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Quantum cluster algebra, braid moves and quantum virtual Grothendieck ring

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Abstract

In this paper, we study the quantum virtual Grothendieck ring, denoted by $\mathfrak{K}_q(\mathfrak{g})$, which was introduced in [39], and further investigated in [25, 26]. Our approach involves examining this ring from two perspectives: first, by considering its connection to quantum cluster algebras of non-skew-symmetric types; and second, by exploring its relevance to categorification theory. We specifically focus on (i) the homomorphisms that arise from braid moves, particularly 4-moves and 6-moves, in the braid group; and (ii) the quantum Laurent positivity phenomena, which have not yet been proven for non-skew-symmetric types. As applications of our results, we derive the substitution formulas for non-skew-symmetric types discussed in [11] for skew-symmetric types, and demonstrate that any truncated element in a heart subring, denoted by $\mathfrak{K}_{q,Q}(\mathfrak{g})$, which corresponds to a simple module over the quiver Hecke algebra $R^{\mathfrak{g}}$, possesses coefficients in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}]$. This result is particularly interesting because it implies that each truncated Kirillov–Reshetikhin polynomial in $\mathfrak{K}_{q,Q}(\mathfrak{g})$ and each element in the standard basis $\mathsf{E}_q(\mathfrak{g})$ of the entire ring $\mathfrak{K}_q(\mathfrak{g})$ have coefficients also in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}]$. Since (truncated) Kirillov–Reshetikhin polynomials can be obtained using a quantum cluster algebra algorithm and appear as quantum cluster variables, they provide compelling evidence in support of the quantum Laurent positivity conjecture in non-skew-symmetric types.

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1 Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra, which has an index set I and a Cartan matrix C. The quantum loop algebra $U_q(\mathcal{L}\mathfrak{g})$ associated with $\mathfrak{g}$ has been extensively studied since its emergence. This is mainly due to the fact that its representation theory exhibits a rich structure characterized by monoidality and rigidity. Although the category $\mathscr{C}_\mathfrak{g}$ of type $\mathbf{1}$-representations over $U_q(\mathcal{L}\mathfrak{g})$ is not braided, it is well-known that the Grothendieck ring $K(\mathscr{C}_\mathfrak{g})$ is commutative.

Motivated by $\mathcal{W}$-algebras, Frenkel–Reshetikhin [9] constructed the *q-character monomorphism* $\chi_q : K(\mathscr{C}_\mathfrak{g}) \hookrightarrow \mathbb{Z}[Y_{i,w}^{\pm 1}]_{i\in I, w\in\mathbb{C}(q)^\times}$ which also explains the commutativity of $K(\mathscr{C}_\mathfrak{g}) \simeq \chi_q(K(\mathscr{C}_\mathfrak{g})) =: \mathcal{K}(\mathscr{C}_\mathfrak{g})$. Nakajima [48] and Varagnolo–Vasserot [56] then employed the geometry of quiver varieties to construct the non-commutative t-quantization, denoted as $\mathcal{K}_t(\mathscr{C}_\mathbf{g})$, of $\mathcal{K}(\mathscr{C}_\mathbf{g})$ for a simply-laced type $\mathbf{g}$. This construction is known as the *quantum Grothendieck ring* and highlights the non-braided property of $\mathscr{C}_\mathbf{g}$. Moreover, $\mathcal{K}_t(\mathscr{C}_\mathbf{g})$ recovers $\mathcal{K}(\mathscr{C}_\mathbf{g})$ when $t = 1$.

The quantum Grothendieck ring admits a remarkable Kazhdan–Lusztig algorithm for $\mathscr{C}_{\mathbf{g}}$, offering insights into the Jordan–Hölder multiplicity $P_{m,m'}$ of a simple module $L(m')$ in a standard module $E(m)$ in $K(\mathscr{C}_{\mathbf{g}})$. Nakajima established this connection through the formulation of the Kazhdan–Lusztig type polynomial $P_{m,m'}(t)$ in $\mathcal{K}_t(\mathscr{C}_{\mathbf{g}})$, with its specialization at $t = 1$ yielding the desired $P_{m,m'}$.

To be more specific, he constructed bases $\mathbf{E}_t(\mathbf{g}) = \{E_t(m)\}$ and $\mathbf{L}_t(\mathbf{g}) = \{L_t(m)\}$ for $\mathcal{K}_t(\mathscr{C}_{\mathbf{g}})$, consisting of (q, t)-*characters* of standard modules and simple modules, respectively. He proved the uni-triangular property that exists between these two bases:

$$E_t(m) = L_t(m) + \sum_{m';m' <_N m} P_{m,m'}(t) L_t(m') \quad \text{for some } P_{m,m'}(t) \in t\mathbb{Z}[t] \tag{1.1}$$

in $\mathcal{K}_t(\mathscr{C}_{\mathbf{g}})$. We call $\mathbf{E}_t(\mathbf{g})$ the *standard basis* and $\mathbf{L}_t(\mathbf{g})$ the *canonical basis*.

Taking a significant stride forward, Nakajima went on to establish that every coefficient polynomial of $L_t(m)$, $E_t(m)$, and the polynomial $P_{m,m'}(t)$ resides in $\mathbb{Z}_{\geqslant 0}[t^{\pm 1/2}]$. Moreover, he proved that the specializations of $L_t(m)$ and $E_t(m)$ at $t = 1$ yield the q-character $\chi_q(L(m))$ for the simple module $L(m)$ and the q-character $\chi_q(E(m))$ for the standard module $E(m)$ in $\mathscr{C}_{\mathbf{g}}$, respectively. These findings collectively underscore the profound nature of the algorithm he unveiled.

In cases where $\mathfrak{g}$ is non-simply-laced, a fully-developed geometry is lacking, rendering the direct application of the framework for $\mathbf{g}$ impractical. To address this, Hernandez [18] introduced a conjectural Kazhdan–Lusztig algorithm for all $\mathfrak{g}$, using the *quantum Cartan matrix* $\mathsf{C}(q)$ associated with $\mathfrak{g}$. He devised a non-commutative t-quantization, also denoted as $\mathcal{K}_t(\mathfrak{g})$, of $\mathcal{K}(\mathfrak{g})$ and constructed a basis $\mathbf{F}_q(\mathfrak{g})$ for $\mathcal{K}_t(\mathfrak{g})$ through a special algorithm known as the t-*algorithm*. It is noteworthy that the t-algorithm can be viewed as a t-quantization of the Frenkel–Mukhin algorithm in [8], utilizing $\mathsf{C}(q)$.

Subsequently, Hernandez constructed bases $\mathbf{E}_t(\mathfrak{g})$ and $\mathbf{L}_t(\mathfrak{g})$ for $\mathcal{K}_t(\mathfrak{g})$, adhering to (1.1), and expected them to also exhibit the same positivity as $L_t(m)$, $E_t(m)$, and $P_{m,m'}(t)$. Particularly, in [19], he showed that the (q, t)-character $L_t(Y_{i,w})$ of a *fundamental representation* $L(Y_{i,w})$ manifests the desired *quantum positivity*. This implies that every coefficient polynomial of $L_t(Y_{i,w})$ resides in $\mathbb{Z}_{\geqslant 0}[t^{\pm 1/2}]$, consequently establishing the quantum positivity of elements in $\mathbf{E}_t(\mathfrak{g})$ as well.

The quantum cluster algebra $\mathscr{A}_q$, introduced by Berenstein–Fomin–Zelevinsky in [2, 6], serves as a crucial framework for the investigation of the dual-canonical/upper-global basis $\mathbf{B}^{\mathrm{up}}$ introduced by Lusztig/Kashiwara, through a combinatorial point of view. The algebra $\mathscr{A}_q$ is a non-commutative $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of the skew field of $\mathbb{Q}(q^{1/2})(Z_i)_{i \in \mathsf{K}}$, generated by cluster variables. These variables are derived from the initial cluster $\{Z_i\}_{i \in \mathsf{K}}$ through a sequence of quantum exchange relations known as *mutations*. Despite the non-trivial nature of these mutations, the *quantum Laurent phenomenon*, proven in [2, 6], assures that $\mathscr{A}_q$ remains contained in the (non-commutative) Laurent polynomial ring $\mathbb{Z}[q^{\pm 1/2}][Z_i^{\pm 1}]_{i \in \mathsf{K}}$.

Another interesting feature of $\mathscr{A}_q$, which has been confirmed for skew-symmetric cases (see [4]), is the *quantum Laurent positivity conjecture*. This conjecture asserts that every cluster variable indeed belongs to $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}][Z_i^{\pm 1}]_{i \in \mathsf{K}}$.

Building upon the earlier discussion, it is worthwhile to emphasize that the algebras $\mathcal{K}_t(\mathfrak{g})$ and $\mathscr{A}_q$ exhibit similar quantum Laurent positivity phenomena. The connection between these two structures is further underscored by the notable works of Hernandez–Leclerc in [21–23]. This relationship gains additional support from the results in [3, 10, 27, 33, 37, 38, 49], which collectively establish that $\mathcal{K}_t(\mathfrak{g})$ possesses a quantum cluster algebra structure.

In a recent development, Fujita, Hernandez, Oya, and the second named author of this paper have successfully resolved the quantum positivity conjectures for $L_t(m)$ and $P_{m,m'}(t)$ in $\mathcal{K}_t(\mathfrak{g})$ for non-simply-laced Lie algebras $\mathfrak{g}$. This achievement was made possible through the utilization of the quantum cluster algebra structure of $\mathcal{K}_t(\mathfrak{g})$ and the application of a technique known as the *propagation of positivity* [10, 11]. Let us explain their results more precisely.

For each specific $\mathfrak{g}$, we define $\mathbf{g}_{\text{fin}}$ of simply-laced type as follows:

$$(\mathfrak{g}, \mathbf{g}_{\text{fin}}) = (B_n, A_{2n-1}),\ (C_n, D_{n+1}),\ (F_4, E_6),\ (G_2, D_4) \text{ and } \mathfrak{g} = \mathbf{g}_{\text{fin}} \text{ otherwise.}$$

In their work [10], it was established that the algebras $\mathcal{K}_t(\mathscr{C}^0_{\mathfrak{g}})$ and $\mathcal{K}_t(\mathscr{C}^0_{\mathbf{g}_{\text{fin}}})$ are isomorphic, with their presentations characterized as a *boson-extension* of the negative half of the quantum group $U_q^-(\mathbf{g}_{\text{fin}})$ (refer also to [25, (1.7)]). Here, $\mathscr{C}^0_{\mathfrak{g}}$ denotes the *skeleton category*, also known as the *Hernandez–Leclerc* subcategory. Taking a further step in [11], they proved that the isomorphisms between $\mathcal{K}_t(\mathscr{C}^0_{\mathfrak{g}})$ and $\mathcal{K}_t(\mathscr{C}^0_{\mathbf{g}_{\text{fin}}})$ can be interpreted as sequences of mutations-permutations between *adapted* sequences of indices. Moreover, they extended these isomorphisms to the skew fields of their quantum tori, introducing the *substitution formula*.

The implications of these isomorphisms are far-reaching, leading to several applications:

(a) **Propagating positivity**: Using the isomorphisms to propagate positivity from type $\mathbf{g}_{\text{fin}}$ to type $\mathfrak{g}$, they proved the quantum positivity conjectures for $L_t(m)$ and $P_{m,m'}(t)$ in non-simply-laced type $\mathfrak{g}$.
(b) **Recovery of q-character**: Using the monoidal categorification result presented in [37] and the isomorphisms, they proved that $L_t(m)$ recovers the q-character of a simple module $L(m)$ at $q = 1$, provided it is *reachable*.
(c) **Interplay between categories**: The substitution formula revealed intriguing behaviors among elements in $\mathbf{L}_t(\mathfrak{g})$ and $\mathbf{L}_t(\mathbf{g}_{\text{fin}})$.

In summary, the algebraic structure of $\mathcal{K}_t(\mathscr{C}^0_{\mathfrak{g}})$ exhibits (skew-)symmetry, irrespective of the inherent asymmetry in $\mathfrak{g}$. Moreover, the observed quantum positivity phenomena of $\mathcal{K}_t(\mathscr{C}^0_{\mathfrak{g}})$ are connected to those witnessed in the context of the *skew-symmetric* quantum cluster algebra $\mathscr{A}_q$.

In light of the aforementioned observations, Kashiwara and the second named author of this paper introduced what is now called the *quantum virtual Grothendieck ring* $\mathfrak{K}_q(\mathfrak{g})$: a distinguished subring of a non-commutative quantum torus over $\mathbb{A} := \mathbb{Z}[q^{\pm 1/2}]$, generated by the variables $\widetilde{X}_{i,p}$ associated with $\mathfrak{g}$, and characterized as the common kernel of t-deformed screening operators $S_{i,t}$ (see Definition 5.2 for more details). Note that the construction of $\mathfrak{K}_q(\mathfrak{g})$ can be seen as a $\mathsf{C}(t)$-analogue of

Hernandez's construction of $\mathcal{K}_t(\mathscr{C}^0_{\mathfrak{g}})$, where $\mathsf{C}(t)$ is another quantization of C known as the *t-quantized Cartan matrix*.

The main results established in [26, 39] include:

(i) **Quantum cluster algebra structures**: The algebra $\mathfrak{K}_q(\mathfrak{g})$, along with various subrings, possesses quantum cluster algebra structures. Importantly, the (a)symmetries of these structures coincide with those of the underlying Lie algebra $\mathfrak{g}$.

(ii) **Bases and relations**: Bases $\mathsf{F}_q(\mathfrak{g})$, $\mathsf{E}_q(\mathfrak{g})$, and $\mathsf{L}_q(\mathfrak{g})$ are identified in $\mathfrak{K}_q(\mathfrak{g})$, and they exhibit similar relations to those in $\mathcal{K}_t(\mathfrak{g})$, including (1.1). Notably, $\mathsf{E}_q(\mathfrak{g})$ is termed the *standard basis*, and $\mathsf{L}_q(\mathfrak{g})$ the *canonical basis*, mirroring the naming in the $\mathcal{K}_t(\mathfrak{g})$ cases.

Conjecture I ([26])

(a) *In the case where $L_q(m)$ is real, meaning that $L_q(m)^2 = q^{a/2}L_q(m^2)$ for some $a \in \mathbb{Z}$, every coefficient polynomial of $L_q(m)$ resides in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}]$.*

(b) *For a* Kirillov–Reshetikhin (KR) *monomial $m \in \mathfrak{K}_q(\mathfrak{g})$, the element $L_q(m)$ in* $\mathsf{L}_q(\mathfrak{g})$ *coincides with $F_q(m)$ in* $\mathsf{F}_q(\mathfrak{g})$ *consisting of real elements, where $F_q(m)$ is a KR-polynomial* (*see* § 5.3 *for the definition*).

In particular, combining (a) *and* (b), *we expect that each coefficient polynomial of a KR-polynomial $F_q(m)$ is in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}]$.*

We remark here that each basis element $F_q(m)$ in $\mathsf{F}_q(\mathfrak{g})$ can be obtained from the q-algorithm, which is a $\mathsf{C}(t)$-analogue of the t-algorithm and is defined inductively. Thus determining whether the coefficient polynomials of $F_q(m)$ are in $\mathbb{Z}_{\geqslant 0}[q^{\pm 1/2}]$ or not is quite intricate.

In this paper, we further study the quantum virtual Grothendieck ring $\mathfrak{K}_q(\mathfrak{g})$, utilizing its quantum cluster algebra structure. It is important to note that $\mathfrak{K}_q(\mathfrak{g})$ is generated by a countably infinite set of *fundamental* polynomials, denoted as $\{F_q(X_{i,p})\}_{(i,p)\in\widehat{\triangle}_0}$ and parameterized by a specific index set $\widehat{\triangle}_0$ (see (5.2)).

For each Dynkin quiver $Q = (\triangle, \xi)$ associated with the Lie algebra $\mathfrak{g}$ (see § 5.5), the *heart* subring $\mathfrak{K}_{q,Q}(\mathfrak{g})$ of $\mathfrak{K}_q(\mathfrak{g})$ is generated by the fundamental polynomials $\{F_q(X_{i,p})\}_{(i,p)\in\Gamma_0^Q\subset\widehat{\triangle}_0}$ (see [12, 25]). The utilization of the heart subrings yields the following homomorphisms [10, 22, 26]:

(a) **Isomorphism** Ψ_Q: An $\mathbb{A}$-algebra isomorphism, denoted by Ψ_Q, is established between the integral form $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ of the quantum unipotent coordinate ring $\mathcal{A}_q(\mathfrak{n})$ and the heart subring $\mathfrak{K}_{q,Q}(\mathfrak{g})$. Importantly, this isomorphism maps the *normalized dual-canonical/upper-global basis* $\widetilde{\mathbf{B}}^{\mathrm{up}}$ to the canonical basis $\mathsf{L}_{q,Q}(\mathfrak{g}) := \mathsf{L}_q(\mathfrak{g}) \cap \mathfrak{K}_{q,Q}(\mathfrak{g})$ of $\mathfrak{K}_{q,Q}(\mathfrak{g})$.

(b) **Automorphism** $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$: For any pair of Dynkin quivers $Q = (\triangle, \xi)$ and $Q' = (\triangle, \xi)$, there exists an automorphism denoted by $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ of $\mathfrak{K}_q(\mathfrak{g})$. This automorphism sends the heart subring $\mathfrak{K}_{q,Q'}(\mathfrak{g})$ to $\mathfrak{K}_{q,Q}(\mathfrak{g})$ and respects the canonical basis L_q.

Note that the integral form $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ of the quantum unipotent coordinate ring $\mathcal{A}_q(\mathfrak{n})$ is isomorphic to the quantum cluster algebra $\mathcal{A}_{\boldsymbol{i}}$ ([13, 15]). The initial seed of $\mathcal{A}_{\boldsymbol{i}}$ can be constructed through the combinatorics of a reduced sequence $\boldsymbol{i}$ associated with the longest element $w_\circ$ of the Weyl group W.

As a main result, Theorem 5.21 establishes, through the use of isomorphisms and the degrees of quantum cluster monomials, that the isomorphism $\Psi_{\xi,\xi'}$: $\mathfrak{K}_{q,\xi'}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{K}_{q,\xi}(\mathfrak{g})$ obtained by restricting $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ to the ξ'*-negative subring* $\mathfrak{K}_{q,\xi'}(\mathfrak{g})$ of $\mathfrak{K}_q(\mathfrak{g})$ (refer to Definition 5.8) is compatible with sequences of mutations-permutations between the seeds associated with infinite sequences $\widehat{\boldsymbol{i}}'$ and $\widehat{\boldsymbol{i}}$ in $I^{(\infty)}$ respectively adapted to Q' and Q (see (5.11)).

The proof of Theorem 5.21 involves the following steps:

(a) **Braid moves as mutations-permutations**: The interpretation of braid moves as mutations-permutations is established.
(b) **Analysis of degrees of quantum cluster monomials**: The behaviors of degrees of quantum cluster monomials concerning mutations-permutations corresponding to braid moves are analyzed. This analysis includes the study of 4-moves for types B_n, C_n, F_4 and 6-moves for type G_2, involving more complicated computations compared to the investigations of 2-moves and 3-moves presented in [11].

Through this detailed analysis, it is proved that the isomorphism $\Psi_{\xi,\xi'}$ induces a corresponding bijection between their quantum cluster monomials, establishing the compatibility highlighted in Theorem 5.21.

As an application, we extend the automorphisms $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ to substitution formulas $\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'}$ for the variables in the fractional skew field $\mathbb{F}(\mathcal{X}_q)$ of $\mathfrak{K}_q(\mathfrak{g})$ (Theorem 5.30). These substitution formulas, derived from mutations-permutations and consequently being automorphisms, involve non-trivial fractions among Laurent polynomials. Independent of the quantum Laurent phenomena, these substitution formulas yield Laurent polynomials in $\mathfrak{K}_q(\mathfrak{g})$ and respect the canonical basis L_q.

Exploring these substitution formulas reveals highly non-trivial relations among elements in L_q. For concrete examples of these formulas, particularly for type B_2, we refer the reader to Appendix A. The investigation of the substitution formulas offers valuable insights into relationships among elements in the canonical basis L_q.

Another main result (Theorem 7.1) of this paper is a proof of quantum positivity for: (a) fundamental polynomials $F_q(X_{i,p}) = L_q(X_{i,p})$, (b) elements in the standard basis E_q, (c) truncated KR-polynomials in $(\mathfrak{K}_{q,Q}(\mathfrak{g}))_{\leqslant\xi}$. The quantum cluster algebra algorithm generates fundamental polynomials $L_q(X_{i,p})$ and truncated KR-polynomials $F_q(m)_{\leqslant\xi}$, starting from their highest dominant monomials $\underline{X_{i,p}}$ and $\underline{m}$ ([3, 23, 26]). This result serves as robust evidence supporting the quantum positivity conjecture for non-skew-symmetric types.

The main argument of the proof involves the categorification theory of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ through the category R-gmod of finite-dimensional graded modules over the quiver Hecke algebra R ([41, 52]). Recently, Kashiwara, Kim, Park, and the second named author introduced the concept of a *Laurent family* of simple modules over R and proved that each Laurent family exhibits the quantum Laurent positivity phenomena. This positivity is observed with respect to the basis of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ that corresponds to the isomorphism classes of simple modules over R through the process of categorification ([36]).

The proof of Theorem 7.1 employs the following key ingredients:

(i) The isomorphisms among $\mathcal{K}_{\mathbb{A}}(R\text{-gmod})$, $\mathfrak{K}_{q,Q}(\mathfrak{g})$, and $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ (see (7.2)).
(ii) The truncation isomorphism $(\cdot)_{\leqslant\xi} : \mathfrak{K}_{q,Q}(\mathfrak{g}) \xrightarrow{\sim} (\mathfrak{K}_{q,Q}(\mathfrak{g}))_{\leqslant\xi}$ (see (5.8)).
(iii) The structure of KR-polynomials (see Theorem 5.6).
(iv) The fact that KR-polynomials in $\mathfrak{K}_{q,Q}(\mathfrak{g})$ *can be understood as certain characters* of determinantal modules over R ([25, 32]).

With these ingredients, we show in Theorem 7.1 that:

(a) The set of KR-polynomials $\{F_q(m^{(i)}[p, \xi_i]) \mid (i, p) \in \Gamma_0^Q\}$ is the set of *such characters* of the Laurent family $\mathcal{L} := \{M(w_k^{\boldsymbol{i}}\varpi_{i_k}, \varpi_{i_k})\}$ of *determinantal modules* for any reduced sequence $\boldsymbol{i}$ of $w_\circ$ *adapted to* $Q = (\triangle, \xi)$.
(b) Any element $\mathsf{X}_{\leqslant\xi} \in (\mathfrak{K}_{q,Q}(\mathfrak{g}))_{\leqslant\xi}$ corresponding to a simple module X over R is quantum positive (see Example 7.3 for instances).

As special cases, we obtain the desired result (Corollary 7.2) that fundamental polynomials, elements in the standard basis $\mathsf{E}_q(\mathfrak{g})$, and truncated KR-polynomials in $(\mathfrak{K}_{q,Q})(\mathfrak{g})_{\leqslant\xi}$ are quantum positive.

This paper is structured as follows: Section 2 provides a review of the fundamental concepts in the theory of quantum cluster algebras, covering essential topics such as the degrees of quantum cluster monomials and the valued quiver associated with an exchange matrix. Section 3 explores the mutations-permutations corresponding to moves in the braid group of $\mathfrak{g}$. It contains intricate computations and makes crucial observations concerning the behaviors of degrees of quantum cluster monomials with respect to these mutations-permutations. We allocate considerable space to this section because the computations are quite important for later parts. Section 4 investigates the connection between mutations-permutations corresponding to moves and the basis $\widetilde{\mathbf{B}}^{\mathrm{up}}$ in $\mathcal{A}_{\mathbb{A}}(\mathfrak{g})$ through the isomorphism $\mathcal{A}_{\boldsymbol{i}} \xrightarrow{\sim} \mathcal{A}_{\mathbb{A}}(\mathfrak{n})$. In Section 5, we establish the compatibility between the isomorphism $\Psi_{\xi,\xi'}$ and mutations-permutations corresponding to moves. This compatibility is then applied to the analysis of substitution formulas. Section 6 reviews the categorification of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ via R-gmod and discusses the results from [36] related to the concept of a Laurent family. Section 7 presents the proof of the quantum positivity of fundamental polynomials, the standard basis $\mathsf{E}_q(\mathfrak{g})$, and truncated KR-polynomials in $(\mathfrak{K}_{q,Q})(\mathfrak{g})_{\leqslant\xi}$.

Convention

Throughout this paper, we use the following convention.

- For a statement $\mathtt{P}$, we set $\delta(\mathtt{P})$ to be 1 or 0 depending on whether $\mathtt{P}$ is true or not. In particular, we set $\delta_{i,j} := \delta(i = j)$ (the Kronecker's delta).
- For a totally ordered set $J = \{\cdots < j_{-1} < j_0 < j_1 < j_2 \cdots\}$, write

$$\prod_{j\in J}^{\to} A_j := \cdots A_{j_{-1}} A_{j_0} A_{j_1} A_{j_2} \cdots .$$

- For $a \in \mathbb{Z}$, we set $[a]_+ := \max(a, 0)$. Note that $a = [a]_+ - [-a]_+$.
- We set $\mathbb{N} := \mathbb{Z}_{\geqslant 1}$ and $\mathbb{N}_0 := \mathbb{Z}_{\geqslant 0}$.
- For a set A, we denote by $\mathfrak{S}_A$ the group of permutations of A. In particular, if $A = \mathbb{N}$ and $k \in \mathbb{N}$, we write σ_k for the simple transposition of k and $k+1$ in $\mathfrak{S}_{\mathbb{N}}$.
- We set q to be an indeterminate and $\mathbb{A} := \mathbb{Z}[q^{\pm 1/2}]$, where $q^{1/2}$ denotes the formal square root of q. For an $\mathbb{A}$-algebra $\mathcal{R}$ and elements X and Y in $\mathcal{R}$, we write $X \cong Y$ if there exists $a \in \mathbb{Z}$ such that $X = q^{a/2} Y$.

2 Quantum cluster algebra structure

2.1 Quantum cluster algebra

Let K be a (possibly infinite) countable index set with a decomposition $\mathsf{K} = \mathsf{K}_{\mathrm{ex}} \sqcup \mathsf{K}_{\mathrm{fr}}$. We call K_{ex} the set of exchangeable indices and K_{fr} the set of frozen indices. Let $\Lambda = (\Lambda_{i,j})_{i,j \in \mathsf{K}}$ be a skew-symmetric $\mathbb{Z}$-valued matrix.

Definition 2.1 We denote by $\mathcal{T}(\Lambda)$ the $\mathbb{A}$-algebra generated by $\{Z_k^{\pm 1}\}_{k \in \mathsf{K}}$ subject to the following defining relations:

$$Z_j Z_j^{-1} = Z_j^{-1} Z_j = 1 \quad \text{and} \quad Z_i Z_j = q^{\Lambda_{i,j}} Z_j Z_i \quad \text{for } i, j \in \mathsf{K}.$$

We define the $\mathbb{Z}$-algebra anti-involution $\overline{(\cdot)}$ of $\mathcal{T}(\Lambda)$, called the *bar-involution*, by

$$\overline{q^{1/2}} := q^{-1/2}, \qquad \overline{Z_k} := Z_k, \tag{2.1}$$

for all $k \in \mathsf{K}$. For $\mathbf{a} = (\mathbf{a}_i)_{i \in \mathsf{K}} \in \mathbb{Z}^{\oplus \mathsf{K}}$, we define the element $Z^{\mathbf{a}}$ of $\mathcal{T}(\Lambda)$ as

$$Z^{\mathbf{a}} := q^{\frac{1}{2} \sum_{i > j} \mathbf{a}_i \mathbf{a}_j \Lambda_{i,j}} \prod_{i \in \mathsf{K}}^{\rightarrow} Z_i^{\mathbf{a}_i},$$

where $>$ denotes a total order on K. Note that $Z^{\mathbf{a}}$ does not depend on the choice of a total order. The set $\{Z^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^{\oplus \mathsf{K}}\}$ forms an $\mathbb{A}$-basis of $\mathcal{T}(\Lambda)$. Since $\mathcal{T}(\Lambda)$ is an Ore domain, it is embedded into the skew field of fraction $\mathbb{F}(\mathcal{T}(\Lambda))$.

Definition 2.2 We call a $\mathbb{Z}$-valued matrix $\widetilde{B} = (b_{i,j})_{i \in \mathsf{K}, j \in \mathsf{K}_{\mathrm{ex}}}$ an *exchange* matrix if it satisfies the following properties:

(a) For each $j \in \mathsf{K}$, there exist finitely many $i \in \mathsf{K}$ such that $b_{i,j} \neq 0$.
(b) Its principal part $B = (b_{i,j})_{i,j \in \mathsf{K}_{\mathrm{ex}}}$ is *skew-symmetrizable* with an $\mathbb{N}_0$-valued diagonal matrix D; i.e., DB is skew-symmetric.

For an exchange matrix $\widetilde{B}$, we associate a *valued quiver* $\Gamma^{\widetilde{B}}$ whose set of vertices is K, and whose arrows between vertices are assigned by the following rules:

$$\begin{cases} \underset{k}{\bullet} \xrightarrow{\ulcorner a,b \lrcorner} \underset{l}{\bullet} & \text{if } l,k \in \mathsf{K}_{\mathrm{ex}}, l \neq k, b_{kl} = a \geqslant 0 \text{ and } b_{lk} = b \leqslant 0, \\ \underset{k}{\circ} \xrightarrow{\ulcorner a,0 \lrcorner} \underset{l}{\bullet} \ (\text{resp. } \underset{k}{\circ} \xleftarrow{\ulcorner 0,b \lrcorner} \underset{l}{\bullet}) & \text{if } l \in \mathsf{K}_{\mathrm{ex}}, k \in \mathsf{K}_{\mathrm{fr}} \text{ and } b_{kl} = a \geqslant 0 \ (\text{resp. } b_{kl} = b \leqslant 0). \end{cases} \tag{2.2}$$

Here we do not draw an arrow between k and l if $b_{kl} = 0$ (and $b_{lk} = 0$ when $l, k \in \mathsf{K}_{\mathrm{ex}}$). Note that $\circ$ denotes a vertex in K_{fr}, and we call $\ulcorner a, b \lrcorner$ the *value* of the arrow.

Convention 1 In this paper, we adopt the following scheme for depiction of a valued quiver:

(i) if $b_{kl} = 1$ and $b_{lk} = -b < 0$, use $k \xrightarrow{<b} l$,
(ii) if $b_{kl} = 2$ and $b_{lk} = -b < 0$, use $k \xRightarrow{<b} l$,
(iii) if $b_{kl} = 3$ and $b_{lk} = -b < 0$, use $k \Rrightarrow_{<b} l$,
(iv) we usually skip <1 in an arrow (when $\ulcorner a, -1 \lrcorner$ and $1 \leqslant a \leqslant 3$) for notational simplicity,
(v) if $b_{kl} = -b_{lk} = \alpha > 1$, use $k \xrightarrow{\alpha} l$.

We say that a pair $(\Lambda, \widetilde{B})$ is *compatible* if

$$\sum_{k \in \mathsf{K}} b_{ki} \Lambda_{kj} = d_i \delta_{i,j} \text{ for any } i \in \mathsf{K}_{\mathrm{ex}},\ j \in \mathsf{K},$$

where $\{d_i\}_{i \in \mathsf{K}_{\mathrm{ex}}}$ are positive integers. In this case, B is skew-symmetrizable with the diagonal matrix $D = \mathrm{diag}(d_i \mid i \in \mathsf{K}_{\mathrm{ex}})$. Also, for $k \in \mathsf{K}_{\mathrm{ex}}$, the *mutation* of $(\Lambda, \widetilde{B})$ *in direction* k is a compatible pair $\mu_k(\Lambda, \widetilde{B}) := (\mu_k(\Lambda), \mu_k(\widetilde{B}))$ defined as follows ([2]):

$$\mu_k(\Lambda) := E^T \Lambda E \quad \text{and} \quad \mu_k(\widetilde{B}) := E \widetilde{B} F,$$

where the matrices $E = (e_{i,j})_{i,j \in \mathsf{K}}$ and $F = (f_{i,j})_{i,j \in \mathsf{K}_{\mathrm{ex}}}$ are given by

$$e_{i,j} := \begin{cases} \delta_{i,j} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ [-b_{ik}]_+ & \text{if } i \neq j = k, \end{cases} \qquad f_{i,j} := \begin{cases} \delta_{i,j} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ [b_{kj}]_+ & \text{if } i = k \neq j. \end{cases}$$

Note that the mutation μ_k is involutive; i.e., $\mu_k(\mu_k(\Lambda, \widetilde{B})) = (\Lambda, \widetilde{B})$. The mutation μ_k on $\widetilde{B}$ can be described as the operation μ_k on its corresponding valued quiver $\Gamma^{\widetilde{B}}$ in the following algorithm:

Algorithm 2.3 *For $k \in \mathsf{K}_{ex}$, the* valued quiver mutation μ_k *transforms* $\widetilde{\Gamma}^{\widetilde{B}}$ *into a new valued quiver* $\mu_k(\widetilde{\Gamma}^{\widetilde{B}}) = \widetilde{\Gamma}^{\mu_k(\widetilde{B})}$ *via the following rules, where we assume (i) $ac > 0$ or $bd > 0$, and (ii) we do not perform ($\mathcal{NC}$) and ($\mathcal{C}$) below, if i and j are frozen at the same time.*

($\mathcal{NC}$) For each full-subquiver $\begin{tikzcd} i \arrow[r, "\ulcorner a{,}b \lrcorner"'] \arrow[rr, bend left, "\ulcorner e{,}f \lrcorner"] & k \arrow[r, "\ulcorner c{,}d \lrcorner"'] & j \end{tikzcd}$ *in* $\widetilde{\Gamma}^{\widetilde{B}}$*, we change the value of the arrow from i to j into* $\ulcorner e + ac, f - bd \lrcorner$ *:*

$$i \xrightarrow[\ulcorner e+ac, f-bd \lrcorner]{} j \ .$$

($\mathcal{C}$) *For each full-subquiver* $\begin{tikzcd} i \arrow[r, "\ulcorner a,b \lrcorner"'] & k \arrow[r, "\ulcorner c,d \lrcorner"'] & j \arrow[ll, bend right, "\ulcorner e,f \lrcorner"'] \end{tikzcd}$ *with* $(e,f) \neq (0,0)$ *in* $\widetilde{\Gamma}^{\widetilde{B}}$, *we change the valued arrow between* i *and* j *as follows:*

$$\begin{cases} i \xleftarrow[\ulcorner e-bd,\, f+ac \lrcorner]{} j & \text{if } f+ac \leqslant 0 \leqslant e-bd, \\ i \xrightarrow[\ulcorner f+ac,\, e-bd \lrcorner]{} j & \text{if } f+ac \geqslant 0 \geqslant e-bd. \end{cases}$$

($\mathcal{R}$) *Reverse the direction of each arrow incident to vertex* k *and change the value* $\ulcorner a, b \lrcorner$ *of each arrow into* $\ulcorner -b, -a \lrcorner$.

Here if there is no arrow between i *and* j *in* ($\mathcal{NC}$) *and* ($\mathcal{C}$), *then put* $e = f = 0$ *and follow the same rule.*

We define an isomorphism of $\mathbb{Q}(q^{1/2})$-algebras $\mu_k^* : \mathbb{F}(\mathcal{T}(\mu_k \Lambda)) \simeq \mathbb{F}(\mathcal{T}(\Lambda))$ by

$$\mu_k^*(Z_j) := \begin{cases} Z^{\mathbf{a}'} + Z^{\mathbf{a}''} & \text{if } j = k, \\ Z_j & \text{if } j \neq k, \end{cases}$$

where

$$\mathbf{a}'_i = -\delta_{i,k} + \delta(i \neq k)[b_{ik}]_+ \quad \text{and} \quad \mathbf{a}''_i = -\delta_{i,k} + \delta(i \neq k)[-b_{ik}]_+ \quad \text{for all } i \in \mathsf{K}.$$

Definition 2.4 For a compatible pair $(\Lambda, \widetilde{B})$, an element z in $\mathbb{F}(\mathcal{T}(\Lambda))$ is called a *quantum cluster variable* (resp. *quantum cluster monomial*) if

$$\mathsf{z} = \mu_{k_l}^* \cdots \mu_{k_2}^* \mu_{k_1}^*(Z_j) \qquad (\text{resp. } \mu_{k_l}^* \cdots \mu_{k_2}^* \mu_{k_1}^*(Z^{\mathbf{a}})),$$

for some finite sequence $(k_1, k_2, \ldots, k_l)$ in K_{ex} and $j \in \mathsf{K}$ (resp. $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^{\oplus \mathsf{K}}$). The *quantum cluster algebra* $\mathscr{A}_q(\Lambda, \widetilde{B})$ is the $\mathbb{A}$-subalgebra of $\mathbb{F}(\mathcal{T}(\Lambda))$ generated by all the quantum cluster variables. We call the triple $\mathcal{S} := (\{Z_k\}_{k \in \mathsf{K}}, \Lambda, \widetilde{B})$ the *initial seed* of $\mathscr{A}_q(\Lambda, \widetilde{B})$.

Theorem 2.5 (The quantum Laurent phenomenon [2]) *The quantum cluster algebra* $\mathscr{A}_q(\Lambda, \widetilde{B})$ *is contained in* $\mathcal{T}(\Lambda)$.

Remark 2.6 For a permutation $\pi \in \mathfrak{S}_{\mathsf{K}}$ of K preserving K_{fr}, we denote by $\pi(\Lambda, \widetilde{B}) = (\pi\Lambda, \pi\widetilde{B})$ where $(\pi\Lambda)_{i,j} := \Lambda_{\pi^{-1}(i)\pi^{-1}(j)}$ for $i, j \in \mathsf{K}$ and $(\pi\widetilde{B})_{i,j} := b_{\pi^{-1}(i)\pi^{-1}(j)}$ for $(i,j) \in \mathsf{K} \times \mathsf{K}_{\mathrm{ex}}$. The operation π on a compatible pair $(\Lambda, \widetilde{B})$ obviously induces the $\mathbb{A}$-algebra isomorphism

$$\pi^* : \mathscr{A}_q(\pi(\Lambda, \widetilde{B})) \simeq \mathscr{A}_q(\Lambda, \widetilde{B}) \quad \text{given by } \pi^*(Z_j) = Z_{\pi^{-1}(j)} \text{ for all } j \in \mathsf{K}$$

and bijections between the sets of quantum cluster monomials.

