}(\{{}^s\mathfrak{v}_{k,p}\})\big)_{(j,a)} = F_q\big(\underline{m}^{(j)}[{}^{s'}o_{j,a'}, {}^{s'}o_{j,a'} + 2\,{}^{s'}l_{j,a'}]\big),$$

where $s' = s - 2(u+1)$ and $a' = a - 2(u+1)$ for $u \in \mathbb{Z}_{\geqslant 0}$ in (9.18).

Proof. Since $\mu_{(j,a)}$ appears $(u+1-e)$-times in $\mu_{\langle t,s\rangle\!\rangle}$ and ${}^s\mathfrak{v}_{j,a} = F_q(\underline{m}^{(j)}[a, s - \delta({}^{(s)}\xi_j \neq s)])$, it tells that Proposition 9.15 that

$$\big(\mu_{\langle t,s\rangle\!\rangle}(\{{}^s\mathfrak{v}_{k,p}\})\big)_{(j,a)} = F_q(\underline{m}^{(j)}[a' + 2e, s' + 2e - \delta(a \not\equiv_2 s)]),$$

On the other hand, we have

$$\big({}^{s'}o_{j,a'}, {}^{s'}l_{j,a'}\big) = \begin{cases} \big(s' - 2(e+1), 2e+1\big) & \text{if } s - a = 4e+2,\\ \big(s' - 2e, 2e\big) & \text{if } s - a = 4e,\\ \big(s' - 2e, 2e\big) & \text{if } (s-1) - a = 4e,\\ \big(s' - 2(e-1), 2e-1\big) & \text{if } (s-1) - a = 4e-2,\end{cases}$$

where the integers on the left-hand side are defined in (9.5). Then one can easily check that

$${}^{s'}o_{j,a'} = a' + 2e \quad \text{and} \quad {}^{s'}o_{j,a'} + 2\,{}^{s'}l_{j,a'} = s' + 2e - \delta(a \not\equiv_2 s),$$

which implies our assertion. □

Now, we are ready to prove Proposition 9.7.

Proof of Proposition 9.7. Write μ in Proposition 9.7 as in (9.17). Let us set

$$Z := (\mu(\{{}^{s}\mathfrak{u}_{k,p}\}))_{(i_l,p_l)}.$$

By Proposition 9.14 and Proposition 9.16, we have

$$(\{{}^{s'}\mathfrak{v}_{k',p'}\})_{\langle t',s'\rangle} = (\{{}^{s}\mathfrak{u}_{k,p}\})_{\langle t,s\rangle},$$

where $t' = t + 2(u+1)$, $s' = s + 2(u+1)$ and (k', p') denotes an element in ${}^{s'}\widetilde{\triangle}_0$. That is, Z can be understood as a mutated variable from $\{{}^{s'}\mathfrak{v}_{k',p'}\}$ as follows:

$$Z = \big(\mu \circ \mu_{\langle t',s'\rangle\!\rangle}(\{{}^{s'}\mathfrak{v}_{k',p'}\})\big)_{(i_l,p'_l)},$$

where $p'_l = p_l + 2(u+1)$ for $u \in \mathbb{Z}_{\geqslant 0}$ in (9.18).

Since

$$\mathscr{S}_{s'} = \big(\{F_q(\underline{m}^{(i)}[p, {}^{(s')}\xi_i])\}_{(i,p)\in {}^{s'}\widetilde{\triangle}_0}, {}^{s'}L, {}^{s'}\widetilde{B}\big)$$

is an initial quantum seed of the quantum cluster algebra $\mathfrak{K}_{q,s'}(\mathfrak{g}) \subset \mathfrak{K}_q(\mathfrak{g})$, the element Z is contained in $\mathfrak{K}_q(\mathfrak{g})$, which completes the proof. □

In the above proof, we show that any finite sequence μ of mutations acting on $\mathfrak{S}_s$ can be understood as a sequence μ' of mutations acting on $\mathscr{S}_{s'}$ for some $s' \in \mathbb{Z}$. Since we proved the corresponding statement for $\mathscr{S}_{s'}$ in Sect. 8 (Proposition 8.8 and Theorem 8.9), the assertion for $\mathfrak{S}_s$ follows. This kind of idea is also presented in [45, Definition 8.3 (1)], which can be understood as a *local* isomorphism of infinite quivers.

9.3. *Proof of Theorem 9.6: Step 2: proof of Proposition 9.8.*

For $k \leqslant s$, we set

$$\langle k\rangle^- := \{(i,k) \in {}^{s}\overleftarrow{\varnothing}_0 \mid (i,k) \text{ is vertically sink and horizontally source in } {}^{s}\overleftarrow{\varnothing}_0\},$$
$$\langle k\rangle^+ := \{(i,k) \in {}^{s}\overleftarrow{\varnothing}_0 \mid (i,k) \text{ is vertically source and horizontally sink in } {}^{s}\overleftarrow{\varnothing}_0\}.$$

For $k \in \mathbb{Z}_{\leqslant s} \sqcup \{-\infty\}$,

$$\langle k,s\rangle^- := \bigsqcup_{k\leqslant t\leqslant s} \langle t\rangle^- \quad \text{and} \quad \langle k,s\rangle^+ := \bigsqcup_{k\leqslant t\leqslant s} \langle t\rangle^+.$$

If $k > s$, then we understand those sets as empty set. Note that there is no arrows between vertices within $\langle k,s\rangle^{\pm}$ for any $k \in \mathbb{Z}_{\leqslant s} \sqcup \{-\infty\}$.

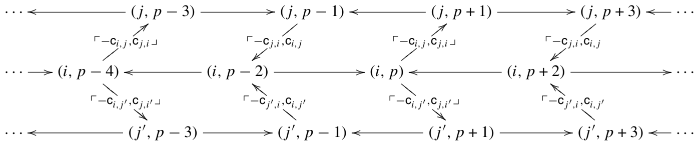
Lemma 9.17. *For $\{(i_t, p_t)\}_{1\leqslant t\leqslant r} = \{(j_t, q_t)\}_{1\leqslant t\leqslant r} = \langle k,s\rangle^{\pm}$, as a valued quiver,*

$$\mu_{(i_1,p_1)} \circ \mu_{(i_2,p_2)} \circ \cdots \circ \mu_{(i_r,p_r)}({}^{s}\overleftarrow{\varnothing}) \simeq \mu_{(j_1,q_1)} \circ \mu_{(j_2,q_2)} \circ \cdots \circ \mu_{(j_r,q_r)}({}^{s}\overleftarrow{\varnothing}),$$

that is, $\mu_{\langle k,s\rangle^{\pm}}({}^{s}\overleftarrow{\varnothing})$ is uniquely determined.

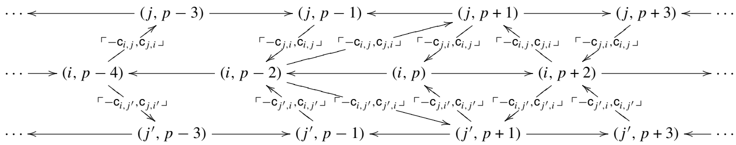
Proof. In this proof, we only consider the case of $\langle k, s\rangle^+$ since the proof of $\langle k, s\rangle^-$ is similar. Let us take $(i, p), (j, s) \in \langle k, s\rangle^+$ such that $(i, p) \neq (j, s)$ and $d(i, j) = 1$. The neighborhood of (i, p) on the valued quiver $\overleftarrow{{}^{s}\text{▱}}$ is depicted as follows:

$\overleftarrow{{}^{s}\text{▱}} =$



By Algorithm 7.3, we have

$\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}}) =$

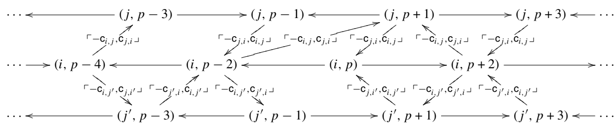


Here one can observe that

- $\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}})$ has arrows between $(i, p \pm 2)$ and $(j, p+1)$ for $d(i, j) = 1$, where $(i, p \pm 2), (j, p+1) \in \langle k, s\rangle^-$,
- the arrows adjacent to $(j, p-1)$ and $(j, p+3)$ are not changed by $\mu_{(i,p)}$.

Hence, for $(x, y) \in \{(j, p-1), (j, p+3) \mid d(i, j) = 1\}$, the mutation $\mu_{(x,y)}(\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}}))$ yields arrows between $(x, y \pm 2)$ and $(k, y+1)$ for $d(x, k) = 1$, one of which disappears due to an arrow from $\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}})$. For instance,

$\mu_{(j',p-1)}(\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}})) =$



Here the arrow from $(i, p-2)$ to $(j', p+1)$ on $\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}})$ disappeared by the new arrow from $(j', p+1)$ to $(i, p-2)$ generated when we apply the mutation $\mu_{(j',p-1)}$ to $(\mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}}))$. In fact, one may observe that

$$\mu_{(i,p)} \circ \mu_{(j',p-1)} = \mu_{(j',p-1)} \circ \mu_{(i,p)} \quad \text{on} \quad \overleftarrow{{}^{s}\text{▱}},$$

and the arrows among $(i, p), (i, p-2), (j', p+1)$ and $(j', p-1)$ in $\mu_{(j',p-1)} \circ \mu_{(i,p)}(\overleftarrow{{}^{s}\text{▱}})$ are reversed.

Furthermore, one may generalize the above as follows:

$$\mu_{(i,p)} \circ \mu_{(j,s)} = \mu_{(j,s)} \circ \mu_{(i,p)} \quad \text{on} \quad \mu_{(i_k,p_k)} \circ \cdots \circ \mu_{(i_r,p_r)}(\overleftarrow{{}^{s}\text{▱}}) \tag{9.21}$$

for $(i, p), (j, s) \in \langle k, s\rangle^+ \backslash \{(i_k, p_k), (i_{k+1}, p_{k+1}), \ldots, (i_r, p_r)\}$. This proves that the order of mutations is not important and completes the proof. $\square$

We remark that an analog of Lemma 9.17 by replacing ${}^s\overleftarrow{\varnothing}$ with ${}^s\overrightarrow{\varnothing}$ also holds.

For $s \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\leqslant s} \sqcup \{-\infty\}$, put

$$\Theta_s(\langle k, s\rangle^{\pm}) := \mu_{\langle k,s\rangle^{\pm}}({}^s\overleftarrow{\varnothing}), \quad \Omega_s(\langle k, s\rangle^{\pm}) := \mu_{\langle k,s\rangle^{\pm}}({}^s\overrightarrow{\varnothing}).$$

Lemma 9.18. *We have*

$$\Omega_s(\langle -\infty, s\rangle^{\pm}) \simeq \Theta_s \quad \textit{and} \quad \Theta_s(\langle -\infty, s\rangle^{\pm}) \simeq \Omega_s \quad \textit{as valued quivers}.$$

Proof. We only prove the second isomorphism for $\langle k, s\rangle^+$ since the proof of the other cases is almost identical. In the proof of Lemma 9.17, we have seen that a mutation $\mu_{(i,p)}$ for $(i, p) \in \langle k, s\rangle^+$ generates arrows between vertices in $\langle k, s\rangle^-$ and then they disappear in the course of the mutations $\mu_{(j,p')}$'s for $(j, p') \in \langle k, s\rangle^+$ located near (i, p). Moreover, the arrows adjacent to (i, p) are reversed during the mutations. Hence we have $\Theta_s(\langle -\infty, s\rangle^+) \simeq \Omega_s$. □

Proposition 9.19. *Every mutation $\mu_{(i,p)}$ in $\mu_{\langle -\infty,s\rangle^{\pm}}$ on the cluster $\{{}^s\mathfrak{u}_{i,p}\}$ corresponds to the quantum folded T-system in Theorem 6.9. Furthermore, each mutation $\mu_{(i,p)}$ on ${}^s\mathfrak{u}_{i,p}$ in $\mu_{\langle -\infty,s\rangle^{\pm}}$ corresponds to $\mathsf{T}_{\pm 2}$.*