Let $(\Lambda, \widetilde{B})$ be a compatible pair. An element z in $\mathcal{T}(\Lambda)$ is *pointed* if it is of the following form:

$$\mathsf{z} = q^a Z^{\boldsymbol{g}^R} + \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geqslant 0}^{\oplus \mathsf{K}_{\mathrm{ex}}} \setminus \{0\}} p_{\boldsymbol{n}} Z^{\boldsymbol{g}^R + \widetilde{B}\boldsymbol{n}} \tag{2.3}$$

for some $a \in \frac{1}{2}\mathbb{Z}$, $\boldsymbol{g}^R \in \mathbb{Z}^{\oplus \mathsf{K}}$ and $p_{\boldsymbol{n}} \in \mathbb{A}$. In this case, we call $\boldsymbol{g}^R$ the *degree* of the pointed element z and denote it by $\mathbf{deg}(\mathsf{z})$. Note that the notion of degree *does depend* on $\widetilde{B}$.

Theorem 2.7 [5, 7, 17, 54]

(a) *Every quantum cluster monomial* z *in* $\mathscr{A}_q(\Lambda, \widetilde{B})$ *is pointed.*
(b) *If two quantum cluster monomials* $\mathsf{z}, \mathsf{z}' \in \mathscr{A}_q(\Lambda, \widetilde{B})$ *have the same degree, then* $\mathsf{z} = \mathsf{z}'$.

Let $\mathsf{z} \in \mathscr{A}_q(\Lambda, \widetilde{B})$ and $\mathsf{z}' \in \mathscr{A}_q(\mu_k(\Lambda, \widetilde{B}))$ be pointed elements such that $\mathbf{deg}(\mathsf{z}) = (\boldsymbol{g}_i) \in \mathbb{Z}^{\oplus \mathsf{K}}$ and $\mathbf{deg}(\mathsf{z}') = (\boldsymbol{g}'_i) \in \mathbb{Z}^{\oplus \mathsf{K}}$. If $\mu_k^*(\mathsf{z}') = \mathsf{z}$, we have ([5, 7, 17])

$$\boldsymbol{g}_j = \begin{cases} -\boldsymbol{g}'_k & \text{if } j = k, \\ \boldsymbol{g}'_j + [b'_{jk}]_+ \boldsymbol{g}'_k & \text{if } j \neq k \text{ and } \boldsymbol{g}'_k \geqslant 0, \\ \boldsymbol{g}'_j + [-b'_{jk}]_+ \boldsymbol{g}'_k & \text{if } j \neq k \text{ and } \boldsymbol{g}'_k \leqslant 0. \end{cases} \tag{2.4}$$

3 Behaviours of degrees with respect to braid moves and forward shifts

In this section, we study the behaviours of degrees with respect to several moves among sequences of simple reflections. For 2-moves, 3-moves and forward shifts, they are investigated in [11]. We mainly focus on the 4-moves and 6-moves, which have not been investigated before as far as the authors know, and quite complicated. We mainly follow the notations and frameworks in [11].

3.1 Setup

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. We denote by I the index set of simple (co)roots $\Pi = \{\alpha_i\}_{i \in I}$ (resp. $\Pi^\vee = \{h_i\}_{i \in I}$), by $\triangle$ the Dynkin diagram of $\mathfrak{g}$, by h the Coxeter number of $\mathfrak{g}$, by $\mathsf{C} = (\mathsf{c}_{i,j})_{i,j \in I}$ the Cartan matrix of $\mathfrak{g}$, by $\mathsf{D} = \mathrm{diag}(\mathsf{d}_i \in \mathbb{Z}_{\geqslant 1} \mid i \in I)$ the minimal (left) symmetrizer of C, by $\Phi^\pm$ the set of positive (resp. negative) roots of $\mathfrak{g}$, and by P the weight lattice of $\mathfrak{g}$. We write $i \sim j$ for $i, j \in I$ if $\mathsf{c}_{i,j} < 0$, and let ϖ_i be the fundamental weight for each $i \in I$. We set $\mathsf{Q} := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\mathsf{Q}^\pm := \pm \sum_{i \in I} \mathbb{N}_0 \alpha_i$. For $\beta = \sum_{i \in I} n_i \alpha_i \in \mathsf{Q}^+$, we set $|\beta| := \sum_{i \in I} n_i$. We denote by $d(i, j)$ the number of edges between i and j in $\triangle$ (see [26] for examples of $d(i, j)$).

We take the symmetric bilinear form $(-,-)$ on P such that

$$(\alpha_i,\alpha_j)=\mathsf{d}_i\mathsf{c}_{i,j}=\mathsf{d}_j\mathsf{c}_{j,i}\quad\text{and}\quad\langle h_i,\alpha_j\rangle=\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}=\mathsf{c}_{i,j}.$$

Let W be the Weyl group of $\mathfrak{g}$ generated by simple reflections $\{s_i\}_{i\in I}$ such that $s_i(\lambda)=\lambda-\langle h_i,\lambda\rangle\alpha_i$ for $\lambda\in\mathsf{P}$. For a finite sequence $\boldsymbol{i}=(i_1,\ldots,i_r)$ in the Weyl group W, we call $\boldsymbol{i}$ *reduced*, if $s_{i_1}\ldots s_{i_r}$ is a reduced expression of some w in W. Note that there exists a unique longest element $w_\circ$ in W and $w_\circ$ induces the automorphism $*$ on I by $w_\circ(\alpha_i)=-\alpha_{i^*}$.

We say that two sequences $\boldsymbol{i}=(i_1,\ldots,i_r)$ and $\boldsymbol{i}'=(i'_1,\ldots,i'_r)$ of indices are *commutation equivalent*, denoted by $\boldsymbol{i}\overset{c}{\sim}\boldsymbol{i}'$, if $\boldsymbol{i}'$ can be obtained from $\boldsymbol{i}$ by applying 2-moves (or commutation moves) $(i_k,i_{k+1})\to(i_{k+1}=i'_k,i_k=i'_{k+1})$ for $1\leqslant k<r$ and $d(i_k,i_{k+1})>1$. We denote by $[\boldsymbol{i}]$ the equivalence class of $\boldsymbol{i}$ under $\overset{c}{\sim}$.

Let $\boldsymbol{i}=(i_1,\ldots,i_\ell)$ be a reduced sequence of $w_\circ$. Then it is well-known that

$$\Phi^+=\{\beta^{\boldsymbol{i}}_k:=s_{i_1}\ldots s_{i_{k-1}}(\alpha_{i_k})\mid 1\leqslant k\leqslant\ell\}\quad\text{and}\quad|\Phi^+|=\ell.\tag{3.1}$$

Example 3.1 For $\mathfrak{g}$ of type B_2, there are only two reduced sequences $\boldsymbol{i}$ and $\boldsymbol{i}'$ of $w_\circ$ given by $\boldsymbol{i}=(1,2,1,2)$ and $\boldsymbol{i}'=(2,1,2,1)$, respectively. Then

$$\begin{aligned}\Phi^+&=\{\beta^{\boldsymbol{i}}_1=\alpha_1,\beta^{\boldsymbol{i}}_2=\alpha_1+\alpha_2,\beta^{\boldsymbol{i}}_3=\alpha_1+2\alpha_2,\beta^{\boldsymbol{i}}_4=\alpha_2\},\\&=\{\beta^{\boldsymbol{i}'}_1=\alpha_2,\beta^{\boldsymbol{i}'}_2=\alpha_1+2\alpha_2,\beta^{\boldsymbol{i}'}_3=\alpha_1+\alpha_2,\beta^{\boldsymbol{i}'}_4=\alpha_1\}.\end{aligned}$$

3.2 Sequences of indices

Let $\boldsymbol{i}=(i_u)_{u\in\mathbb{N}}\in I^{\mathbb{N}}$ be an arbitrary sequence satisfying

$$|\{u\in\mathbb{N}\mid i_u=i\}|=\infty\quad\text{for any } i\in I.\tag{3.2}$$

We denote by $I^{(\infty)}$ the set of sequences in $I^{\mathbb{N}}$ satisfying (3.2).

For $\boldsymbol{i}\in I^r$ $(r\in\mathbb{N}\sqcup\{\infty\})$, $u\in[1,r]$ and $j\in I$, we set

$$\begin{aligned}u^+_{\boldsymbol{i}}(j)&:=\min(\{k\in\mathbb{N}\mid k>u,\ i_k=j\}\sqcup\{r+1\}), & u^+_{\boldsymbol{i}}&:=u^+_{\boldsymbol{i}}(i_u),\\u^-_{\boldsymbol{i}}(j)&:=\max(\{k\in\mathbb{N}\mid k<u,\ i_k=j\}\sqcup\{0\}), & u^-_{\boldsymbol{i}}&:=u^-_{\boldsymbol{i}}(i_u),\\u^{\min}_{\boldsymbol{i}}(j)&:=\min(k\in\mathbb{N}\mid\ i_k=j), & u^{\min}_{\boldsymbol{i}}&:=u^{\min}_{\boldsymbol{i}}(i_u).\end{aligned}$$

We drop $_{\boldsymbol{i}}$ if there is no danger of confusion.

For $\boldsymbol{i}\in I^{(\infty)}$, we define an $\mathbb{N}\times\mathbb{N}$ exchange matrix $\widetilde{B}^{\boldsymbol{i}}=(b^{\boldsymbol{i}}_{u,v})_{u,v\in\mathbb{N}}$ as follows:

$$b^{\boldsymbol{i}}_{u,v}:=\begin{cases}\pm1 & \text{if } v=u^{\pm},\\ \pm\mathsf{c}_{i_u,i_v} & \text{if } u<v<u^+<v^+ \text{ (resp. } v<u<v^+<u^+\text{)},\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\widetilde{B}^{\boldsymbol{i}}$ is skew-symmetrizable with $\mathrm{diag}(\mathsf{d}_{i_u} \mid u \in \mathbb{N})$ and satisfies the conditions in Definition 2.2.

Let $\Lambda^{\boldsymbol{i}} = (\Lambda^{\boldsymbol{i}}_{u,v})$ be a skew-symmetric matrix given by

$$\Lambda^{\boldsymbol{i}}_{u,v} = -\Lambda^{\boldsymbol{i}}_{v,u} := (\varpi_{i_u} - w^{\boldsymbol{i}}_u \varpi_{i_u}, \varpi_{i_v} + w^{\boldsymbol{i}}_v \varpi_{i_v}) \quad \text{for } u \leqslant v,$$

where

$$w^{\boldsymbol{i}}_k := s_{i_1} \dots s_{i_k} \in \mathsf{W} \quad \text{for } k \in \mathbb{N}.$$

For $s \in \mathbb{N}$, we set

$$\mathsf{K} := [1, s], \qquad \mathsf{K}_{\mathrm{fr}} := \{k \in \mathsf{K} \mid k^+ > s\}, \qquad \mathsf{K}_{\mathrm{ex}} := \mathsf{K} \setminus \mathsf{K}_{\mathrm{fr}}.$$

Lemma 3.2 [11, Lemma 1.1] *For $u, v \in \mathbb{N}$ with $u < v^+$, we have $\Lambda^{\boldsymbol{i}}_{u,v} = (\varpi_{i_u} - w^{\boldsymbol{i}}_u \varpi_{i_u}, \varpi_{i_v} + w^{\boldsymbol{i}}_v \varpi_{i_v})$.*

Proposition 3.3 [11, Proposition 1.2] *Let $\boldsymbol{i} \in I^{(\infty)}$.*

(a) *The pair $(\Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}})$ is* compatible with $\mathrm{diag}(\mathsf{d}_{i_u})_{u\in\mathbb{N}}$*; i.e., for any $u, v \in \mathbb{N}$, we have*

$$\sum_{k\in\mathbb{N}} b^{\boldsymbol{i}}_{k,u} \Lambda^{\boldsymbol{i}}_{k,v} = 2\mathsf{d}_{i_u} \delta_{u,v}.$$

(b) *For each $s \in \mathbb{N}$, we set*

$$\Lambda^{\boldsymbol{i},s} := (\Lambda^{\boldsymbol{i}}_{u,v})_{u,v\in\mathsf{K}} \quad \textit{and} \quad \widetilde{B}^{\boldsymbol{i},s} := (b^{\boldsymbol{i}}_{u,v})_{u\in\mathsf{K}, v\in\mathsf{K}_{ex}}.$$

Then the pair $(\Lambda^{\boldsymbol{i},s}, \widetilde{B}^{\boldsymbol{i},s})$ is compatible with diag$(\mathsf{d}_{i_k})_{k\in\mathsf{K}_{ex}}$.

We define $\widetilde{\Gamma}^{\boldsymbol{i}} = (\widetilde{\Gamma}^{\boldsymbol{i}}_0, \widetilde{\Gamma}^{\boldsymbol{i}}_1)$ as the valued quiver associated with $\widetilde{B}^{\boldsymbol{i}}$, where $\widetilde{\Gamma}^{\boldsymbol{i}}_0$ denotes the set of vertices and $\widetilde{\Gamma}^{\boldsymbol{i}}_1$ denotes the set of arrows. For a subset $\mathcal{K}$ of $\widetilde{\Gamma}^{\boldsymbol{i}}_0$, we denote by $\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}}$ the full subquiver of $\widetilde{\Gamma}^{\boldsymbol{i}}$ whose set of vertices is $\mathcal{K}$.

We denote by $\mathcal{A}_{\boldsymbol{i}}$ the quantum cluster algebra associated with the compatible pair $(\Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}})$, which is an $\mathbb{A}$-subalgebra of the quantum torus $\mathcal{T}(\Lambda^{\boldsymbol{i}})$. For each $s \in \mathbb{N}$, we also have the quantum cluster algebra $\mathcal{A}^s_{\boldsymbol{i}} = \mathscr{A}_q(\Lambda^{\boldsymbol{i},s}, \widetilde{B}^{\boldsymbol{i},s})$ in the quantum torus $\mathcal{T}(\Lambda^{\boldsymbol{i},s})$. Then we have

$$\begin{tikzcd} \mathcal{A}^1_{\boldsymbol{i}} \arrow[r, hook] \arrow[d, hook] & \mathcal{A}^2_{\boldsymbol{i}} \arrow[r, hook] \arrow[d, hook] & \cdots \arrow[r, hook] & \mathcal{A}_{\boldsymbol{i}} \arrow[d, hook] \\ \mathcal{T}(\Lambda^{\boldsymbol{i},1}) \arrow[r, hook] & \mathcal{T}(\Lambda^{\boldsymbol{i},2}) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathcal{T}(\Lambda^{\boldsymbol{i}}) \end{tikzcd} \quad \text{and} \quad \bigcup_{s\in\mathbb{N}} \mathcal{A}^s_{\boldsymbol{i}} = \mathcal{A}_{\boldsymbol{i}}.$$

For $\boldsymbol{i} \in I^{\mathbb{N}}$, let $C_{\boldsymbol{i}} \subset \mathbb{Z}^{\oplus\mathbb{N}}$ denote the cone defined by

$$C_{\boldsymbol{i}} := \left\{ \boldsymbol{g} = (g_u)_{u\in\mathbb{N}} \in \mathbb{Z}^{\oplus\mathbb{N}} \;\middle|\; \sum_{v\geqslant u, i_v = i_u} g_v \geqslant 0, \forall u \in \mathbb{N} \right\} = \sum_{u\in\mathbb{N}} \mathbb{N}_0(\boldsymbol{e}_u - \boldsymbol{e}_{u^-_{\boldsymbol{i}}}), \tag{3.3}$$

where $\{\boldsymbol{e}_u\}_{u\in\mathbb{N}}$ is the natural basis of $\mathbb{Z}^{\oplus\mathbb{N}}$ and we understand $\boldsymbol{e}_0 = 0$.

For $\boldsymbol{g} = (g_u)_{u\in\mathbb{N}} \in \mathbb{Z}^{\oplus\mathbb{N}}$ and $\boldsymbol{i} \in I^{\mathbb{N}}$, write

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) := \sum_{u\in\mathbb{N}, i_u=i} g_u. \tag{3.4}$$

3.3 2-moves

In this and next subsections, we recall the results on 2-moves and 3-moves investigated in [11].

For sequences $\boldsymbol{i} = (i_u)_{u\in\mathbb{N}}, \boldsymbol{i}' = (i'_u)_{u\in\mathbb{N}} \in I^{(\infty)}$, we say that $\boldsymbol{i}'$ can be obtained from $\boldsymbol{i}$ via a 2-*move* if there exists a unique $k \in \mathbb{N}$ such that

$$i_k = i'_{k+1}, \quad i_{k+1} = i'_k, \quad i_u = i'_u \ \ (u \notin \{k, k+1\})$$

and

$$\mathsf{c}_{i_{k+1}, i_k} = \mathsf{c}_{i_k, i_{k+1}} = 0.$$

In this case, we write $\boldsymbol{i}' = \gamma_k \boldsymbol{i}$.

Proposition 3.4 [11, §2.1] *Let $\boldsymbol{i}, \boldsymbol{i}' \in I^{\mathbb{N}}$ be related by $\boldsymbol{i}' = \gamma_k \boldsymbol{i}$ for some $k \in \mathbb{N}$.*

(i) $\widetilde{B}^{\boldsymbol{i}'} = \sigma_k \widetilde{B}^{\boldsymbol{i}}$ *and* $\Lambda^{\boldsymbol{i}'} = \sigma_k \Lambda^{\boldsymbol{i}}$.

(ii) *We have an $\mathbb{A}$-algebra isomorphism*

$$\gamma_k^* (= \sigma_k^*) : \mathcal{A}_{\boldsymbol{i}'} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}} \quad \text{given by } Z_u \mapsto Z_{\sigma_k(u)} \text{ for all } u \in \mathbb{N},$$

which induces a bijection between the sets of quantum cluster monomials.

(iii) *If $\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}'}$ is a quantum cluster monomial with $\mathbf{deg}(\mathsf{z}) = \boldsymbol{g}' = (g'_u)_{u\in\mathbb{N}}$, then the quantum cluster monomial $\gamma_k^* \mathsf{z} \in \mathcal{A}_{\boldsymbol{i}}$ has $\mathbf{deg}(\gamma_k^* \mathsf{z}) = \boldsymbol{g} = (g_u)_{u\in\mathbb{N}}$ as follows:*

$$g_u = \begin{cases} g'_{\sigma_k(u)} & \text{if } u \in \{k, k+1\}, \\ g'_u & \text{otherwise}. \end{cases} \tag{3.5}$$

(iv) *The map $\boldsymbol{g}' \mapsto \boldsymbol{g}$ given by (3.5) sends the cone $C_{\boldsymbol{i}'} \subset \mathbb{Z}^{\oplus\mathbb{N}}$ into the cone $C_{\boldsymbol{i}} \subset \mathbb{Z}^{\oplus\mathbb{N}}$.*

(v) *The elements $\boldsymbol{g}, \boldsymbol{g}' \in \mathbb{Z}^{\oplus\mathbb{N}}$ related by (3.5) satisfy*

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) = \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) \quad \text{for all } i \in I.$$

3.4 3-moves

For sequences $\boldsymbol{i} = (i_u)_{u\in\mathbb{N}}, \boldsymbol{i}' = (i'_u)_{u\in\mathbb{N}} \in I^{\infty}$, we say that $\boldsymbol{i}'$ can be obtained from $\boldsymbol{i}$ via a 3-*move*, if there exists a unique $k \in \mathbb{N}$ such that

$$i_k = i_{k+2} = i'_{k+1}, \quad i_{k+1} = i'_k = i'_{k+2}, \quad i_u = i'_u \ \ (u \notin [k, k+2])$$

and

$$\mathsf{c}_{i_{k+1},i_k}\mathsf{c}_{i_k,i_{k+1}} = 1.$$

In this case, we write $\boldsymbol{i}' = \beta_k \boldsymbol{i}$.

Proposition 3.5 [11, §2.2] *Let $\boldsymbol{i}, \boldsymbol{i}' \in I^{(\infty)}$ be related by $\boldsymbol{i}' = \beta_k \boldsymbol{i}$ for some $k \in \mathbb{N}$.*

(i) $\widetilde{B}^{\boldsymbol{i}'} = \sigma_{k+1}\mu_k \widetilde{B}^{\boldsymbol{i}}$ *and* $\Lambda^{\boldsymbol{i}'} = \sigma_{k+1}\mu_k \Lambda^{\boldsymbol{i}}$.

(ii) *We have an $\mathbb{A}$-algebra isomorphism*

$$\beta_k^*(= \mu_k^*\sigma_{k+1}^*) : \mathcal{A}_{\boldsymbol{i}'} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}} \quad \text{given by } Z_u \mapsto \mu_k(Z_{\sigma_{k+1}(u)}) \text{ for all } u \in \mathbb{N},$$

which induces a bijection between the sets of quantum cluster monomials.

(iii) *If $\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}'}$ is a quantum cluster monomial with $\boldsymbol{deg}(\mathsf{z}) = \boldsymbol{g}' = (g'_u)_{u\in\mathbb{N}}$, then the quantum cluster monomial $\beta_k^*\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}}$ has $\boldsymbol{deg}(\beta_k^*\mathsf{z}) = \boldsymbol{g} = (g_u)_{u\in\mathbb{N}}$ as follows:*

$$g_u = \begin{cases} -g'_k & \text{if } u = k, \\ g'_{\sigma_{k+1}(u)} + [g'_k]_+ & \text{if } u \in \{k+2, (k+1)^-_{\boldsymbol{i}}\}, \\ g'_{\sigma_{k+1}(u)} - [-g'_k]_+ & \text{if } u \in \{k+1, k^-_{\boldsymbol{i}}\}, \\ g'_u & \text{otherwise}. \end{cases} \tag{3.6}$$

(iv) *The map $\boldsymbol{g}' \mapsto \boldsymbol{g}$ given by (3.6) sends the cone $C_{\boldsymbol{i}'} \subset \mathbb{Z}^{\oplus\mathbb{N}}$ into the cone $C_{\boldsymbol{i}} \subset \mathbb{Z}^{\oplus\mathbb{N}}$.*

(v) *The elements $\boldsymbol{g}, \boldsymbol{g}' \in \mathbb{Z}^{\oplus\mathbb{N}}$ related by (3.6) satisfy*

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) = \begin{cases} \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) + [-g'_k]_+ & \text{if } i = i_k \text{ and } k^-_{\boldsymbol{i}} = 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - [g'_k]_+ & \text{if } i = i_{k+1} \text{ and } (k+1)^-_{\boldsymbol{i}} = 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) & \text{otherwise}. \end{cases}$$

3.5 4-moves

For sequences $\boldsymbol{i} = (i_u)_{u\in\mathbb{N}}, \boldsymbol{i}' = (i'_u)_{u\in\mathbb{N}} \in I^{(\infty)}$, we say that $\boldsymbol{i}'$ can be obtained from $\boldsymbol{i}$ via a 4-*move* if there exists a unique $k \in \mathbb{N}$ such that

$$i_k = i_{k+2} = i'_{k+1} = i'_{k+3}, \quad i_{k+1} = i_{k+3} = i'_k = i'_{k+2}, \quad i_u = i'_u \quad (u \notin [k, k+3])$$

and

$$\mathsf{c}_{i_{k+1},i_k}\mathsf{c}_{i_k,i_{k+1}} = 2.$$

In this case, we write $\boldsymbol{i}' = \eta_k \boldsymbol{i}$. Throughout this subsection, we assume that $\boldsymbol{i}, \boldsymbol{i}' \in I^{\mathbb{N}}$ are related by $\boldsymbol{i}' = \eta_k \boldsymbol{i}$ for some $k \in \mathbb{N}$, and sometimes we write $i := i_k$ and $j := i_{k+1}$.

Lemma 3.6 *We have*

$$\widetilde{B}^{\boldsymbol{i}'} = \sigma_{k+2}\sigma_k\mu_k\mu_{k+1}\mu_k \widetilde{B}^{\boldsymbol{i}}. \tag{3.7}$$

Proof By condition (a) in Definition 2.2, it is enough to consider the local behaviour under the operation described in (3.7). Note that

(1) the set $\mathcal{K} = \{k, k+1, k+2, k+3, k_{\boldsymbol{i}}^- = (k+1)_{\boldsymbol{i}'}^-, (k+1)_{\boldsymbol{i}}^- = k_{\boldsymbol{i}'}^-\}$ consists of vertices adjacent to $\{k, k+1\}$ in $\widetilde{\Gamma}^{\boldsymbol{i}}$ (resp. $\widetilde{\Gamma}^{\boldsymbol{i}'}$) and themselves,
(2) the sequence of mutations $\mu_k\mu_{k+1}\mu_k$ on $\widetilde{\Gamma}^{\boldsymbol{i}}$ only affects the full subquiver $\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}}$ (resp. $\widetilde{\Gamma}^{\boldsymbol{i}'}|_{\mathcal{K}}$).

Thus it is enough to check that

$$\mu_k\mu_{k+1}\mu_k\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}} = \sigma_k\sigma_{k+2}\widetilde{\Gamma}^{\boldsymbol{i}'}|_{\mathcal{K}}.$$

In this proof, we shall only consider the case when both $k_{\boldsymbol{i}}^-$ and $(k+1)_{\boldsymbol{i}}^-$ exist, since the proofs for the other cases are similar. In that case, the valued quiver $\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}}$ can be depicted as follows:

$$\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}} = \begin{tikzcd} & k+2 \arrow[dr, "{\ulcorner b,-a \lrcorner}"] & & k \arrow[ll] \arrow[dr, "{\ulcorner b,-a \lrcorner}"] & & k_{\boldsymbol{i}}^- \arrow[ll] \\ k+3 \arrow[ur, dotted, "{\ulcorner a,-b \lrcorner}"] & & k+1 \arrow[ll] \arrow[ur, "{\ulcorner a,-b \lrcorner}"] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll] \arrow[ur, dotted, "{\ulcorner a,-b \lrcorner}"] & \end{tikzcd} \tag{3.8}$$

where $a = -\mathsf{c}_{i_{k+1}, i_k}$ and $b = -\mathsf{c}_{i_k, i_{k+1}}$. Here

- the dotted arrow between $k_{\boldsymbol{i}}^-$ and $(k+1)_{\boldsymbol{i}}^-$ exists if and only if $k_{\boldsymbol{i}}^- < (k+1)_{\boldsymbol{i}}^-$,
- the dotted arrow between $k+3$ and $k+2$ exists if and only if $(k+2)_{\boldsymbol{i}}^+ < (k+3)_{\boldsymbol{i}}^+$.

Let us consider the case when $i_k = n-1$, $i_{k+1} = n$ and $\mathfrak{g} = B_n$. Then (3.8) can be drawn as follows with Convention 1:

$$\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}} = \begin{tikzcd} & k+2 \arrow[dr, "\ulcorner 2"] & & k \arrow[ll] \arrow[dr, "\ulcorner 2"] & & k_{\boldsymbol{i}}^- \arrow[ll] \\ k+3 \arrow[ur, Rightarrow, dotted] & & k+1 \arrow[ll] \arrow[ur, Rightarrow] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll] \arrow[ur, Rightarrow, dotted] & \end{tikzcd}$$

Then, by following Algorithm 2.3, one can see that $\mu_k\mu_{k+1}\mu_k\Gamma^{\boldsymbol{i}}|_{\mathcal{K}}$ can be depicted as follows:

$$\mu_k\mu_{k+1}\mu_k\Gamma^{\boldsymbol{i}}|_{\mathcal{K}} = \begin{tikzcd} & k+2 \arrow[dl, dotted, "2\urcorner"'] & & k \arrow[ll] \arrow[dl, "2\urcorner"'] & & k_{\boldsymbol{i}}^- \arrow[ll] \arrow[dl, dotted, "2\urcorner"'] \\ k+3 \arrow[urrr, Rightarrow] & & k+1 \arrow[ll] \arrow[urrr, Rightarrow] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll] & \end{tikzcd}$$

Here the dotted arrow between $k+3$ and $k+2$ exists if and only if $(k+2)_{\boldsymbol{i}}^+ > (k+3)_{\boldsymbol{i}}^+$.

On the other hand, the valued quiver $\Gamma^{\boldsymbol{i}'}|_{\mathcal{K}}$ is depicted as:

$$\Gamma^{\boldsymbol{i}'}|_{\mathcal{K}} = \begin{tikzcd} k+3 \arrow[dr, dotted, "\ulcorner 2"] & & k+1 \arrow[ll] \arrow[dr, "\ulcorner 2"] & & (k+1)_{\boldsymbol{i}'}^- \arrow[ll] \arrow[dr, dotted, "\ulcorner 2"] & \\ & k+2 \arrow[ur, Rightarrow] & & k \arrow[ll] \arrow[ur, Rightarrow] & & k_{\boldsymbol{i}'}^- \arrow[ll] \end{tikzcd}$$

Here

- the dotted arrow between $(k+1)^-_{i'}$ and $k^-_{i'}$ exists if and only if

$$(k+1)^-_{i'} = k^-_{i} > (k+1)^-_{i} = k^-_{i'},$$

- the dotted arrow between $k+3$ and $k+2$ exists if and only if

$$(k+3)^+_{i} = (k+2)^+_{i'} < (k+3)^+_{i'} = (k+2)^+_{i}.$$

Thus our assertion follows. For C_n and F_4 cases, one can prove in a similar way. □

Lemma 3.7 *We have*

$$\Lambda^{i'} = \sigma_{k+2}\sigma_k\mu_k\mu_{k+1}\mu_k\Lambda^{i}.$$

Proof Let us denote by

$$\Lambda := \Lambda^{i} \quad \text{and} \quad \Lambda''' := \mu_k\mu_{k+1}\mu_k\Lambda^{i} \qquad \text{for notational simplicity.}$$

Note that

$$\Lambda'''_{u,v} = \Lambda_{u,v} \quad \text{if } u, v \notin \{k, k+1\}.$$

(A) For the cases when $|\{u \neq v\} \cap \{k, k+1\}| = 1$, we have

$$\begin{aligned}
\Lambda'''_{k+1,v} &= \Lambda''_{k+1,v} = -\Lambda_{k+1,v} + \Lambda_{k+3,v} + \Lambda_{(k+1)^-_i,v} \quad (\text{for } v > k+1),\\
\Lambda'''_{u,k+1} &= \Lambda''_{u,k+1} = -\Lambda_{u,k+1} + \Lambda_{u,k+3} + \Lambda_{u,(k+1)^-_i} \quad (\text{for } u < k),\\
\Lambda'''_{k,v} &= \Lambda_{k,v} + \mathsf{c}_{i_{(k+1)^-_i},i_k}\Lambda_{(k+1)^-_i,v} - \Lambda_{k+2,v} - \mathsf{c}_{i_{k+3},i_k}\Lambda_{k+3,v} + \Lambda_{k^-_i,v} \quad (\text{for } v > k+1),\\
\Lambda'''_{u,k} &= \Lambda_{u,k} + \mathsf{c}_{i_{(k+1)^-_i},i_k}\Lambda_{u,(k+1)^-_i} - \Lambda_{u,k+2} - \mathsf{c}_{i_{k+3},i_k}\Lambda_{u,k+3} + \Lambda_{u,k^-_i} \quad (\text{for } u < k),
\end{aligned}$$

by direct calculations.
(B) For the cases when $\{u, v\} = \{k, k+1\}$, we have

$$\begin{aligned}
\Lambda'''_{k,k+1} = &-\Lambda_{k,k+1} - \mathsf{c}_{i_{(k+1)^-_i},i_k}\Lambda_{(k+1)^-_i,k+1} - \Lambda_{k+1,k+2}\\
&+ \Lambda_{k,k+3} + \mathsf{c}_{i_{(k+1)^-_i},i_k}\Lambda_{(k+1)^-_i,k+3}\\
&- \Lambda_{k+2,k+3} - \Lambda_{(k+1)^-_i,k} + \Lambda_{(k+1)^-_i,k+2} - \mathsf{c}_{i_{k+3},i_k}\Lambda_{k+1,k+3}\\
&+ \mathsf{c}_{i_{k+3},i_k}\Lambda_{(k+1)^-_i,k+3} - \Lambda_{k^-_i,k+1} + \Lambda_{k^-_i,k+3} + \Lambda_{k^-_i,(k+1)^-_i},\\
\Lambda'''_{k+1,k} = &-\Lambda'''_{k,k+1}.
\end{aligned}$$

Recall that $(i_k, i_{k+1}, i_{k+2}, i_{k+3}) = (i, j, i, j) \overset{\eta_k}{\rightsquigarrow} (i'_k, i'_{k+1}, i'_{k+2}, i'_{k+3}) = (j, i, j, i)$. Thus it suffices to prove that the following equation holds:

$$\Lambda'''_{\sigma_{k+2}\sigma_k(u),\sigma_{k+2}\sigma_k(v)} = \Lambda^{i'}_{u,v}.$$

Since only $\{k, k+1, k+2, k+3\}$ are changed by $\sigma_{k+2}\sigma_k$, it is enough to show that the following 15 equations hold:

(i) $\Lambda^{i'}_{k,v} = \Lambda'''_{k+1,v} = -\Lambda_{k+1,v} + \Lambda_{k+3,v} + \Lambda_{(k+1)^-_i,v}$ for $v > k+3$,
(ii) $\Lambda^{i'}_{k+1,v} = \Lambda'''_{k,v} = \Lambda_{k,v} + \mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,v} - \Lambda_{k+2,v} - \mathsf{c}_{j,i}\Lambda_{k+3,v} + \Lambda_{k^-_i,v}$ for $v > k+3$,
(iii) $\Lambda^{i'}_{k+2,v} = \Lambda'''_{k+3,v} = \Lambda_{k+3,v}$ for $v > k+3$,
(iv) $\Lambda^{i'}_{k+3,v} = \Lambda'''_{k+2,v} = \Lambda_{k+2,v}$ for $v > k+3$,
(v) $\Lambda^{i'}_{u,k} = \Lambda'''_{u,k+1} = -\Lambda_{u,k+1} + \Lambda_{u,k+3} + \Lambda_{u,(k+1)^-_i}$ for $u < k$,
(vi) $\Lambda^{i'}_{u,k+1} = \Lambda'''_{u,k} = \Lambda_{u,k} + \mathsf{c}_{j,i}\Lambda_{u,(k+1)^-_i} - \Lambda_{u,k+2} - \mathsf{c}_{j,i}\Lambda_{u,k+3} + \Lambda_{u,k^-_i}$ for $u < k$,
(vii) $\Lambda^{i'}_{u,k+2} = \Lambda'''_{u,k+3} = \Lambda_{u,k+3}$ for $u < k$,
(viii) $\Lambda^{i'}_{u,k+3} = \Lambda'''_{u,k+2} = \Lambda_{u,k+2}$ for $u < k$,
(ix) $\Lambda^{i'}_{k,k+1} = \Lambda'''_{k+1,k}$,
(x) $\Lambda^{i'}_{k+2,k+3} = \Lambda'''_{k+3,k+2} = \Lambda_{k+3,k+2}$,
(xi) $\Lambda^{i'}_{k,k+3} = \Lambda'''_{k+1,k+2} = -\Lambda_{k+1,k+2} + \Lambda_{k+3,k+2} + \Lambda_{(k+1)^-_i,k+2}$,
(xii) $\Lambda^{i'}_{k,k+2} = \Lambda'''_{k+1,k+3} = -\Lambda_{k+1,k+3} + \Lambda_{(k+1)^-_i,k+3}$,
(xiii) $\Lambda^{i'}_{k+1,k+2} = \Lambda'''_{k,k+3} = \Lambda_{k,k+3} + \mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+3} - \Lambda_{k+2,k+3} + \Lambda_{k^-_i,k+3}$,
(xiv) $\Lambda^{i'}_{k+1,k+3} = \Lambda'''_{k,k+2} = \Lambda_{k,k+2} + \mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+2} - \mathsf{c}_{j,i}\Lambda_{k+3,k+2} + \Lambda_{k^-_i,k+2}$,
(xv) $\Lambda^{i'}_{u,v} = \Lambda'''_{u,v}$ for $u, v \notin \{k, k+1, k+2, k+3\}$.

(iii), (iv), (vii), (viii) and (x)-cases: Let us recall the following facts:

(a) $w^{\boldsymbol{i}}_{k+3}\varpi_j = w^{\boldsymbol{i}'}_{k+3}\varpi_j = w^{\boldsymbol{i}'}_{k+2}\varpi_j$,
(b) $w^{\boldsymbol{i}'}_{k+3}\varpi_i = w^{\boldsymbol{i}}_{k+3}\varpi_i = w^{\boldsymbol{i}}_{k+2}\varpi_i$.

The assertion for (iii), (iv), (vii) and (viii)-cases follows directly. Let us check (x)-cases. By Lemma 3.2, we have

$$\begin{aligned}\Lambda_{k+3,k+2} &= (\varpi_j - w^{\boldsymbol{i}}_{k+3}\varpi_j, \varpi_i + w^{\boldsymbol{i}}_{k+2}\varpi_i)\\ &= (\varpi_j - w^{\boldsymbol{i}'}_{k+2}\varpi_j, \varpi_i + w^{\boldsymbol{i}'}_{k+3}\varpi_i) = \Lambda^{\boldsymbol{i}'}_{k+3,k+2},\end{aligned}$$

which completes our assertions for these cases.

(i), (v), (xi) and (xii)-cases: Since the proofs for these cases are all similar, we shall give a proof for (i)-case only. Note that

$$\Lambda^{\boldsymbol{i}'}_{k,v} = (\varpi_j - w^{\boldsymbol{i}}_{k-1}s_j\varpi_j, \varpi_{i_v} + w^{\boldsymbol{i}}_v\varpi_{i_v}) = (\varpi_j - w^{\boldsymbol{i}}_{k-1}(\varpi_j - \alpha_j), \varpi_{i_v} + w^{\boldsymbol{i}}_v\varpi_{i_v}).$$

On the other hand

$$\begin{aligned}-\Lambda_{k+1,v} &= -(\varpi_j - w^{\boldsymbol{i}}_{k-1}s_is_j\varpi_j, \varpi_{i_v} + w^{\boldsymbol{i}}_v\varpi_{i_v})\\ &= (-\varpi_j - w^{\boldsymbol{i}}_{k-1}(-\varpi_j + \alpha_j - \mathsf{c}_{i,j}\alpha_i), \varpi_{i_v} + w^{\boldsymbol{i}}_v\varpi_{i_v}),\\ \Lambda_{k+3,v} &= (\varpi_j - w^{\boldsymbol{i}}_{k-1}s_is_js_is_j\varpi_j, \varpi_{i_v} + w^{\boldsymbol{i}}_v\varpi_{i_v})\\ &= (\varpi_j - w^{\boldsymbol{i}}_{k-1}(\varpi_j + \mathsf{c}_{i,j}\alpha_i - 2\alpha_j), \varpi_{i_v} + w^{\boldsymbol{i}}_v\varpi_{i_v}),\end{aligned}$$

$$\Lambda_{(k+1)_i^-,v} = (\varpi_j - w_{k-1}^{\boldsymbol{i}}\varpi_j, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}).$$

Thus the sum of the three given above is

$$(\varpi_j - w_{k-1}^{\boldsymbol{i}}(\varpi_j - \alpha_j), \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}) = \Lambda_{k,v}^{\boldsymbol{i}'},$$

which shows the assertion (i).