Proof. For $(i, p) \neq (j, t) \in \langle -\infty, s\rangle^{\pm}$, recall that the mutation $\mu_{(i,p)}$ does not affect the arrows adjacent to (j, s). Thus it suffices to consider (i, p) and vertices connected to (i, p) by arrows. Assume first that $(i, p) \in \langle -\infty, s\rangle^+$. Then by replacing vertices in ${}^s\overleftarrow{\varnothing}$ with ${}^s\mathfrak{u}_{k,q}$'s, we have the following:

$$\begin{tikzcd}[column sep=small]
 & F_q(\underline{m}^{(k)}[a+1, b+1]) & \\
F_q(\underline{m}^{(i)}[a, b+2]) \arrow[r] & F_q(\underline{m}^{(i)}[a, b]) \arrow[u, "\ulcorner -\mathsf{c}_{i,k}, \mathsf{c}_{k,i} \lrcorner"'] \arrow[d, "\ulcorner -\mathsf{c}_{i,j}, \mathsf{c}_{j,i} \lrcorner"] & F_q(\underline{m}^{(i)}[a+2, b]) \arrow[l] \\
 & F_q(\underline{m}^{(j)}[a+1, b+1]) &
\end{tikzcd} \quad \text{for } j, k \text{ with } d(i, j), d(i, k) \leqslant 1$$

where ${}^s\mathfrak{u}_{i,p} = F_q(\underline{m}^{(i)}[a, b])$. Note that ${}^s\mathfrak{u}_{k,q}$ for $(k, q) \in \langle -\infty, s\rangle^-$ never mutate by $\mu_{\langle -\infty,s\rangle^+}$. Hence the mutation rule for cluster variables can be expressed as

$$\begin{aligned} F_q(\underline{m}^{(i)}[a, b]) * \mu_{(i,p)}(F_q(\underline{m}^{(i)}[a, b])) &= q^{\alpha(i,(b+2-a)/2)} F_q(\underline{m}^{(i)}[a+2, b]) \cdot F_q(\underline{m}^{(i)}[a, b+2]) \\ &\quad + q^{\gamma(i,(b+2-a)/2)} \prod_{j;d(i,j)=1} F_q(\underline{m}^{(j)}[a+1, b+1])^{-\mathsf{c}_{j,i}}. \end{aligned}$$

Here $q^{\alpha(i,(b+2-a)/2)}$ and $q^{\gamma(i,(b+2-a)/2)}$ are computed by bar-invariance. Hence, as in Proposition 8.6, and the above equation coincides with the formula in Theorem 6.9. Thus we have

$$\mu_{(i,p)}(F_q(\underline{m}^{(i)}[a, b])) = F_q(\underline{m}^{(i)}[a+2, b+2]).$$

Thus the assertion for $\langle -\infty, s\rangle^+$ follows.

Similarly, the arrows adjacent to (i, p) for $(i, p) \in \langle -\infty, s\rangle^-$ can be depicted as follows:

$$\begin{tikzcd}[column sep=small]
 & F_q(\underline{m}^{(k)}[a-1, b-1]) \arrow[d, "\ulcorner -\mathsf{c}_{k,i}, \mathsf{c}_{i,k} \lrcorner"] & \\
F_q(\underline{m}^{(i)}[a-2, b]) & F_q(\underline{m}^{(i)}[a, b]) \arrow[l] \arrow[r] & F_q(\underline{m}^{(i)}[a, b-2]) \\
 & F_q(\underline{m}^{(j)}[a-1, b-1]) \arrow[u, "\ulcorner -\mathsf{c}_{j,i}, \mathsf{c}_{i,j} \lrcorner"'] &
\end{tikzcd} \quad \text{for } j, k \text{ with } d(i, j), d(i, k) \leqslant 1.$$

Then as in $\langle -\infty, s\rangle^{+}$, we can conclude that

$$\mu_{(i,p)}(F_q(\underline{m}^{(i)}[a,b])) = F_q(\underline{m}^{(i)}[a-2, b-2]),$$

which proves our assertion. $\square$

Example 9.20. By replacing vertices (i, p) in $\mu_{\langle -\infty,0\rangle^{\pm}}({}^{0}\overleftarrow{\unicode{x25B1}})$ with $\mu_{\langle -\infty,0\rangle^{\pm}}({}^{0}\mathfrak{u}_{i,p})$ obtained from Example 9.3 (1) for type B_3 and $s = 0$, we have the following by Lemma 9.18 and Proposition 9.19:

Case 1. $\mu_{\langle -\infty,0\rangle^{+}}({}^{0}\overleftarrow{\unicode{x25B1}})$.

$$\begin{tikzcd}
\cdots \arrow[rr] & & F_q(m^{(1)}[-2,2]) \arrow[dl] & & F_q(m^{(1)}[0,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(1)}[0,0]) \arrow[dl] \\
\cdots & F_q(m^{(2)}[-1,3]) \arrow[l] \arrow[rr] & & F_q(m^{(2)}[-1,1]) \arrow[ur] \arrow[dr, "2"] & & F_q(m^{(2)}[1,1]) \arrow[ll] & \\
\cdots \arrow[rr] & & F_q(m^{(3)}[-2,2]) \arrow[ul, Rightarrow] & & F_q(m^{(3)}[0,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(3)}[0,0]) \arrow[ul, Rightarrow]
\end{tikzcd}
\simeq
\begin{tikzcd}
\cdots \arrow[r] & F_q(m^{(1)}[-2,2]) \arrow[dr] & & F_q(m^{(1)}[0,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(1)}[0,0]) \arrow[dr] & \\
\cdots & & F_q(m^{(2)}[-1,3]) \arrow[ll] \arrow[rr] & & F_q(m^{(2)}[-1,1]) \arrow[ul] \arrow[dl, "2"'] & & F_q(m^{(2)}[1,1]) \arrow[ll] \\
\cdots \arrow[r] & F_q(m^{(3)}[-2,2]) \arrow[ur, Rightarrow] & & F_q(m^{(3)}[0,2]) \arrow[ll] \arrow[rr] & & F_q(m^{(3)}[0,0]) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

where the parameters of quantum cluster variables located at vertices that are vertically sink and horizontally source are shifted by 2, and the orientation of all arrows is reversed.

Case 2. $\mu_{\langle -\infty,0\rangle^{-}}({}^{0}\overleftarrow{\unicode{x25B1}})$.

$$\begin{tikzcd}
\cdots \arrow[rr] & & F_q(m^{(1)}[-4,0]) \arrow[dl] & & F_q(m^{(1)}[-2,0]) \arrow[ll] \arrow[rr] & & F_q(m^{(1)}[-2,-2]) \arrow[dl] \\
\cdots & F_q(m^{(2)}[-3,1]) \arrow[l] \arrow[rr] & & F_q(m^{(2)}[-3,-1]) \arrow[ur] \arrow[dr, "2"] & & F_q(m^{(2)}[-1,-1]) \arrow[ll] & \\
\cdots \arrow[rr] & & F_q(m^{(3)}[-4,0]) \arrow[ul, Rightarrow] & & F_q(m^{(3)}[-2,0]) \arrow[ll] \arrow[rr] & & F_q(m^{(3)}[-2,-2]) \arrow[ul, Rightarrow]
\end{tikzcd}
\simeq
\begin{tikzcd}
\cdots \arrow[r] & F_q(m^{(1)}[-4,0]) \arrow[dr] & & F_q(m^{(1)}[-2,0]) \arrow[ll] \arrow[rr] & & F_q(m^{(1)}[-2,-2]) \arrow[dr] & \\
\cdots & & F_q(m^{(2)}[-3,1]) \arrow[ll] \arrow[rr] & & F_q(m^{(2)}[-3,-1]) \arrow[ul] \arrow[dl, "2"'] & & F_q(m^{(2)}[-1,-1]) \arrow[ll] \\
\cdots \arrow[r] & F_q(m^{(3)}[-4,0]) \arrow[ur, Rightarrow] & & F_q(m^{(3)}[-2,0]) \arrow[ll] \arrow[rr] & & F_q(m^{(3)}[-2,-2]) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

where the parameters of quantum cluster variables located at vertices that are vertically sink and horizontally source are shifted by -2, and the orientation of all arrows is reversed.

Thus we can conclude that

$$\mu_{\langle -\infty,0\rangle^{+}}(\mathfrak{S}_0) \simeq \mathfrak{S}_1 \qquad \text{and} \qquad \mu_{\langle -\infty,0\rangle^{-}}(\mathfrak{S}_0) \simeq \mathfrak{S}_{-1}.$$

Following Example 9.20, it is straightforward to check the following proposition.

Proposition 9.21. *For $s \in \mathbb{Z}$, we have*

$$\mu_{\langle -\infty,s\rangle^{\pm}}(\mathfrak{S}_s) \simeq \mathfrak{S}_{s\pm 1}.$$

For $s \in \mathbb{Z}$, put

$$\mathfrak{U} := \mu_{\langle -\infty,s\rangle^{\pm}}(\{{}^{s}\mathfrak{u}_{i,p}\}).$$

Proposition 9.22. *Every mutation $\mu_{(i,p)}$ in $\mu_{\langle -\infty,s\rangle^{\pm}}$ on the cluster $\mathfrak{U}$ corresponds to the quantum folded T-system in Theorem 6.9. Furthermore, each mutation $\mu_{(i,p)}$ on the quantum cluster variable at (i, p) in $\mu_{\langle -\infty,s\rangle^{\pm}}$ corresponds to $\mathsf{T}_{\mp 2}$.*

Proof. Set

$$\{{}^{s}\mathfrak{z}^{+}_{i,p}\} := \mu_{\langle -\infty,s\rangle^{+}}(\{{}^{s}\mathfrak{u}_{i,p}\}) \quad \text{and} \quad \{{}^{s}\mathfrak{z}^{-}_{i,p}\} := \mu_{\langle -\infty,s\rangle^{-}}(\{{}^{s}\mathfrak{u}_{i,p}\}).$$

In this proof, we only consider the case of $\{{}^{s}\mathfrak{z}^{+}_{i,p}\}$ since the proof of $\{{}^{s}\mathfrak{z}^{-}_{i,p}\}$ is parallel. Let $(i, p) \in \langle -\infty, s\rangle^{\pm}$.

Case 1. $(i, p) \in \langle -\infty, s\rangle^+$. By replacing vertices in ${}^s\overrightarrow{\varDelta}$ with ${}^s\mathfrak{z}^+_{k,q}$'s, we have the following:

$$\begin{tikzcd}
 & F_q(\underline{m}^{(k)}[a+1,b+1]) \arrow[d, "\ulcorner -\mathsf{c}_{k,i}, \mathsf{c}_{i,k} \lrcorner"] & \\
F_q(\underline{m}^{(i)}[a,b+2]) & F_q(\underline{m}^{(i)}[a+2,b+2]) \arrow[l] \arrow[r] & F_q(\underline{m}^{(i)}[a+2,b]) \\
 & F_q(\underline{m}^{(j)}[a+1,b+1]) \arrow[u, "\ulcorner -\mathsf{c}_{j,i}, \mathsf{c}_{i,j} \lrcorner"'] &
\end{tikzcd} \qquad \text{for } j, k \text{ with } d(i, j), d(i, k) \leqslant 1$$

where ${}^s\mathfrak{z}^+_{i,p} = F_q(\underline{m}^{(i)}[a+2, b+2])$. Hence the mutation rule for quantum cluster variables can be expressed as

$$\begin{aligned}
\mu_{(i,p)}(F_q(\underline{m}^{(i)}[a+2,b+2])) * F_q(\underline{m}^{(i)}[a+2,b+2]) &= q^{\alpha(i,(b+2-a)/2)} F_q(\underline{m}^{(i)}[a+2,b]) \\
&\quad \cdot F_q(\underline{m}^{(i)}[a,b+2]) \\
&\quad + q^{\gamma(i,(b+2-a)/2)} \prod_{j;d(i,j)=1} F_q(\underline{m}^{(j)}[a+1,b+1])^{-\mathsf{c}_{j,i}}.
\end{aligned}$$

Thus we have

$$\mu_{(i,p)}(F_q(\underline{m}^{(i)}[a+2,b+2])) = F_q(\underline{m}^{(i)}[a,b]).$$