(ii), (vi), (xiii) and (xiv)-cases: Since the proofs for these cases are all similar, we shall give a proof for (ii)-case only. Note that

$$\begin{aligned}\Lambda_{k+1,v}^{\boldsymbol{i}'} &= (\varpi_i - w_{k-1}^{\boldsymbol{i}}s_j s_i\varpi_i, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v})\\ &= (\varpi_i - w_{k-1}^{\boldsymbol{i}}(\varpi_i - \alpha_i + \mathsf{c}_{j,i}\alpha_j), \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}).\end{aligned}$$

First let us consider the following two terms:

$$\begin{aligned}\mathsf{c}_{j,i}\Lambda_{(k+1)_i^-,v} &= (\mathsf{c}_{j,i}\varpi_j - w_{k-1}^{\boldsymbol{i}}\mathsf{c}_{j,i}\varpi_j, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v})\\ -\mathsf{c}_{j,i}\Lambda_{k+3,v} &= (-\mathsf{c}_{j,i}\varpi_j - w_{k-1}^{\boldsymbol{i}}(-\mathsf{c}_{j,i}\varpi_j - 2\alpha_i + 2\mathsf{c}_{j,i}\alpha_j)\varpi_j, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}),\end{aligned}$$

whose sum is

$$(-w_{k-1}^{\boldsymbol{i}}(-2\alpha_i + 2\mathsf{c}_{j,i}\alpha_j), \varpi_{\boldsymbol{i}_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}). \tag{3.9}$$

Next, let us consider the following three terms:

$$\begin{aligned}\Lambda_{k,v} &= (\varpi_i - w_{k-1}^{\boldsymbol{i}}s_i\varpi_i, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}) = (\varpi_i - w_{k-1}^{\boldsymbol{i}}(\varpi_i - \alpha_i), \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v})\\ -\Lambda_{k+2,v} &= -(\varpi_i - w_{k-1}^{\boldsymbol{i}}s_i s_j s_i\varpi_i, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v})\\ &= (-\varpi_i - w_{k-1}^{\boldsymbol{i}}(-\varpi_i - \mathsf{c}_{j,i}\alpha_j + 2\alpha_i), \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}),\\ \Lambda_{k_i^-,v} &= (\varpi_i - w_{k-1}^{\boldsymbol{i}}\varpi_i, \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}),\end{aligned}$$

whose sum is

$$(\varpi_i - w_{k-1}^{\boldsymbol{i}}(\varpi_i - \mathsf{c}_{j,i}\alpha_j + \alpha_i), \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}). \tag{3.10}$$

Hence, RHS of (ii) becomes

$$(3.9) + (3.10) = (\varpi_i - w_{k-1}^{\boldsymbol{i}}(\varpi_i - \alpha_i + \mathsf{c}_{j,i}\alpha_j), \varpi_{i_v} + w_v^{\boldsymbol{i}}\varpi_{i_v}) = \Lambda_{k+1,v}^{\boldsymbol{i}'},$$

which implies assertion (ii).

(ix)-case: This is the most complicated case. We have to show thats

$$\begin{aligned}\Lambda^{i'}_{k,k+1} = \Lambda_{k,k+1} + \mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+1} + \Lambda_{k+1,k+2} - \Lambda_{k,k+3}\\ - \mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+3} + \Lambda_{k+2,k+3} + \Lambda_{(k+1)^-_i,k} - \Lambda_{(k+1)^-_i,k+2} + \mathsf{c}_{j,i}\Lambda_{k+1,k+3}\\ - \mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+3} + \Lambda_{k^-_i,k+1} - \Lambda_{k^-_i,k+3} - \Lambda_{k^-_i,(k+1)^-_i},\end{aligned} \tag{3.11}$$

whose RHS consists of 13 terms.

On the other hand, $\Lambda^{i'}_{k,k+1}$ can be rewritten as follows:

$$\begin{aligned}\Lambda^{i'}_{k,k+1} &= (\varpi_j - w^{i}_{k-1}s_j\varpi_j, \varpi_i + w^{i}_{k-1}s_js_i\varpi_i)\\ &= (\varpi_j - w^{i}_{k-1}(\varpi_j - \alpha_j), \varpi_i + w^{i}_{k-1}(\varpi_i - \alpha_i + \mathsf{c}_{j,i}\alpha_j))\\ &= (\varpi_j, w^{i}_{k-1}(\varpi_i)) - (\varpi_j, w^{i}_{k-1}(\alpha_i)) + \mathsf{c}_{j,i}(\varpi_j, w^{i}_{k-1}(\alpha_j))\\ &\quad - (w^{i}_{k-1}(\varpi_j), \varpi_i) + (w^{i}_{k-1}(\alpha_j), \varpi_i).\end{aligned}$$

Let us analyze the 13 terms in (3.11):

(1) $\Lambda_{k,k+1} = -\Lambda_{k+1,k} \overset{Lemma\ 3.2}{=} -(\varpi_j - w^{i}_{k-1}s_is_j\varpi_j, \varpi_i + w^{i}_{k-1}s_i\varpi_i)$,
(2) $\Lambda_{(k+1)^-_i,k} = (\varpi_j - w^{i}_{k-1}\varpi_j, \varpi_i + w^{i}_{k-1}s_i\varpi_i)$,
(3) $\mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+1} = \mathsf{c}_{j,i}(\varpi_j - w^{i}_{k-1}\varpi_j, \varpi_j + w^{i}_{k-1}s_is_j\varpi_j)$,
(4) $-\mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+3} = -\mathsf{c}_{j,i}(\varpi_j - w^{i}_{k-1}\varpi_j, \varpi_j + w^{i}_{k-1}s_is_js_is_j\varpi_j)$,
(5) $\Lambda_{k+1,k+2} = (\varpi_j - w^{i}_{k-1}s_is_j\varpi_j, \varpi_i + w^{i}_{k-1}s_is_js_i\varpi_i)$,
(6) $-\Lambda_{(k+1)^-_i,k+2} = -(\varpi_j - w^{i}_{k-1}\varpi_j, \varpi_i + w^{i}_{k-1}s_is_js_i\varpi_i)$,
(7) $-\Lambda_{k,k+3} = -(\varpi_i - w^{i}_{k-1}s_i\varpi_i, \varpi_j + w^{i}_{k-1}s_is_js_is_j\varpi_j)$,
(8) $\Lambda_{k+2,k+3} = (\varpi_i - w^{i}_{k-1}s_is_js_i\varpi_i, \varpi_j + w^{i}_{k-1}s_is_js_is_j\varpi_j)$,
(9) $\mathsf{c}_{j,i}\Lambda_{k+1,k+3} = \mathsf{c}_{j,i}(\varpi_j - w^{i}_{k-1}s_is_j\varpi_j, \varpi_j + w^{i}_{k-1}s_is_js_is_j\varpi_j)$,
(10) $-\mathsf{c}_{j,i}\Lambda_{(k+1)^-_i,k+3} = -\mathsf{c}_{j,i}(\varpi_j - w^{i}_{k-1}\varpi_j, \varpi_j + w^{i}_{k-1}s_is_js_is_j\varpi_j)$,
(11) $\Lambda_{k^-_i,k+1} = (\varpi_i - w^{i}_{k-1}\varpi_i, \varpi_j + w^{i}_{k-1}s_is_j\varpi_j)$,
(12) $-\Lambda_{k^-_i,k+3} = -(\varpi_i - w^{i}_{k-1}\varpi_i, \varpi_j + w^{i}_{k-1}s_is_js_is_j\varpi_j)$,
(13) $-\Lambda_{k^-_i,(k+1)^-_i} \overset{Lemma\ 3.2}{=} -(\varpi_i - w^{i}_{k-1}\varpi_i, \varpi_j + w^{i}_{k-1}\varpi_j)$,

Using the techniques applied to the previous cases, we have

$$\begin{aligned}(1) + (2) &= -(w^{i}_{k-1}s_i(\alpha_j), \varpi_i),\\ (3) + (4) &= \mathsf{c}_{j,i}(\varpi_j, w^{i}_{k-1}(\alpha_j)) - (\alpha_i, \alpha_j),\\ (5) + (6) &= (\alpha_j, \alpha_i) + (w^{i}_{k-1}s_i(\alpha_j), \varpi_i),\\ (7) + (8) &= (w^{i}_{k-1}(\alpha_j), \varpi_j) - \mathsf{c}_{j,i}(w^{i}_{k-1}\alpha_j, \varpi_j) + (\alpha_i, \alpha_j),\\ (9) + (10) &= \mathsf{c}_{j,i}(w^{i}_{k-1}\alpha_j, \varpi_j) - 2(w^{i}_{k-1}(\alpha_i), \varpi_j) - \mathsf{c}_{i,j}(\alpha_i, \varpi_i),\\ (11) + (12) + (13) &= -(\varpi_i, w^{i}_{k-1}(\varpi_j)) + (\varpi_i, w^{i}_{k-1}(\alpha_j)) + (w^{i}_{k-1}\varpi_i, \varpi_j).\end{aligned}$$

Hence the summation of the 13 terms becomes

$$\sum_{s=1}^{13}(s) = -(\varpi_i, w^{\boldsymbol{i}}_{k-1}(\varpi_j)) + (\varpi_i, w^{\boldsymbol{i}}_{k-1}(\alpha_j)) + (w^{\boldsymbol{i}}_{k-1}\varpi_i, \varpi_j) \\ - (w^{\boldsymbol{i}}_{k-1}(\alpha_j), \varpi_j) + \mathsf{c}_{j,i}(\varpi_j, w^{\boldsymbol{i}}_{k-1}(\alpha_j)) = \Lambda^{\boldsymbol{i}'}_{k,k+1},$$

which completes the proof of assertion (ix). □

As a consequence of the previous two lemmas, we have the following proposition:

Proposition 3.8 *We have an isomorphism of* $\mathbb{A}$*-algebras*

$$\eta_k^*(= \mu_k^*\mu_{k+1}^*\mu_k^*\sigma_k^*\sigma_{k+2}^*) \colon \mathcal{A}_{\boldsymbol{i}'} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}}$$

given by

$$Z_u \mapsto \mu_k^*\mu_{k+1}^*\mu_k^*(Z_{\sigma_{k+2}\sigma_k(u)}) \quad \text{for all } u \in \mathbb{N},$$

which induces a bijection between the sets of quantum cluster monomials.

Lemma 3.9 *If* $\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}'}$ *is a quantum cluster monomial with* $\boldsymbol{deg}(\mathsf{z}) = \boldsymbol{g}' = (g'_u)_{u\in\mathbb{N}}$, *then the quantum cluster monomial* $\eta_k^*\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}}$ *has* $\boldsymbol{deg}(\eta_k^*\mathsf{z}) = \boldsymbol{g} = (g_u)_{u\in\mathbb{N}}$ *as follows:*

$$g_u = \begin{cases} g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + [A]_+ & \text{if } u = k+3, \\ g'_{k+3} - [-g'_{k+1}]_+ + [B]_+ & \text{if } u = k+2, \\ -A + \mathsf{c}_{j,i}[-B]_+ & \text{if } u = k+1, \\ -B & \text{if } u = k, \\ g'_{(k+1)^-_{\boldsymbol{i}'}} + [g'_{k+1}]_+ - [-B]_+ & \text{if } u = k^-_{\boldsymbol{i}}, \\ g'_{k^-_{\boldsymbol{i}'}} + [A]_+ - \mathsf{c}_{j,i}[B]_+ & \text{if } u = (k+1)^-_{\boldsymbol{i}}, \\ g'_u & \text{otherwise}, \end{cases} \tag{3.12}$$

where

$$A = g'_k + \mathsf{c}_{j,i}[-g'_{k+1}]_+ \quad \text{and} \quad B = -g'_{k+1} + \mathsf{c}_{i,j}[-A]_+. \tag{3.13}$$

Proof By applying (2.4) repeatedly, we obtain the assertion. □

Lemma 3.10 *The map* $\boldsymbol{g}' \mapsto \boldsymbol{g}$ *given by* (3.12) *sends the cone* $C_{\boldsymbol{i}'} \subset \mathbb{Z}^{\oplus\mathbb{N}}$ *into the cone* $C_{\boldsymbol{i}} \subset \mathbb{Z}^{\oplus\mathbb{N}}$.

Proof To simplify the notation, for $\boldsymbol{g} = (g_u)_{u\in\mathbb{N}}$ and $\boldsymbol{g}' = (g'_u)_{u\in\mathbb{N}}$, we set

$$\Sigma_u := \sum_{v\geqslant u, i_v = i_u} g_v, \quad \Sigma'_u := \sum_{v\geqslant u, i'_v = i'_u} g'_v \tag{3.14}$$

for each $u \in \mathbb{N}$. Throughout this proof, we shall use the notations A and B in (3.13).

We have to show that $\Sigma'_u \geqslant 0$ ($\forall u \in \mathbb{N}$) implies $\Sigma_u \geqslant 0$ ($\forall u \in \mathbb{N}$).

(a) If $u > k+3$, we have $\Sigma_u = \Sigma'_u \geqslant 0$.
(b) If $u = k+3$, we have $\Sigma_{k+3} = \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + [A]_+ \geqslant 0$.
(c) If $u = k+2$, the assertion holds when $g'_{k+1} \geqslant 0$ obviously. Let us assume further $g'_{k+1} < 0$. Then we have

$$\Sigma_{k+2} = \Sigma'_{k+3} + g'_{k+1} + [B]_+ = \Sigma'_{k+1} + [B]_+,$$

which implies the assertion for $u = k+2$.
(d) If $u = k+1$, we have

$$\begin{aligned}\Sigma_{k+1} &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + [A]_+ - A + \mathsf{c}_{j,i}[-B]_+ \\ &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + [-A]_+ + \mathsf{c}_{j,i}[-B]_+.\end{aligned} \tag{3.15}$$

Thus the assertion holds when $B \geqslant 0$. Assume that $B < 0$. Then we have

$$\begin{aligned}\Sigma_{k+1} &= \Sigma'_{k+2} - \mathsf{c}_{j,i}([g'_{k+1}]_+ - g'_{k+1}) + [-A]_+ - 2[-A]_+ \\ &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[-g'_{k+1}]_+ - [-A]_+.\end{aligned}$$

Thus the assertion holds when $B < 0$ and $A > 0$. Let us consider the case when $B < 0$ and $A < 0$. Then we have

$$\begin{aligned}\Sigma_{k+1} &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ - A - \mathsf{c}_{j,i}B \\ &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ - A + \mathsf{c}_{j,i}g'_{k+1} + 2A \\ &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + \mathsf{c}_{j,i}g'_{k+1} + g'_k + \mathsf{c}_{j,i}[-g'_{k+1}]_+ = \Sigma'_k,\end{aligned}$$

which completes the assertion.
(e) If $u = k$, we have

$$\begin{aligned}\Sigma_k &= \Sigma'_{k+1} - [g'_{k+1}]_+ + [-B]_+ \\ &= \delta(g'_{k+1} \geqslant 0)\Sigma'_{k+3} + \delta(g'_{k+1} < 0)\Sigma'_{k+1} + [-B]_+ \geqslant 0,\end{aligned} \tag{3.16}$$

which implies the assertion.
(f) If $u < k$, we have

$$\begin{aligned}g_{k+3} + g_{k+1} + g_{(k+1)_i^-} &= g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + 2[A]_+ - A + g'_{k_{i'}^-} - \mathsf{c}_{j,i}B \\ &= g'_{k+2} + g'_k + g'_{k_{i'}^-}\end{aligned}$$

and

$$g_{k+2} + g_k + g_{k_i^-} = g'_{k+3} + g'_{k+1} + g'_{(k+1)_{i'}^-},$$

and thus $\Sigma_u = \Sigma'_u$, which implies the assertion. □

By (f) in the proof of Lemma 3.10, we have the following corollary:

Corollary 3.11 *Assume that $k^- \in \mathbb{N}$. Then we have*

$$\mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) = \mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) \quad \textit{for all } i \in I,$$

where $\boldsymbol{g}' \mapsto \boldsymbol{g}$ under the map given by (3.12).

Lemma 3.12 *The elements $\boldsymbol{g}, \boldsymbol{g}' \in \mathbb{Z}^{\oplus \mathbb{N}}$ related by* (3.12) *satisfy the followings:*

(1) *When $i = i_{k+1}$ and $(k+1)_{\boldsymbol{i}}^- = 0$, we have*

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) = \begin{cases} \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - [g'_k]_+ & \textit{if } g'_{k+1} \geqslant 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - g'_k & \textit{if } g'_{k+1} < 0,\ g'_k \geqslant 0 \textit{ and } A \geqslant 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - 2g'_k + \mathsf{c}_{j,i} g'_{k+1} & \textit{if } g'_{k+1} < 0,\ g'_k \geqslant 0,\ A < 0 \textit{ and } B \geqslant 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) & \textit{otherwise}. \end{cases}$$

(2) *When $i = i_k$ and $k_{\boldsymbol{i}}^- = 0$, we have*

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) = \begin{cases} \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - g'_{k+1} & \textit{if } g'_{k+1} \geqslant 0,\ B \geqslant 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - \mathsf{c}_{i,j}[-g'_k]_+ & \textit{if } g'_{k+1} \geqslant 0,\ B < 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) & \textit{if } g'_{k+1} < 0,\ B \geqslant 0, \\ \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) - g'_{k+1} + \mathsf{c}_{i,j}[-g'_k]_+ & \textit{if } g'_{k+1} < 0,\ B < 0. \end{cases}$$

(3) *Except above two cases, we have*

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) = \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i).$$

Here

$$A = g'_k + \mathsf{c}_{j,i}[-g'_{k+1}]_+ \quad \textit{and} \quad B = -g'_{k+1} + \mathsf{c}_{i,j}[-A]_+.$$

Proof Since the proofs for γ_k and β_k are given in [11, Lemma 2.18], let us focus on the case η_k.

(1) Let us first consider when $i = i_{k+1}$ and $(k+1)_{\boldsymbol{i}}^- = 0$. In (3.15), we see that

$$\Sigma_{k+1} = \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + [-A]_+ + \mathsf{c}_{j,i}[-B]_+.$$

(i) If $g'_{k+1} \geqslant 0$, then $A = g'_k, [-B]_+ = g'_{k+1} - \mathsf{c}_{i,j}[-g'_k]_+$ and

$$\begin{aligned} \Sigma_{k+1} &= \Sigma'_{k+2} - \mathsf{c}_{j,i} g'_{k+1} + [-g'_k]_+ + \mathsf{c}_{j,i}(g'_{k+1} - \mathsf{c}_{i,j}[-g'_k]_+) \\ &= \Sigma'_{k+2} - [-g'_k]_+ = \Sigma'_k - [g'_k]_+. \end{aligned}$$

(ii) If $g'_{k+1} < 0$, then $A = g'_k - \mathsf{c}_{j,i} g'_{k+1}$ and $B = -g'_{k+1} + \mathsf{c}_{i,j}[-A]_+$.

(ii-a) Let us assume that $g'_k < 0$ and $A \leqslant 0$. Then $B = g'_{k+1} - \mathsf{c}_{i,j} g'_k < 0$ and

$$\Sigma_{k+1} = \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + [-A]_+ + \mathsf{c}_{j,i}[-B]_+ = \Sigma'_{k+2} + g'_k = \Sigma'_k$$

(ii-b) Let us assume that $g'_k \geqslant 0$ and $A \geqslant 0$. Then $B = -g'_{k+1}$ and

$$\Sigma_{k+1} = \Sigma'_{k+2} = \Sigma'_k - g'_k.$$

(ii-c) Finally, assume that $g'_k \geqslant 0$ and $A < 0$. Then $B = g'_{k+1} - \mathsf{c}_{i,j} g'_k$.

$$\Sigma_{k+1} = \Sigma'_{k+2} - g'_k + \mathsf{c}_{j,i} g'_{k+1} + \mathsf{c}_{j,i}[-B]_+ = \begin{cases} \Sigma'_k - 2g'_k + \mathsf{c}_{j,i} g'_{k+1} & \text{if } B \geqslant 0, \\ \Sigma'_k & \text{if } B < 0. \end{cases}$$

(2) Let us consider when $i = i_k$. In (3.16), we see that

$$\Sigma_k = \Sigma'_{k+1} - [g'_{k+1}]_+ + [-B]_+.$$

Then we have

$$\Sigma_k = \begin{cases} \Sigma'_{k+1} - g'_{k+1} & \text{if } g'_{k+1} \geqslant 0,\ B \geqslant 0, \\ \Sigma'_{k+1} - \mathsf{c}_{i,j}[-g'_k]_+ & \text{if } g'_{k+1} \geqslant 0,\ B < 0, \\ \Sigma'_{k+1} & \text{if } g'_{k+1} < 0,\ B \geqslant 0, \\ \Sigma'_{k+1} - g'_{k+1} + \mathsf{c}_{i,j}[-g'_k]_+ & \text{if } g'_{k+1} < 0,\ B < 0, \end{cases}$$

which implies our assertion. □

3.6 6-moves

For sequences $\boldsymbol{i} = (i_u)_{u \in \mathbb{N}}$ and $\boldsymbol{i}' = (i'_u)_{u \in \mathbb{N}} \in I^{(\infty)}$, we say that $\boldsymbol{i}'$ can be obtained from $\boldsymbol{i}$ via a 6-*move*, if there exists a unique $k \in \mathbb{N}$ such that

$$\begin{aligned} & i_k = i_{k+2} = i_{k+4} = i'_{k+1} = i'_{k+3} = i'_{k+5}, \\ & i_{k+1} = i_{k+3} = i_{k+5} = i'_k = i'_{k+2} = i'_{k+4}, \\ & i_u = i'_u \ (u \notin [k, k+5]) \end{aligned}$$

and

$$\mathsf{c}_{i_{k+1}, i_k} \mathsf{c}_{i_k, i_{k+1}} = 3.$$

In this case, we write $\boldsymbol{i}' = \zeta_k \boldsymbol{i}$. Throughout this subsection, we assume that $\boldsymbol{i}, \boldsymbol{i}' \in I^{(\infty)}$ are related by $\boldsymbol{i}' = \zeta_k \boldsymbol{i}$ for some $k \in \mathbb{N}$ and we sometimes write $i := i_k$ and $j := i_{k+1}$.

Lemma 3.13 *We have*

$$\widetilde{B}^{\boldsymbol{i}'} = \sigma_{k+4}\sigma_{k+2}\sigma_k \mu_k \mu_{k+3} \mu_{k+2} \mu_k \mu_{k+1} \mu_{k+3} \mu_k \mu_{k+2} \mu_{k+1} \mu_k \widetilde{B}^{\boldsymbol{i}}. \tag{3.17}$$

Proof As in the 4-move case, it is enough to show that

$$\mu_k \mu_{k+3} \mu_{k+2} \mu_k \mu_{k+1} \mu_{k+3} \mu_k \mu_{k+2} \mu_{k+1} \mu_k \widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}} = \sigma_{k+4}\sigma_{k+2}\sigma_k \widetilde{\Gamma}^{\boldsymbol{i}'}|_{\mathcal{K}}$$

where $\mathcal{K} = [k, k+5] \sqcup \{k_{\boldsymbol{i}}^-, (k+1)_{\boldsymbol{i}}^-\}$.

We shall only consider the case when both $k_{\boldsymbol{i}}^-$ and $(k+1)_{\boldsymbol{i}}^-$ are nonzero since the proofs for the other cases are similar. In that case, the valued quiver $\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}}$ is depicted as:

$$\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}} = \begin{tikzcd}[column sep=small] & k+4 \arrow[dr, "\ulcorner b,-a \lrcorner"] & & k+2 \arrow[ll] \arrow[dr, "\ulcorner b,-a \lrcorner"] & & k \arrow[ll] \arrow[dr, "\ulcorner b,-a \lrcorner"] & & k_{\boldsymbol{i}}^- \arrow[ll] \\ k+5 \arrow[ur, dotted, "\ulcorner a,-b \lrcorner"] & & k+3 \arrow[ll] \arrow[ur, "\ulcorner a,-b \lrcorner"] & & k+1 \arrow[ll] \arrow[ur, "\ulcorner a,-b \lrcorner"] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll] \arrow[ur, dotted, "\ulcorner a,-b \lrcorner"] & \end{tikzcd} \tag{3.18}$$

where $a = -\mathsf{c}_{i_{k+1},i_k}$ and $b = -\mathsf{c}_{i_k,i_{k+1}}$. Here

- the dotted arrow between $k_{\boldsymbol{i}}^-$ and $(k+1)_{\boldsymbol{i}}^-$ exists if and only if $k_{\boldsymbol{i}}^- < (k+1)_{\boldsymbol{i}}^-$,
- the dotted arrow between $k+5$ and $k+4$ exists if and only if $(k+4)_{\boldsymbol{i}}^+ < (k+5)_{\boldsymbol{i}}^+$.

Let us consider the case when $i_k = 1$, $i_{k+1} = 2$, $\mathfrak{g} = G_2$ and dotted arrows exist. Then (3.18) can be drawn as follows with Convention 1:

$$\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}} = \begin{tikzcd}[column sep=small] & k+4 \arrow[dr, Rrightarrow] & & k+2 \arrow[ll] \arrow[dr, Rrightarrow] & & k \arrow[ll] \arrow[dr, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[ll] \\ k+5 \arrow[ur, "\llcorner 3"'] & & k+3 \arrow[ll] \arrow[ur, "\llcorner 3"'] & & k+1 \arrow[ll] \arrow[ur, "\llcorner 3"'] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll] \arrow[ur, "\llcorner 3"'] & \end{tikzcd}$$

From now on, we shall depict the sequence of mutations on $\widetilde{\Gamma}^{\boldsymbol{i}}|_{\mathcal{K}}$ in turn:

$$\xrightarrow{\mu_k} \begin{tikzcd}[column sep=small] & k+4 \arrow[dr, Rrightarrow] & & k+2 \arrow[ll] \arrow[rr] & & k \arrow[rr] \arrow[dl, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[llll, bend right] \\ k+5 \arrow[ur, "\llcorner 3"'] & & k+3 \arrow[ll] \arrow[ur, "\llcorner 3"'] & & k+1 \arrow[ll] \arrow[rr, "2"'] & & (k+1)_{\boldsymbol{i}}^- \arrow[ul, "3 \lrcorner"] & \end{tikzcd}$$

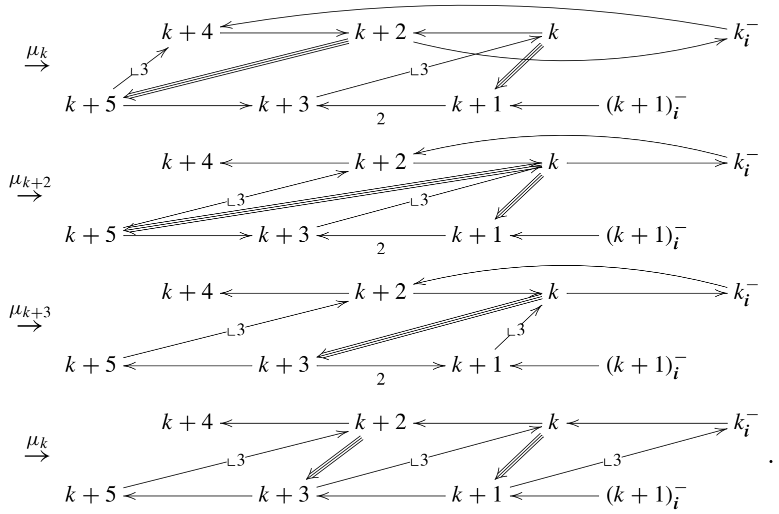
$$\xrightarrow{\mu_{k+1}} \begin{tikzcd}[column sep=small] & k+4 \arrow[dr, Rrightarrow] & & k+2 \arrow[ll] \arrow[rr] & & k \arrow[rr] \arrow[dlll, Rrightarrow] \arrow[dr, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[llll, bend right] \\ k+5 \arrow[ur, "\llcorner 3"'] & & k+3 \arrow[ll] \arrow[ur, "\llcorner 3"'] \arrow[rr] & & k+1 \arrow[ur, "\llcorner 3"'] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll, "2"] & \end{tikzcd}$$

$$\xrightarrow{\mu_{k+2}} \begin{tikzcd}[column sep=small] & k+4 \arrow[rr] & & k+2 \arrow[dl, Rrightarrow] \arrow[rrrr, bend right] & & k \arrow[ll] \arrow[dr, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[llllll, bend right] \\ k+5 \arrow[ur, "\llcorner 3"'] & & k+3 \arrow[ll] \arrow[rr] & & k+1 \arrow[ur, "\llcorner 3"'] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll, "2"] & \end{tikzcd}$$

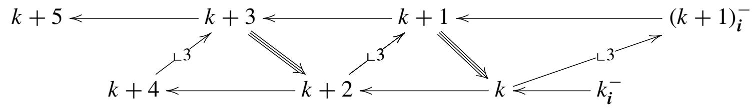
$$\xrightarrow{\mu_k} \begin{tikzcd}[column sep=small] & k+4 \arrow[rr] & & k+2 \arrow[dl, Rrightarrow] \arrow[rr] \arrow[rrrr, bend right] & & k \arrow[dl, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[llllll, bend right] \\ k+5 \arrow[ur, "\llcorner 3"'] & & k+3 \arrow[ll] \arrow[rr] & & k+1 \arrow[ul, "3 \lrcorner"] \arrow[rr] & & (k+1)_{\boldsymbol{i}}^- \arrow[ul, "3 \lrcorner"] & \end{tikzcd}$$

$$\xrightarrow{\mu_{k+3}} \begin{tikzcd}[column sep=small] & k+4 \arrow[rr] & & k+2 \arrow[dlll, Rrightarrow] \arrow[rr] \arrow[rrrr, bend right] & & k \arrow[dl, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[llllll, bend right] \\ k+5 \arrow[ur, "\llcorner 3"'] \arrow[rr] & & k+3 \arrow[ur, "\llcorner 3"'] & & k+1 \arrow[ll] \arrow[rr] & & (k+1)_{\boldsymbol{i}}^- \arrow[ul, "3 \lrcorner"] & \end{tikzcd}$$

$$\xrightarrow{\mu_{k+1}} \begin{tikzcd}[column sep=small] & k+4 \arrow[rr] & & k+2 \arrow[dlll, Rrightarrow] \arrow[rr] \arrow[rrrr, bend right] & & k \arrow[dlll, Rrightarrow] & & k_{\boldsymbol{i}}^- \arrow[llllll, bend right] \\ k+5 \arrow[ur, "\llcorner 3"'] \arrow[rr] & & k+3 \arrow[ur, "\llcorner 3"'] \arrow[rr] & & k+1 \arrow[ur, "\llcorner 3"'] & & (k+1)_{\boldsymbol{i}}^- \arrow[ll] & \end{tikzcd}$$



On the other hand, the valued quiver $\widetilde{\Gamma}^{\boldsymbol{i}'}$ is depicted as:

$\widetilde{\Gamma}^{\boldsymbol{i}'}|_{\mathcal{K}} =$ 

Thus our assertion follows. The remaining cases can be proved in a similar way. $\square$

Lemma 3.14 *We have*

$$\Lambda^{\boldsymbol{i}'} = \sigma_{k+4}\sigma_{k+2}\sigma_k\mu_k\mu_{k+3}\mu_{k+2}\mu_k\mu_{k+1}\mu_{k+3}\mu_k\mu_{k+2}\mu_{k+1}\mu_k\Lambda^{\boldsymbol{i}}. \tag{3.19}$$

Proof To simplify notation, let us denote $\Lambda^{\boldsymbol{i}}$ as Λ. Unless $k \le u \le k+5$ or $k \le v \le k+5$, the expressions $\Lambda_{u,v} = \Lambda^{\boldsymbol{i}'}_{u,v}$ hold true, and the values of $\Lambda_{u,v}$ remain unchanged under the operations $\mu_k, \mu_{k+1}, \mu_{k+2}, \mu_{k+3}, \sigma_k, \sigma_{k+2}$, and σ_{k+4}. Therefore, for simplicity, we will only consider $\Lambda_{u,v}$ when $k \le u \le k+5$ or $k \le v \le k+5$, which we will assume from now on. Furthermore, due to the skew-symmetry of Λ and $\Lambda^{\boldsymbol{i}'}$, we can assume further $u < v$.

Recall the definition
t

$$\Lambda_{u,v} = (\varpi_{i_u} - w_u\varpi_{i_u}, \varpi_{i_v} + w_v\varpi_{i_v}) \quad \text{for } u < v.$$

Since the number of elements in the set W is 12 and the sequence $\boldsymbol{i}$ contains only 1 and 2, we will demonstrate that there exist only a limited number of finite subsequences of $\boldsymbol{i}$ that need to be examined to directly verify the assertion.

Consider a fixed set $\underline{\mathsf{W}}$ consisting of reduced sequences of elements in W. Let us examine how $\Lambda_{u,v}$ is determined. To avoid minor exceptions, assume $k \geqslant 4$. If $u < k$

and $k \leqslant v \leqslant k+5$, then $\Lambda_{u,v}$ is determined by the subsequence

$$w_1\, a\, w_2\, i_k\, i_{k+1} \cdots i_{k+5},$$

where $w_1, w_2 \in \underline{\mathsf{W}}$, $i_u = a$, and $i_v = i_\ell$ for some $\ell = k, k+1, \ldots, k+5$. Note that it is sufficient to consider $w_1, w_2 \in \underline{\mathsf{W}}$, as $\Lambda_{u,v}$ does not change with other expressions of the Weyl group elements represented by w_1 and w_2.

If $k \leqslant u, v \leqslant k+5$ then $\Lambda_{u,v}$ is determined by

$$i_k\, i_{k+1} \cdots i_{k+5}.$$

If $k \leqslant u \leqslant k+5$ and $v > k+5$ then $\Lambda_{u,v}$ is determined by the subsequence

$$i_k\, i_{k+1} \cdots i_{k+5} w_3 d,$$

where $w_3 \in \underline{\mathsf{W}}$, $i_v = d$, $i_u = i_\ell$ for some $\ell = k, k+1, \ldots, k+5$.

All three cases mentioned above are encompassed in the sequences given by

$$w_1\, a\, w_2\, i_k\, i_{k+1} \cdots i_{k+5}\, w_3\, d, \tag{3.20}$$

where $w_1, w_2, w_3 \in \underline{\mathsf{W}}$, $a, d \in \{1, 2\}$ and

$$(i_k, i_{k+1}, \ldots, i_{k+5}) \in \{(1, 2, 1, 2, 1, 2), (2, 1, 2, 1, 2, 1)\}.$$

However, to apply $\mu_k, \mu_{k+1}, \mu_{k+2}, \mu_{k+3}$, it is crucial to keep track of the corresponding mutations of $\widetilde{B}^{\boldsymbol{i}}$. As demonstrated in the proof of Lemma 3.13, we only need to consider $k_{\boldsymbol{i}}^-$ and $(k+1)_{\boldsymbol{i}}^-$ in addition to $k, k+1, \ldots, k+5$ when dealing with mutations (note that $(k+4)_{\boldsymbol{i}}^+$ and $(k+5)_{\boldsymbol{i}}^+$ are not relevant). In the matrices obtained from the mutations of $\widetilde{B}^{\boldsymbol{i}}$ associated with the mutations of $\Lambda^{\boldsymbol{i}}$ in the statement of this lemma, we observe that

$$\text{the } (k, j)\text{- and } (k+1, j)\text{-entries are zero for } j < k \text{ unless } j = k_{\boldsymbol{i}}^- \text{ or } j = (k+1)\boldsymbol{i}^-. \tag{3.21}$$

Therefore, these entries do not contribute to the mutations of $\Lambda_{u,v}$ by definition.

Now consider a sequence

$$\boldsymbol{j}_1 a \boldsymbol{j}_2\, i_k\, i_{k+1} \cdots i_{k+5} \boldsymbol{j}_3 d,$$

where $\boldsymbol{j}_1 = i_1\, i_2 \cdots i_{u-1}$, $i_u = a$, $i_v = d$ and $\boldsymbol{j}_2$ and $\boldsymbol{j}_3$ are finite sequences. Let $\{b, c\} = \{i_{k_{\boldsymbol{i}}^-}, i_{(k+1)_{\boldsymbol{i}}^-}\}$. Without loss of generality, we may consider the following three cases:

$$\begin{aligned}
&\text{(i)}\, \boldsymbol{j}_1 a \cdots bc \cdots c\, i_k\, i_{k+1} \cdots i_{k+5} \boldsymbol{j}_3 d, \\
&\text{(ii)}\, \boldsymbol{j}_1 ac \cdots c\, i_k\, i_{k+1} \cdots i_{k+5} \boldsymbol{j}_3 d \qquad \text{with } a = b, \\
&\text{(iii)} \cdots ba \cdots a\, i_k\, i_{k+1} \cdots i_{k+5} \boldsymbol{j}_3 d \qquad \text{with } a = c\ .
\end{aligned}$$

Applying the observations (3.20) and (3.21) in determining $\Lambda_{u,v}$ and their mutations, we only need to consider

$$\text{(i)} \rightsquigarrow \; w_1 a w_2 bc \, i_k \, i_{k+1} \cdots i_{k+5} \, w_3 d, \tag{3.22}$$

$$\text{(ii)} \rightsquigarrow \; w_1 ac \, i_k \, i_{k+1} \cdots i_{k+5} \, w_3 d \quad \text{or} \quad w_1 acc \, i_k \, i_{k+1} \cdots i_{k+5} \, w_3 d, \tag{3.23}$$

$$\text{(iii)} \rightsquigarrow \; w_1 a \, i_k \, i_{k+1} \cdots i_{k+5} \, w_3 d \ , \tag{3.24}$$

where $w_1, w_2, w_3 \in \underline{\mathsf{W}}$, $a, d \in \{1, 2\}$, $(b, c) \in \{(1, 2), (2, 1)\}$ and

$$(i_k, i_{k+1}, \ldots, i_{k+5}) \in \{(1, 2, 1, 2, 1, 2), (2, 1, 2, 1, 2, 1)\}.$$

We consider the submatrices of $\Lambda^{\boldsymbol{i}}$ given by the finite sequences in (3.22)-(3.24). These sequences are not mutually exclusive, and we deliberately maintain this redundancy for convenience, as it is unnecessary to remove it when directly checking using a computer. The total number of sequences in (3.22)-(3.24) is 62,208. If $k \leqslant 3$, it is easy to list all possible sequences. We have used a computer to verify the assertion of the lemma for all these finite subsequences, which took less than 12 minutes on a regular laptop (single CPU). □

Remark 3.15 The above lemma assumes a general situation. For our purpose in later sections, we need only special $\boldsymbol{i}$ and $\boldsymbol{i}'$ in this remark (see Remark 5.22 below also), and hence we *do not* need to use a computer. Consider $\boldsymbol{i} = (1, 2, 1, 2, 1, 2, 1, 2, \ldots)$ and $\boldsymbol{i}' = (1, 2, 2, 1, 2, 1, 2, 1, \ldots)$ with $k = 3$. Then we obtain

$$\Lambda^{\boldsymbol{i},8} = \begin{pmatrix} 0 & -3 & -1 & -3 & 0 & 0 & 2 & 3 \\ 3 & 0 & 0 & -3 & 0 & 0 & 3 & 6 \\ 1 & 0 & 0 & -3 & 0 & 0 & 3 & 6 \\ 3 & 3 & 3 & 0 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ -2 & -3 & -3 & -3 & -2 & 0 & 0 & 3 \\ -3 & -6 & -6 & -9 & -6 & -6 & -3 & 0 \end{pmatrix} \xrightarrow{\mu_3} \begin{pmatrix} 0 & -3 & -2 & -3 & 0 & 0 & 2 & 3 \\ 3 & 0 & 0 & -3 & 0 & 0 & 3 & 6 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 3 & 3 & 0 & 0 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ -2 & -3 & -2 & -3 & -2 & 0 & 0 & 3 \\ -3 & -6 & -6 & -9 & -6 & -6 & -3 & 0 \end{pmatrix} \xrightarrow{\mu_3\mu_5\mu_4}$$

$$\begin{pmatrix} 0 & -3 & -1 & -3 & 0 & 0 & 2 & 3 \\ 3 & 0 & 3 & 3 & 3 & 0 & 3 & 6 \\ 1 & -3 & 0 & 0 & 1 & 0 & 1 & 3 \\ 3 & -3 & 0 & 0 & 3 & 0 & 3 & 9 \\ 0 & -3 & -1 & -3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ -2 & -3 & -1 & -3 & 0 & 0 & 0 & 3 \\ -3 & -6 & -3 & -9 & -3 & -6 & -3 & 0 \end{pmatrix} \xrightarrow{\mu_3\mu_4\mu_6} \begin{pmatrix} 0 & -3 & 1 & 0 & 0 & 0 & 2 & 3 \\ 3 & 0 & 6 & 6 & 3 & 9 & 3 & 6 \\ -1 & -6 & 0 & 0 & -1 & -3 & -1 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & -9 & 3 & 0 & 0 & 0 & 0 & 3 \\ -2 & -3 & 1 & 0 & 0 & 0 & 0 & 3 \\ -3 & -6 & 0 & 0 & -3 & -3 & -3 & 0 \end{pmatrix} \xrightarrow{\mu_3\mu_6\mu_5}$$

$$\begin{pmatrix} 0 & -3 & 2 & 0 & 3 & 3 & 2 & 3 \\ 3 & 0 & 6 & 6 & 6 & 9 & 3 & 6 \\ -2 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -6 & 0 & 0 & 0 & 3 & -1 & 0 \\ -3 & -9 & 0 & 0 & -3 & 0 & -3 & -3 \\ -2 & -3 & 0 & 0 & 1 & 3 & 0 & 3 \\ -3 & -6 & 0 & 0 & 0 & 3 & -3 & 0 \end{pmatrix} \xrightarrow{\sigma_7\sigma_5\sigma_3} \begin{pmatrix} 0 & -3 & 0 & 2 & 3 & 3 & 3 & 2 \\ 3 & 0 & 6 & 6 & 9 & 6 & 6 & 3 \\ 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & -9 & 0 & 0 & 0 & -3 & -3 & -3 \\ -3 & -6 & 0 & 0 & 3 & 0 & 0 & -1 \\ -3 & -6 & 0 & 0 & 3 & 0 & 0 & -3 \\ -2 & -3 & 0 & 0 & 3 & 1 & 3 & 0 \end{pmatrix} = \Lambda^{\boldsymbol{i}',8}.$$

As a consequence of the previous two lemmas, we have the following proposition:

Proposition 3.16 *We have an isomorphism of* $\mathbb{A}$*-algebras*

$$\zeta_k^* (= \mu_k^* \mu_{k+1}^* \mu_{k+2}^* \mu_k^* \mu_{k+3}^* \mu_{k+1}^* \mu_k^* \mu_{k+2}^* \mu_{k+3}^* \mu_k^* \sigma_k^* \sigma_{k+2}^* \sigma_{k+4}^*) \colon \mathcal{A}_{\boldsymbol{i}'} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}} \quad (3.25)$$

given by

$$Z_u \mapsto \mu_k^* \mu_{k+1}^* \mu_{k+2}^* \mu_k^* \mu_{k+3}^* \mu_{k+1}^* \mu_k^* \mu_{k+2}^* \mu_{k+3}^* \mu_k^* (Z_{\sigma_{k+4}\sigma_{k+2}\sigma_k(u)}) \quad \text{for all } u \in \mathbb{N},$$

which induces a bijection between the sets of quantum cluster monomials.

Remark 3.17 There are another 31 sequences of mutations of length 10, which give rise to mutation equivalence from $\mathcal{A}_{\boldsymbol{i}'} \to \mathcal{A}_{\boldsymbol{i}}$. For instance, we have

$$\mu_{k+3}^* \mu_{k+2}^* \mu_{k+1}^* \mu_{k+3}^* \mu_k^* \mu_{k+2}^* \mu_{k+3}^* \mu_{k+1}^* \mu_k^* \mu_{k+3}^* \sigma_k^* \sigma_{k+2}^* \sigma_{k+4}^* \colon \mathcal{A}_{\boldsymbol{i}'} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}}.$$

In Appendix B, we record all the 32 sequences of mutations of length 10 that produce the equivalence with index permutation $\sigma_k^* \sigma_{k+2}^* \sigma_{k+4}^*$. Recall that **deg** does not depend on the choice of sequences of mutations [17, 50] .

Lemma 3.18 *If* $\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}'}$ *is a quantum cluster monomial with* $\boldsymbol{deg}(\mathsf{z}) = \boldsymbol{g}' = (g'_u)_{u\in\mathbb{N}}$*, then the quantum cluster monomial* $\zeta_k^* \mathsf{z} \in \mathcal{A}_{\boldsymbol{i}}$ *has* $\boldsymbol{deg}(\zeta_k^* \mathsf{z}) = \boldsymbol{g} = (g_u)_{u\in\mathbb{N}}$ *as follows:*

$$g_j = \begin{cases} g'_{(k+1)^-_{\boldsymbol{i}'}} + [g'_{k+1}]_+ + [B]_+ - [-G]_+ - [-I]_+ & \text{if } j = k^-_{\boldsymbol{i}}, \\ g'_{k^-_{\boldsymbol{i}'}} + [D]_+ - \mathsf{c}_{j,i}[F]_+ + 2[H]_+ - \mathsf{c}_{j,i}[I]_+ & \text{if } j = (k+1)^-_{\boldsymbol{i}}, \\ -I & \text{if } j = k, \\ -H + \mathsf{c}_{j,i}[-I]_+ & \text{if } j = k+1, \\ -G + [I]_+ & \text{if } j = k+2, \\ -E + \mathsf{c}_{j,i}[-G]_+ + [H]_+ & \text{if } j = k+3, \\ g'_{k+5} - [-B]_+ + [G]_+ & \text{if } j = k+4, \\ g'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [E]_+ & \text{if } j = k+5, \\ g'_j & \text{otherwise}, \end{cases} \quad (3.26)$$

where

$$\begin{aligned} &A := g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+, \quad B := g'_{k+3} - [-g'_{k+1}]_+, \quad C := -g'_{k+1} - \mathsf{c}_{i,j}[A]_+ - [-B]_+, \\ &D := g'_k + \mathsf{c}_{j,i}[-g'_{k+1}]_+ - 2[-A]_+ + \mathsf{c}_{j,i}[-C]_+, \qquad E := -A - \mathsf{c}_{j,i}[C]_+ + [D]_+, \\ &F := -C + \mathsf{c}_{i,j}[-D]_+, \qquad G := -B - [-C]_+ + \mathsf{c}_{i,j}[-E]_+ + [F]_+, \\ &H := -D + [E]_+ + \mathsf{c}_{j,i}[-F]_+, \qquad I := -F + [G]_+ + \mathsf{c}_{i,j}[-H]_+. \end{aligned} \quad (3.27)$$

Proof By applying (2.4) repeatedly, we can obtain the assertion. □

Lemma 3.19 *The map $\boldsymbol{g}' \mapsto \boldsymbol{g}$ given by (3.26) sends the cone $C_{\boldsymbol{i}'} \subset \mathbb{Z}^{\oplus \mathbb{N}}$ into the cone $C_{\boldsymbol{i}} \subset \mathbb{Z}^{\oplus \mathbb{N}}$.*

Proof Let us use similar arguments to those in Lemma 3.10 and notations in (3.27). As in the 4–move case, we check positivity by verifying that all partial sums $\Sigma_u = \sum_{v \leqslant u} g_v$ remain nonnegative. Because the 6–move only modifies entries between k and $k+5$, we only need to check those indices.

(a) If $u > k+5$, we have $\Sigma_u = \Sigma'_u \geqslant 0$.

(b) If $u = k+5$, we have

$$\Sigma_{k+5} = \Sigma'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [E]_+.$$

In the above equation, the only term that can contribute a negative amount is $-[-A]_+$. Therefore, to check the non-negativity of Σ_{k+5} it is enough to distinguish whether A is nonnegative or negative.

Note that if $A < 0$, then $E > 0$ and $A + E \geqslant 0$ since

$$A + E = A + (-A - \mathsf{c}_{j,i}[C]_+ + [D]_+) = -\mathsf{c}_{j,i}[C]_+ + [D]_+ \geqslant 0.$$

Thus $\Sigma_{k+5} \geqslant 0$. Assume that $A \geqslant 0$. In this case, we have

$$\Sigma_{k+5} = \Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + [E]_+,$$

which proves the assertion for $k+5$.

(c) If $u = k+4$, we have

$$\Sigma_{k+4} = \Sigma'_{k+5} - [-B]_+ + [G]_+.$$

As in the case (b), it suffices to consider when $B < 0$. If $g'_{k+1} < 0$, then $\Sigma'_{k+5} + B = \Sigma'_{k+1} \geqslant 0$, and if $g'_{k+1} \geqslant 0$, then $\Sigma'_{k+5} + B = \Sigma'_{k+3} \geqslant 0$. Thus the assertion follows.

(d) If $u = k+3$, we have

$$\Sigma_{k+3} = \Sigma'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [E]_+ - E + \mathsf{c}_{j,i}[-G]_+ + [H]_+. \tag{3.28}$$

Applying $[E]_+ - E = [-E]_+$, (3.28) becomes

$$\Sigma_{k+3} = \Sigma'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+ + \mathsf{c}_{j,i}[-G]_+ + [H]_+. \tag{3.29}$$

Now we divide into cases according to the signs of A and E.

<u>($A < 0$, $E > 0$)</u>: In this case, we have

$$[-E]_+ = 0, \quad -[-A]_+ = A = g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+$$

and hence (3.29) becomes

$$\begin{aligned}&\underline{\Sigma'_{k+4}+g'_{k+2}}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}[B]_++\mathsf{c}_{j,i}[-G]_++[H]_+\\&\quad=\underline{\Sigma'_{k+2}}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}[B]_++\mathsf{c}_{j,i}[-G]_++[H]_+.\end{aligned}$$

Since the only term that can contribute a negative amount is $[-G]_+$, it is enough to consider when

$$0>G=-B-[-C]_++\mathsf{c}_{i,j}[-E]_++[F]_+.$$

Note that it suffices to show that

$$\Sigma'_{k+2}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}[B]_++\mathsf{c}_{j,i}[-G]_+\geqslant 0. \tag{3.30}$$

Replacing G with the above formula, (3.29) turns into

$$\begin{aligned}&\Sigma'_{k+2}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}[B]_+-\mathsf{c}_{j,i}(-B-[-C]_++\mathsf{c}_{i,j}[-E]_++[F]_+)\\&\quad=\Sigma'_{k+2}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}([B]_+-B)+\mathsf{c}_{j,i}[-C]_+-3[-E]_+-\mathsf{c}_{j,i}[F]_+\\&\quad\overset{*}{=}\Sigma'_{k+2}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}[-B]_++\mathsf{c}_{j,i}[-C]_+-\mathsf{c}_{j,i}[F]_+.\end{aligned}$$

Here $\overset{*}{=}$ follows from

$$\text{(1) } E\geqslant 0 \text{ by the assumption} \quad \text{and} \quad \text{(2) } [B]_+-B=[-B]_+.$$

Finally assume

$$0>C=-g'_{k+1}-[-B]_+.$$

Then we have

$$\begin{aligned}&\Sigma'_{k+2}-\mathsf{c}_{j,i}[g'_{k+1}]_+-\mathsf{c}_{j,i}[-B]_+-\mathsf{c}_{j,i}[F]_++\mathsf{c}_{j,i}g'_{k+1}+\mathsf{c}_{j,i}[-B]_+\\&\quad\overset{\dagger}{=}\Sigma'_{k+2}-\mathsf{c}_{j,i}[-g'_{k+1}]_+-\mathsf{c}_{j,i}[F]_+.\end{aligned}$$

Here $\overset{\dagger}{=}$ follows from (3) $[g'_{k+1}]_+-g'_{k+1}=[-g'_{k+1}]_+$ and (4) $[-B]_+-[-B]_+=0$. Then our assertion for this case follows.

$\underline{(A\geqslant 0,\ E<0)}$: In this case, we have

$$[-E]_+=-E,\quad -[-A]_+=0$$

and hence (3.29) becomes

$$\Sigma'_{k+4}-\mathsf{c}_{j,i}[B]_+-E+\mathsf{c}_{j,i}[-G]_++[H]_+$$

Since $-E>0$, it suffices to consider when

$$0>G=-B-[-C]_+-\mathsf{c}_{i,j}E+[F]_+,$$

as in the previous case. Replacing G with the above formula, (3.29) turns into

$$\begin{aligned}
&\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ - E + [H]_+ + \mathsf{c}_{j,i}B + \mathsf{c}_{j,i}[-C]_+ + 3E - \mathsf{c}_{j,i}[F]_+ \\
&\overset{\ddagger}{=} \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + 2E + [H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}[-C]_+.
\end{aligned}$$

Here $\overset{\ddagger}{=}$ follows from (1) $[B]_+ - B = [-B]_+$. Finally assume

$$0 > C = -g'_{k+1} - \mathsf{c}_{i,j}A - [-B]_+.$$

Then we have

$$\begin{aligned}
&\Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + 2E + [H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}g'_{k+1} + 3A + \mathsf{c}_{j,i}[-B]_+ \\
&\overset{\star}{=} \Sigma'_{k+4} - 2A + 2[D]_+ + [H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}g'_{k+1} + 3A \\
&\overset{\sharp}{=} \Sigma'_{k+4} + g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + 2[D]_+ + [H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}g'_{k+1} \\
&\overset{\diamond}{=} \Sigma'_{k+2} - \mathsf{c}_{j,i}[-g'_{k+1}]_+ + 2[D]_+ + [H]_+ - \mathsf{c}_{j,i}[F]_+.
\end{aligned}$$

Here

(2) $\overset{\star}{=}$ follows from the replacement of E with $-A + [D]_+$,
(3) $\overset{\sharp}{=}$ follows from the replacement of A with $g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+$,
(4) $\overset{\diamond}{=}$ follows from $\Sigma'_{k+4} + g'_{k+2} = \Sigma'_{k+2}$ and $[g'_{k+1}]_+ - g'_{k+1} = [-g'_{k+1}]_+$.

($A \geqslant 0$, $E \geqslant 0$): In this case, we have

$$[-E]_+ = 0, \quad -[-A]_+ = 0$$

and hence (3.29) becomes

$$\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}[-G]_+ + [H]_+.$$

Similarly, it suffices to consider when

$$0 > G = -B - [-C]_+ + [F]_+.$$

Replacing G with the above formula, (3.29) turns into

$$\begin{aligned}
&\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + [H]_+ + \mathsf{c}_{j,i}B + \mathsf{c}_{j,i}[-C]_+ - \mathsf{c}_{j,i}[F]_+ \\
&= \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [H]_+ + \mathsf{c}_{j,i}[-C]_+ - \mathsf{c}_{j,i}[F]_+.
\end{aligned}$$

Finally assume

$$0 > C = -g'_{k+1} - \mathsf{c}_{i,j}A - [-B]_+.$$

Then we have

$$\begin{aligned}&\Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}g'_{k+1} + 3A + \mathsf{c}_{j,i}[-B]_+\\ &\overset{\triangle}{=} \Sigma'_{k+4} + [H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}g'_{k+1} + 2A + g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+\\ &\overset{\oplus}{=} \Sigma'_{k+2} + [H]_+ - \mathsf{c}_{j,i}[F]_+ - \mathsf{c}_{j,i}[-g'_{k+1}]_+ + 2A \geqslant 0,\end{aligned}$$

which completes a proof of the assertion for $k+3$. Here

(i) $\overset{\triangle}{=}$ follows from $[-B]_+ - [-B]_+ = 0$ and $A = g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+$,
(ii) $\overset{\oplus}{=}$ follows from $\Sigma'_{k+2} = \Sigma'_{k+4} + g'_{k+2}$ and $[g'_{k+1}]_+ - g'_{k+1} = [-g'_{k+1}]_+$.

(e) If $u = k+2$, we have

$$\begin{aligned}\Sigma_{k+2} &= \Sigma'_{k+5} - [-B]_+ + [G]_+ - G + [I]_+\\ &= \Sigma'_{k+5} - [-B]_+ + [-G]_+ + [I]_+.\end{aligned}$$

Thus the assertion for $k+2$ is a direct consequence of the assertion for $k+4$.

(f) If $u = k+1$, we have

$$\begin{aligned}\Sigma_{k+1} &= \Sigma'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+\\ &\qquad + \mathsf{c}_{j,i}[-G]_+ + \underline{[H]_+ - H} + \mathsf{c}_{j,i}[-I]_+\\ &= \Sigma'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+\\ &\qquad + \mathsf{c}_{j,i}[-G]_+ + \underline{[-H]_+} + \mathsf{c}_{j,i}[-I]_+.\end{aligned} \tag{3.31}$$

If $I \geqslant 0$, the assertion for $k+1$ follows from the one for $k+3$. Thus it suffices to assume that $I < 0$. As in the case (c), we divide into cases according to the signs of A and E.

$\underline{(A < 0,\ E \geqslant 0,\ I < 0)}$: By replacing I with $-F + [G]_+ + \mathsf{c}_{i,j}[-H]_+$, (3.31) becomes

$$\begin{aligned}&\Sigma'_{k+4} + A - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}[-G]_+ + [-H]_+ + \mathsf{c}_{j,i}F - \mathsf{c}_{j,i}[G]_+ - 3[-H]_+\\ &\overset{\square}{=} \Sigma'_{k+4} + g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}(F - G) - 2[-H]_+\\ &= \Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}(F - G) - 2[-H]_+.\end{aligned}$$

Here $\overset{\square}{=}$ follows from $-[-G]_+ + [G]_+ = G$. Note that the assumption $I < 0$ is equivalent to

$$[G]_+ + \mathsf{c}_{i,j}[-H]_+ < F. \tag{3.32}$$

Using (3.32), we have

$$\mathsf{c}_{j,i}(F - G) - 2[-H]_+ > \mathsf{c}_{j,i}([G]_+ + \mathsf{c}_{i,j}[-H]_+ - G) - 2[-H]_+$$

$$= \mathsf{c}_{j,i}[-G]_+ + [-H]_+ = \mathsf{c}_{j,i}[-G]_+ + [-H]_+.$$

Then

$$\Sigma_{k+1} > \left\{\Sigma'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}[-G]_+\right\} + [-H]_+. \tag{3.33}$$

Then (3.30) implies that the {}-part in (3.33) is non-negative and hence our assertion for this case follows.

$(A \geqslant 0,\, E < 0,\, I < 0)$: By replacing I with $-F+[G]_+ + \mathsf{c}_{i,j}[-H]_+$, (3.31) becomes

$$\begin{aligned} &\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ - E + \mathsf{c}_{j,i}[-G]_+ + [-H]_+ + \mathsf{c}_{j,i}F - \mathsf{c}_{j,i}[G]_+ - 3[-H]_+ \\ &\quad = \Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ - E + \mathsf{c}_{j,i}(F-G) - 2[-H]_+ \end{aligned} \tag{3.34}$$

as in the former case. Here we remark that $-E > 0$.

Using (3.32) again and (3.34), we have an inequality

$$\begin{aligned} &\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}(F-G) - 2[-H]_+ - E \\ &\quad > \Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ - E + \mathsf{c}_{j,i}([G]_+ + \mathsf{c}_{i,j}[-H]_+ - G) - 2[-H]_+ \\ &\quad = \Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}[-G]_+ + [-H]_+ - E. \end{aligned} \tag{3.35}$$

Now let us assume that

$$0 > G = -B - [-C]_+ + \mathsf{c}_{i,j}[-E]_+ + [F]_+.$$

Then (3.35) becomes

$$\begin{aligned} &\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + [-H]_+ - E + \mathsf{c}_{j,i}B + \mathsf{c}_{j,i}[-C]_+ + 3E - \mathsf{c}_{j,i}[F]_+ \\ &\quad \overset{\triangle}{=} \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [-H]_+ + 2E + \mathsf{c}_{j,i}[-C]_+ - \mathsf{c}_{j,i}[F]_+ \\ &\quad \overset{\circ}{=} \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [-H]_+ - 2A - 2\mathsf{c}_{j,i}[C]_+ + [D]_+ + \mathsf{c}_{j,i}[-C]_+ - \mathsf{c}_{j,i}[F]_+ \\ &\quad \overset{\bullet}{=} \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [-H]_+ - 2A - \mathsf{c}_{j,i}[C]_+ + [D]_+ - \mathsf{c}_{j,i}C - \mathsf{c}_{j,i}[F]_+ \\ &\quad \overset{*}{=} \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [-H]_+ - 2A - \mathsf{c}_{j,i}[C]_+ \\ &\qquad - \mathsf{c}_{j,i}[F]_+ + [D]_+ + \mathsf{c}_{j,i}g'_{k+1} + 3A + \mathsf{c}_{j,i}[-B]_+ \\ &\quad \overset{\triangledown}{=} \Sigma'_{k+4} + [-H]_+ - \mathsf{c}_{j,i}[C]_+ - \mathsf{c}_{j,i}[F]_+ + [D]_+ + A + \mathsf{c}_{j,i}g'_{k+1} \\ &\quad \overset{\odot}{=} \Sigma'_{k+4} + [-H]_+ - \mathsf{c}_{j,i}[C]_+ - \mathsf{c}_{j,i}[F]_+ + [D]_+ + g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + \mathsf{c}_{j,i}g'_{k+1} \\ &\quad \overset{\vee}{=} \Sigma'_{k+2} + [-H]_+ - \mathsf{c}_{j,i}[C]_+ - \mathsf{c}_{j,i}[F]_+ + [D]_+ - \mathsf{c}_{j,i}[-g'_{k+1}]_+. \end{aligned}$$

Here

(i) $\overset{\triangle}{=}$ follows from $[B]_+ - B = [-B]_+$,
(ii) $\overset{\circ}{=}$ follows from $E = -A - \mathsf{c}_{j,i}[C]_+ + [D]_+$,
(iii) $\overset{\bullet}{=}$ follows from $[C]_+ - [-C]_+ = C$,
(iv) $\overset{*}{=}$ follows from $C = -g'_{k+1} - \mathsf{c}_{i,j}[A]_+ - [-B]_+$,

(v) $\overset{\nabla}{=}$ follows from $[-B]_+ - [-B]_+ = 0$,
(vi) $\overset{\odot}{=}$ follows from $A = g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+$,
(vii) $\overset{\vee}{=}$ follows from $\Sigma'_{k+2} = \Sigma'_{k+4} + g'_{k+2}$ and $[g'_{k+1}]_+ - g'_{k+1} = [-g'_{k+1}]_+$.

<u>$(A \geqslant 0, E \geqslant 0, I < 0)$</u>: By replacing I with $-F+[G]_+ + \mathsf{c}_{i,j}[-H]_+$, (3.31) becomes

$$\begin{aligned}&\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}[-G]_+ + [-H]_+\\ &\quad + \mathsf{c}_{j,i}F - \mathsf{c}_{j,i}[G]_+ - 3[-H]_+\\ &\quad = \Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ - 2[-H]_+ + \mathsf{c}_{j,i}(F-G)\end{aligned} \tag{3.36}$$

as in the previous cases. Using (3.32) and (3.36), we have

$$\begin{aligned}&\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ - 2[-H]_+ + \mathsf{c}_{j,i}(F-G)\\ &\qquad > \Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + \mathsf{c}_{j,i}[-G]_+ + [-H]_+.\end{aligned} \tag{3.37}$$

Now let us assume that

$$0 > G = -B - [-C]_+ + [F]_+.$$

Then (3.37) becomes

$$\begin{aligned}&\Sigma'_{k+4} - \mathsf{c}_{j,i}[B]_+ + [-H]_+ + \mathsf{c}_{j,i}B + \mathsf{c}_{j,i}[-C]_+ - \mathsf{c}_{j,i}[F]_+\\ &\quad = \Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [-H]_+ + \mathsf{c}_{j,i}[-C]_+ - \mathsf{c}_{j,i}[F]_+\end{aligned}$$

Finally, it suffices to assume that

$$0 > C = -g'_{k+1} - \mathsf{c}_{i,j}A - [-B]_+.$$

Then we have

$$\begin{aligned}&\Sigma'_{k+4} - \mathsf{c}_{j,i}[-B]_+ + [-H]_+ - \mathsf{c}_{j,i}[F]_+ + \mathsf{c}_{j,i}g'_{k+1} + 3A + \mathsf{c}_{j,i}[-B]_+\\ &\quad = \Sigma'_{k+4} + [-H]_+ - \mathsf{c}_{j,i}[F]_+ + 2A + \mathsf{c}_{j,i}g'_{k+1} + g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+\\ &\quad = \Sigma'_{k+2} + [-H]_+ - \mathsf{c}_{j,i}[F]_+ + 2A - \mathsf{c}_{j,i}[-g'_{k+1}]_+.\end{aligned}$$

Since $A \geqslant 0$, it proves the assertion for $k+1$.

(g) If $u = k$, we have

$$\begin{aligned}\Sigma_k &= \Sigma'_{k+5} - [-B]_+ + [G]_+ - G + [I]_+ - I\\ &= \Sigma'_{k+5} - [-B]_+ + [-G]_+ + [-I]_+.\end{aligned} \tag{3.38}$$

Thus the assertion for k is also a direct consequence of the assertion for $k+4$.

(h) For $u < k$, the assertion follows from Lemma 3.20 below. □

Lemma 3.20 *Assume that $k^- \in \mathbb{N}$. Then we have*

$$\mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) = \mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) \quad \textit{for all } i \in I,$$

where $\boldsymbol{g}' \mapsto \boldsymbol{g}$ under the map given by (3.26).

Proof It suffices to show that

(i) $g_{k+5} + g_{k+3} + g_{k+1} + g_{(k+1)_i^-} = g'_{k+4} + g'_{k+2} + g'_k + g'_{k_{i'}^-}$,

(ii) $g_{k+4} + g_{k+2} + g_k + g_{k_i^-} = g'_{k+5} + g'_{k+3} + g'_{k+1} + g'_{(k+1)_i^-}$.

(i): By (3.31) and applying the techniques we have used,

$$\begin{aligned}
\Sigma_{(k+1)_i^-} &= (3.31) + g_{(k+1)_i^-} \\
&= \Sigma'_{k+4} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+ + \mathsf{c}_{j,i}[-G]_+ \\
&\quad + [-H]_+ + \mathsf{c}_{j,i}[-I]_+ + \big(g'_{k_{i'}^-} + [D]_+ \\
&\qquad\qquad -\mathsf{c}_{j,i}[F]_+ + 2[H]_+ - \mathsf{c}_{j,i}[I]_+\big) \quad (\because (3.26)) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+ + \mathsf{c}_{j,i}[-G]_+ + [-H]_+ \\
&\qquad + [D]_+ - \mathsf{c}_{j,i}[F]_+ + 2[H]_+ - \mathsf{c}_{j,i}I \qquad (\because [I]_+ - [-I]_t = I) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+ + \mathsf{c}_{j,i}[-G]_+ \\
&\quad + [-H]_+ + [D]_+ - \mathsf{c}_{j,i}[F]_+ \\
&\qquad + 2[H]_+ + \mathsf{c}_{j,i}F - \mathsf{c}_{j,i}[G]_+ - 3[-H]_+ \quad (\because I = -F + [G]_+ \\
&\quad + \mathsf{c}_{i,j}[-H]_+) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+ + [D]_+ - \mathsf{c}_{j,i}[F]_+ \\
&\quad + 2H + \mathsf{c}_{j,i}F - \mathsf{c}_{j,i}G \\
&\quad\ (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [-E]_+ + [D]_+ - \mathsf{c}_{j,i}[F]_+ - 2D \\
&\quad + 2[E]_+ + 2\mathsf{c}_{j,i}[-F]_+ - \mathsf{c}_{j,i}C + 3[-D]_+ + \mathsf{c}_{j,i}B + \mathsf{c}_{j,i}[-C]_+ \\
&\quad - 3[-E]_+ - \mathsf{c}_{j,i}[F]_+ \\
&\quad (\because (3.27) \text{ for } H, F \text{ and } G) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [D]_+ + 3[-D]_+ + \mathsf{c}_{j,i}[-C]_+ \\
&\quad - 2D + 2E - 2\mathsf{c}_{j,i}F - \mathsf{c}_{j,i}C + \mathsf{c}_{j,i}B \qquad (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - [-A]_+ - \mathsf{c}_{j,i}[B]_+ + [D]_+ + 3[-D]_+ + \mathsf{c}_{j,i}[-C]_+ - 2g'_k \\
&\quad - 2\mathsf{c}_{j,i}[-g'_{k+1}]_+ + 4[-A]_+ - 2\mathsf{c}_{j,i}[-C]_+ - 2A - 2\mathsf{c}_{j,i}[C]_+ \\
&\quad + 2[D]_+ + 2\mathsf{c}_{j,i}C - 6[-D]_+ + \mathsf{c}_{j,i}g'_{k+1} + 3[A]_+ \\
&\quad + \mathsf{c}_{j,i}[-B]_+ + \mathsf{c}_{j,i}g'_{k+3} - \mathsf{c}_{j,i}[-g'_{k+1}]_+ \\
&\quad (\because (3.27) \text{ for } B, C, D, E \text{ and } F) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - 2g'_k - 3\mathsf{c}_{j,i}[-g'_{k+1}]_+ + 3[-A]_+ - 3\mathsf{c}_{j,i}[-C]_+ + \mathsf{c}_{j,i}g'_{k+1} \\
&\quad + 3[A]_+ + \mathsf{c}_{j,i}g'_{k+3} - 2A - \mathsf{c}_{j,i}B + 3D \qquad (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{k+4} + g'_{k_{i'}^-} - 2g'_k - 3\mathsf{c}_{j,i}[-g'_{k+1}]_+ + 3[-A]_+ - 3\mathsf{c}_{j,i}[-C]_+
\end{aligned}$$

$$
\begin{aligned}
&+ \mathsf{c}_{j,i} g'_{k+1} + 3[A]_+ \\
&+ \mathsf{c}_{j,i} g'_{k+3} - 2A - \mathsf{c}_{j,i} B + 3g'_k + 3\mathsf{c}_{j,i}[-g'_{k+1}]_+ - 6[-A]_+ + 3\mathsf{c}_{j,i}[-C]_+ \\
&(\because (3.27) \text{ for } D) \\
&= \Sigma'_{k+4} + g'_{k^-_{i'}} + g'_k + A + \mathsf{c}_{j,i} g'_{k+1} + \mathsf{c}_{j,i} g'_{k+3} - \mathsf{c}_{j,i} B \\
&\quad (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{k+4} + g'_{k^-_{i'}} + g'_k + g'_{k+2} - \mathsf{c}_{j,i}[g'_{k+1}]_+ + \mathsf{c}_{j,i} g'_{k+1} \\
&\qquad + \mathsf{c}_{j,i} g'_{k+3} - \mathsf{c}_{j,i} g'_{k+3} + \mathsf{c}_{j,i}[-g'_{k+1}]_+ \quad (\because (3.27) \text{ for } A \text{ and } B) \\
&= \Sigma'_{k+4} + g'_{k^-_{i'}} + g'_k + g'_{k+2} (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{k^-_{i'}}
\end{aligned}
$$

In the above computation, we repeatedly substitute the relations in (3.27) and use $[x]_+ - [-x]_+ = x$ to cancel all auxiliary terms.

(ii): By (3.38), we have

$$
\begin{aligned}
\Sigma_{k^-_i} &= (3.38) + g_{k^-_i} \\
&= \Sigma'_{k+5} - [-B]_+ + [-G]_+ + [-I]_+ + (g'_{(k+1)^-_{i'}} + [g'_{k+1}]_+ \\
&\quad + [B]_+ - [-G]_+ - [-I]_+) \\
&= \Sigma'_{k+5} + B + g'_{(k+1)^-_{i'}} + [g'_{k+1}]_+ \qquad (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{k+5} + g'_{k+3} - [-g'_{k+1}]_+ + [g'_{k+1}]_+ + g'_{(k+1)^-_{i'}} \qquad (\because (3.27) \text{ for } B) \\
&= \Sigma'_{k+5} + g'_{k+3} + g'_{k+1} + g'_{(k+1)^-_{i'}} \qquad (\because [x]_+ - [-x]_+ = x) \\
&= \Sigma'_{(k+1)^-_{i'}}.
\end{aligned}
$$

□

The following lemma can be obtained as in other moves and we omit the proof.

Lemma 3.21 *Let $\boldsymbol{g}, \boldsymbol{g}' \in \mathbb{Z}^{\oplus \mathbb{N}}$ be elements related by (3.12). If $\boldsymbol{g}'_s = 0$ for $s \in [k, k+3]$, we have*

$$\mathsf{p}_{\boldsymbol{i}}(\boldsymbol{g}; i) = \mathsf{p}_{\boldsymbol{i}'}(\boldsymbol{g}'; i) \qquad \textit{for any } i \in I.$$

Example 3.22 Take $\boldsymbol{i}' = (2, 1, 2, 1, 2, 1, \ldots)$ and $\boldsymbol{i} = (1, 2, 1, 2, 1, 2, \ldots)$. Using the above formulae, the map $\theta_1^{(1)*} : \boldsymbol{g}' \mapsto \boldsymbol{g}$ in (3.26) that is induced by the isomorphism $\zeta_1^{(1)*} : \mathcal{A}_{\boldsymbol{i}'} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}}$ can be descried for special $\boldsymbol{g}$'s as follows:

$$
\begin{aligned}
&\theta_1^{(1)*}(\boldsymbol{e}_1) = \boldsymbol{e}_6 - \boldsymbol{e}_4, && \theta_1^{(1)*}(\boldsymbol{e}_2) = (\boldsymbol{e}_6 - \boldsymbol{e}_4) + 3\boldsymbol{e}_1, \\
&\theta_1^{(1)*}(\boldsymbol{e}_3 - \boldsymbol{e}_1) = 2(\boldsymbol{e}_6 - \boldsymbol{e}_4) + 3\boldsymbol{e}_1, && \theta_1^{(1)*}(\boldsymbol{e}_4 - \boldsymbol{e}_2) = (\boldsymbol{e}_6 - \boldsymbol{e}_4) + 2\boldsymbol{e}_1, \\
&\theta_1^{(1)*}(\boldsymbol{e}_5 - \boldsymbol{e}_3) = (\boldsymbol{e}_6 - \boldsymbol{e}_4) + 3\boldsymbol{e}_1, && \theta_1^{(1)*}(\boldsymbol{e}_6 - \boldsymbol{e}_4) = \boldsymbol{e}_1.
\end{aligned}
\tag{3.39}
$$

Similarly, the map $\theta_1^{(2)*}$ induced by the isomorphism $\zeta_1^{(2)*} : \mathcal{A}_{\boldsymbol{i}} \xrightarrow{\sim} \mathcal{A}_{\boldsymbol{i}'}$ can be descried as follows:

$$\begin{aligned} &\theta_1^{(2)*}(\boldsymbol{e}_1) = (\boldsymbol{e}_6 - \boldsymbol{e}_4), && \theta_1^{(2)*}(\boldsymbol{e}_2) = 3(\boldsymbol{e}_6 - \boldsymbol{e}_4) + \boldsymbol{e}_1, \\ &\theta_1^{(2)*}(\boldsymbol{e}_3 - \boldsymbol{e}_1) = 2(\boldsymbol{e}_6 - \boldsymbol{e}_4) + \boldsymbol{e}_1, && \theta_1^{(2)*}(\boldsymbol{e}_4 - \boldsymbol{e}_2) = 3(\boldsymbol{e}_6 - \boldsymbol{e}_4) + 2\boldsymbol{e}_1, \\ &\theta_1^{(2)*}(\boldsymbol{e}_5 - \boldsymbol{e}_3) = (\boldsymbol{e}_6 - \boldsymbol{e}_4) + \boldsymbol{e}_1, && \theta_1^{(2)*}(\boldsymbol{e}_6 - \boldsymbol{e}_4) = \boldsymbol{e}_1. \end{aligned} \tag{3.40}$$

3.7 Forward shifts

In this subsection, we consider a forward shift of $\boldsymbol{i} \in I^{\mathbb{N}}$ introduced in [11] when $\mathfrak{g}$ is of simply-laced type. However, in this subsection, we do not impose any restriction on the type of $\mathfrak{g}$.