Case 2. $(i, p) \in \langle -\infty, s\rangle^-$. The arrows adjacent to (i, p) for $(i, p) \in \langle -\infty, s\rangle^-$ are depicted as follows:

$$\begin{tikzcd}
 & F_q(\underline{m}^{(k)}[a+1,b+1]) & \\
F_q(\underline{m}^{(i)}[a,b+2]) \arrow[r] & F_q(\underline{m}^{(i)}[a,b]) \arrow[u, "\ulcorner -\mathsf{c}_{i,k}, \mathsf{c}_{k,i} \lrcorner"'] \arrow[d, "\ulcorner -\mathsf{c}_{i,j}, \mathsf{c}_{j,i} \lrcorner"] & F_q(\underline{m}^{(i)}[a+2,b]) \arrow[l] \\
 & F_q(\underline{m}^{(j)}[a+1,b+1]) &
\end{tikzcd} \qquad \text{for } j, k \text{ with } d(i, j), d(i, k) \leqslant 1$$

where ${}^s\mathfrak{z}^+_{i,p} = F_q(\underline{m}^{(i)}[a, b])$. Then as in *Case 1*, we have

$$\mu_{(i,p)}(F_q(\underline{m}^{(i)}[a,b])) = F_q(\underline{m}^{(i)}[a+2,b+2]).$$

□

Now, we are ready to prove Proposition 9.8.

Proof of Proposition 9.8. Let us define

$$\boldsymbol{\mu}_+ := \mu_{\langle -\infty, s+1\rangle^-} \circ \mu_{\langle -\infty, s\rangle^+} \quad \text{and} \quad \boldsymbol{\mu}_- := \mu_{\langle -\infty, s-1\rangle^+} \circ \mu_{\langle -\infty, s\rangle^-}.$$

It follows from Propositions 9.19, 9.21 and 9.22 that

$$\boldsymbol{\mu}_+(\mathfrak{S}_s) \simeq \mathfrak{S}_{s+2} \qquad \text{and} \qquad \boldsymbol{\mu}_-(\mathfrak{S}_s) \simeq \mathfrak{S}_{s-2}.$$

By applying $\boldsymbol{\mu}_+$ repeatedly, we obtain $F_q(\mathsf{X}_{i,p})$ for $(i, p) \in \widetilde{\triangle}_0$ with $p \geqslant s$ as a cluster variable of ${}^s\mathscr{A}_q(\mathfrak{g})$. Similarly, we obtain every $F_q(\mathsf{X}_{i,p})$ for $(i, p) \in \widetilde{\triangle}_0$ with $p \leqslant s$ as a cluster variable of ${}^s\mathscr{A}_q(\mathfrak{g})$ by using the repetition of $\boldsymbol{\mu}_-$. Thus the cluster algebra ${}^s\mathscr{A}_q(\mathfrak{g})$ contains every $F_q(\mathsf{X}_{i,p})$ associated to $\widetilde{\triangle}_0$ as its cluster variables. □

Conjecture 5. *Let s be an arbitrary integer. If* $F_q(\underline{m}^{(j)}[a, b]) \in \mathfrak{K}_q(\mathsf{g})$ *q-commutes with* ${}^s\mathfrak{u}_{i,p}$ *for all* ${}^s\mathfrak{u}_{i,p} \in \mathfrak{S}_s$*, then there exists* $(j, l) \in \widetilde{\mathbf{\Delta}}_0$ *such that*

$$ {}^s\mathfrak{u}_{j,l} = F_q(\underline{m}^{(j)}[a, b]). $$

When we replace g in Conjecture 5 with $\mathbf{g}$ of type ADE, the conjecture can be interpreted as the problem on the *maximal commuting families* of Kirillov–Reshetikhin modules over the quantum affine algebras $U_q'(\mathbf{g})$. It is proved in [45] for type ADE by using the *monoidal categorification* (see [38,53] for the notion of maximal commuting family). As a non-symmetric analog, we propose the above conjecture.

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Availability of data and materials Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no Conflict of interest to declare.

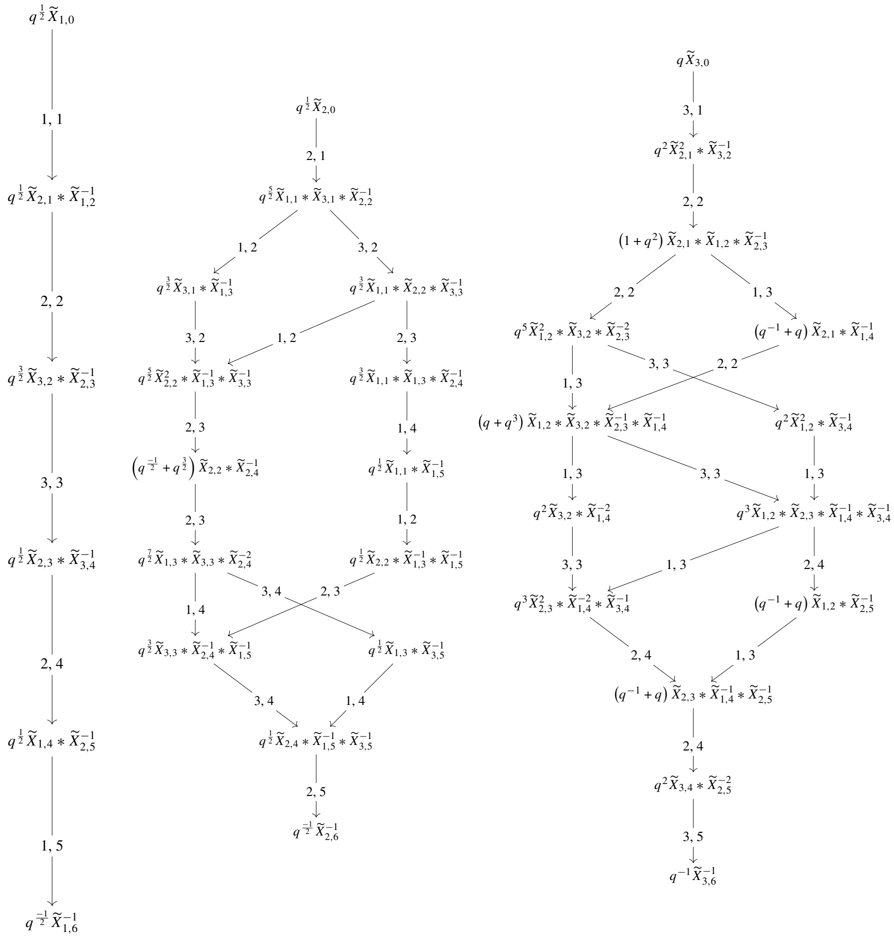
Ethical Approval Ethical approval is not applicable to this article.