We say that $\boldsymbol{i}' = (i'_k)_{k\in\mathbb{N}}$ is obtained from $\boldsymbol{i} = (i_k)_{k\in\mathbb{N}}$ via a forward shift if

$$i'_k = i_{k+1} \quad \text{for all } k \in \mathbb{N}.$$

In this case, we write $\boldsymbol{i}' = \partial_+\boldsymbol{i}$. Throughout this subsection, we assume that $\boldsymbol{i}, \boldsymbol{i}' \in I^{(\infty)}$ are related by $\boldsymbol{i}' = \partial_+\boldsymbol{i}$ and $i_1 := i$. Define

(a) a sequence $(x_u)_{u\in\mathbb{N}}$ such that $x_1 = 1$ and $x_u = (x_{u-1})_{\boldsymbol{i}}^+$ for $u \geqslant 2$,

(b) a permutation $\sigma_+ \in \mathfrak{S}_{\mathbb{N}}$ by $\sigma_+(k) = \begin{cases} k_{\boldsymbol{i}}^+ - 1 & \text{if } i_k = i, \\ k - 1 & \text{if } i_k \neq i. \end{cases}$

Remark 3.23 Even though [11] considered only simply-laced types, the statements and proofs in [11, §2.3] can be extended to non simply-laced types. Thus we only give the statements and leave their proofs to the reader.

Proposition 3.24 (a) *The limit* $\lim_{n\to\infty} \mu_{x_n} \cdots \mu_{x_2}\mu_{x_1} \widetilde{B}^{\boldsymbol{i}}$ *yields a well-defined matrix* $\mu_+\widetilde{B}^{\boldsymbol{i}}$ *and we have* $\sigma_+\mu_+\widetilde{B}^{\boldsymbol{i}} = \widetilde{B}^{\boldsymbol{i}'}$.

(b) *The limit* $\lim_{n\to\infty} \mu_{x_n} \cdots \mu_{x_2}\mu_{x_1} \Lambda^{\boldsymbol{i}}$ *yields a well-defined matrix* $\mu_+\Lambda^{\boldsymbol{i}}$ *and we have* $\sigma_+\mu_+\Lambda^{\boldsymbol{i}} = \Lambda^{\boldsymbol{i}'}$.

(c) *We have a homomorphism of* $\mathbb{A}$*-algebras*

$$\partial_+^*(= \mu_+^*\sigma_+^*) \colon \mathcal{A}_{\boldsymbol{i}'} \to \mathcal{A}_{\boldsymbol{i}} \qquad \textit{given by } Z_u \mapsto \mu_+(Z_{\sigma_+^{-1}(u)}) \textit{ for all } u \in \mathbb{N},$$

which sends each cluster monomial in $\mathcal{A}_{\boldsymbol{i}'}$ *to a cluster monomial in* $\mathcal{A}_{\boldsymbol{i}}$.

(d) *If* $\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}'}$ *is a quantum cluster monomial with* $\boldsymbol{deg}(\mathsf{z}) = \boldsymbol{g}' = (g'_u)_{u\in\mathbb{N}}$, *then the quantum cluster monomial* $\partial_+^*\mathsf{z} \in \mathcal{A}_{\boldsymbol{i}}$ *has* $\boldsymbol{deg}(\partial_+^*\mathsf{z}) = \boldsymbol{g} = (g_u)_{u\in\mathbb{N}}$ *as follows:*

$$g_u = \begin{cases} -\sum_{v\in\mathbb{N};\, i'_v = i_1} g'_v & \textit{if } u = 1, \\ g'_{u-1} & \textit{if } u > 1. \end{cases} \tag{3.41}$$

In other words, the assignment $\boldsymbol{g}' \mapsto \boldsymbol{g}$ *is given by the* $\mathbb{N}_0$*-linear map* $C_{\boldsymbol{i}'} \to C_{\boldsymbol{i}}$ *which sends* $\boldsymbol{e}_u - \boldsymbol{e}_{u_{\boldsymbol{i}'}^-} \in C_{\boldsymbol{i}'}$ *to* $\boldsymbol{e}_{u+1} - \boldsymbol{e}_{(u+1)_{\boldsymbol{i}}^-} \in C_{\boldsymbol{i}}$ *for all* $u \in \mathbb{N}$.

4 Bases and braid moves

In this section, we first recall the quantum unipotent coordinate algebra $\mathcal{A}_q(\mathfrak{n})$ of $\mathfrak{g}$ and its bases. Then we apply the result in the previous section to investigate the relationship among braid moves and the bases.

4.1 Quantum unipotent coordinate algebra

Let $\mathcal{U}_q(\mathfrak{g})$ be the quantum group of $\mathfrak{g}$ over $\mathbb{Q}(q^{1/2})$. It is generated by e_i, f_i $(i \in I)$ and q^h, where h belongs to the dual weight lattice $\mathsf{P}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}, \mathbb{Z})$. We denote by $\mathcal{U}_q^+(\mathfrak{g})$ the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by e_i $(i \in I)$ and $\mathcal{U}_{\mathbb{A}}^+(\mathfrak{g})$ the $\mathbb{A}$-subalgebra of $\mathcal{U}_q^+(\mathfrak{g})$ generated by $e_i/[n]_i!$, where

$$q_i := q^{\mathsf{d}_i}, \quad [k]_i := \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad \text{and} \quad [k]_i! := \prod_{s=1}^{k} [s_i] \quad \text{for } k \in \mathbb{N}.$$

Set

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta \in \mathsf{Q}^-} \mathcal{A}_q(\mathfrak{n})_\beta \quad \text{where } \mathcal{A}_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q^{1/2})}(\mathcal{U}_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q^{1/2})),$$

and $\mathcal{U}_q^+(\mathfrak{g})_{-\beta}$ denotes the $(-\beta)$-root space of $\mathcal{U}_q^+(\mathfrak{g})$. For an element $a \in \mathcal{A}_q(\mathfrak{n})_\beta$, we set $\mathrm{wt}(a) := \beta$. It is known that $\mathcal{A}_q(\mathfrak{n})$ has an algebra structure and is called the *quantum unipotent coordinate algebra* of $\mathfrak{g}$. We denote by $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ the $\mathbb{A}$-submodule of $\mathcal{A}_q(\mathfrak{n})$ generated by $\psi \in \mathcal{A}_q(\mathfrak{n})$ such that $\psi(\mathcal{U}_{\mathbb{A}}^+(\mathfrak{g})) \subset \mathbb{A}$. Then $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ also has an $\mathbb{A}$-algebra structure.

For each $\lambda \in \mathsf{P}^+ := \sum_{i \in I} \mathbb{N}\varpi_i$ and Weyl group elements $w, w' \in \mathsf{W}$, we can define a specific homogeneous element $D(w\lambda, w'\lambda)$ of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$, called a *quantum unipotent minor* (see for example [29, Section 9]). Note that $D(w\lambda, w'\lambda)$ vanishes if $w\lambda - w'\lambda \notin \mathsf{Q}^-$. We set

$$\widetilde{D}(w\lambda, w'\lambda) := q^{-(w'\lambda - w\lambda, w'\lambda - w\lambda)/4 - (w'\lambda - w\lambda, \rho)/2} D(w\lambda, w'\lambda)$$

where $\rho := \sum_{i \in I} \varpi_i \in \mathsf{P}^+$, and call it a *normalized quantum unipotent minor*.

Let us fix a reduced sequence $\boldsymbol{i}$ of $w_\circ$. For $1 \leqslant k \leqslant \ell$, we set

$$\widetilde{F}_{\boldsymbol{i}}(k) := \widetilde{D}(w_k^{\boldsymbol{i}} \varpi_{i_k}, w_{k^-}^{\boldsymbol{i}} \varpi_{i_k}).$$

Note that $\mathrm{wt}(\widetilde{F}_{\boldsymbol{i}}(k)) = -\beta_k^{\boldsymbol{i}} \in \Phi^-$ as an element in $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$. For $\boldsymbol{c} = (c_1, \ldots, c_\ell) \in \mathbb{N}_0^\ell$, we set

$$\nu(\boldsymbol{c}) = -\frac{1}{2} \sum_{1 \leqslant k, l \leqslant \ell} c_k c_l (\beta_k^{\boldsymbol{i}}, \beta_l^{\boldsymbol{i}}) + \sum_{s=1}^{\ell} c_s^2 \in \mathbb{Z}$$

and define

$$\widetilde{F}_{\boldsymbol{i}}(\boldsymbol{c}) := q^{\nu(\boldsymbol{c})/2} \prod_{1 \leqslant k \leqslant \ell}^{\rightarrow} \widetilde{F}_{\boldsymbol{i}}(k)^{c_k}.$$

Proposition 4.1 (see, for instance, [46, Chapters 40,41]) *The set* $\{\widetilde{F}_{\boldsymbol{i}}(\boldsymbol{c}) \mid \boldsymbol{c} \in \mathbb{N}_0^{\ell}\}$ *forms an* $\mathbb{A}$*-basis of* $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$.

There exist an anti-involution $\overline{(\cdot)}$ on $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ defined by

$$q^{\pm 1/2} \longmapsto q^{\mp 1/2} \quad \text{and} \quad \widetilde{F}_{\boldsymbol{i}}(k) \longmapsto \widetilde{F}_{\boldsymbol{i}}(k)$$

for all $1 \leqslant k \leqslant \ell$, which is called the *twisted bar-involution*.

4.2 Comparison with braid moves

Lusztig [44, 45] and Kashiwara [30] have constructed a distinguished $\mathbb{A}$-basis $\mathbf{B}^{\mathrm{up}}$, which is called the *dual-canonical/upper-global basis*. In this paper, we consider the normalized one $\widetilde{\mathbf{B}}^{\mathrm{up}}$ of $\mathbf{B}^{\mathrm{up}}$ (see [43] for more details). Note that the normalized quantum unipotent minor $\widetilde{D}(w\lambda, w'\lambda)$ belongs to $\widetilde{\mathbf{B}}^{\mathrm{up}}$ whenever it is non-zero.

Theorem 4.2 ([43, Theorem 4.29]) *For a reduced sequence* $\boldsymbol{i}$ *of* $w_\circ$*, we have*

$$\widetilde{\boldsymbol{B}}^{up} = \{\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c}) \mid \boldsymbol{c} \in \mathbb{N}_0^{\ell}\}$$

where $\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c})$ *is the unique element of* $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ *satisfying the following properties for each* $\boldsymbol{c} \in \mathbb{N}_0^{\Phi^+}$*:*

(a) $\overline{\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c})} = \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c})$,
(b) $\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c}) - \widetilde{F}_{\boldsymbol{i}}(\boldsymbol{c}) \in \sum_{\boldsymbol{c}' <_{\boldsymbol{i}} \boldsymbol{c}} q\mathbb{Z}[q] \widetilde{F}_{\boldsymbol{i}}(\boldsymbol{c}')$,

where $<_{\boldsymbol{i}}$ *is a certain bi-lexicographical order determined by* $\boldsymbol{i}$*. In particular,* $\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_k) = \widetilde{D}(w_k^{\boldsymbol{i}} \varpi_{i_k}, w_{k^-}^{\boldsymbol{i}} \varpi_{i_k})$.

Proposition 4.3 ([46, Chapter 42], [1, Theorem 3.1]) *Let* $\boldsymbol{i}$ *and* $\boldsymbol{i}'$ *be two reduced sequences for* $w_\circ$*, and* $\boldsymbol{c} = (c_1, \ldots, c_\ell)$, $\boldsymbol{c}' = (c'_1, \ldots, c'_\ell) \in \mathbb{N}_0^{\ell}$.

(1) *Assume* $\boldsymbol{i}' = \gamma_k \boldsymbol{i}$ *for some* $1 \leqslant k \leqslant \ell - 1$*. Then we have*

$$\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c}) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{c}) \quad \textit{if and only if} \quad \begin{cases} (c_k, c_{k+1}) = (c'_{k+1}, c'_k), \\ c_u = c'_u \ \textit{if}\ u \notin \{k, k+1\}. \end{cases}$$

(2) *Assume* $\boldsymbol{i}' = \beta_k \boldsymbol{i}$ *for some* $1 \leqslant k \leqslant \ell - 2$*. Then we have*

$$\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c}) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{c}) \quad \textit{if and only if} \quad \begin{cases} c'_k = c_{k+1} + c_{k+2} - \min(c_k, c_{k+2}), \\ c'_{k+1} = \min(c_k, c_{k+2}), \\ c'_{k+2} = c_{k+1} + c_k - \min(c_k, c_{k+2}), \\ c_u = c'_u \ \textit{if}\ u \notin \{k, k+1, k+2\}. \end{cases}$$

(3) *Assume* $\boldsymbol{i}' = \eta_k \boldsymbol{i}$ *for some* $1 \leqslant k \leqslant \ell - 3$. *Note that* $(i_k, i_{k+1}, i_{k+2}, i_{k+3})$ *and* $(i'_k, i'_{k+1}, i'_{k+2}, i'_{k+3})$ *can be regarded as reduced sequences of* $w_\circ$ *of type* B_2, *and hence we have parametrizations of these sequences in terms of* $\Phi^+_{B_2}$ (*see Example 3.1*). *Then we have* (*see also [53, Lemma 2.15]*)

$$\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c}) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{c}') \quad \Longleftrightarrow \quad \begin{cases} c'_{\alpha_2} = c_{\alpha_1+\alpha_2} + 2c_{\alpha_1+2\alpha_2} + c_{\alpha_2} - \pi_2, \\ c'_{\alpha_1+2\alpha_2} = \pi_2 - \pi_1, \\ c'_{\alpha_1+\alpha_2} = 2\pi_1 - \pi_2, \\ c'_{\alpha_1} = c_{\alpha_1} + c_{\alpha_1+\alpha_2} + c_{\alpha_1+2\alpha_2} - \pi_1, \\ c'_u = c_u \ \text{if}\ u \notin \{k, k+1, k+2, k+3\}, \end{cases}$$

where

$$\pi_1 = \min\{c_{\alpha_1} + c_{\alpha_1+\alpha_2}, c_{\alpha_1} + c_{\alpha_2}, c_{\alpha_1+2\alpha_2} + c_{\alpha_2}\},$$
$$\pi_2 = \min\{2c_{\alpha_1} + c_{\alpha_1+\alpha_2}, 2c_{\alpha_1} + c_{\alpha_2}, 2c_{\alpha_1+2\alpha_2} + c_{\alpha_2}\}.$$

(4) *For* $\boldsymbol{i}' = (2, 1, 2, 1, 2, 1)$ *and* $\boldsymbol{i} = (1, 2, 1, 2, 1, 2)$ *of* G_2*-type, we have*

$$\begin{aligned} &\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_1) = \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_6), && \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_2) = \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_1 + \boldsymbol{e}_6), \\ &\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_3) = \widetilde{G}_{\boldsymbol{i}}(3\boldsymbol{e}_1 + 2\boldsymbol{e}_6), && \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_4) = \widetilde{G}_{\boldsymbol{i}}(2\boldsymbol{e}_1 + 3\boldsymbol{e}_6), \\ &\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_5) = \widetilde{G}_{\boldsymbol{i}}(3\boldsymbol{e}_1 + \boldsymbol{e}_6), && \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_6) = \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_1), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_1) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_6), && \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_2) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_1 + 3\boldsymbol{e}_6), \\ &\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_3) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_1 + 2\boldsymbol{e}_6), && \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_4) = \widetilde{G}_{\boldsymbol{i}'}(2\boldsymbol{e}_1 + 3\boldsymbol{e}_6), \\ &\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_5) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_6 + \boldsymbol{e}_1), && \widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_6) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_1). \end{aligned} \tag{4.2}$$

Theorem 4.4 ([13, 15, 16, 29, 31, 36, 51]) *For each reduced sequence* $\boldsymbol{i} = (i_1, \ldots, i_\ell)$ *of* $w_\circ$, *there is an isomorphism of* $\mathbb{A}$*-algebras*

$$\varphi_{\boldsymbol{i}} \colon \mathcal{A}_{\boldsymbol{i}} \xrightarrow{\sim} \mathcal{A}_{\mathbb{A}}(\mathfrak{n})$$

which satisfies the following properties:

(1) *For any* $1 \leqslant v < u \leqslant \ell$ *with* $i_u = i_v$, *there exists a cluster variable which corresponds to* $\widetilde{D}(w^{\boldsymbol{i}}_u \varpi_{i_u}, w^{\boldsymbol{i}}_v \varpi_{i_v})$ *under* $\varphi_{\boldsymbol{i}}$.

(2) *The initial seed of* $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ *corresponding to that of* $\mathcal{A}_{\boldsymbol{i}}$ *is given by*

$$\big(\{\widetilde{D}(w^{\boldsymbol{i}}_k \varpi_{i_k}, \varpi_{i_k})\}_{1 \leqslant k \leqslant \ell}, \Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}}\big). \tag{4.3}$$

(3) *Every cluster monomial of* $\mathcal{A}_{\boldsymbol{i}}$ *corresponds to an element of the basis* $\widetilde{\boldsymbol{B}}^{up}$ *under* $\varphi_{\boldsymbol{i}}$.

(4) *For each* $\boldsymbol{c} = (c_u) \in \mathbb{N}_0^{\ell}$, *the element* $\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c})$ *is pointed and we have*

$$\boldsymbol{deg}\big(\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{c})\big) = \sum_{1 \leqslant u \leqslant \ell} c_u(\boldsymbol{e}_u - \boldsymbol{e}_{u^-}) \in \mathbb{Z}^{\ell}, \tag{4.4}$$

where we understand $\boldsymbol{e}_0 = 0$. *In particular, the map* $\boldsymbol{deg} \circ \varphi_{\boldsymbol{i}}^{-1} \colon \widetilde{\boldsymbol{B}}^{up} \to \mathbb{Z}^{\ell}$ *is injective.*

The following proposition is proved for 2-moves and 3-moves in [11, Proposition 3.3], and we extend it to 4-moves and 6-moves:

Proposition 4.5 *Let* $\boldsymbol{i}, \boldsymbol{i}'$ *be two reduced sequences for* $w_\circ$. *Assume that we have* $\boldsymbol{i}' = \tau \boldsymbol{i}$ *for some* $\tau \in \{\gamma_1, \ldots, \gamma_{\ell-1}\} \sqcup \{\beta_1, \ldots, \beta_{\ell-2}\} \sqcup \{\eta_1, \ldots, \eta_{\ell-3}\} \cup \{\zeta_1\}$. *Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}_{\boldsymbol{i}'} \arrow[rr, "\tau^*"] \arrow[dr, "\varphi_{\boldsymbol{i}'}"'] & & \mathcal{A}_{\boldsymbol{i}} \arrow[dl, "\varphi_{\boldsymbol{i}}"] \\ & \mathcal{A}_{\mathbb{A}}(\mathfrak{n}) & \end{tikzcd}$$

Proof Since $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$ is generated by $\{\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u) \mid 1 \leqslant u \leqslant \ell\}$, it is enough to show that

$$\varphi_{\boldsymbol{i}}\tau^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u) = \widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u) \tag{4.5}$$

for all $u \in [1, \ell]$. By Theorem 4.4 and the fact that τ^* preserves cluster monomials, we have

$$\varphi_{\boldsymbol{i}}\tau^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u) \in \widetilde{\mathbf{B}}^{\mathrm{up}} \quad \text{and} \quad \mathbf{deg}\big(\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u)\big) = (\boldsymbol{e}_u - \boldsymbol{e}_{u_{\boldsymbol{i}'}^-}).$$

$\underline{(\eta)}$: Let us consider $\tau = \eta_k$ for some $1 \leqslant u \leqslant \ell - 2$. Recall A and B in (3.13).

(i) $u = k + 3$. In this case

$$\mathbf{deg}\big(\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_{k+3})\big) = (\boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1}), \quad A = \mathsf{c}_{j,i} < 0 \quad \text{and } B = -1.$$

Then we have $g_k = -1 + \mathsf{c}_{j,i}\mathsf{c}_{i,j} = 1$, $g_{k_{\boldsymbol{i}}^-} = -1$, $g_{k+3} = g_{k+2} = g_{k+1} = g_{(k+1)_{\boldsymbol{i}}^-} = 0$ by (3.12). Hence

$$\begin{aligned} \mathbf{deg}\big(\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_{k+3})\big) &= \boldsymbol{e}_k - \boldsymbol{e}_{k_{\boldsymbol{i}}^-} \\ &= \mathbf{deg}\big(\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_k)\big). \end{aligned}$$

(ii) $u = k + 2$. In this case

$$\mathbf{deg}\big(\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_{k+2})\big) = \boldsymbol{e}_{k+2} - \boldsymbol{e}_k, \quad A = -1 \quad \text{and} \quad B = \mathsf{c}_{i,j} < 0.$$

Then we have $g_k = -g_{k_i^-} = -\mathsf{c}_{i,j}$, $g_{k+3} = -g_{k+1} = 1$, $g_{k+2} = g_{(k+1)_i^-} = 0$. Hence

$$\begin{aligned}\mathbf{deg}\big(\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_{k+2})\big) &= (\boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1}) - \mathsf{c}_{i,j}(\boldsymbol{e}_k - \boldsymbol{e}_{k_i^-}) \\ &= \mathbf{deg}\big(\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(-\mathsf{c}_{i,j}\boldsymbol{e}_k + \boldsymbol{e}_{k+3})\big).\end{aligned}$$

(iii) $u = k+1$. In this case

$$\mathbf{deg}\big(\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_{k+1})\big) = (\boldsymbol{e}_{k+1} - \boldsymbol{e}_{(k+1)_{i'}^-}) = (\boldsymbol{e}_{k+1} - \boldsymbol{e}_{k_i^-}), \quad A = 0 \quad \text{and} \quad B = -1.$$

Then we have $g_k = -g_{k_i^-} = 1$, $g_{k+3} = -g_{k+1} = -\mathsf{c}_{j,i}$, $g_{k+2} = g_{(k+1)_i^-} = 0$. Hence

$$\begin{aligned}\mathbf{deg}\big(\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_{k+1})\big) &= (\boldsymbol{e}_k - \boldsymbol{e}_{k_i^-}) - \mathsf{c}_{j,i}(\boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1}) \\ &= \mathbf{deg}\big(\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_k - \mathsf{c}_{j,i}\boldsymbol{e}_{k+3})\big).\end{aligned}$$

(iv) $u = k$. In this case

$$\mathbf{deg}\big(\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_k)\big) = (\boldsymbol{e}_k - \boldsymbol{e}_{k_{i'}^-}) = (\boldsymbol{e}_k - \boldsymbol{e}_{(k+1)_i^-}),$$

Then we have $g_{k+3} = 1$, $g_{k+1} = -1$, $g_{k+2} = g_k = g_{k_{i'}^-} = g_{(k+1)_i^-} = g_{k_i^-} = 0$, by (3.12) and (3.13). Hence

$$\mathbf{deg}\big(\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_k)\big) = (\boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1}) = \big(\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_{k+3})\big).$$

(v) $u \notin \{k, k+1, k+2, k+3\}$. In this case, one can easily see that

$$\mathbf{deg}\big(\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u)\big) = (\boldsymbol{e}_u - \boldsymbol{e}_{u_i^-}) = \mathbf{deg}\varphi_{\boldsymbol{i}}^{-1}\widetilde{G}_{\boldsymbol{i}}(\boldsymbol{e}_u).$$

In conclusion, we have

$$\mathbf{deg}\big(\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u)\big) = \begin{cases} \boldsymbol{e}_k - \boldsymbol{e}_{k_i^-} & \text{if } u = k+3, \\ (\boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1}) - \mathsf{c}_{i,j}(\boldsymbol{e}_k - \boldsymbol{e}_{k_i^-}) & \text{if } u = k+2, \\ (\boldsymbol{e}_k - \boldsymbol{e}_{k_i^-}) - \mathsf{c}_{j,i}(\boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1}) & \text{if } u = k+1, \\ \boldsymbol{e}_{k+3} - \boldsymbol{e}_{k+1} & \text{if } u = k, \\ \boldsymbol{e}_u - \boldsymbol{e}_{u_i^-} & \text{otherwise,} \end{cases}$$

and get $\varphi_{\boldsymbol{i}}\eta_k^*\varphi_{\boldsymbol{i}'}^{-1}\widetilde{G}_{\boldsymbol{i}'}(\boldsymbol{e}_u) = \widetilde{G}_{\boldsymbol{i}}(\mathbf{c}_u)$, where

$$\mathbf{c}_u = \begin{cases} \boldsymbol{e}_k & \text{if } u = k+3, \\ \boldsymbol{e}_{k+3} - \mathsf{c}_{i,j}\boldsymbol{e}_k & \text{if } u = k+2, \\ \boldsymbol{e}_k - \mathsf{c}_{j,i}\boldsymbol{e}_{k+3} & \text{if } u = k+1, \\ \boldsymbol{e}_{k+3} & \text{if } u = k, \\ \boldsymbol{e}_u & \text{otherwise.} \end{cases}$$

Then our assertion for η_k follows from Proposition 4.3 (3).
$\underline{(\zeta)}$: The assertion for ζ-case follows from (3.39), (3.40), (4.1), (4.2) and Proposition 4.3 (4). □

Note that any two reduced sequences of $w_\circ$ are connected by braid moves; i.e., there exists a finite sequence $\boldsymbol{\tau} = (\tau_1, \dots, \tau_l)$ in $\{\gamma_1, \dots, \gamma_{\ell-1}\} \sqcup \{\beta_1, \dots, \beta_{\ell-2}\} \sqcup \{\eta_1, \dots, \eta_{\ell-3}\} \cup \{\zeta_1\}$ such that $\boldsymbol{i}' = \tau_1 \cdots \tau_l \boldsymbol{i}$. Then we have the isomorphism

$$\boldsymbol{\tau}^* = \tau_l^* \circ \cdots \circ \tau_1^* : \mathcal{A}_{\boldsymbol{i}'} \simeq \mathcal{A}_{\boldsymbol{i}}.$$

Corollary 4.6 *With the above notation, the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}_{\boldsymbol{i}'} \arrow[rr, "\boldsymbol{\tau}^*"] \arrow[dr, "\varphi_{\boldsymbol{i}'}"'] & & \mathcal{A}_{\boldsymbol{i}} \arrow[dl, "\varphi_{\boldsymbol{i}}"] \\ & \mathcal{A}_{\mathbb{A}}(\mathfrak{n}) & \end{tikzcd}$$

Moreover, the isomorphism $\boldsymbol{\tau}^* = \varphi_{\boldsymbol{i}'}^{-1} \circ \varphi_{\boldsymbol{i}}$ depends only on the pair $(\boldsymbol{i}, \boldsymbol{i}')$, and not on the sequence $\boldsymbol{\tau}$ satisfying $\boldsymbol{i}' = \tau_1 \cdots \tau_l \boldsymbol{i}$.

5 Homomorphisms and substitution formulas

In this section, we first review the quantum (virtual) Grothendieck ring $\mathfrak{K}_q(\mathfrak{g})$ associated with $\mathfrak{g}$, its bases and subrings. For $\mathfrak{g}$ of simply-laced type, the ring $\mathfrak{K}_q(\mathfrak{g})$ is introduced in [18, 48, 55] while it is introduced in [25, 26, 39] for $\mathfrak{g}$ of non-simply-laced type. Then we study homomorphisms on subrings of $\mathfrak{K}_q(\mathfrak{g})$ and the skew field of fraction $\mathbb{F}(\mathcal{X}_q)$, which are closely related to braid moves and the theory of quantum cluster algebras.

5.1 Quantum torus $\mathcal{X}_q(\mathfrak{g})$ and the ring $\mathfrak{K}_q(\mathfrak{g})$

Let t be an indeterminate. For $i, j \in I$, we set

$$\underline{\mathsf{c}}_{i,j}(t) := \delta_{i,j}(t + t^{-1}) + \delta(i \neq j)\mathsf{c}_{i,j}$$

and call the matrix $\underline{\mathsf{C}}(t) := (\underline{\mathsf{c}}_{i,j}(t))$ the *t-quantized Cartan matrix*. Then

$$\underline{\mathsf{B}}(t) := \underline{\mathsf{C}}(t)\mathsf{D}^{-1} \text{ is a symmetric matrix in } \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}]).$$

Regarding $\underline{\mathsf{B}}(t)$ as a $\mathbb{Q}(t)$-valued matrix, it is invertible and we denote the inverse of $\underline{\mathsf{B}}(t)$ by $\widetilde{\underline{\mathsf{B}}}(t) = (\widetilde{\underline{\mathsf{B}}}_{i,j}(t))_{i,j \in I}$. Write

$$\widetilde{\underline{\mathsf{B}}}_{i,j}(t) = \sum_{u \in \mathbb{Z}} \widetilde{\underline{\mathsf{b}}}_{i,j}(u) t^u \in \mathbb{Z}(\!(t)\!)$$

for the Laurent expansion of $\widetilde{\underline{\mathsf{B}}}_{i,j}(t)$ at $t = 0$. It is proved in [22, 39] that, for any $i, j \in I$ and $u \in \mathbb{Z}$, we have

$$\widetilde{\underline{\mathsf{b}}}_{i,j}(u) = 0 \quad \text{if (i) } u \leqslant d(i,j) \quad \text{or} \quad \text{(ii) } d(i,j) \equiv u \pmod 2. \tag{5.1}$$

We fix a parity function $\epsilon : I \to \{0, 1\}$ such that $\epsilon_i \neq \epsilon_j$ for $i \sim j$, based on (5.1), and define

$$\widehat{\triangle}_0 := \{(i,p) \in I \times \mathbb{Z} \mid p \equiv \epsilon_i \pmod 2\} \subsetneq I \times \mathbb{Z}. \tag{5.2}$$

Definition 5.1 *([18, 39, 48, 56])* We define a quantum torus $(\mathcal{X}_q(\mathfrak{g}), *)$ to be the $\mathbb{A}$-algebra generated by $\{\widetilde{X}^{\pm}_{i,p} \mid (i,p) \in \widehat{\triangle}_0\}$ subject to the following relations:

$$\widetilde{X}_{i,p} * \widetilde{X}^{-1}_{i,p} = \widetilde{X}^{-1}_{i,p} * \widetilde{X}_{i,p} = 1 \quad \text{and} \quad \widetilde{X}_{i,p} * \widetilde{X}_{j,s} = q^{\underline{\mathcal{N}}(i,p;j,s)} \widetilde{X}_{j,s} * \widetilde{X}_{i,p},$$

where

$$\underline{\mathcal{N}}(i,p;j,s) = \widetilde{\underline{\mathsf{b}}}_{i,j}(p-s-1) - \widetilde{\underline{\mathsf{b}}}_{i,j}(s-p-1) - \widetilde{\underline{\mathsf{b}}}_{i,j}(p-s+1) \\ + \widetilde{\underline{\mathsf{b}}}_{i,j}(s-p+1).$$

Note that there exists a $\mathbb{Z}$-algebra anti-involution on $\mathcal{X}_q(\mathfrak{g})$ given by

$$q^{1/2} \longmapsto q^{-1/2} \quad \text{and} \quad \widetilde{X}_{i,p} \longmapsto q^{\mathsf{d}_i} \widetilde{X}_{i,p}.$$

We will write $\mathcal{X}_q$ instead of $\mathcal{X}_q(\mathfrak{g})$ if there is no danger of confusion. We denote by $\mathcal{X}$ the commutative Laurent polynomial ring obtained from $\mathcal{X}_q$ by specializing q at 1. We write the map $\mathrm{ev}_{q=1} : \mathcal{X}_q \to \mathcal{X}$ and set $X_{i,p} := \mathrm{ev}_{q=1}(q^{a/2}\widetilde{X}_{i,p})$ for any $a \in \mathbb{Z}$. We call a product $\widetilde{m}$ of $\widetilde{X}^{\pm 1}_{i,p}$ and $q^{\pm 1/2}$ in $\mathcal{X}_q$ a *monomial*; i.e., it is of the form

$$\widetilde{m} = q^{a/2} \overset{\to}{\underset{(i,p)\in\widehat{\triangle}_0}{*}} \widetilde{X}^{u_{i,p}(\widetilde{m})}_{i,p} \quad \text{for } a, u_{i,p}(\widetilde{m}) \in \mathbb{Z}.$$

We also call a monomial $\widetilde{m} \in \mathcal{X}_q$ *commutative* if $\widetilde{m} = \overline{\widetilde{m}}$, and *dominant* (resp.*anti-dominant*) if it is a product of $\widetilde{X}_{i,p}$ (resp. $\widetilde{X}^{-1}_{i,p}$) and $q^{\pm 1/2}$.

For a monomial $m \in \mathcal{X}$, we can associate the commutative monomial $\underline{m} \in \mathcal{X}_q$ such that $\mathrm{ev}_{q=1}(\underline{m}) = m$. Thus we write $\underline{m}$ for commutative monomials in $\mathcal{X}_q$. Let $\mathcal{M}$ be the set of all monomials in $\mathcal{X}$, and $\mathcal{M}_+$ be the set of all dominant monomials in $\mathcal{X}$.

For monomials $\widetilde{m}, \widetilde{m}'$ in $\mathcal{X}_q$, define

$$\underline{\mathcal{N}}(\widetilde{m}, \widetilde{m}') := \sum_{(i,p),(j,s)\in\widehat{\triangle}_0} u_{i,p}(\widetilde{m}) u_{j,s}(\widetilde{m}') \underline{\mathcal{N}}(i,p;j,s).$$

For each $(i,p) \in I \times \mathbb{Z}$ such that $(i, p \pm 1) \in \widehat{\triangle}_0$, define the commutative monomial

$$\widetilde{\mathscr{B}}_{i,p} := \underline{\mathscr{B}}_{i,p},$$

where $\mathscr{B}_{i,p} = X_{i,p-1}X_{i,p+1}\prod_{j\sim i} X_{j,p}^{\mathsf{c}_{j,i}} \in \mathcal{X}$.

Definition 5.2 *([18, 26, 39])*

(a) For each $i \in I$, denote by $\mathfrak{K}_{i,q}(\mathfrak{g})$ the $\mathbb{A}$-subalgebra of $\mathcal{X}_q$ generated by

$$\widetilde{X}_{i,p} * (1 + q^{-\mathsf{d}_i}\widetilde{\mathscr{B}}_{i,p+1}^{-1}) \text{ and } \widetilde{X}_{j,s}^{\pm 1} \quad \text{for } j \in I \setminus \{i\} \text{ and } (i,p),(j,s) \in \widehat{\Delta}_0.$$

(b) Define

$$\mathfrak{K}_q(\mathfrak{g}) := \bigcap_{i \in I} \mathfrak{K}_{i,q}(\mathfrak{g})$$

and call it the *quantum virtual Grothendieck ring* associated with $\underline{\mathsf{C}}(t)$. If there is no confusion, we write $\mathfrak{K}_q$ instead of $\mathfrak{K}_q(\mathfrak{g})$.

5.2 Bases

For $m, m' \in \mathcal{M}$, there exists a partial order $\preccurlyeq_{\mathrm{N}}$, called *Nakajima order*, defined by

$$m \preccurlyeq_{\mathrm{N}} m' \text{ if and only if } m^{-1}m' \text{ is a product of elements in } \{\mathscr{B}_{i,p} \mid (i, p \pm 1) \in \widehat{\Delta}_0\}.$$

For monomials $\widetilde{m}, \widetilde{m}' \in \mathcal{X}_q$, we write $\widetilde{m} \preccurlyeq_{\mathrm{N}} \widetilde{m}'$ if $\mathrm{ev}_{q=1}(\widetilde{m}) \preccurlyeq_{\mathrm{N}} \mathrm{ev}_{q=1}(\widetilde{m}')$.

Theorem 5.3 ([18, 26], cf. [8]) *For each dominant monomial $m \in \mathcal{M}_+$, there exists a unique element $F_q(m)$ of $\mathfrak{K}_q$ such that*

$$\underline{m} \text{ is the unique dominant monomial appearing in } F_q(m) \text{ and } \overline{F_q(m)} = F_q(m).$$

Furthermore,

(a) *a monomial appearing in $F_q(m) - \underline{m}$ is strictly less than $\underline{m}$ with respect to $\preccurlyeq_N$,*
(b) *$\mathsf{F}_q := \{F_q(m) \mid m \in \mathcal{M}_+\}$ forms an $\mathbb{A}$-basis of $\mathfrak{K}_q$,*
(c) *$\mathfrak{K}_q$ is generated by the set $\{F_q(X_{i,p}) \mid (i,p) \in \widehat{\Delta}_0\}$ as an $\mathbb{A}$-algebra.*

For $m \in \mathcal{M}_+$, we set

$$E_q(m) := q^b \left(\overset{\to}{\underset{p\in\mathbb{Z}}{*}} \left(\underset{i;\in I,(i,p)\in\widehat{\Delta}_0}{*} F_q(X_{i,p})^{u_{i,p}(m)} \right) \right),$$

where b is an element in $\frac{1}{2}\mathbb{Z}$ such that $\underline{m}$ appears in $E_q(m)$ with coefficient 1. Then the set $\mathsf{E}_q := \{E_q(m) \mid m \in \mathcal{M}_+\}$ also forms an $\mathbb{A}$-basis of $\mathfrak{K}_q(\mathfrak{g})$, called the *standard basis*, and satisfies

$$E_q(m) = F_q(m) + \sum_{m' \prec_{\mathrm{N}} m} C_{m,m'} F_q(m') \quad \text{for some } C_{m,m'} \in \mathbb{A}. \tag{5.3}$$

Theorem 5.4 ([18, 26, 48]) *For each $m \in \mathcal{M}_+$, there exists a unique element $L_q(m)$ in $\mathfrak{K}_q(\mathfrak{g})$ such that*

$$\text{(i) } \overline{L_q(m)} = L_q(m) \text{ and}$$

$$\text{(ii) } L_q(m) = E_q(m) + \sum_{m' \prec_N m} Q_{m,m'}(q) E_q(m') \text{ for some } Q_{m,m'}(q) \in q\mathbb{Z}[q]. \tag{5.4}$$

Furthermore,

(a) *a monomial appearing in $L_q(m) - \underline{m}$ is strictly less than $\underline{m}$ with respect to $\preccurlyeq_N$,*
(b) $\mathsf{L}_q := \{L_q(m) \mid m \in \mathcal{M}_+\}$ *forms an $\mathbb{A}$-basis of $\mathfrak{K}_q$ called the* canonical basis *of $\mathfrak{K}_q(\mathfrak{g})$,*
(c) $L_q(X_{i,p}) = F_q(X_{i,p})$ *for all $(i, p) \in \widehat{\triangle}_0$.*

Let $\mathfrak{D}_q^{\pm 1}$ be the $\mathbb{A}$-algebra automorphism of $\mathcal{X}_q$ given by $\widetilde{X}_{i,p} \mapsto \widetilde{X}_{i^*,p\pm\mathsf{h}}$, where h is the Coxeter number of $\mathfrak{g}$. It is well-known that it induces a $\mathbb{Z}$-algebra automorphism $\mathfrak{D}$ on $\mathcal{X}$ satisfying $\mathrm{ev}_{q=1} \circ \mathfrak{D}_q^{\pm 1} = \mathfrak{D}^{\pm 1} \circ \mathrm{ev}_{q=1}$ and preserves the subalgebra $\mathfrak{K}_q$ of $\mathcal{X}_q$. Moreover, we have

$$\mathfrak{D}_q^{\pm 1}(F_q(m)) = F_q(\mathfrak{D}^{\pm 1}m) \quad \text{and} \quad \mathfrak{D}_q^{\pm 1}(L_q(m)) = L_q(\mathfrak{D}^{\pm 1}m) \quad \text{for any } m \in \mathcal{M}_+. \tag{5.5}$$

Remark 5.5 Note that, for $r \in 2\mathbb{Z}$, there exists an automorphism T_r of $\mathcal{X}_q$ sending $\widetilde{X}_{i,p}$ to $\widetilde{X}_{i,p+r}$ satisfying the following properties:

(a) The restriction to $\mathfrak{K}_q$ gives an automorphism of $\mathfrak{K}_q$.
(b) For each $m \in \mathcal{M}_+$, we have $\mathsf{T}_r(F_q(m)) = F_q(\mathtt{T}_r(m))$, $\mathsf{T}_r(E_q(m)) = E_q(\mathtt{T}_r(m))$ and $\mathsf{T}_r(L_q(m)) = L_q(\mathtt{T}_r(m))$. Here $\mathtt{T}_r$ denotes the induced automorphism on $\mathcal{X}$.

5.3 Kirillov–Reshetikhin and fundamental polynomials

For a pair $(i, p), (i, s) \in \widehat{\triangle}_0$ with $p \leqslant s$, we set

$$m^{(i)}[p, s] := \prod_{(i,u) \in \widehat{\triangle}_0,\ p \leqslant u \leqslant s} X_{i,s} \in \mathcal{M}_+. \tag{5.6}$$

We call $m^{(i)}[p, s]$ the *Kirillov–Reshetikhin(KR) monomial* and $F_q(m^{(i)}[p, s]) \in \mathfrak{K}_q(\mathfrak{g})$ the *KR-polynomial* for the pair (i, p) and (i, s), respectively. In particular, we call $F_q(X_{i,p}) = F_q(m^{(i)}[p, p])$ the *fundamental polynomial* for each $(i, p) \in \widehat{\triangle}_0$.

Theorem 5.6 ([10, 20, 47]) *For each KR-monomial $m^{(i)}[p, s]$, the KR-polynomial $F_q(m^{(i)}[p, s])$ satisfies the following properties:*

(a) *$F_q(m^{(i)}[p, s])$ contains the unique dominant monomial $\underline{m^{(i)}[p, s]}$.*
(b) *$F_q(m^{(i)}[p, s])$ contains the unique anti-dominant monomial $\underline{m^{(i^*)}[p + \mathsf{h}, s + \mathsf{h}]}$.*

(c) *Each* $\mathcal{X}_q$*-monomial of* $F_q(\underline{m^{(i)}[p,s]}) - \underline{m^{(i)}[p,s]} - \underline{m^{(i^*)}[p+\mathsf{h}, s+\mathsf{h}]}$ *is a product of* $\widetilde{X}_{j,u}^{\pm 1}$ *with* $p \leqslant u \leqslant s+\mathsf{h}$*, having at least one of its factors from* $p < u < s+\mathsf{h}$*. In particular, each monomial of* $F_q(\underline{\widetilde{X}_{i,p}}) - \underline{\widetilde{X}_{i,p}} - \underline{\widetilde{X}_{i^*,p+\mathsf{h}}}$ *is a product of* $\widetilde{X}_{j,u}^{\pm 1}$ *with* $p < u < p+\mathsf{h}$*.*

5.4 Height functions and their negative subrings

A *height function* ξ *on* $\triangle$ is a function $\xi : I \to \mathbb{Z}$ given by $i \mapsto \xi_i$ satisfying

$$\xi_i \equiv \epsilon_i \pmod 2 \quad \text{and} \quad |\xi_i - \xi_j| = 1 \quad \text{if } i \sim j.$$

For a height function ξ on $\triangle$, we set

$${}^{\xi}\widehat{\triangle}_0 := \{(i,p) \in \widehat{\triangle}_0 \mid p \leqslant \xi_i\}.$$

We call a bijective function $\widetilde{\boldsymbol{\rho}} : \mathbb{N} \to {}^{\xi}\widehat{\triangle}_0$ *a compatible reading of* ${}^{\xi}\widehat{\triangle}_0$ if $k > l$ when $\widetilde{\boldsymbol{\rho}}(k) = (i,p)$, $\widetilde{\boldsymbol{\rho}}(l) = (j,p+1) \in {}^{\xi}\widehat{\triangle}_0$ and $i \sim j$. For a compatible reading $\widetilde{\boldsymbol{\rho}}$ of ${}^{\xi}\widehat{\triangle}_0$, we set $\boldsymbol{\rho} : \mathbb{N} \to I^{(\infty)}$ where $\boldsymbol{\rho}(k) = i_k$ for $\widetilde{\boldsymbol{\rho}}(k) = (i_k, p_k) \in {}^{\xi}\widehat{\triangle}_0$. We abuse $\boldsymbol{\rho}$ to be the image $(i_1, i_2, \ldots) \in I^{\mathbb{N}}$ of $\mathbb{N}$ under $\boldsymbol{\rho}$ and call it an I*-compatible reading of* ${}^{\xi}\widehat{\triangle}_0$.

For each $(i,p) \in {}^{\xi}\widetilde{\triangle}_0$, set the KR-monomial

$$Z^{\xi}_{i,p} := \underline{m^{(i)}[p, \xi_i]} \in \mathcal{X}_q.$$

When there is no danger of confusion, we write $Z_{i,p}$ instead of $Z^{\xi}_{i,p}$ for notational simplicity. Denote by $\mathcal{X}_{q,\leqslant\xi}$ the quantum subtorus of $\mathcal{X}_q$ generated by $\{\widetilde{X}_{i,p}\}_{(i,p)\in{}^{\xi}\widehat{\triangle}_0}$.

Proposition 5.7 ([39, Theorem 7.1], see also [3, Proposition 5.1.1]) *Let* $\boldsymbol{\rho} = (i_1, i_2, \ldots)$ *and* $\boldsymbol{\rho}' = (j_1, j_2, \ldots)$ *be* I*-compatible readings of* ${}^{\xi}\widehat{\triangle}_0$*.*

(i) *For* $u \leqslant v$*, we have*

$$\underline{\mathcal{N}}(Z^{\xi}_{i_u,p_u}, Z^{\xi}_{i_v,p_v}) = (\varpi_{i_u} - w^{\boldsymbol{\rho}}_u \varpi_{i_u}, \varpi_{i_v} + w^{\boldsymbol{\rho}}_u \varpi_{i_v}),$$

where $\widetilde{\boldsymbol{\rho}}(u) = (i_u, p_u)$ *and* $\widetilde{\boldsymbol{\rho}}(v) = (i_v, p_v)$*. Hence we have an* $\mathbb{A}$*-algebra isomorphism*

$$\widetilde{\kappa}_{\boldsymbol{\rho}} : \mathcal{T}(\Lambda^{\boldsymbol{\rho}}) \xrightarrow{\sim} \mathcal{X}_{q,\leqslant\xi} \quad \text{given by} \quad Z_u \longmapsto Z^{\xi}_{i_u,p_u} = \underline{m^{(i_u)}[p_u, \xi_{i_u}]}. \tag{5.7}$$

(ii) *We have* $\boldsymbol{\rho} \overset{c}{\sim} \boldsymbol{\rho}'$ *and there exists* $\pi \in \mathfrak{S}_{\mathbb{N}}$ *such that*

$$(\Lambda^{\boldsymbol{\rho}}, \widetilde{B}^{\boldsymbol{\rho}}) = \pi(\Lambda^{\boldsymbol{\rho}'}, \widetilde{B}^{\boldsymbol{\rho}'}).$$

For an element $\widetilde{x} \in \mathcal{X}_q$, we denote by $\widetilde{x}_{\leqslant \xi}$ the element of $\mathcal{X}_{q,\leqslant \xi}$ obtained from $\widetilde{x}$ by discarding all the monomials containing factors $\widetilde{X}^{\pm 1}_{i,p}$ with $(i,p) \in \widehat{\triangle}_0 \setminus {}^{\xi}\widehat{\triangle}_0$. We write the $\mathbb{A}$-linear map

$$(\cdot)_{\leqslant \xi} : \mathcal{X}_q \longrightarrow \mathcal{X}_{q,\leqslant \xi}, \qquad \widetilde{x} \longmapsto \widetilde{x}_{\leqslant \xi}.$$

Definition 5.8 Let $\mathfrak{K}_{q,\xi}$ be the subring of $\mathfrak{K}_q$ generated by $\{F_q(X_{i,p}) \mid (i,p) \in {}^{\xi}\widehat{\triangle}_0\}$ and call it the ξ*-negative subring* of $\mathfrak{K}_q$.

We set
$$(\Lambda^{\xi}, \widetilde{B}^{\xi}) := \left(\Lambda^{[\boldsymbol{\rho}]}, \widetilde{B}^{[\boldsymbol{\rho}]}\right).$$

Remark 5.9 In [25], the subring $\mathfrak{K}_{q,\xi}$ was written as $\mathfrak{K}_{q,\leqslant \xi}$. We hope there is no confusion by these notations.

Proposition 5.10 ([21, Proposition 6.1], [26, Proposition 6.3]) *For a height function ξ on $\triangle$, the $\mathbb{A}$-linear map $(\cdot)_{\leqslant \xi}$ induces the injective $\mathbb{A}$-algebra homomorphism*

$$(\cdot)_{\leqslant \xi} : \mathfrak{K}_{q,\xi} \hookrightarrow \mathcal{X}_{q,\leqslant \xi}. \tag{5.8}$$

We call $(\cdot)_{\leqslant \xi}$ in (5.8) the *truncation homomorphism*. Note that, for a monomial $m^{(i)}[p, \xi_i]$ with $p \leqslant \xi_i$ and $p \equiv_2 \xi_i$, we have

$$(F_q(m^{(i)}[p, \xi_i]))_{\leqslant \xi} = \underline{m^{(i)}[p, \xi_i]} \tag{5.9}$$

by Theorem 5.6 (c).

Proposition 5.11 (cf. [23, 39] see also [11, Proposition 5.14]) *For a Dynkin quiver $Q = (\triangle, \xi)$, $\boldsymbol{e}_u$ $(u \in \mathbb{N})$ and an I-compatible reading $\boldsymbol{\rho}$ of ${}^{\xi}\widehat{\triangle}_0$, let us set*

$$\boldsymbol{b}^{\boldsymbol{\rho}}_u = \widetilde{B}^{[\boldsymbol{\rho}]}\boldsymbol{e}_u = \sum_{k \in \mathbb{N}} b_{k,u}\boldsymbol{e}_k.$$

Then we have

$$\widetilde{\kappa}_{\boldsymbol{\rho}}(Z^{\boldsymbol{b}^{\boldsymbol{\rho}}_u}) = \widetilde{\mathscr{B}}^{-1}_{i,p-1} \tag{5.10}$$

where $\widetilde{\boldsymbol{\rho}}(u) = (i,p)$.

Proof We claim that
$$\widetilde{\kappa}_{\boldsymbol{\rho}}(Z^{\boldsymbol{b}^{\boldsymbol{\rho}}_u}) \cong \widetilde{\mathscr{B}}^{-1}_{i,p-1}.$$

Note that $b_{u^+,u} = -1$, $b_{u^-,u} = 1$, $\widetilde{\boldsymbol{\rho}}(u^+) = (i, p-2)$ and $\widetilde{\boldsymbol{\rho}}(u^-) = (i, p+2)$ by the assumption on $\boldsymbol{\rho}$. Then we have

$$\widetilde{\kappa}_{\boldsymbol{\rho}}(Z^{-\boldsymbol{e}_{u^+}+\boldsymbol{e}_{u^-}}) \cong \widetilde{X}^{-1}_{i,p}\widetilde{X}^{-1}_{i,p-2}.$$

For $j \sim i$, we see $u^{\pm}(j)$ satisfy $u < u^{+}(j) < u^{+} < u^{+}(j)^{+}$, $u^{-}(j) < u < u^{-}(j)^{+} = u^{+}(j) < u^{+}$, $\widetilde{\boldsymbol{\rho}}(u^{+}(j)) = (j, p-1)$ and $\widetilde{\boldsymbol{\rho}}(u^{-}(j)) = (j, p+1)$. Since $b_{u^{+}(j),u} = -\mathsf{c}_{j,i}$ and $b_{u^{-}(j),u} = \mathsf{c}_{j,i}$, we have

$$\widetilde{\kappa}_{\boldsymbol{\rho}}(Z^{-\mathsf{c}_{j,i}\boldsymbol{e}_{u^{+}(j)}+\mathsf{c}_{j,i}\boldsymbol{e}_{u^{-}(j)}}) \cong \widetilde{X}_{j,p-1}^{-\mathsf{c}_{j,i}}.$$

Hence the claim follows. Since $\widetilde{\kappa}_{\boldsymbol{\rho}}$ is compatible with the bar involutions on $\mathcal{T}(\Lambda^{\boldsymbol{\rho}})$ and $\mathcal{X}_{q,\leqslant\xi}$, we obtain (5.10). □

Theorem 5.12 ([11, Theorem 5.15], [26, Theorem 8.8])

(i) *For an I-compatible reading $\boldsymbol{\rho}$ of ${}^{\xi}\widehat{\triangle}_0$, there exists a unique $\mathbb{A}$-algebra isomorphism $\kappa_{\boldsymbol{\rho}}$ which makes the following diagram commutative:*

$$\begin{tikzcd} \mathcal{A}_{\boldsymbol{\rho}} \arrow[r, "\kappa_{\boldsymbol{\rho}}"] \arrow[d, hook] & \mathfrak{K}_{q,\xi} \arrow[d, hook, "(\cdot)_{\leqslant\xi}"] \\ \mathcal{T}(\Lambda^{\boldsymbol{\rho}}) \arrow[r, "\widetilde{\kappa}_{\boldsymbol{\rho}}"] & \mathcal{X}_{q,\leqslant\xi} \end{tikzcd}$$

(ii) *For $k \in \mathbb{N}$, we have $\kappa_{\boldsymbol{\rho}}(Z_k) = F_q(\underline{m^{(i)}[p,\xi_i]})$ where $\widetilde{\boldsymbol{\rho}}(k) = (i,p)$. Moreover every KR-polynomial $F_q(m)$ in $\mathfrak{K}_{q,\xi}$ appears as a cluster variable.*

(iii) $\mathsf{F}_{q,\xi} := \mathsf{F}_q \cap \mathfrak{K}_{q,\xi}$, $\mathsf{E}_{q,\xi} := \mathsf{E}_q \cap \mathfrak{K}_{q,\xi}$ *and* $\mathsf{L}_{q,\xi} := \mathsf{L}_q \cap \mathfrak{K}_{q,\xi}$ *are bases of* $\mathfrak{K}_{q,\xi}$.

Remark 5.13

(a) The entire ring $\mathfrak{K}_q(\mathfrak{g})$ has also a quantum cluster algebra structure (see [11, Theorem 6.17], [26, Theorem 9.6]), but we will not use the structure in this paper.

(b) The image $(\mathfrak{K}_{q,\xi})_{\leqslant\xi}$ of $\mathfrak{K}_{q,\xi}$ under $(\cdot)_{\leqslant\xi}$ has a cluster algebra structure with initial cluster variables $\{\underline{m^{(i)}[p,\xi_i]} \mid i \in I,\ p \leqslant \xi_i\}$ by (5.9) and Theorem 5.12 (ii).

Theorem 5.14 ([3, 23, 26]) *For each KR-monomial $m^{(i)}[p,s]$ and any height function ξ on $\triangle$ such that $\xi_i \geqslant s$, there exists a quantum cluster algebra algorithm for calculating $(F_q(m^{(i)}[p,s]))_{\leqslant\xi}$, which can be obtained from the initial cluster variable $\underline{m^{(i)}[p+2k,\xi_i]}$ of $(\mathfrak{K}_{q,\xi})_{\leqslant\xi}$ with $\xi_i = s+2k$ $(k \in \mathbb{N}_0)$ via a certain sequence of mutations.*

Remark 5.15 By Theorem 5.6 (c), $(F_q(m^{(i)}[p,s]))_{\leqslant\xi} = F_q(m^{(i)}[p,s])$ if k in Theorem 5.14 is big enough.

Conjecture 1 ([26])

(i) *For each KR-monomial $m^{(i)}[p,s]$, the element $F_q(m^{(i)}[p,s])$ is quantum positive; i.e., each coefficient of $F_q(m^{(i)}[p,s])$ is contained in $\mathbb{N}_0[q^{\pm 1/2}]$.*

(ii) *For each KR-monomial $m^{(i)}[p,s]$, we have $L_q(m^{(i)}[p,s]) = F_q(m^{(i)}[p,s])$.*

The conjecture in the above is proved by Nakajima in [48] for symmetric cases (see also [11]). By Theorem 5.14, Conjecture 1 is closely related to the quantum Laurent positivity conjecture suggested by Berenstein–Fomin–Zelevinsky.

5.5 Dynkin quivers and their heart subrings

A *Dynkin quiver* Q is a pair $(\triangle, \xi)$ consisting of (i) a Dynkin diagram $\triangle$ of $\mathfrak{g}$ and (ii) a height function ξ on $\triangle$.

For a Dynkin quiver $Q = (\triangle, \xi)$, we call $i \in I$ a *source* (resp. *sink*) of Q, if $\xi_i > \xi_j$ (resp. $\xi_i < \xi_j$) for all $j \in I$ with $i \sim j$. For a Dynkin quiver $Q = (\triangle, \xi)$ and its source i, we denote by $s_i Q$ the Dynkin quiver $(\triangle, s_i\xi)$, where $s_i\xi$ is the height function defined by

$$(s_i\xi)_j := \xi_j - 2 \times \delta_{i,j}.$$

We call the operation from Q to $s_i Q$ the reflection of Q at the source i of Q. For any Dynkin quivers Q and Q' with the same Dynkin diagram $\triangle$, there exists a sequence of reflections, including s_i^{-1}, from Q to Q'.

For a sequence of indices $\boldsymbol{i} = (i_1, i_2, \ldots, i_l) \in I^l$ ($l \in \mathbb{N} \sqcup \{\infty\}$) and a Dynkin quiver $Q = (\triangle, \xi)$, we call $\boldsymbol{i}$ *Q-adapted* if

$$i_k \text{ is a source of } s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1} Q \text{ for all } 1 \leqslant k \leqslant l.$$

The followings are known (see [22, 37, 39]):

(A) Any I-compatible reading of ${}^{\xi}\widehat{\triangle}_0$ is Q-adapted.

(B) For each Dynkin quiver Q, there exists a unique Coxeter element $\tau_Q \in \mathsf{W}$, all of whose reduced sequences are adapted to Q. Conversely, for any Coxeter element τ, there exists a unique Dynkin quiver Q such that $\tau = \tau_Q$.

(C) For each Dynkin quiver Q, there exists a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to Q and the set of all reduced sequence of $w_\circ$ adapted to Q forms a single commutation class of $w_\circ$, denoted by $[Q]$.

(D) For a reduced sequence $\boldsymbol{i} = (i_1, \ldots, i_\ell)$ of $w_\circ$ adapted to Q, define $\widehat{\boldsymbol{\imath}} = (\imath_1, \imath_2, \ldots) \in I^{(\infty)}$ by

$$\imath_k = i_k \quad \text{for } 1 \leqslant k \leqslant \ell \quad \text{and} \quad \imath_k = \imath^*_{k-\ell} \quad \text{for all } k > \ell. \tag{5.11}$$

Then $\widehat{\boldsymbol{\imath}}$ is an I-compatible reading of ${}^{\xi}\widehat{\triangle}_0$ and we have a compatible reading $\widetilde{\boldsymbol{\imath}}$ of ${}^{\xi}\widehat{\triangle}_0$ corresponding to $\widehat{\boldsymbol{\imath}}$. Furthermore, we have $\widehat{\boldsymbol{\imath}} \overset{c}{\sim} \boldsymbol{\rho}$ for any I-compatible reading $\boldsymbol{\rho}$ of ${}^{\xi}\widehat{\triangle}_0$.

(E) For each Dynkin quiver $Q = (\triangle, \xi)$, there exists a bijection $\phi_Q : \widehat{\triangle}_0 \to \Phi^+ \times \mathbb{Z}$ defined recursively as follows: Set $\gamma_i^Q := (1 - \tau_Q)\varpi_i$ and

(i) $\phi_Q(i, \xi_i) := (\gamma_i^Q, 0)$ for each $i \in I$,

(ii) if $\phi_i(i, p) = (\alpha, k)$, we have

$$\phi_Q(i, p \pm 2) := \begin{cases} (\tau_Q^{\mp 1}\alpha, k) \text{ if } \tau_Q^{\mp 1} \in \Phi^+, \\ (-\tau_Q^{\mp 1}\alpha, k \pm 1) \text{ if } \tau_Q^{\mp 1} \in \Phi^-. \end{cases} \tag{5.12}$$

For a Dynkin quiver $Q = (\triangle, \xi)$, we denote by $\mathfrak{D}^{\pm 1} Q$ the Dynkin quiver $(\triangle, \mathfrak{D}^{\pm 1}\xi)$, where $\mathfrak{D}^{\pm 1}\xi$ is the height function defined as follows:

$$(\mathfrak{D}^{\pm 1}\xi)_j := \xi_{j^*} \pm \mathsf{h} \quad \text{for } j \in I.$$

Proposition 5.16 ([12, 22, 39]) *Let Q be a Dynkin quiver and set $\Gamma_0^Q := \phi_Q^{-1}(\Phi^+ \times \{0\})$. Then Γ_0 is a subset of $\widehat{\triangle}_0$ characterized by*

$$\Gamma_0^Q = \{(i, p) \in \widehat{\triangle}_0 \mid \xi_{i^*} - \mathsf{h} < p \leqslant \xi_i\}.$$

Furthermore, for any $(j, s) \in \widehat{\triangle}_0$, there exists unique $k \in \mathbb{Z}$ and $(i, p) \in \Gamma_0^Q$ such that

$$\mathfrak{D}^k(X_{i,p}) = X_{j,s}.$$

We define $\mathcal{X}_{q,Q}$ the quantum subtorus of $\mathcal{X}_q$ generated by $\{\widetilde{X}_{i,p} \mid (i, p) \in \Gamma_0^Q\}$, and set

$$\mathcal{M}^Q := \mathcal{M} \cap \mathcal{X}_{q,Q} \quad \text{and} \quad \mathcal{M}_+^Q := \mathcal{M}_+ \cap \mathcal{X}_{q,Q}.$$

We denote by $\mathfrak{K}_{q,Q}$ the subring of $\mathfrak{K}_{q,\xi}$ which is generated by $\{F_q(X_{i,p}) \mid (i, p) \in \Gamma_0^Q\}$. We call it the *heart* subring associated with the Dynkin quiver $Q = (\triangle, \xi)$.

Theorem 5.17 ([10, 11, 22, 24–26]) *Let $\boldsymbol{i}$ be a reduced sequence of $w_\circ$ adapted to $Q = (\triangle, \xi)$.*

(i) *We have $\mathbb{A}$-algebra isomorphisms $\widetilde{\kappa}_{\boldsymbol{i}} : \mathcal{T}(\Lambda^{\boldsymbol{i}}) \to \mathcal{X}_{q,Q}$ and $\kappa_{\boldsymbol{i}} : \mathcal{A}_{\boldsymbol{i}} \to \mathfrak{K}_{q,Q}$, and an $\mathbb{A}$-algebra monomorphism $(\cdot)_{\leqslant\xi}|_{\mathfrak{K}_{q,Q}}$ which make the following diagram commutative:*

$$\begin{tikzcd}
 & \mathcal{A}_{\boldsymbol{i}} \arrow[rr, "\kappa_{\boldsymbol{i}}"] \arrow[dl, hook'] \arrow[dd, hook] & & \mathfrak{K}_{q,Q} \arrow[dl, hook'] \arrow[dd, hook, "(\cdot)_{\leqslant\xi}|_{\mathfrak{K}_{q,Q}}"] \\
\mathcal{A}_{\widehat{\boldsymbol{i}}} \arrow[rr, "\kappa_{\widehat{\boldsymbol{i}}}"] \arrow[dd, hook] & & \mathfrak{K}_{q,\xi} \arrow[dd, hook, "(\cdot)_{\leqslant\xi}"] & \\
 & \mathcal{T}(\Lambda^{\boldsymbol{i}}) \arrow[rr, "\widetilde{\kappa}_{\boldsymbol{i}}"'] \arrow[dl, hook'] & & \mathcal{X}_{q,Q} \arrow[dl, hook'] \\
\mathcal{T}({}^{\widehat{\boldsymbol{i}}}\Lambda) \arrow[rr, "\widetilde{\kappa}_{\widehat{\boldsymbol{i}}}"'] & & \mathcal{X}_{q,\leqslant\xi} &
\end{tikzcd}$$

(ii) *The set $\mathsf{F}_{q,Q} := \mathsf{F}_q \cap \mathfrak{K}_{q,Q}$, $\mathsf{E}_{q,Q} := \mathsf{E}_q \cap \mathfrak{K}_{q,Q}$ and $\mathsf{L}_{q,Q} := \mathsf{L}_q \cap \mathfrak{K}_{q,Q}$ are bases of $\mathfrak{K}_{q,Q}$.*

The following theorem is proved in [25, Corollary 5.4] which gives a partial answer for Conjecture 1 (ii).

Theorem 5.18 ([25, Corollary 5.4]) *For a KR-monomial m contained in $\mathcal{M}_+^Q$ for some Dynkin quiver $Q = (\triangle, \xi)$, we have*

$$L_q(m) = F_q(m).$$

Recall that, for a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to a Dynkin quiver Q, we have an $\mathbb{A}$-algebra isomorphism

$$\varphi_{\boldsymbol{i}} : \mathcal{A}_{\boldsymbol{i}} \xrightarrow{\sim} \mathcal{A}_{\mathbb{A}}(\mathfrak{n}).$$

Hence we have an $\mathbb{A}$-algebra isomorphism

$$\Psi_{\boldsymbol{i}} := \kappa_{\boldsymbol{i}} \circ \varphi_{\boldsymbol{i}}^{-1} : \mathcal{A}_{\mathbb{A}}(\mathfrak{n}) \xrightarrow{\sim} \mathfrak{K}_{q,Q}. \tag{5.13}$$

Moreover, for another reduced sequence $\boldsymbol{i}'$ of $w_\circ$ adapted to Q, we have $\Psi_{\boldsymbol{i}} = \Psi_{\boldsymbol{i}'}$ (see [10, 25] for more details about the independence of the choice of $\boldsymbol{i} \in [Q]$ for the isomorphism $\Psi_{\boldsymbol{i}}$). Thus the notation Ψ_Q make sense.

Theorem 5.19 ([22], [10, §10.3], [25, §7])

(a) *For each Dynkin quiver Q, the isomorphism Ψ_Q sends $\widetilde{\boldsymbol{B}}^{up}$ to $\mathsf{L}_{q,Q}$.*
(b) *For a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to Q and a KR-monomial $m^{(i)}[p,s]$ in $\mathcal{M}_+^Q$, there exists a pair of integers (t,u) such that $0 \leqslant t < u \leqslant \ell$,*

$$\text{(i) } i_t = i_u \quad \text{if } t > 0 \quad \text{and} \quad \text{(ii) } \Psi_Q(\widetilde{D}(w_u^{\boldsymbol{i}}\varpi_{i_u}, w_t^{\boldsymbol{i}}\varpi_{i_u})) = F_q(m^{(i)}[p,s]).$$

Here we understand $w_0^{\boldsymbol{i}} = id \in \mathsf{W}$.
(c) *Let $Q = (\triangle, \xi)$ and $Q' = (\triangle, \xi')$ be Dynkin quivers defined on the same $\triangle$. Then we have a unique $\mathbb{A}$-algebra automorphism*

$$\Psi_{\mathfrak{Q},\mathfrak{Q}'} : \mathfrak{K}_q \longrightarrow \mathfrak{K}_q \tag{5.14}$$

satisfying the following properties:

(i) *The automorphisms $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ restricted to $\mathfrak{K}_{q,Q'}$ and $\mathfrak{K}_{q,\xi}$ respectively induce $\mathbb{A}$-algebra isomorphisms*

$$\Psi_{Q,Q'} : \mathfrak{K}_{q,Q'} \xrightarrow{\sim} \mathfrak{K}_{q,Q} \quad \text{and} \quad \Psi_{\xi,\xi'} : \mathfrak{K}_{q,\xi'} \xrightarrow{\sim} \mathfrak{K}_{q,\xi} \tag{5.15}$$

such that $\Psi_{Q,Q'} = \Psi_Q \circ \Psi_{Q'}^{-1}$.
(ii) *We have $\Psi_{\mathfrak{Q},\mathfrak{Q}'} \circ \mathfrak{D}_q^{\pm 1} = \mathfrak{D}_q^{\pm 1} \circ \Psi_{\mathfrak{Q},\mathfrak{Q}'}$.*
(iii) *$\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ sends L_q to L_q bijectively.*

Theorem 5.20 ([25, 34]) *Let Q be a Dynkin quiver with a source $i \in I$ and write $\boldsymbol{T}_i$ for the automorphism $\Psi_{s_i\mathfrak{Q},\mathfrak{Q}}$ of $\mathfrak{K}_q$. Then the collection of automorphism $\{\boldsymbol{T}_i\}_{i \in I}$ satisfies the braid relations of $\mathfrak{g}$.*

For Dynkin quivers $Q = (\triangle, \xi)$ and $Q' = (\triangle, \xi')$ on the same $\triangle$, let $\boldsymbol{i} = (i_1, \ldots, i_\ell)$ and $\boldsymbol{i}' = (i'_1, \ldots, i'_\ell)$ be reduced sequences of $w_\circ$ adapted to Q and Q', respectively. Then we have a sequence

$$\boldsymbol{\tau} = (\tau_1, \ldots, \tau_r) \text{ in } \{\gamma_1, \ldots, \gamma_{\ell-1}\} \sqcup \{\beta_1, \ldots, \beta_{\ell-2}\} \sqcup \{\eta_1, \ldots, \eta_{\ell-3}\} \cup \{\zeta_1\}$$

such that

$$\boldsymbol{i}' = \tau_1 \cdots \tau_r \boldsymbol{i}.$$

Recall we have $\widehat{\boldsymbol{i}} = (i_1, i_2, \ldots)$, $\widehat{\boldsymbol{i}}' = (i'_1, i'_2, \ldots) \in I^{(\infty)}$ corresponding to $\boldsymbol{i}$ and $\boldsymbol{i}'$ respectively, which are constructed in (5.11). We extend the isomorphism $\boldsymbol{\tau}^* : \mathcal{A}_{\boldsymbol{i}'} \to \mathcal{A}_{\boldsymbol{i}}$ to the one from $\mathcal{A}_{\widehat{\boldsymbol{i}}'}$ to $\mathcal{A}_{\widehat{\boldsymbol{i}}}$ as follows: For each $n \in \mathbb{N}$, consider the $\mathbb{A}$-algebra isomorphism

$$\widehat{\boldsymbol{\tau}}^{(n)*} := \widehat{\tau}_r^{(n)*} \circ \cdots \circ \widehat{\tau}_1^{(n)*} : \mathcal{A}^{n\ell}_{\widehat{\boldsymbol{i}}'} \to \mathcal{A}^{n\ell}_{\widehat{\boldsymbol{i}}},$$

where $\ell = \ell(w_\circ)$ and

$$\widehat{\gamma}_k^{(n)*} := \gamma_k^* \gamma_{k+\ell}^* \cdots \gamma_{k+(n-1)\ell}^*, \qquad \widehat{\beta}_k^{(n)*} := \beta_k^* \beta_{k+\ell}^* \cdots \beta_{k+(n-1)\ell}^*,$$
$$\widehat{\eta}_k^{(n)*} := \eta_k^* \eta_{k+\ell}^* \cdots \eta_{k+(n-1)\ell}^*, \qquad \widehat{\zeta}_1^{(n)*} := \zeta_1^* \zeta_{1+\ell}^* \cdots \zeta_{1+(n-1)\ell}^*.$$

Recalling $\tau_k \circ \tau_l = \tau_l \circ \tau_k$ when $|k - l| \geqslant \ell$, we define

$$\widehat{\boldsymbol{\tau}}^* := \lim_{n \to \infty} \widehat{\boldsymbol{\tau}}^{(n)*} : \mathcal{A}_{\widehat{\boldsymbol{i}}'} \to \mathcal{A}_{\widehat{\boldsymbol{i}}}.$$

The following theorem tells us that the isomorphism $\widehat{\boldsymbol{\tau}}^*$ is compatible with the isomorphisms $\Psi_{\xi,\xi'}$, $\kappa_{\widehat{\boldsymbol{i}}'}$ and $\kappa_{\widehat{\boldsymbol{i}}}$. The argument of a proof can be almost the same as the one in [11, Theorem 6.3], which investigates simply-laced types only. For the reader's convenience, we give a proof of the theorem.

Theorem 5.21 *We have the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{A}_{\widehat{\boldsymbol{i}}'} \arrow[r, "\kappa_{\widehat{\boldsymbol{i}}'}"] \arrow[d, "\widehat{\boldsymbol{\tau}}^*"] & \mathfrak{K}_{q,\xi'} \arrow[d, "\Psi_{\xi,\xi'}"] \\
\mathcal{A}_{\widehat{\boldsymbol{i}}} \arrow[r, "\kappa_{\widehat{\boldsymbol{i}}}"] & \mathfrak{K}_{q,\xi}
\end{tikzcd}$$

Proof Setting $\Psi := \kappa_{\widehat{\boldsymbol{i}}} \circ \widehat{\boldsymbol{\tau}}^* \circ \kappa_{\widehat{\boldsymbol{i}}}^{-1}$, it is enough to prove that

$$\Psi = \Psi_{\xi,\xi'}.$$

By Theorem 5.19, we have to show the following:

(1) $\Psi|_{\mathfrak{K}_{q,Q}} = \Psi_{Q,Q'}$,

(2) $\Psi \circ \mathfrak{D}_q^{-1} = \mathfrak{D}_q^{-1} \circ \Psi$.

Noting that $\mathcal{A}^{\ell}_{\widehat{\boldsymbol{i}}'} \simeq \mathcal{A}_{\boldsymbol{i}'}$, $\mathcal{A}^{\ell}_{\widehat{\boldsymbol{i}}} \simeq \mathcal{A}_{\boldsymbol{i}}$, $\widehat{\boldsymbol{\tau}}^*|_{\mathcal{A}_{\boldsymbol{i}'}} = \boldsymbol{\tau}^*$, $\kappa_{\widehat{\boldsymbol{i}}'}|_{\mathcal{A}_{\boldsymbol{i}'}} = \kappa_{\boldsymbol{i}'}$, $\kappa_{\widehat{\boldsymbol{i}}}|_{\mathcal{A}_{\boldsymbol{i}}} = \kappa_{\boldsymbol{i}}$ and $\Psi_{\xi,\xi'}|_{\mathfrak{K}_{q,Q'}} = \Psi_{Q,Q'}$, we have

$$\begin{tikzcd}
\mathcal{A}_{\boldsymbol{i}'} \arrow[rr, "\kappa_{\boldsymbol{i}'}"] \arrow[dr, "\varphi_{\boldsymbol{i}'}"'] \arrow[dd, "\boldsymbol{\tau}^*"'] & {} \arrow[d, phantom, "\mathrm{A}"] & \mathfrak{K}_{q,Q'} \arrow[dd, "\Psi_{Q,Q'}"] \\
{} \arrow[r, phantom, "\mathrm{B}"] & \mathcal{A}_{\mathbb{A}}(\mathfrak{n}) \arrow[ur, "\Psi_{Q'}"'] \arrow[dr, "\Psi_Q"] \arrow[d, phantom, "\mathrm{A}"] \arrow[r, phantom, "\mathrm{C}"] & {} \\
\mathcal{A}_{\boldsymbol{i}} \arrow[ur, "\varphi_{\boldsymbol{i}}"] \arrow[rr, "\kappa_{\boldsymbol{i}}"'] & {} & \mathfrak{K}_{q,Q}
\end{tikzcd} \tag{5.16}$$

Here the commutativity for Ⓐ-triangles follows from (5.13), the commutativity for Ⓑ-triangle from Lemma 4.6 and the commutativity for Ⓒ-triangle from the definition in (5.15). Thus we have shown (1).