Appendix A. Examples for (Quantum) Positivity

A.1. Quantum positivity and speciality of KR-polynomials. In this subsection, we provide examples for Conjecture 1 and Conjecture 2. Recall that

$$ F_q(\underline{X_{i,p}}) = E_q(\underline{X_{i,p}}) = L_q(\underline{X_{i,p}}), $$

and the quantum positivity of $F_q(\underline{X_{i,p}})$ for types B_3 and G_2 are already verified (up to shift of spectral parameters) in Example 5.16, Example 5.24 (for type G_2), and Example 5.25 (for type B_3). In what follows, we provide the formulas for fundamental polynomials for type C_3. Those elements may be obtained from the q-algorithm (cf. Example 5.16) or the quantum cluster algebra algorithm in Proposition 8.6, so we skip the details. The explicit formulas of $F_q(\underline{X_{i,0}})$'s for type C_3 are given as follows (under the same convention in the previous examples):



(A.1)

Since $F_q(\underline{X_{i,p}}) = \mathsf{T}_p(F_q(\underline{X_{i,0}}))$, we verify the quantum positivity of all fundamental polynomials for type C_3. We further remark that the quantum positivity of $F_q(\underline{X_{i,p}})$ for type F_4 also holds (with the help of computer program).

For type G_2, one may compute

$$\begin{aligned} &L_q(\underline{X_{1,6}^3}) = E_q(\underline{X_{1,6}^3}) = F_q(\underline{X_{1,6}^3}), \quad L_q(\underline{X_{2,5}X_{2,7}}) = F_q(\underline{X_{2,5}X_{2,7}}), \\ &E_q(\underline{X_{2,5}X_{2,7}}) = L_q(\underline{X_{2,5}X_{2,7}}) + P_{X_{2,5}X_{2,7},X_{1,6}^3}(q) L_q(\underline{X_{1,6}^3}), \end{aligned} \tag{A.2}$$

where $P_{X_{2,5}X_{2,7},X_{1,6}^3}(q) = q^3 \in q\mathbb{Z}_{\geqslant 0}[q]$. Then the quantum positivity of $L_q(\underline{X_{1,6}^3})$ follows from Example 5.24 and the definition of $E_q(\underline{X_{1,6}^3})$. Moreover, it follows from (A.2), Example 5.16, and Example 5.24 that the quantum positivity of $L_q(\underline{X_{2,5}X_{2,7}})$ also holds.

In general, for $\underline{m}^{(i)}[p,s]$ with $|p-s| \leqslant 2$, the computation of $L_q(\underline{m}^{(i)}[p,s])$ is similar with the one for (A.2), since $E_q(\underline{m}^{(i)}[p,s])$ has only two dominant monomials thanks to Theorem 6.9. It follows from Proposition 5.23 that this implies that $L_q(\underline{m}^{(i)}[p,s]) = F_q(\underline{m}^{(i)}[p,s])$. For example, when g is of type B_3,

$$\begin{aligned}
E_q(\underline{X_{1,0}X_{1,2}}) &= qF_q(\underline{X_{1,0}}) * F_q(\underline{X_{1,2}}) = L_q(\underline{X_{1,0}X_{1,2}}) + q^2 L_q(\underline{X_{2,1}}), \\
E_q(\underline{X_{2,0}X_{2,2}}) &= q^{-1}F_q(\underline{X_{2,0}}) * F_q(\underline{X_{2,2}}) = L_q(\underline{X_{2,0}X_{2,2}}) + q^2 L_q(\underline{X_{1,1}X_{3,1}^2}), \\
E_q(\underline{X_{3,0}X_{3,2}}) &= F_q(\underline{X_{3,0}}) * F_q(\underline{X_{3,2}}) = L_q(\underline{X_{3,0}X_{3,2}}) + qL_q(\underline{X_{2,1}}).
\end{aligned} \tag{A.3}$$

Here $L_q(\underline{X_{1,1}X_{3,1}^2}) = E_q(\underline{X_{1,1}X_{3,1}^2}) = F_q(\underline{X_{1,1}X_{3,1}^2})$. Furthermore, one may check that the quantum positivity of $L_q(\underline{X_{i,0}X_{i,2}})$ holds from Example 5.25 and (A.3). Similarly, one may have an analog of (A.3) for type C_3 with (A.1), which implies the quantum positivity of $L_q(\underline{X_{i,0}X_{i,2}})$ in this case. However, we cannot use the same argument in general because it does not seem to be easily determined by direct computation how many dominant monomials $E_q(\underline{m}^{(i)}[p,s])$ have for higher levels.

A.2. Quantum positivity of non KR-polynomials. In this subsection, we observe some examples in which $L_q(\underline{m})$ has the quantum positivity for a dominant monomial m different from the KR-monomials.