Now let us consider (2). Setting $\widehat{\boldsymbol{i}}^* := (i_k^*)_{k\in\mathbb{N}}$, we have $\widehat{\boldsymbol{i}} = \partial_+^\ell \widehat{\boldsymbol{i}}^*$, $\widetilde{B}^{\boldsymbol{i}} = \widetilde{B}^{\boldsymbol{i}^*}$ and $\Lambda^{\boldsymbol{i}} = \Lambda^{\boldsymbol{i}^*}$. Thus we have an isomorphism $\nu : \mathcal{A}_{\widehat{\boldsymbol{i}}^*} \simeq \mathcal{A}_{\widehat{\boldsymbol{i}}}$ sending Z_k to itself. By Proposition 3.24 and Theorem 5.12, the cluster variables

$$\kappa_{\widehat{\boldsymbol{i}}}^{-1}\mathfrak{D}_q^{-1}F_q(X_{i,p}) \quad \text{and} \quad \nu(\partial_+^*)^\ell\kappa_{\widehat{\boldsymbol{i}}}^{-1}F_q(X_{i,p})$$

have the same degrees for all $(i,p) \in {}^\xi\widehat{\triangle}_0$, and hence they are the same by Theorem 2.7. Since the set $\{F_q(X_{i,p})\}_{(i,p)\in{}^\xi\widehat{\triangle}_0}$ generates $\mathfrak{K}_{q,\xi}$, we conclude that

$$\nu \circ (\partial_+^*)^\ell \circ \kappa_{\widehat{\boldsymbol{i}}}^{-1} = \kappa_{\widehat{\boldsymbol{i}}}^{-1} \circ \mathfrak{D}_q^{-1}. \tag{5.17}$$

Note that the same assertion for $\widehat{\boldsymbol{i}}'$ holds by the same argument. Furthermore, for a cluster monomial $x \in \mathcal{A}_{\widehat{\boldsymbol{i}}'}$ whose degree belongs to $C_{\widehat{\boldsymbol{i}}'}$, one can see that the degrees of the cluster variables

$$\widehat{\boldsymbol{\tau}}^*\nu(\partial_+^*)^\ell x \quad \text{and} \quad \nu(\partial_+^*)^\ell\widehat{\boldsymbol{\tau}}^* x \quad \text{are the same} \tag{5.18}$$

by Proposition 3.24 and by the result in Section 3 that $\widehat{\boldsymbol{\tau}}^*$ sends $C_{\widehat{\boldsymbol{i}}'}$ to $C_{\widehat{\boldsymbol{i}}}$. Now we have

$$\begin{aligned}
\Psi\mathfrak{D}_q^{-1}F_q(X_{i,p}) &= \kappa_{\widehat{\boldsymbol{i}}}\widehat{\boldsymbol{\tau}}^*\kappa_{\widehat{\boldsymbol{i}}'}^{-1}\mathfrak{D}_q^{-1}F_q(X_{i,p}) \\
&= \kappa_{\widehat{\boldsymbol{i}}}\widehat{\boldsymbol{\tau}}^*\nu(\partial_+^*)^\ell\kappa_{\widehat{\boldsymbol{i}}'}^{-1}F_q(X_{i,p}) && \text{by (5.17) for } \widehat{\boldsymbol{i}}' \\
&= \kappa_{\widehat{\boldsymbol{i}}}\nu(\partial_+^*)^\ell\widehat{\boldsymbol{\tau}}^*\kappa_{\widehat{\boldsymbol{i}}'}^{-1}F_q(X_{i,p}) && \text{by (5.18)} \\
&= \mathfrak{D}_q^{-1}\kappa_{\widehat{\boldsymbol{i}}}\widehat{\boldsymbol{\tau}}^*\kappa_{\widehat{\boldsymbol{i}}'}^{-1}F_q(X_{i,p}) && \text{by (5.17) for } \widehat{\boldsymbol{i}} \\
&= \mathfrak{D}_q^{-1}\Psi F_q(X_{i,p})
\end{aligned}$$

for any $(i,p) \in {}^{\xi'}\widehat{\triangle}_0$. Since the set $\{F_q(X_{i,p})\}_{(i,p)\in{}^{\xi'}\widehat{\triangle}_0}$ generates $\mathfrak{K}_{q,\xi'}$, the assertion (2) follows. □

Remark 5.22 In the above theorem, we have applied a special sequence $(i_1, i_2, \ldots) \in I^{(\infty)}$ such that $(i_k, \ldots, i_{k+\ell-1})$ is a reduced sequence of $w_\circ$ for all $k \in \mathbb{N}$.

Corollary 5.23 *Let Q and Q' be Dynkin quivers on the same Dynkin diagram $\triangle$. For reduced sequences $\boldsymbol{i}$ and $\boldsymbol{i}'$ adapted to Q and Q' respectively, the $\mathbb{A}$-algebra isomorphism $\kappa_{\widehat{\boldsymbol{i}}}^{-1} \circ \Psi_{\mathfrak{Q},\mathfrak{Q}'} \circ \kappa_{\widehat{\boldsymbol{i}}'}$ induces a bijection between the set of quantum cluster monomials of $\mathcal{A}_{\widehat{\boldsymbol{i}}'}$ and that of $\mathcal{A}_{\widehat{\boldsymbol{i}}}$.*

5.6 Substitution formulas

One of the celebrated results of Berenstein–Fomin–Zelevinsky is to show that quantum cluster algebra is still contained in $\mathbb{Z}[q^{\pm 1/2}][Z_k^{\pm 1}]_{k \in \mathsf{K}}$ even though mutations involves non-trivial fractions. In this subsection, we shall show that the automorphism $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ can be interpreted as an restriction of birational auto-transformation of $\mathbb{F}(\mathcal{X}_q)$, which is referred to as *substitution formula* in [11, Section 7]. Since $\Psi_{\mathfrak{Q},\mathfrak{Q}'}$ can be understood as a composition of $\{\boldsymbol{T}_i^{\pm 1}\}_{i \in I}$, it gives an interesting relation among elements in L_q. The result in this section for simply-laced type is given in [11, Section 7] and we apply the same framework and arguments of proofs in this paper for non simply-laced type, based on the results that have been obtained for $\mathfrak{K}_q$, 4-moves and 6-moves. Thus we just give statements, several notations and maps, so that readers may recover the substitution formula on $\mathbb{F}(\mathcal{X}_q)$ and $\mathfrak{K}_q$ by themselves.

Throughout this subsection, we (i) fix Dynkin quivers $Q = (\triangle, \xi)$ and $Q' = (\triangle, \xi')$ defined on the same Dynkin diagram $\triangle$ of non simply-laced type, (ii) denote by $\boldsymbol{i} = (i_1, \dots, i_\ell)$ and $\boldsymbol{j} = (j_1, \dots, j_\ell)$ reduced sequences of $w_\circ$ adapted to Q and Q' respectively. To distinguish generators of quantum tori $\mathcal{T}(\Lambda^{\widehat{\boldsymbol{i}}})$ and $\mathcal{T}(\Lambda^{\widehat{\boldsymbol{j}}})$, we use Z_k for initial cluster variables $\mathcal{T}(\Lambda^{\widehat{\boldsymbol{i}}})$ while Z'_k for ones of $\mathcal{T}(\Lambda^{\widehat{\boldsymbol{j}}})$.

For $k \in \mathbb{Z}$, we set $\xi[k] := \mathfrak{D}^k \xi$ and $Q[k] := \mathfrak{D}^k Q$ for notational simplicity. We recall the following:

(a) We have injective homomorphisms

$$\mathfrak{K}_{q,\xi[-1]} \hookrightarrow (\mathfrak{K}_{q,\xi})_{\leqslant \xi} \hookrightarrow \mathcal{X}_{q,\leqslant \xi}$$

induced from $(\cdot)_{\leqslant \xi}$, whose composition is an identity indeed, by Theorem 5.6 and the fact that $\mathfrak{K}_{q,\xi}$ is generated by $\{F_q(X_{i,p}) \mid (i,p) \in {}^{\xi}\widehat{\triangle}_0\}$.

(b) We have a commutative diagram

$$\begin{tikzcd}
\mathfrak{K}_{q,\xi} \arrow[r, "(\cdot)_{\leqslant \xi}"] \arrow[dr, "\simeq"] & \mathcal{X}_{q,\leqslant \xi} \\
\mathfrak{K}_{q,\xi[-1]} \arrow[u, hook] \arrow[r, hook] & (\mathfrak{K}_{q,\xi})_{\leqslant \xi} \arrow[u, hook]
\end{tikzcd}$$

Then, for I-compatible readings $\widehat{\boldsymbol{i}}$ (resp. $\widehat{\boldsymbol{i}}'$) of ${}^{\xi}\widehat{\triangle}_0$ (resp. ${}^{\xi'}\widehat{\triangle}_0$), we have unique morphisms $\widetilde{\kappa}_{\widehat{\boldsymbol{i}},\mathbb{F}}$, $\widetilde{\kappa}_{\widehat{\boldsymbol{i}}',\mathbb{F}}$, $\widehat{\boldsymbol{\tau}}^*_{\mathbb{F}}$ and $\widetilde{\Psi}_{\xi,\xi}$ which make the diagram below commutative:

$$
\begin{tikzcd}
& \mathbb{F}(\mathcal{X}_{q,\xi'}) \arrow[rrr, bend left, "\widetilde{\Psi}_{\xi,\xi'}"] & \mathbb{F}(\mathcal{T}(\Lambda^{\widehat{\boldsymbol{j}}})) \arrow[l, "\widetilde{\kappa}_{\widehat{\boldsymbol{j}},\mathbb{F}}"', "\sim"] \arrow[r, "\widehat{\boldsymbol{\tau}}^*_{\mathbb{F}}", "\sim"'] & \mathbb{F}(\mathcal{T}(\Lambda^{\widehat{\boldsymbol{i}}})) \arrow[r, "\widetilde{\kappa}_{\widehat{\boldsymbol{i}},\mathbb{F}}", "\sim"'] & \mathbb{F}(\mathcal{X}_{q,\xi}) & \\
& \mathcal{X}_{q,\leqslant\xi'} \arrow[u, hook] & \mathcal{T}(\Lambda^{\widehat{\boldsymbol{j}}}) \arrow[l, "\widetilde{\kappa}_{\widehat{\boldsymbol{j}}}"', "\sim"] \arrow[u, hook] & \mathcal{T}(\Lambda^{\widehat{\boldsymbol{i}}}) \arrow[r, "\widetilde{\kappa}_{\widehat{\boldsymbol{i}}}", "\sim"'] \arrow[u, hook] & \mathcal{X}_{q,\xi} \arrow[u, hook] & \\
(\mathfrak{K}_{q,\xi'})_{\leqslant\xi'} \arrow[ur, hook] & \mathfrak{K}_{q,\xi'} \arrow[l, "\sim"'] \arrow[u, "(\cdot)_{\leqslant\xi'}"'] \arrow[rrr, bend right, "\Psi_{\xi,\xi'}", "\sim"'] & \mathcal{A}_{\widehat{\boldsymbol{j}}} \arrow[l, "\sim"', "\kappa_{\widehat{\boldsymbol{j}}}"] \arrow[u, hook] \arrow[r, "\widehat{\boldsymbol{\tau}}^*", "\sim"'] & \mathcal{A}_{\widehat{\boldsymbol{i}}} \arrow[u, hook] \arrow[r, "\sim", "\kappa_{\widehat{\boldsymbol{i}}}"'] & \mathfrak{K}_{q,\xi} \arrow[u, "(\cdot)_{\leqslant\xi}"] \arrow[r, "\sim"] & (\mathfrak{K}_{q,\xi})_{\leqslant\xi} \arrow[ul, hook] \\
& \mathfrak{K}_{q,\xi'[-1]} \arrow[u, hook] \arrow[ul] \arrow[rrr, bend right, "\sim", "\Psi_{\xi'[-1],\xi[-1]}"'] & & & \mathfrak{K}_{q,\xi[-1]} \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}
\tag{5.19}
$$

Let us set

$$
\begin{aligned}
\widetilde{\Sigma} &:= \widetilde{\kappa}^{-1}_{\widehat{\boldsymbol{i}},\mathbb{F}} \circ \mathfrak{D}_q^{-1} \circ \widetilde{\kappa}_{\widehat{\boldsymbol{i}},\mathbb{F}} : \mathbb{F}(\mathcal{T}(\Lambda^{\widehat{\boldsymbol{i}}})) \to \mathbb{F}(\mathcal{T}(\Lambda^{\widehat{\boldsymbol{i}}})), \\
\widetilde{\Sigma}' &:= \widetilde{\kappa}^{-1}_{\widehat{\boldsymbol{j}},\mathbb{F}} \circ \mathfrak{D}_q^{-1} \circ \widetilde{\kappa}_{\widehat{\boldsymbol{j}},\mathbb{F}} : \mathbb{F}(\mathcal{T}(\Lambda^{\widehat{\boldsymbol{j}}})) \to \mathbb{F}(\mathcal{T}(\Lambda^{\widehat{\boldsymbol{j}}})),
\end{aligned}
$$

which are automorphisms of skew field of fractions. We also define $\mathbb{Z}$-module homomorphisms $\Sigma, \widetilde{\Sigma} : \mathbb{Z}^{\oplus\mathbb{N}} \to \mathbb{Z}^{\oplus\mathbb{N}}$ given by

$$
\Sigma(\boldsymbol{e}_u) = \boldsymbol{e}_{u+\ell} \qquad \widetilde{\Sigma}(\boldsymbol{e}_u) = \boldsymbol{e}_{u+\ell} - \boldsymbol{e}_{(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i_u^*)} \quad \text{for all } u \in \mathbb{N}.
$$

Note that

$$
\widetilde{\Sigma}(\boldsymbol{a}) = \Sigma(\boldsymbol{a}) - \sum_{i\in I} \mathsf{p}_{\widehat{\boldsymbol{i}}}(\boldsymbol{a}; i)\boldsymbol{e}_{(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i^*)} \quad \text{for } \boldsymbol{a} \in \mathbb{Z}^{\mathbb{N}}. \tag{5.20}
$$

The proof of the following lemma utilizes Proposition 5.11.

Lemma 5.24 *For $\boldsymbol{a} \in \mathbb{Z}^{\oplus\mathbb{N}}$, we have*

$$
\widetilde{\Sigma}(Z^{\boldsymbol{a}}) = Z^{\widetilde{\Sigma}(\boldsymbol{a})} \quad \textit{and} \quad \widetilde{\Sigma}'(Z'^{\boldsymbol{a}}) = Z'^{\widetilde{\Sigma}(\boldsymbol{a})}.
$$

In particular,

$$
\widetilde{\Sigma}(\boldsymbol{b}^{\widehat{\boldsymbol{i}}}_u) = \boldsymbol{b}^{\widehat{\boldsymbol{i}}}_{u+\ell} \quad \textit{and} \quad \widetilde{\Sigma}(\boldsymbol{b}^{\widehat{\boldsymbol{j}}}_u) = \boldsymbol{b}^{\widehat{\boldsymbol{j}}}_{u+\ell}. \tag{5.21}
$$

For $u \in \mathbb{N}$ and an initial cluster variable Z'_u of $\mathcal{A}_{\widehat{\boldsymbol{j}}}$, we set $\mathcal{Z}_u := \widehat{\boldsymbol{\tau}}^*(Z'_u)$, which is a cluster variable of $\mathcal{A}_{\widehat{\boldsymbol{i}}}$. Then there exists $\boldsymbol{g}_u \in \mathbb{Z}^{\oplus\mathbb{N}}$ and $c_{u,\boldsymbol{n}}$ satisfying

$$
\mathcal{Z}_u = Z^{\boldsymbol{g}_u}\left(1 + \sum_{\mathbf{n}\in\mathbb{N}_0^{\oplus\mathbb{N}}\setminus\{0\}} c_{u,\mathbf{n}} Z^{\sum_{k\in\mathbb{N}} n_k \boldsymbol{b}^{\widehat{\boldsymbol{i}}}_k}\right)
$$

by Theorem 2.7.

For $k \in \mathbb{N}_0$, we set

$$\Gamma_{0,\mathrm{fr}}^{Q[-k]} := \{(i, \xi[-k-1]_i + 2) \mid i \in I\}, \quad \Gamma_{0,\mathrm{ex}}^{Q[-k]} = \Gamma_0^{Q[-k]} \setminus \Gamma_{0,\mathrm{fr}}^{Q[-k]}.$$

Note that, for each $u \in \mathbb{N}$, there exists a unique $k \in \mathbb{N}_0$ such that $\mathcal{Z}_u$ can be obtained from the initial cluster $\{Z_u \mid u \in \mathbb{N}\}$ by applying the mutations only at the vertices labeled by $\Gamma_{0,\mathrm{ex}}^{Q[-k]}$. By ℓ-periodicity of $\widehat{\boldsymbol{i}}$, $\widehat{\boldsymbol{j}}$ and $\widehat{\boldsymbol{\tau}}^*$, for $u \in \mathbb{N}$, we have

$$\boldsymbol{g}_{u+\ell} \begin{cases} = \Sigma(\boldsymbol{g}_u) & \text{if } u > \ell \text{ or } u = (\ell+1)^-_{\widehat{\boldsymbol{j}}}(i) \text{ for } i \in I, \\ \in \Sigma(\boldsymbol{g}_u) + \sum_{i \in I} \mathbb{Z}\boldsymbol{e}_{(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i)} & \text{otherwise,} \end{cases} \tag{5.22}$$

and

$$c_{u+\ell,\boldsymbol{n}} = \begin{cases} c_{u,\Sigma^{-1}(\boldsymbol{n})} & \text{if } \boldsymbol{n} \in \Sigma(\mathbb{N}_0^{\oplus \mathbb{N}}), \\ 0 & \text{otherwise.} \end{cases} \tag{5.23}$$

Remark 5.25 Any braid move between $\widehat{\boldsymbol{i}}$ and $\widehat{\boldsymbol{j}}$ does not occur at $(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i) + k\ell$ for any $k \in \mathbb{N}_0$ and $i \in I$. Hence $\mathcal{Z}_{(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i)+k\ell} = Z_{(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i)+k\ell} = Z'_{(\ell+1)^-_{\widehat{\boldsymbol{i}}}(i)+k\ell}$ in $\mathcal{A}_{\widehat{\boldsymbol{i}}}$ and they q-commute with all Z_a $(a \in \mathbb{N})$.

When $\mathfrak{g}$ is of non-simply-laced type, we need to employ Lemma 3.12, Lemma 3.20 and Lemma 3.21 to prove the following lemma.

Lemma 5.26 *For $u \in \mathbb{N}$, we have*

$$\widetilde{\Sigma}(\mathcal{Z}_u) = Z^{\widetilde{\Sigma}(\boldsymbol{g}_u)} \left(1 + \sum_{\boldsymbol{n} \in \mathbb{N}_0^{\oplus \mathbb{N}} \setminus \{0\}} c_{u+\ell,\boldsymbol{n}} Z^{\sum_{k \in \mathbb{N}} n_k \boldsymbol{b}_k^{\widehat{\boldsymbol{i}}}} \right) \simeq Z^{\widetilde{\Sigma}(\boldsymbol{g}_u) - \boldsymbol{g}_{u+\ell}} \mathcal{Z}_{u+\ell}. \tag{5.24}$$

Remark 5.27 The following proposition is a non-symmetric analogue of [11, Lemma 7.4 (1)], and it is *not at all* a consequence of Theorem 5.21 as emphasized in the introduction of [11, Section 7].

Proposition 5.28 *We have*

$$\mathfrak{D}_q^{-1}|_{\mathbb{F}(\mathcal{X}_{q,\leqslant \xi})} \circ \widetilde{\Psi}_{\xi,\xi'} = \widetilde{\Psi}_{\xi,\xi'} \circ \mathfrak{D}_q^{-1}|_{\mathbb{F}(\mathcal{X}_{q,\leqslant \xi'})}.$$

Corollary 5.29 *For $k \in \mathbb{N}_0$ and $(i, p) \in \Gamma_0^{Q'[-k]}$, we have*

$$\widetilde{\Psi}_{\xi,\xi'}(\widetilde{X}_{i,p}) \in \mathbb{F}(\mathcal{X}_{q,Q[-k]}).$$

Proof Since $\widetilde{\Psi}_{\xi,\xi'}(\mathbb{F}(\mathcal{X}_{q,Q'})) \subset \mathcal{X}_{q,Q}$, it follows from Proposition 5.28. □

Theorem 5.30 *There exists an automorphism of the skew field* $\mathbb{F}(\mathcal{X}_q)$

$$\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'} : \mathbb{F}(\mathcal{X}_q) \xrightarrow{\sim} \mathbb{F}(\mathcal{X}_q)$$

satisfying the following properties:

(a) $\mathfrak{D}_q \circ \widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'} = \widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'} \circ \mathfrak{D}_q$.
(b) $\widetilde{\Psi}_{\xi,\xi'}(\widetilde{X}_{i,p}) \in \mathbb{F}(\mathcal{X}_{q,Q[k]})$ *for any* $k \in \mathbb{Z}$ *and* $(i,p) \in \Gamma_0^{Q'[k]}$.
(c) *We have the following commutative diagram:*

$$\begin{tikzcd}
\mathbb{F}(\mathcal{X}_q) \arrow[r, "\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'}", "\sim"'] & \mathbb{F}(\mathcal{X}_q) \\
\mathfrak{K}_q \arrow[u, hook] \arrow[r, "\Psi_{\mathfrak{Q},\mathfrak{Q}'}", "\sim"'] & \mathfrak{K}_q \arrow[u, hook]
\end{tikzcd}$$

Proof We first define $\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'}$ from $\widetilde{\Psi}_{\xi,\xi'}$ by using $\mathfrak{D}_q$ as follows: For each $y \in \mathcal{F}(\mathcal{X}_q)$, there exists $k \in \mathbb{N}_0$ such that $\mathfrak{D}_q^{-k}(y) \in \mathbb{F}(\mathcal{X}_{q,\leqslant\xi})$. Then we define

$$\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'}(y) = \mathfrak{D}_q^k(\widetilde{\Psi}_{\xi,\xi'}(\mathfrak{D}_q^{-k}(y))).$$

Now, as in [11, Theorem 7.1], one can prove that it is a well-defined automorphism of $\mathbb{F}(\mathcal{X}_q)$ satisfying the properties in the statement. $\square$

In Appendix A, we provide examples of substitution formulas.

6 Categorification via quiver Hecke algebras and Laurent family

In this section, we first review the results on the categorification theory via the quiver Hecke algebra R related to our purpose. Then we recall the definitions and the results about the Laurent family of simple R-modules, which were introduced and investigated in [36] very recently.

6.1 Quiver Hecke algebra

Let $\mathbf{k}$ be a base field. For $i, j \in I$, we choose polynomials $\mathcal{Q}_{i,j}(u,v) \in \mathbf{k}[u,v]$ such that

(a) $\mathcal{Q}_{i,j}(u,v) = \mathcal{Q}_{j,i}(v,u)$,
(b) $\mathcal{Q}_{i,j}(u,v) = \delta(i \neq j) \sum_{2\mathsf{d}_i p + 2\mathsf{d}_j q = -2(\alpha_i,\alpha_j)} t_{i,j;p,q} u^p v^q$ where $t_{i,j;-a_{i,j},0} \in \mathbf{k}^\times$.

For $\beta \in \mathsf{Q}^+$ with $|\beta| = n$, we set

$$I^\beta := \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \sum_{k=1}^n \alpha_{\nu_k} = \beta\}.$$

For $\mathfrak{g}$ and $\beta \in \mathsf{Q}^+$ with $|\beta| = r$, the quiver Hecke algebra $R(\beta)$ associated with $(\mathcal{Q}_{i,j})_{i,j\in I}$ is the $\mathbb{Z}$-graded algebra over $\mathbf{k}$ generated by

$$x_k \ (1 \leqslant k \leqslant r), \quad e(\nu) \ (\nu \in I^\beta), \quad \tau_s \ (1 \leqslant s < r),$$

subject to certain defining relations (see [29] for the relations). Note that the $\mathbb{Z}$-grading of $R(\beta)$ is determined by the degrees of following elements:

$$\deg(e(\nu)) = 0, \ \deg(x_k e(\nu)) = 2\mathsf{d}_{\nu_k} \ (1 \leqslant k \leqslant n), \ \text{ and } \ \deg(\tau_m e(\nu)) \\ = -(\alpha_{\nu_m}, \alpha_{\nu_{m+1}}) \ (1 \leqslant m < n).$$

We denote by $R(\beta)$-gmod the category of finite-dimensional graded $R(\beta)$-modules with homogeneous homomorphisms. For $M \in R(\beta)$-gmod, we set $\mathrm{wt}(M) = -\beta \in \mathsf{Q}^-$. Denote by q the degree shift functor acting on $R(\beta)$-gmod as follows:

$$q(M)_n = M_{n-1} \qquad \text{for } M = \bigoplus_{k\in\mathbb{Z}} M_k \in R(\beta)\text{-gmod}.$$

For an $R(\beta)$-module M and an $R(\gamma)$-module N, we obtain an $R(\beta+\gamma)$-module by

$$M \circ N := R(\beta+\gamma)e(\beta,\gamma) \otimes_{R(\beta)\otimes R(\gamma)} (M \circ N),$$

where $e(\beta,\gamma) := \sum_{\nu\in I^\beta, \nu'\in I^\gamma} e(\nu * \nu') \in R(\beta+\gamma)$. Here $\nu * \mu$ denotes the concatenation of ν and μ, and $\circ$ is called the *convolution product*. We say that two simple R-modules M and N *strongly commute* if $M \circ N$ is simple. If a simple module M strongly commutes with itself, then M is called *real*.

Set R-gmod$:=\bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)$-gmod. Then R-gmod has the monoidal category structure with the unit object $\mathbf{1} = \mathbf{k} \in R(0)$-gmod and the tensor product $\circ$. Thus the Grothendieck group $K(R\text{-gmod})$ has the $\mathbb{Z}[q^{\pm 1}]$-algebra structure derived from $\circ$ and the degree shift functors $q^{\pm 1}$. We set

$$\mathcal{K}_{\mathbb{A}}(R\text{-gmod}) := \mathbb{A} \otimes_{\mathbb{Z}[q^{\pm 1}]} K(R\text{-gmod}).$$

Theorem 6.1 ([41, 42, 52])*There exists an $\mathbb{A}$-algebra isomorphism*

$$\Omega : \mathcal{K}_{\mathbb{A}}(R\text{-}gmod) \xrightarrow{\sim} \mathcal{A}_{\mathbb{A}}(\mathfrak{n}).$$

Proposition 6.2 ([32]) *For $\lambda \in \mathsf{P}^+$ and $\mu, \zeta \in \mathsf{W}\lambda$ with $\mu \preccurlyeq \zeta$, there exists a self-dual real simple $R(\zeta-\mu)$-module $L(\mu,\zeta)$ such that*

$$\Omega([L(\mu,\zeta)]) = D(\mu,\zeta).$$

Here $[L(\mu,\zeta)]$ denotes the isomorphism class of $L(\mu,\zeta)$.

We call $L(\mu,\zeta)$ the *determinantal module* associated with $D(\mu,\zeta)$.

6.2 *R*-matrices and their $\mathbb{Z}$-invariants

For $\beta \in \mathsf{Q}^+$ and $i \in I$, let

$$\mathfrak{p}_{i,\beta} = \sum_{\nu \in I^\beta} \left(\prod_{a \in [1,|\beta|];\ \nu_a = i} x_a \right) e(\nu) \in \mathfrak{X}(R(\beta)),$$

where $\mathfrak{X}(R(\beta))$ denotes the center of $R(\beta)$.

Definition 6.3 *([40, Definition 2.2])* For an $R(\beta)$-module L, we say that L *admits an affinization* if there exists an $R(\beta)$-module $\widehat{L}$ satisfying the condition: there exists an endomorphism $z_{\widehat{L}}$ of degree $t \in \mathbb{Z}_{>0}$ such that $\widehat{L}/z_{\widehat{L}}\widehat{L} \simeq L$ and

(i) $\widehat{L}$ is a finitely generated free modules over the polynomial ring $\mathbf{k}[z_{\widehat{L}}]$,
(ii) $\mathfrak{p}_{i,\beta}\widehat{L} \neq 0$ for all $i \in I$.

We say that a simple $R(\beta)$-module L is *affreal* if L is real and admits an affinization.

It is known that any $L \in R(\beta)$-gmod admits an affinization if $R(\beta)$ is symmetric. However, when $R(\beta)$ is not symmetric, it is widely open whether an $R(\beta)$-module L admits an affinization or not.

Theorem 6.4 ([35, Theorem 3.26]) *For $\varpi \in \mathsf{P}^+$ and $\mu, \zeta \in \mathsf{W}\varpi$ such that $\mu \preccurlyeq \zeta$, the determinantal module $L(\mu, \zeta)$ is affreal.*

Let $\boldsymbol{i}$ be a reduced sequence of $w_\circ$. For $\beta \in \Phi^+$ such that $\beta = \beta^{\boldsymbol{i}}_k$ in (3.1), we write $S^{\boldsymbol{i}}_k$ for the determinantal $R(\beta)$-module such that

$$\Omega([S^{\boldsymbol{i}}_k]) = D(w^{\boldsymbol{i}}_k \varpi_{i_k}, w^{\boldsymbol{i}}_{k^-} \varpi_{i_k}).$$

We sometimes write $S^{\boldsymbol{i}}(\beta)$ for $S^{\boldsymbol{i}}_k$ to emphasize β. Note that

$$\mathrm{wt}(S^{\boldsymbol{i}}_k) = \mathrm{wt}(D(w^{\boldsymbol{i}}_k \varpi_{i_k}, w^{\boldsymbol{i}}_{k^-} \varpi_{i_k})) = -\beta.$$

The module $S^{\boldsymbol{i}}_k$ is also called the *cuspidal module* associated with $\boldsymbol{i}$ and β. For another reduced sequence $\boldsymbol{i}'$ of $w_\circ$ such that $\boldsymbol{i} \overset{c}{\sim} \boldsymbol{i}'$, we have $S^{\boldsymbol{i}}(\beta) \simeq S^{\boldsymbol{i}'}(\beta)$. Thus the notation $S^{[\boldsymbol{i}]}(\beta)$ make sense.

Proposition 6.5 ([29, 35]) *Let M and N be simple modules such that one of them is affreal. Then there exists a unique R-module homomorphism $\mathbf{r}_{M,N} \in \mathrm{HOM}_R(M, N)$ satisfying*

$$\mathrm{HOM}_R(M \circ N, N \circ M) = \boldsymbol{k}\,\mathbf{r}_{M,N}.$$

We call the homomorphism $\mathbf{r}_{M,N}$ the *R-matrix*.

Definition 6.6 For simple R-modules M and N such that one of them is affreal, we define

$$\begin{aligned}\Lambda(M,N) &:= \deg(\mathbf{r}_{M,N}),\\ \widetilde{\Lambda}(M,N) &:= \frac{1}{2}\Big(\Lambda(M,N) + \big(\mathrm{wt}(M),\mathrm{wt}(N)\big)\Big),\\ \mathfrak{d}(M,N) &:= \frac{1}{2}\big(\Lambda(M,N)+\Lambda(N,M)\big).\end{aligned}$$

It is proved in [29, 35] that the invariants $\widetilde{\Lambda}(M,N)$ and $\mathfrak{d}(M,N)$ in Definition 6.6 belong to $\mathbb{Z}_{\geqslant 0}$.

For simple modules M and N, let $M \nabla N$ and $M \Delta N$ denote the head and the socle of $M \circ N$, respectively.

Proposition 6.7 ([28, Lemma 3.1.4], [29, Lemma 4.1.2]) *Let M and N be self-dual simple R-modules such that one of them is affreal.*

(a) *$M \nabla N$ and $M \Delta N$ are simple modules and they appear exactly once in the composition series of $M \circ N$.*
(b) *$q^{\widetilde{\Lambda}(M,N)} M \nabla N$ is a self-dual simple module.*
(c) *If $M \circ N \simeq N \circ M$ up to a grading shift, then $M \circ N$ is simple, which is equivalent to $\mathfrak{d}(M,N)=0$ and $M \nabla N \simeq M \Delta N$ up to a grading shift.*

For a reduced sequence $\boldsymbol{i}$ of $w_\circ$, we say an interval $[a,b] \subset [1,\ell]$ is an i-*box* if $i_a = i_b$. For each i-box $[a,b]$ and $\boldsymbol{i}$, we define a self-dual simple modules $L^{\boldsymbol{i}}[a,b]$ as follows: $L^{\boldsymbol{i}}[a,a] := S^{\boldsymbol{i}}_a$ and

$$L^{\boldsymbol{i}}[a,b] := q^{\widetilde{\Lambda}\left(S^{\boldsymbol{i}}_b, L^{\boldsymbol{i}}[a,b^-]\right)} S^{\boldsymbol{i}}_b \nabla L^{\boldsymbol{i}}[a,b^-].$$

Proposition 6.8 ([32, Proposition 4.6]) *For a reduced sequence $\boldsymbol{i}=(i_1,\ldots,i_\ell)$ of $w_\circ$ and an i-box $[a,b] \subset [1,\ell]$, we have*

$$\Omega\left(\left[L^{\boldsymbol{i}}[a,b]\right]\right) = D(w^{\boldsymbol{i}}_b \varpi_{i_b}, w^{\boldsymbol{i}}_{a^-}\varpi_{i_a}).$$

Hence $L^{\boldsymbol{i}}[a,b]$ is an affreal simple module.

6.3 Laurent family

Let J be an index set and $\mathscr{C}$ be a full monoidal subcategory of R-gmod stable by taking subquotients, extensions and grading shifts. We say that a family of affreal modules $\mathcal{L} = \{L_j\}$ in $\mathscr{C}$ is a commuting family in $\mathscr{C}$ if

$$L_i \circ L_j \simeq L_j \circ L_i \quad \text{up to a grading shift for any } i,j \in J.$$

Then there exists a family $\{\mathcal{L}(\mathbf{a}) \mid \boldsymbol{a} \in \mathbb{N}_0^{\oplus J}\}$ of simple modules in $\mathscr{C}$ such that

$$\mathcal{L}(0) = \mathbf{1}, \quad \mathcal{L}(\boldsymbol{e}_j) = L_j \quad \text{for any } j \in J,$$
$$\mathcal{L}(\boldsymbol{a}) \circ \mathcal{L}(\boldsymbol{b}) \simeq q^{-\widetilde{\Lambda}(\mathcal{L}(\boldsymbol{a}),\mathcal{L}(\boldsymbol{b}))} \mathcal{L}(\boldsymbol{a}+\boldsymbol{b}) \quad \text{for any } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}_0^{\oplus J}.$$

We say a commuting family $\mathcal{L} = \{L_j\}_{j \in J}$ is *independent* if, when $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}_0^{\oplus J}$ satisfies $\mathcal{L}(\boldsymbol{a}) \simeq q^s \mathcal{L}(\boldsymbol{b})$ for some $s \in \mathbb{Z}$, we have $\boldsymbol{a} = \boldsymbol{b}$.

Definition 6.9 A commuting family $\mathcal{L} = \{L_j \mid j \in J\}$ of affreal simple objects of $\mathscr{C}$ is said to be a *quasi-Laurent family* in $\mathscr{C}$ if $\mathcal{L}$ satisfies the following conditions:

(a) $\mathcal{L}$ is independent, and
(b) if a simple module X commutes with all L_j $(j \in J)$, then there exist $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_{\geqslant 0}^J$ such that
$$X \circ \mathcal{L}(\boldsymbol{a}) \simeq \mathcal{L}(\boldsymbol{b}) \quad \text{up to a grading shift.}$$

If $\mathcal{M}$ satisfies (a) and (c) below, then we say that $\mathcal{L}$ is a *Laurent family*:

(c) if a simple module X commutes with all L_j $(j \in J)$, then there exists $\boldsymbol{b} \in \mathbb{Z}_{\geqslant 0}^J$ such that $X \simeq \mathcal{L}(\boldsymbol{b})$ up to a grading shift.

For an index set $J = J_{\mathrm{ex}} \sqcup J_{\mathrm{fr}}$, assume that $\mathcal{K}_{\mathbb{A}}(\mathscr{C}) := \mathbb{A} \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathscr{C})$ is isomorphic to a quantum cluster algebra $\mathscr{A}$ such that $\Omega(\mathcal{K}_{\mathbb{A}}(\mathscr{C})) \simeq \mathscr{A}$. We say that a commuting family $\mathcal{L} = \{L_i\}_{i \in J}$ in $\mathscr{C}$ is a *monoidal cluster* in $\mathscr{C}$ if there exists a seed $(\{Z_i\}_{i \in J}, \Lambda = (\Lambda_{i,j})_{i,j \in J}, \widetilde{B} = (b_{i,j})_{i \in J, j \in J_{\mathrm{ex}}})$ of $\mathscr{A}$ such that

$$Z_i \cong \Omega([L_i]) \ \text{ in } \mathscr{A} \quad \text{and} \quad \Lambda_{i,j} = -\Lambda(L_i, L_j).$$

Theorem 6.10 ([36, Proposition 3.6]) *Assume that $\mathcal{K}_{\mathbb{A}}(\mathscr{C})$ has a quantum cluster algebra structure, and let $\mathcal{L} = \{L_j \mid j \in J\}$ be a monoidal cluster in $\mathscr{C}$. Then the commuting family $\mathcal{L}$ is a quasi-Laurent family. In particular, the isomorphism class $[X]$ of a module X in $\mathcal{K}_{\mathbb{A}}(\mathscr{C})$ can be expressed as an element in the Laurent polynomial $\mathbb{Z}[q^{\pm 1}][\,[L_j] \mid j \in J]$ with positive coefficients. Moreover, if every $[L_k]$ is prime in $\mathcal{K}_{\mathbb{A}}(\mathscr{C})|_{q=1}$, then $\mathcal{L}$ is a Laurent family.*

Remark 6.11 Note that if $K(\mathscr{C})|_{q=1}$ is factorial, then every $[L_k]$ is prime in $K(\mathscr{C})|_{q=1}$ (see [14]).

For a reduced sequence $\boldsymbol{i}$ of $w_\circ$ and $1 \leqslant k \leqslant \ell$, we set

$$L_k^{\boldsymbol{i}} := L(w_k^{\boldsymbol{i}} \varpi_{i_k}, \varpi_{i_k}) \simeq L^{\boldsymbol{i}}[k^{\min}, k].$$

Proposition 6.12 ([32, Theorem 4.12]) *Let $\boldsymbol{i}$ be a reduced sequence of $w_\circ$. For $1 \leqslant s < t \leqslant \ell$, we have*

$$-\Lambda(L_s^{\boldsymbol{i}}, L_t^{\boldsymbol{i}}) = (\varpi_{i_s} - w_s^{\boldsymbol{i}} \varpi_{i_s}, \varpi_{i_t} + w_t^{\boldsymbol{i}} \varpi_{i_t}).$$

Hence $\mathcal{L}^{\boldsymbol{i}} := \{L_k^{\boldsymbol{i}}\}$ forms a commuting family.

7 Quantum positivity

In this section, we give a partial proof of the quantum positivity conjecture (Conjecture 1 (i)) for non-simply-laced type. Thus throughout this section, we assume that $\mathfrak{g}$ is of non-simply-laced type.

Recall the following:

(a) For any reduced sequence $\boldsymbol{i}$ of $w_\circ$, there exists an $\mathbb{A}$-algebra isomorphism

$$\varphi_{\boldsymbol{i}} : \mathcal{A}_{\boldsymbol{i}} \longrightarrow \mathcal{A}_{\mathbb{A}}(\mathfrak{n})$$

sending Z_k to $\widetilde{D}(w_k^{\boldsymbol{i}}\varpi_{i_k}, \varpi_{i_k})$. Moreover, this isomorphism sends the seed $(\{Z_k\}_{1\leqslant k\leqslant \ell}, \Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}})$ of $\mathcal{A}_{\boldsymbol{i}}$ to the seed $(\{\widetilde{D}(w_k^{\boldsymbol{i}}\varpi_{i_k}, \varpi_{i_k})\}_{1\leqslant k\leqslant \ell}, \Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}})$ of $\mathcal{A}_{\mathbb{A}}(\mathfrak{n})$.

(b) There exists an $\mathbb{A}$-algebra monomorphism $(\cdot)_{\leqslant\xi} : \mathfrak{K}_{q,Q} \to \mathcal{X}_{q,\leqslant\xi}$ for each Dynkin quiver $Q = (\triangle, \xi)$. Thus the image $(\mathfrak{K}_{q,Q})_{\leqslant\xi}$ is isomorphic to $\mathfrak{K}_{q,Q}$ as $\mathbb{A}$-algebras. Moreover $(F_q(m^{(i)}[p,\xi_i])_{\leqslant\xi} = \underline{m^{(i)}[p,\xi_i]}$ for $(i,p) \in \Gamma_0^Q$.

(c) For a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to a Dynkin quiver $Q = (\triangle, \xi)$, there exists an $\mathbb{A}$-algebra isomorphism

$$\kappa_{\boldsymbol{i}} : \mathcal{A}_{\boldsymbol{i}} \longrightarrow \mathfrak{K}_{q,Q}$$

sending the seed $(\{Z_k\}_{1\leqslant k\leqslant \ell}, \Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}})$ of $\mathcal{A}_{\boldsymbol{i}}$ to the one $(\{F_q([p_k, \xi_{i_k}]\}_{1\leqslant k\leqslant \ell}, \Lambda^{\boldsymbol{i}}, \widetilde{B}^{\boldsymbol{i}})$ of $\mathfrak{K}_{q,Q}$, where $\widetilde{\boldsymbol{i}}(k) = (i_k, p_k)$.

(d) For a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to a Dynkin quiver $Q = (\triangle, \xi)$, Proposition 6.8 tells us that

$$\Omega(L_k^{\boldsymbol{i}}) \cong \widetilde{D}(w_k^{\boldsymbol{i}}\varpi_{i_k}, \varpi_{i_k}) \quad \text{for all } 1 \leqslant k \leqslant \ell. \tag{7.1}$$

Thus, for a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to a Dynkin quiver $Q = (\triangle, \xi)$, we have the following commutative diagram of isomorphisms

$$\begin{tikzcd}
& \mathcal{K}_{\mathbb{A}}(R\text{-gmod}) \arrow[r, "\Omega"] \arrow[dl, "\Xi_{Q,\leqslant\xi}"'] \arrow[d, "\Xi_Q"'] & \mathcal{A}_{\mathbb{A}}(\mathfrak{n}) \arrow[dl, "\Psi_Q"'] \\
(\mathfrak{K}_{q,Q})_{\leqslant\xi} & \mathfrak{K}_{q,Q} \arrow[l, "(\cdot)_{\leqslant\xi}"] & \mathcal{A}_{ii} \arrow[l, "\kappa_{\boldsymbol{i}}"] \arrow[u, "\varphi_i"']
\end{tikzcd} \tag{7.2}$$

such that

$$\Xi_Q := \Psi_Q \circ \Omega : \mathcal{K}_{\mathbb{A}}(R\text{-gmod}) \xrightarrow{\sim} \mathfrak{K}_{q,Q} \quad \text{and}$$

$$\Xi_{Q,\leqslant\xi} := (\cdot)_{\leqslant\xi} \circ \Xi_Q : \mathcal{K}_{\mathbb{A}}(R\text{-gmod}) \xrightarrow{\sim} (\mathfrak{K}_{q,Q})_{\leqslant\xi}.$$

Moreover, the isomorphism $\Xi_{Q,\leqslant\xi}$ sends $L(w_k^{\boldsymbol{i}}\varpi_k, \varpi_k)$ for $1 \leqslant k \leqslant \ell$ to $q^{a/2}\underline{m^{(i)}[p,\xi_i]}$ for some $a \in \mathbb{Z}$, where $\widetilde{\boldsymbol{i}}(k) = (i,p) \in \Gamma_0^Q$.

Theorem 7.1 *Let $Q = (\triangle, \xi)$ be a Dynkin quiver. For an element $\mathsf{X}_{\leqslant\xi} \in (\mathfrak{K}_{q,Q})_{\leqslant\xi}$, assume that there exists a module $X \in R$-gmod and $\Xi_{Q,\leqslant\xi}([X]) \cong \mathsf{X}_{\leqslant\xi}$. Write*

$$\mathsf{X}_{\leqslant\xi} = \sum_{m' \in \mathcal{M}^Q} a[m;m']\underline{m'} \quad \textit{for } a[m:m'] \in \mathbb{Z}[q^{\pm 1/2}].$$

Then $\mathsf{X}_{\leqslant\xi}$ is quantum positive; that is

$$a[m;m'] \in \mathbb{N}_0[q^{\pm 1/2}].$$

Proof By Proposition 6.12 and (7.1),

$$\text{the commuting family } \mathcal{L}^{\boldsymbol{i}} \text{ is a monoidal cluster of } R\text{-gmod}.$$

Hence Theorem 6.10 tells us that

$$[X] = \sum_{\boldsymbol{a} \in \mathbb{Z}^\ell} \alpha(X;\boldsymbol{a}) \prod_{r \in [1,\ell]}^{\to} [L_r^{\boldsymbol{i}}]^{\boldsymbol{a}_r} \quad \text{for some } \alpha(X;\boldsymbol{a}) \in \mathbb{N}_0[q^{\pm 1}].$$

Now let us take a reduced sequence $\boldsymbol{i}$ of $w_\circ$ adapted to the Dynkin quiver Q. Then, for $1 \leqslant k \leqslant \ell$ with $\widetilde{\boldsymbol{i}}(k) = (i_k, p_k)$, we have $\Xi_{Q,\leqslant\xi}(L_k^{\boldsymbol{i}}) \cong \underline{m^{(i)}[p_k, \xi_{i_k}]} \in \mathcal{M}_+^Q$. Hence

$$\mathsf{X}_{\leqslant\xi} \cong \Xi_{Q,\leqslant\xi}([X]) = \sum_{\boldsymbol{a} \in \mathbb{Z}^\ell} \alpha'(X;\boldsymbol{a}) \prod_{r \in [1,\ell]}^{\to} \underline{m^{(i)}[p_r, \xi_{i_r}]}^{\boldsymbol{a}_r} \quad \text{where } \alpha'(X;\boldsymbol{a}) \cong \alpha'(X;\boldsymbol{a}).$$

Since $\displaystyle\prod_{r \in [1,\ell]}^{\to} \underline{m^{(i)}[p_r, \xi_{i_r}]}^{\boldsymbol{a}_r} \in \mathcal{M}_Q$ for any $\boldsymbol{a} \in \mathbb{Z}^\ell$, our assertion follows. □

The following corollary provides a partial proof of Conjecture 1 (i).

Corollary 7.2 *Let m be a KR-monomial contained in $\mathcal{M}_+^Q$ for some Dynkin quiver $Q = (\triangle, \xi)$. Then $(F_q(m))_{\leqslant\xi}$ is quantum positive. In particular, $F_q(X_{i,p})$ is quantum positive for every $(i, p) \in \triangle_0$, and hence, so are the elements in the standard basis E_q.*

Proof The first assertion is a direct consequence of Theorem 5.19 (b), Proposition 6.2, (7.1) and Theorem 7.1. Let us focus on the second assertion which gives a complete proof of Conjecture 1 (i) for $m = X_{i,p}$. Recall that $i^* = i$ for all $i \in I$ because $\mathfrak{g}$ is of non-simply-laced type. Hence

$$\Gamma_0^Q = \{(i,p) \in \widehat{\triangle}_0 \mid \xi_i - \mathsf{h} < p \leqslant \xi_i\}.$$

By the first assertion, $(F_q(X_{i,\xi_i-\mathsf{h}+2}))_{\leqslant\xi}$ is quantum positive as an element in $\mathbb{N}_0[q^{\pm1/2}][\widetilde{X}_{i,p}^{\pm1}]_{(i,p)\in\Gamma_0^Q}$. On the other hand, Theorem 5.6 (c) implies that

$$(F_q(X_{i,\xi_i-\mathsf{h}+2}))_{\leqslant\xi} = F_q(X_{i,\xi_i-\mathsf{h}+2}) - \underline{X_{i,\xi_i+2}^{-1}}.$$

Hence our assertion follows from the automorphism T_r in Remark 5.5. □

Example 7.3 (a) Let $\mathfrak{g}$ be of type B_4. Then we have

$$\begin{aligned}F_q^{B_4}(X_{1,0}) = {} & q\widetilde{X}_{1,0} + q\widetilde{X}_{2,1} * \widetilde{X}_{1,2}^{-1} + q\widetilde{X}_{3,2} * \widetilde{X}_{2,3}^{-1} + q^2\widetilde{X}_{4,3}^2 * \widetilde{X}_{3,4}^{-1}\\ & + (q^{-1}+q)\widetilde{X}_{4,3} * \widetilde{X}_{4,5}^{-1} + q^2\widetilde{X}_{3,4} * \widetilde{X}_{4,5}^{-2}\\ & + q\widetilde{X}_{2,5} * \widetilde{X}_{3,6}^{-1} + q\widetilde{X}_{1,6} * \widetilde{X}_{2,7}^{-1} + q^{-1}\widetilde{X}_{1,8}^{-1},\end{aligned}$$

where $q\widetilde{X}_{1,0} = \underline{X_{1,0}}$ and $q^{-1}\widetilde{X}_{1,8}^{-1} = \underline{X_{1,8}^{-1}}$.

(b) Let $\mathfrak{g}$ be of type B_3. Take $Q = (\triangle_{B_3}, \xi)$ with $\xi_1 = \xi_3 = 0$ and $\xi_2 = -1$, and a simple module $X = L(2) \nabla L(1)$ in R^{B_3}-gmod, where $L(1)$ and $L(2)$ are 1-dimensional cuspidal module with their weights α_1 and α_2, respectively. Note that the image X of X under Ξ_Q is neither a fundamental polynomial nor a KR-polynomial, while $\Xi_Q(L(1) \nabla L(2)) \simeq F_q^{B_3}(X_{1,-4})$. Then one can compute that

$$\begin{aligned}\Xi_{Q,\leqslant\xi}([X]) = \mathsf{X}_{\leqslant\xi} = {} & q\widetilde{X}_{2,-5} * \widetilde{X}_{1,0} + q^4\widetilde{X}_{1,-4} * \widetilde{X}_{3,-4}^2 * \widetilde{X}_{2,-3}^{-1} * \widetilde{X}_{1,0}\\ & + q^4\widetilde{X}_{3,-4}^2 * \widetilde{X}_{1,-2}^{-1} * \widetilde{X}_{1,0}\\ & + (q+q^3)\widetilde{X}_{1,-4} * \widetilde{X}_{3,-4} * \widetilde{X}_{3,-2}^{-1} * \widetilde{X}_{1,0}\\ & + q^4\widetilde{X}_{1,-4} * \widetilde{X}_{2,-3} * \widetilde{X}_{3,-2}^{-2} * \widetilde{X}_{1,0}\\ & + (q^3+q^5)\widetilde{X}_{3,-4} * \widetilde{X}_{2,-3} * \widetilde{X}_{1,-2}^{-1} * \widetilde{X}_{3,-2}^{-1} * \widetilde{X}_{1,0}\\ & + q^8\widetilde{X}_{2,-3}^2 * \widetilde{X}_{1,-2}^{-1} * \widetilde{X}_{3,-2}^{-2} * \widetilde{X}_{1,0}\\ & + q^3\widetilde{X}_{1,-4} * \widetilde{X}_{1,-2} * \widetilde{X}_{2,-1}^{-1} * \widetilde{X}_{1,0}\\ & + (q^2+q^4)\widetilde{X}_{3,-4} * \widetilde{X}_{3,-2} * \widetilde{X}_{2,-1}^{-1} * \widetilde{X}_{1,0}\\ & + (q+q^5)\widetilde{X}_{2,-3} * \widetilde{X}_{2,-1}^{-1} * \widetilde{X}_{1,0}\\ & + q^8\widetilde{X}_{1,-2} * \widetilde{X}_{3,-2}^2 * \widetilde{X}_{2,-1}^{-2} * \widetilde{X}_{1,0}\\ & + (q+q^3)\widetilde{X}_{3,-4} * \widetilde{X}_{1,0} * \widetilde{X}_{3,0}^{-1}\\ & + (q^2+q^4)\widetilde{X}_{2,-3} * \widetilde{X}_{3,-2}^{-1} * \widetilde{X}_{1,0} * \widetilde{X}_{3,0}^{-1}\\ & + (q^3+q^5)\widetilde{X}_{1,-2} * \widetilde{X}_{3,-2} * \widetilde{X}_{2,-1}^{-1} * \widetilde{X}_{1,0} * \widetilde{X}_{3,0}^{-1}\\ & + q^4\widetilde{X}_{1,-2} * \widetilde{X}_{1,0} * \widetilde{X}_{3,0}^{-2},\end{aligned}$$

all of whose coefficients are contained in $\mathbb{Z}_{\geqslant0}[q^{\pm1/2}]$ and $q\widetilde{X}_{2,-5} * \widetilde{X}_{1,0} = \underline{X_{2,-5}X_{1,1}}$.

8 Appendix A. Examples of substitution formula

In this appendix, we provide several examples of substitution formulas for B_2. We consider the situation when $q = 1$ for simplicity. Its quantum analogue $\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}'}$ can be calculated in a similar manner.

Consider the height functions given by

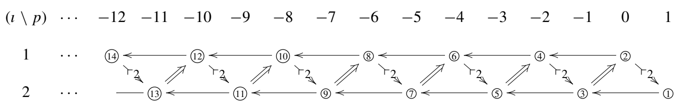
$$\begin{aligned} \xi' &: \{1,2\} \to \mathbb{Z}, \quad 1 \mapsto 0, \quad 2 \mapsto -1, \\ \xi &: \{1,2\} \to \mathbb{Z}, \quad 1 \mapsto 0, \quad 2 \mapsto 1. \end{aligned}$$

Then the following infinite sequences $\boldsymbol{i}' = (i'_u)_{u\in\mathbb{N}}, \boldsymbol{i} = (i_u)_{u\in\mathbb{N}}$ are of the form in (5.11):

$$\boldsymbol{i}' = (\underbrace{1,2,1,2,1,2,1,2}_{\text{reduced word for } w_\circ}, \underbrace{1,2,1,2}, \dots),$$

$$\boldsymbol{i} = (\underbrace{2,1,2,1,2,1,2,1}_{\text{reduced word for } w_\circ}, \underbrace{2,1,2,1}, , \dots).$$

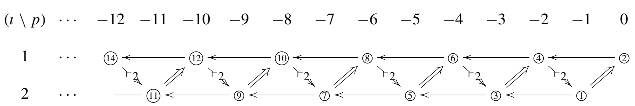
Then $\widetilde{\Gamma}^{\boldsymbol{i}}$ is the following quiver:



Note that $\boldsymbol{i}$ can be obtained from $\boldsymbol{i}'$, and vise versa, by applying the braid moves in the red parts of $\boldsymbol{i}$ and $\boldsymbol{i}'$ above. Hence, by applying the mutations at the following vertices

$$(2,1), (1,0), (2,1), (2,-3), (1,-4), (2,-3), (2,-7), (1,-8), (2,-7), \dots \tag{A.1}$$

from left to right to $\widetilde{\Gamma}^{\boldsymbol{i}}$, we obtain the quiver $\widetilde{\Gamma}^{\boldsymbol{i}'}$.



Here the correspondence of labelling of vertices is given by

$$\begin{aligned} (1,-4m) &\mapsto (1,-4m), & (1,-2-4m) &\mapsto (1,-2-4m), \\ (2,1-4m) &\mapsto (2,-1-4m), & (2,-1-4m) &\mapsto (2,-3-4m). \end{aligned} \tag{A.2}$$

Let us write the initial cluster variable for $\boldsymbol{i}$ as $\{Z_{i,p} := Z^{\xi}_{i,p}\}$ and the one for $\boldsymbol{i}'$ as $\{Z'_{i,p} := Z^{\xi'}_{i,p}\}$. Similarly, we write the generators of $\mathcal{X}_{q,\leqslant\xi}$ as $\{X_{i,p}\}$ and the generators of $\mathcal{X}_{q,\leqslant\xi'}$ as $\{X'_{i,p}\}$.

Using Keller's mutation applet, the isomorphism $\widehat{\boldsymbol{\tau}}^*(Z'_{(i,p)})$ is computed as follows:

$$
\begin{aligned}
Z'_{1,-4m} &\mapsto \Big(Z^2_{2,-4m+3}Z^2_{1,-4m} + 2Z_{2,-4m+3}Z_{1,-4m+2}Z_{1,-4m}Z_{2,-4m-1} \\
&\qquad + Z^2_{1,-4m+2}Z^2_{2,-4m-1} + Z_{1,-4m+2}Z^2_{2,-4m+1}Z_{1,-4m-2}\Big)/(Z^2_{2,-4m+1}Z_{1,-4m}), \\
Z'_{2,-4m-1} &\mapsto \frac{Z_{2,-4m+3}Z_{1,-4m}Z_{2,-4m-1} + Z_{1,-4m+2}Z^2_{2,-4m-1} + Z^2_{2,-4m+1}Z_{1,-4m-2}}{Z_{2,-4m+1}Z_{1,-4m}}, \\
Z'_{1,-4m-2} &\mapsto Z_{1,-4m-2}, \qquad Z'_{2,-4m-3} \mapsto Z_{2,-4m-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\widetilde{\Psi}_{\xi',\xi}(m^{(i)}[p,\xi'_i]) \\
&= \begin{cases}
m^{(1)}[-4m-2,0] & \text{if } (i,p)=(1,-4m-2), \\
m^{(2)}[-4m-1,1] & \text{if } (i,p)=(2,-4m-3), \\
\Big(X^{-2}_{2,-4m+1}X_{1,-4m} + 2X_{2,-4m-1}X^{-1}_{2,-4m+1} & \\
\qquad + X^{-1}_{1,-4m}X^2_{2,-4m-1} + X_{1,-4m-2}\Big)m^{(1)}[-4m+2,0] & \text{if } (i,p)=(1,-4m), \\
(X^{-1}_{2,-4m+1} + X^{-1}_{1,-4m}X_{2,-4m-1} + X_{1,-4m-2}X^{-1}_{2,-4m-1})m^{(2)}[-4m-1,1] & \text{if } (i,p)=(2,-4m-1).
\end{cases}
\end{aligned}
$$

This calculation implies that, for $(i,p) \in {}^{\xi'[-2]}\widehat{\triangle}_0$,

$$
\begin{aligned}
&\widetilde{\Psi}_{\xi',\xi}(X'_{i,p}) = \widetilde{\Psi}_{\xi',\xi}(m^{(i)}[p,\xi'_i])\widetilde{\Psi}_{\xi',\xi}(m^{(i)}[p+2,\xi'_i])^{-1} \\
&= \begin{cases}
\Big(X^{-1}_{1,-4m-2}X^{-2}_{2,-4m+1} + 2X_{2,-4m-1}X^{-1}_{2,-4m+1}X^{-1}_{1,-4m}X^{-1}_{1,-4m-2} & \\
\qquad + X^{-2}_{1,-4m}X^{-1}_{1,-4m-2}X^2_{2,-4m-1} + X^{-1}_{1,-4m}\Big)^{-1} & \text{if } (i,p)=(1,-4m-2), \\
\Big(X_{1,-4m}X^{-2}_{2,-4m+1} + 2X_{2,-4m-1}X^{-1}_{2,-4m+1} + X^{-1}_{1,-4m}X^2_{2,-4m-1} + X_{1,-4m-2}\Big) & \text{if } (i,p)=(1,-4m), \\
(X^{-1}_{2,-4m+1} + X^{-1}_{1,-4m}X_{2,-4m-1} + X_{1,-4m-2}X^{-1}_{2,-4m-1})^{-1} & \text{if } (i,p)=(2,-4m-3), \\
(X_{2,-4m-1} + X^{-1}_{1,-4m}X^2_{2,-4m-1}X_{2,-4m+1} + X_{1,-4m-2}X_{2,-4m+1}) & \text{if } (i,p)=(2,-4m-1),
\end{cases}
\end{aligned}
$$

which shows the substitution formula representing T_2 on $\mathfrak{K}(B_2)$. Then we can observe the following:

$$
\begin{aligned}
\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}}|_{q=1}(X_{1,-4m-2})\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}}|_{q=1}(X_{1,-4m}) &= X_{1,-4m}X_{1,-4m-2}, \\
\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}}|_{q=1}(X_{2,-4m-3})\widetilde{\Psi}_{\mathfrak{Q},\mathfrak{Q}}|_{q=1}(X_{2,-4m-1}) &= X_{2,-4m+1}X_{2,-4m-1}.
\end{aligned}
$$

(I) Note that we have

$$
L_{q=1}(X_{2,-7}) = X_{2,-7} + X_{1,-6}X^{-1}_{2,-5} + X_{2,-5}X^{-1}_{1,-4} + X^{-1}_{2,-3} \tag{A.3}
$$

(see [26] for more details). Here, we set $L_{q=1}(m) := \mathrm{ev}_{q=1}(L_q(m))$. We set

(1) $\boxed{1} = X^{-1}_{2,-3} + X^{-1}_{1,-4}X_{2,-5} + X_{1,-6}X^{-1}_{2,-5} \leftrightarrow (X^{-1}_{2,-7})'$,
(2) $\boxed{2} = X_{2,-5} + X^{-1}_{1,-4}X_{2,-3}X^2_{2,-5} + X_{1,-6}X_{2,-3} \leftrightarrow (X_{2,-5})'$,
(3) $\boxed{3} = X^{-1}_{1,-6}X^{-2}_{2,-3} + 2X^{-1}_{1,-4}X^{-1}_{1,-6}X^{-1}_{2,-3}X_{2,-5} + X^{-2}_{1,-4}X^{-1}_{1,-6}X^2_{2,-5} + X^{-1}_{1,-4} \leftrightarrow (X^{-1}_{1,-6})'$,
(4) $\boxed{4} = X_{1,-4}X^{-2}_{2,-3} + 2X^{-1}_{2,-3}X_{2,-5} + X^{-1}_{1,-4}X^2_{2,-5} + X_{1,-6} \leftrightarrow (X_{1,-4})'$,
(5) $\boxed{5} = X^{-1}_{2,1} + X^{-1}_{1,0}X_{2,-1} + X_{1,-2}X^{-1}_{2,-1} \leftrightarrow (X^{-1}_{2,-3})'$.

Note that

$$\boxed{1} \times X_{2,-3}X_{2,-5} = \boxed{2} \quad \text{and} \quad \boxed{3} \times X_{1,-4}X_{1,-6} = \boxed{4}.$$

Then the first three terms of (A.3) become

$$\frac{\boxed{3} \times X_{2,-3}X_{2,-5} + 1 + \boxed{2}^2 \times X_{1,-4}^{-1}X_{1,-6}^{-1}}{\boxed{2} \times \boxed{3}}. \tag{A.4}$$

Let us compute each term in numerator:

(i) $\boxed{3} \times X_{2,-3}X_{2,-5} = (X_{1,-6}^{-1}X_{2,-3}^{-2} + 2X_{1,-4}^{-1}X_{1,-6}^{-1}X_{2,-3}^{-1}X_{2,-5}$
$\quad + X_{1,-4}^{-2}X_{1,-6}^{-1}X_{2,-5}^{2} + X_{1,-4}^{-1}) \times X_{2,-3}X_{2,-5}$
$= X_{1,-6}^{-1}X_{2,-3}^{-1}X_{2,-5} + 2X_{1,-4}^{-1}X_{1,-6}^{-1}X_{2,-5}^{2} + X_{1,-4}^{-2}X_{1,-6}^{-1}X_{2,-3}X_{2,-5}^{3}$
$\quad + X_{1,-4}^{-1}X_{2,-3}X_{2,-5},$

(ii) 1,

(iii) $\boxed{2}^2 \times X_{1,-4}^{-1}X_{1,-6}^{-1} = (X_{2,-5} + X_{1,-4}^{-1}X_{2,-3}X_{2,-5}^{2} + X_{1,-6}X_{2,-3})^2 \times X_{1,-4}^{-1}X_{1,-6}^{-1}$
$= \Big(X_{2,-5}^{2} + X_{1,-4}^{-2}X_{2,-3}^{2}X_{2,-5}^{4} + X_{1,-6}^{2}X_{2,-3}^{2} + 2X_{1,-4}^{-1}X_{2,-3}X_{2,-5}^{3}$
$\quad +2X_{1,-6}X_{2,-3}X_{2,-5} + 2X_{1,-4}^{-1}X_{1,-6}X_{2,-3}^{2}X_{2,-5}^{2}\Big) \times X_{1,-4}^{-1}X_{1,-6}^{-1}$
$= X_{1,-4}^{-1}X_{1,-6}^{-1}X_{2,-5}^{2} + X_{1,-4}^{-3}X_{1,-6}^{-1}X_{2,-3}^{2}X_{2,-5}^{4} + X_{1,-4}^{-1}X_{1,-6}^{1}X_{2,-3}^{2}$
$\quad + 2X_{1,-4}^{-2}X_{1,-6}^{-1}X_{2,-3}X_{2,-5}^{3} + 2X_{1,-4}^{-1}X_{2,-3}X_{2,-5} + 2X_{1,-4}^{-2}X_{2,-3}^{2}X_{2,-5}^{2}.$

On the other hand, the denominator is

$$\begin{aligned}
\boxed{2} \times \boxed{3} &= (X_{2,-5} + X_{1,-4}^{-1}X_{2,-3}X_{2,-5}^{2} + X_{1,-6}X_{2,-3}) \\
&\quad \times (X_{1,-6}^{-1}X_{2,-3}^{-2} + 2X_{1,-4}^{-1}X_{1,-6}^{-1}X_{2,-3}^{-1}X_{2,-5} + X_{1,-4}^{-2}X_{1,-6}^{-1}X_{2,-5}^{2} + X_{1,-4}^{-1}) \\
&= X_{1,-6}^{-1}X_{2,-3}^{-2}X_{2,-5} + 2X_{1,-4}^{-1}X_{1,-6}^{-1}X_{2,-3}^{-1}X_{2,-5}^{2} + X_{1,-4}^{-2}X_{1,-6}^{-1}X_{2,-5}^{3} \\
&\quad + X_{1,-4}^{-1}X_{2,-5} + X_{1,-4}^{-1}X_{1,-6}^{-1}X_{2,-3}^{-1}X_{2,-5}^{2} + 2X_{1,-4}^{-2}X_{1,-6}^{-1}X_{2,-5}^{3} \\
&\quad + X_{1,-4}^{-3}X_{1,-6}^{-1}X_{2,-3}X_{2,-5}^{4} + X_{1,-4}^{-2}X_{2,-3}X_{2,-5}^{2} + X_{2,-3}^{-1} + 2X_{1,-4}^{-1}X_{2,-5} \\
&\quad + X_{1,-4}^{-2}X_{2,-3}X_{2,-5}^{2} + X_{1,-4}^{-1}X_{1,-6}X_{2,-3}.
\end{aligned}$$

Thus (A.4) becomes

$$\frac{\boxed{3} \times X_{2,-3}X_{2,-5} + 1 + \boxed{2}^2 \times X_{1,-4}^{-1}X_{1,-6}^{-1}}{\boxed{2} \times \boxed{3}} = X_{2,-3}.$$

Consequently, (A.3) is the same as

$$X_{2,1}^{-1} + X_{1,0}^{-1}X_{2,-1} + X_{1,-2}X_{2,-1}^{-1} + X_{2,-3} = L_{q=1}(X_{2,-3}) = \mathfrak{D}_q|_{q=1}(L_{q=1}(X_{2,-7})). \tag{A.5}$$

(II) Now let us observe

$$L_{q=1}(X_{1,-4}) = X_{1,-4} + X_{2,-3}^2 X_{1,-2}^{-1} + 2X_{2,-3}X_{2,-1}^{-1} + X_{1,-2}X_{2,-1}^{-2} + X_{1,0}^{-1}. \tag{A.6}$$

We set

(1) $\boxed{1} = X_{2,-1} + X_{1,0}^{-1}X_{2,1}X_{2,-1}^2 + X_{1,-2}X_{2,1} \leftrightarrow (X_{2,-1})'$,
(2) $\boxed{2} = X_{1,-4}X_{2,-3}^{-2} + 2X_{2,-3}^{-1}X_{2,-5} + X_{1,-4}^{-1}X_{2,-5}^2 + X_{1,-6} \leftrightarrow (X_{1,-4})'$,
(3) $\boxed{3} = X_{1,-2}^{-1}X_{2,1}^{-2} + 2X_{1,0}^{-1}X_{1,-2}^{-1}X_{2,1}^{-1}X_{2,-1} + X_{1,0}^{-2}X_{1,-2}^{-1}X_{2,-1}^2 + X_{1,0}^{-1} \leftrightarrow (X_{1,-2}^{-1})'$,
(4) $\boxed{4} = X_{2,1}^{-1} + X_{1,0}^{-1}X_{2,-1} + X_{1,-2}X_{2,-1}^{-1} \leftrightarrow (X_{2,-3}^{-1})'$,
(5) $\boxed{5} = X_{1,0}X_{2,1}^{-2} + 2X_{2,1}^{-1}X_{2,-1} + X_{1,0}^{-1}X_{2,-1}^2 + X_{1,-2} \leftrightarrow (X_{1,0})'$.

Note that

$$\boxed{3} \times X_{1,-2}X_{1,0} = \boxed{5} \quad \text{and} \quad \boxed{4} \times X_{2,-1}X_{2,1} = \boxed{1}.$$

Thus (A.6) becomes

$$\begin{aligned} &\boxed{2} + \frac{\boxed{3} \times X_{2,-1}^2 X_{2,1}^2}{\boxed{1}^2} + 2 \times \frac{X_{2,-1}X_{2,1}}{\boxed{1}^2} + \frac{X_{1,-2}X_{1,0}}{\boxed{1}^2 \times \boxed{5}} + \frac{1}{\boxed{5}} \\ &= \boxed{2} + \frac{\boxed{3} \times X_{2,-1}^2 X_{2,1}^2 \times \boxed{5} + 2 \times \boxed{5} \times X_{2,-1}X_{2,1} + X_{1,-2}X_{1,0} + \boxed{1}^2}{\boxed{1}^2 \times \boxed{5}}. \end{aligned} \tag{A.7}$$

After a miraculous reduction of the fraction, the second term in (A.7) become $X_{1,-2}^{-1}$ as follows: Here are the four terms in the numerator

(i) $2 \times \boxed{5} \times X_{2,-1}X_{2,1} = 2X_{1,0}X_{2,-1}X_{2,1}^{-1} + 4X_{2,-1}^2$
$\quad + 2X_{1,0}^{-1}X_{2,-1}^3X_{2,1} + 2X_{1,-2}X_{2,-1}X_{2,1}$,

(ii) $\boxed{1}^2 = X_{2,-1}^2 + X_{1,0}^{-2}X_{2,1}^2X_{2,-1}^4 + X_{1,-2}^2X_{2,1}^2$
$\quad + 2X_{1,0}^{-1}X_{2,1}X_{2,-1}^3 + 2X_{1,-2}X_{2,-1}X_{2,1} + 2X_{1,0}^{-1}X_{1,-2}X_{2,1}^2X_{2,-1}^2$,

(iii) $\boxed{3} \times X_{2,-1}^2X_{2,1}^2 \times \boxed{5} = (X_{1,-2}^{-1}X_{2,-1}^2 + 2X_{1,0}^{-1}X_{1,-2}^{-1}X_{2,1}^3X_{2,-1}$
$\quad + X_{1,0}^{-2}X_{1,-2}^{-1}X_{2,1}^2X_{2,-1}^4 + X_{1,0}^{-1}X_{2,-1}^2X_{2,1}^2)$
$\quad \times (X_{1,0}X_{2,1}^{-2} + 2X_{2,1}^{-1}X_{2,-1} + X_{1,0}^{-1}X_{2,-1}^2 + X_{1,-2})$

$$\begin{aligned} &= X_{1,-2}^{-1}X_{1,0}X_{2,1}^{-2}X_{2,-1}^2 + 2X_{1,-2}^{-1}X_{2,1}X_{2,-1} + X_{1,0}^{-1}X_{1,-2}^{-1}X_{2,-1}^4 + X_{2,-1}^2 \\ &\quad + 2X_{2,1}^{-1}X_{1,-2}^{-1}X_{2,-1}^3 + 4X_{1,0}^{-1}X_{1,-2}^{-1}X_{2,1}^2X_{2,-1}^2 + 2X_{1,0}^{-2}X_{1,-2}^{-1}X_{2,1}X_{2,-1}^5 \\ &\quad + 2X_{1,0}^{-1}X_{2,-1}^3X_{2,1} + X_{1,-2}^{-1}X_{1,0}^{-1}X_{2,-1}^4 + 2X_{1,0}^{-2}X_{2,-1}^3X_{1,-2}^{-1}X_{2,1}^3 \\ &\quad + X_{1,0}^{-3}X_{2,-1}^6X_{1,-2}^{-1}X_{2,1}^2 + X_{1,0}^{-2}X_{2,-1}^4X_{2,1}^2 \\ &\quad + X_{2,-1}^2 + 2X_{1,0}^{-1}X_{2,1}^3X_{2,-1} + X_{1,0}^{-2}X_{2,1}^2X_{2,-1}^4 + X_{1,0}^{-1}X_{1,-2}X_{2,-1}^2X_{2,1}^2, \end{aligned}$$

(iv) $X_{1,-2}X_{1,0}$,

while the denominator is

$$
\begin{aligned}
&\boxed{1}^2 \times \boxed{5} = \\
&X_{1,0}X_{2,1}^{-2}X_{2,-1}^{2} + X_{1,0}^{-1}X_{2,-1}^{4} + X_{1,-2}^{2}X_{1,0} + 2X_{2,1}^{-1}X_{2,-1}^{3} \\
&\quad + 2X_{1,-2}X_{1,0}X_{2,1}^{-1}X_{2,-1} + 2X_{1,-2}X_{2,-1}^{2} \\
&\quad + 2X_{2,1}^{-1}X_{2,-1}^{3} + 2X_{1,0}^{-2}X_{2,1}^{3}X_{2,-1}^{3} + 2X_{2,-1}X_{1,-2}^{2}X_{2,1} \\
&\quad + 4X_{1,0}^{-1}X_{2,1}^{2}X_{2,-1}^{2} + 4X_{1,-2}X_{2,-1}^{2} \\
&\quad + 4X_{1,0}^{-1}X_{1,-2}X_{2,1}X_{2,-1}^{3} + X_{1,0}^{-1}X_{2,-1}^{4} + X_{1,0}^{-3}X_{2,1}^{2}X_{2,-1}^{6} \\
&\quad + X_{1,-2}^{2}X_{1,0}^{-1}X_{2,-1}^{2}X_{2,1}^{2} + 2X_{1,0}^{-2}X_{2,1}X_{2,-1}^{5} \\
&\quad + 2X_{1,-2}X_{1,0}^{-1}X_{2,-1}^{3}X_{2,1} + 2X_{1,0}^{-2}X_{1,-2}X_{2,1}^{2}X_{2,-1}^{4} + X_{1,-2}X_{2,-1}^{2} \\
&\quad + X_{1,-2}X_{1,0}^{-2}X_{2,1}^{2}X_{2,-1}^{4} + X_{1,-2}^{3}X_{2,1}^{2} + 2X_{1,-2}X_{1,0}^{-1}X_{2,1}X_{2,-1}^{3} \\
&\quad + 2X_{1,-2}^{2}X_{2,-1}X_{2,1} + 2X_{1,-2}^{2}X_{1,0}^{-1}X_{2,1}^{2}X_{2,-1}^{2}.
\end{aligned}
$$

Thus (A.6) is transformed into

$$X_{1,-4}X_{2,-3}^{-2} + 2X_{2,-3}^{-1}X_{2,-5} + X_{1,-4}^{-1}X_{2,-5}^{2} + X_{1,-6} + X_{1,-2}^{-1} = L_{q=1}(X_{1,-6}).$$

Appendix B. G_2-equivalence

The sequences of mutations that give the equivalence in (3.25) are listed below:

$$
\begin{aligned}
&\mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*, && \mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+2}^*\mu_k^*, \\
&\mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*\mu_k^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*, && \mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+2}^*\mu_k^*, \\
&\mu_k^*\mu_{k+2}^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*, && \mu_k^*\mu_{k+2}^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+2}^*\mu_k^*, \\
&\mu_k^*\mu_{k+2}^*\mu_{k+1}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*\mu_k^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*, && \mu_k^*\mu_{k+2}^*\mu_{k+1}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+2}^*\mu_k^*, \\
&\mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*, && \mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*, \\
&\mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+2}^*\mu_{k+1}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*, && \mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_{k+1}^*\mu_k^*\mu_{k+2}^*\mu_{k+1}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*, \\
&\mu_{k+1}^*\mu_{k+3}^*\mu_{k+2}^*\mu_{k+1}^*\mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_k^*\mu_{k+3}^*\mu_{k+1}^*, && \mu_{k+1}^*\mu_{k+3}^*\mu_{k+2}^*\mu_{k+1}^*\mu_k^*\mu_{k+1}^*\mu_{k+2}^*\mu_{k+3}^*\mu_k^*\mu_{k+1}^*,
\end{aligned}
$$

$$
\begin{array}{ll}
\mu^*_{k+1}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+1}, & \mu^*_{k+1}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+1}, \\
\mu^*_{k+2}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}, & \mu^*_{k+2}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}, \\
\mu^*_{k+2}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}, & \mu^*_{k+2}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}, \\
\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2} & \mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}, \\
\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}, & \mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}, \\
\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+3}, & \mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}, \\
\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+3}, & \mu^*_{k+3}\mu^*_{k+1}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}, \\
\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+3}, & \mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}, \\
\mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+1}\mu^*_{k+3}, & \mu^*_{k+3}\mu^*_{k+2}\mu^*_{k+1}\mu^*_{k+3}\mu^*_{k}\mu^*_{k+2}\mu^*_{k+3}\mu^*_{k+1}\mu^*_{k}\mu^*_{k+3}.
\end{array}
$$

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