Example A.1. Let us consider the case of type C_2. Then the fundamental polynomials $F_q(\underline{X_{1,2}})$ and $F_q(\underline{X_{2,5}})$ are given as follows:

$$\begin{aligned}
F_q(\underline{X_{1,2}}) &= q^{\frac{1}{2}}\widetilde{X}_{1,2} + q^{\frac{3}{2}}\widetilde{X}_{2,3} * \widetilde{X}_{1,4}^{-1} + q^{\frac{1}{2}}\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} + q^{-\frac{1}{2}}\widetilde{X}_{1,6}^{-1}, \\
F_q(\underline{X_{2,5}}) &= q\widetilde{X}_{2,5} + q^2\widetilde{X}_{1,6}^2 * \widetilde{X}_{2,7}^{-1} + (q^{-1}+q)\widetilde{X}_{1,6}\widetilde{X}_{1,8}^{-1} + q^2\widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2} + q^{-1}\widetilde{X}_{2,9}^{-1}.
\end{aligned} \tag{A.4}$$

It follows from (A.4) that

$$E_q(\underline{X_{1,2}X_{2,5}}) = qF_q(\underline{X_{1,2}}) * F_q(\underline{X_{2,5}}) = L_q(\underline{X_{1,2}X_{2,5}}) + q^2 L_q(\underline{X_{1,4}}), \tag{A.5}$$

where $L_q(\underline{X_{1,2}X_{2,5}}) = F_q(\underline{X_{1,2}X_{2,5}})$ and $P_{X_{1,2}X_{2,5},\,X_{1,4}}(q) = q^2 \in q\mathbb{Z}[q]$. Then the quantum positivity of $L_q(\underline{X_{1,2}X_{2,5}})$ follows from the formula (that may be computed with (A.4) and (A.5)) as shown below:

$$\begin{aligned}
& q^{\frac{5}{2}}\widetilde{X}_{1,2} * \widetilde{X}_{2,5} + q^{\frac{7}{2}}\widetilde{X}_{2,3} * \widetilde{X}_{1,4}^{-1}\widetilde{X}_{2,5} + q^{\frac{7}{2}}\widetilde{X}_{1,2} * \widetilde{X}_{1,6}^2 * \widetilde{X}_{2,7}^{-1} + q^{\frac{9}{2}}\widetilde{X}_{2,3} * \widetilde{X}_{1,4}^{-1} * \widetilde{X}_{1,6}^2 * \widetilde{X}_{2,7}^{-1} \\
&\quad + q^{\frac{7}{2}}\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{1,6}^2 * \widetilde{X}_{2,7}^{-1} \\
&\quad + (q^{\frac{1}{2}} + q^{\frac{5}{2}})\widetilde{X}_{1,2} * \widetilde{X}_{1,6} * \widetilde{X}_{1,8}^{-1} + q^{\frac{7}{2}}\widetilde{X}_{1,2} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2} + (q^{\frac{3}{2}} + q^{\frac{7}{2}})\widetilde{X}_{2,3} * \widetilde{X}_{1,4}^{-1} * \widetilde{X}_{1,6} * \widetilde{X}_{1,8}^{-1} \\
&\quad + q^{\frac{9}{2}}\widetilde{X}_{2,3} * \widetilde{X}_{1,4}^{-1} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2} \\
&\quad + (q^{\frac{1}{2}} + q^{\frac{5}{2}})\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{1,6} * \widetilde{X}_{1,8}^{-1} + q^{\frac{7}{2}}\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2} + q^{-\frac{1}{2}}\widetilde{X}_{1,8}^{-1} + q^{\frac{5}{2}}\widetilde{X}_{1,6}^{-1} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-2} \\
&\quad + q^{\frac{1}{2}}\widetilde{X}_{1,2} * \widetilde{X}_{2,9}^{-1} \\
&\quad + q^{\frac{3}{2}}\widetilde{X}_{2,3} * \widetilde{X}_{1,4}^{-1} * \widetilde{X}_{2,9}^{-1} + q^{\frac{1}{2}}\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{2,9}^{-1} + q^{-\frac{1}{2}}\widetilde{X}_{1,6}^{-1}\widetilde{X}_{2,9}^{-1}
\end{aligned}$$

Example A.2. Let us consider the case of type B_2. Then the fundamental polynomials $F_q(\underline{X_{1,2}})$ and $F_q(\underline{X_{2,5}})$ are given as follows (cf. (A.4)):

$$\begin{aligned} F_q(\underline{X_{1,2}}) &= q\widetilde{X}_{1,2} + q^2\widetilde{X}_{2,3}^2 * \widetilde{X}_{1,4}^{-1} + (q^{-1}+q)\widetilde{X}_{2,3} * \widetilde{X}_{2,5}^{-1} + q^2\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-2} + q^{-1}\widetilde{X}_{1,6}^{-1}, \\ F_q(\underline{X_{2,5}}) &= q^{\frac{1}{2}}\widetilde{X}_{2,5} + q^{\frac{3}{2}}\widetilde{X}_{1,6} * \widetilde{X}_{2,7}^{-1} + q^{\frac{1}{2}}\widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} + q^{-\frac{1}{2}}\widetilde{X}_{2,9}^{-1}. \end{aligned} \tag{A.6}$$

It follows from (A.6) that

$$E_q(\underline{X_{1,2}X_{2,5}}) = qF_q(\underline{X_{1,2}}) * F_q(\underline{X_{2,5}}) = L_q(\underline{X_{1,2}X_{2,5}}) + q^2L_q(\underline{X_{2,3}}), \tag{A.7}$$

where $L_q(\underline{X_{1,2}X_{2,5}}) = F_q(\underline{X_{1,2}X_{2,5}}) + F_q(\underline{X_{2,3}})$ and $P_{X_{1,2}X_{2,5},\, X_{2,3}}(q) = q^2 \in q\mathbb{Z}[q]$. As in Example A.1, it follows from (A.6) and (A.7) that the quantum positivity of $L_q(\underline{X_{1,2}X_{2,5}})$ holds. Note that $L_q(\underline{X_{1,2}X_{2,5}})$ has two dominant monomials.

Let us also consider $E_q(\underline{X_{1,2}X_{2,5}^2})$. By (A.6), $E_q(\underline{X_{1,2}X_{2,5}^2})$ has three dominant monomials, namely,

$$\underline{X_{1,2}X_{2,5}^2} = q^4\widetilde{X}_{1,2}*\widetilde{X}_{2,5}^2, \quad (q^2+q^4)\widetilde{X}_{2,3}*\widetilde{X}_{2,5} = (q+q^3)\underline{X_{2,3}X_{2,5}}, \quad q^5\widetilde{X}_{1,4} = q^4\underline{X_{1,4}}.$$

Then we have

$$E_q(\underline{X_{1,2}X_{2,5}^2}) = L_q(\underline{X_{1,2}X_{2,5}^2}) + (q+q^3)L_q(\underline{X_{2,3}X_{2,5}}) + q^4L_q(\underline{X_{1,4}}), \tag{A.8}$$

where $L_q(\underline{X_{1,2}X_{2,5}^2}) = F_q(\underline{X_{1,2}X_{2,5}^2})$, $L_q(\underline{X_{2,3}X_{2,5}}) = F_q(\underline{X_{2,3}X_{2,5}})$, and

$$P_{X_{1,2}X_{2,5}^2,\, X_{2,3}X_{2,5}}(q) = q + q^3, \quad P_{X_{1,2}X_{2,5}^2,\, X_{1,4}}(q) = q^4 \in q\mathbb{Z}[q].$$

We provide the formula of $L_q(\underline{X_{2,3}X_{2,5}}) = F_q(\underline{X_{2,3}X_{2,5}})$ as follows:

$$\begin{aligned} & q\widetilde{X}_{2,3} * \widetilde{X}_{2,5} + q^2\widetilde{X}_{2,3} * \widetilde{X}_{1,6} * \widetilde{X}_{2,7}^{-1} + q^3\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{1,6} * \widetilde{X}_{2,7}^{-1} + q\widetilde{X}_{2,3} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} \\ &\quad + q^2\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} \\ &\quad + q\widetilde{X}_{2,5} * \widetilde{X}_{1,6}^{-1} * \widetilde{X}_{2,7} * \widetilde{X}_{1,8}^{-1} + \widetilde{X}_{2,3} * \widetilde{X}_{2,9}^{-1} + q\widetilde{X}_{1,4} * \widetilde{X}_{2,5}^{-1} * \widetilde{X}_{2,9}^{-1} + \widetilde{X}_{2,5} * \widetilde{X}_{1,6}^{-1} * \widetilde{X}_{2,9}^{-1} + q^{-1}\widetilde{X}_{2,7}^{-1} * \widetilde{X}_{2,9}^{-1} \end{aligned} \tag{A.9}$$

Then one may compute the formula of $L_q(\underline{X_{1,2}X_{2,5}^2})$ by using (A.8) with (A.6) and (A.9) (or the q-algorithm directly), and then the quantum positivity of $L_q(\underline{X_{1,2}X_{2,5}^2})$ also follows.

A.3. Positivity of Kazhdan–Lusztig polynomials. This subsection presents examples for the positivity of the KL-type polynomials $P_{m,m'}(q) \in q\mathbb{Z}[q]$ with at least 2 terms.

Example A.3. In Example A.2 (for type B_2), we have seen the positivity of KL-type polynomial given by

$$P_{X_{1,2}X_{2,5}^2,\, X_{2,3}X_{2,5}}(q) = q + q^3 \in q\mathbb{Z}[q],$$

which is an example for the positivity of KL-type polynomials with 2-terms. Let us consider the case of type G_2 to investigate more complicated examples.

For $\underline{m} = \underline{X_{2,0}X_{1,5}^2}$, we have

$$E_q(\underline{m}) = L_q(\underline{m}) + (q^2 + q^4)L_q(\underline{X_{1,1}X_{1,5}}) + q^6 L_q(\underline{X_{1,3}}),$$

where $L_q(\underline{m}) = F_q(\underline{m}) + L_q(\underline{X_{1,1}X_{1,5}})$ and $L_q(\underline{X_{1,1}X_{1,5}}) = F_q(\underline{X_{1,1}X_{1,5}}) + L_q(\underline{X_{1,3}})$ and the KL-type polynomials are

$$P_{X_{2,0}X_{1,5}^2,\, X_{1,1}X_{1,5}}(q) = q^2 + q^4, \quad P_{X_{2,0}X_{1,5}^2,\, X_{1,3}}(q) = q^6 \in q\mathbb{Z}[q].$$

For $\underline{m} = \underline{X_{2,0}^2 X_{1,1}X_{1,3}}$, we have

$$E_q(\underline{m}) = L_q(\underline{m}) + qL_q(\underline{X_{2,0}^2 X_{2,2}}) + \left(2q^4 + q^6 + q^8 + q^{10}\right) L_q(\underline{X_{2,0}X_{1,1}^3}),$$

where $L_q(\underline{m}) = F_q(\underline{m}) + (q^{-2} + 1 + q^2)F_q(\underline{X_{2,0}X_{1,1}^3})$ and the KL-type polynomials are

$$P_{X_{2,0}^2X_{1,1}X_{1,3},\, X_{2,0}^2X_{2,2}}(q) = q, \quad P_{X_{2,0}^2X_{1,1}X_{1,3},\, X_{2,0}X_{1,1}^3}(q) = 2q^4 + q^6 + q^8 + q^{10} \in q\mathbb{Z}[q].$$

For $\underline{m} = \underline{X_{2,0}^2 X_{2,4}}$, we have the expansion of $E_q(\underline{m}) - L_q(\underline{m})$ in terms of L_q as follows:

$$(q^4 + q^6 + q^8)L_q(\underline{X_{2,0}X_{1,1}X_{1,3}}) + (q^3 + q^6)L_q(\underline{X_{2,0}X_{2,2}}) + (2q^2 + 6q^4 + 6q^6 + 4q^8 + 2q^{10} + q^{12}) L_q(\underline{X_{1,1}^3}),$$

where $L_q(\underline{X_{2,0}X_{1,1}X_{1,3}}) = F_q(\underline{X_{2,0}X_{2,2}}) + (q^{-2} + 1 + q^2)L_q(\underline{X_{1,1}^3})$ and the KL-type polynomials (in $q\mathbb{Z}[q]$) are

$$P_{X_{2,0}^2X_{2,4},\, X_{2,0}X_{1,1}X_{1,3}}(q) = q^4 + q^6 + q^8, \quad P_{X_{2,0}^2X_{2,4},\, X_{2,0}X_{2,2}}(q) = q^3 + q^6,$$
$$P_{X_{2,0}^2X_{2,4},\, X_{1,1}^3}(q) = 2q^2 + 6q^4 + 6q^6 + 4q^8 + 2q^{10} + q^{12}.$$

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