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Rank 2 symmetric hyperbolic Kac–Moody algebras and Hilbert modular forms

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ABSTRACT

In this paper we study rank 2 symmetric hyperbolic Kac–Moody algebras $\mathcal{H}(a)$ with the Cartan matrices $\left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$, $a \geqslant 3$, and their automorphic correction in terms of Hilbert modular forms. We associate a family of $\mathcal{H}(a)$'s to the quadratic field $\mathbb{Q}(\sqrt{p})$ for each odd prime p and show that there exists a chain of embeddings in each family. When $p = 5, 13, 17$, we show that the first $\mathcal{H}(a)$ in each family, i.e. $\mathcal{H}(3)$, $\mathcal{H}(11)$, $\mathcal{H}(66)$, is contained in a generalized Kac–Moody superalgebra whose denominator function is a Hilbert modular form given by a Borcherds product. Hence, our results provide automorphic correction for those $\mathcal{H}(a)$'s. We also compute asymptotic formulas for the root multiplicities of the generalized Kac–Moody superalgebras using the fact that the exponents in the Borcherds products are Fourier coefficients of weakly holomorphic modular forms of weight 0.

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Introduction

Rank 2 symmetric Kac–Moody algebras $\mathcal{H}(a)$ are the Lie algebras with a Cartan matrix of the form $\left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$, $a \geqslant 1$. When $a = 1$, the Lie algebra $\mathcal{H}(1)$ is nothing but $\mathfrak{sl}_3(\mathbb{C})$; when $a = 2$, we obtain the affine Lie algebra $\widehat{\mathfrak{sl}_2}(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ which is a central extension of the loop algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$. These Lie algebras are fundamental objects and their structures and representations are quite well-known. Surprisingly enough, when $a \geqslant 3$, we still do not know much about the Lie algebra $\mathcal{H}(a)$. What makes one intrigued is that there seem to be hidden connections of these algebras $\mathcal{H}(a)$ to automorphic forms.

Lepowsky and Moody [20] showed that there are remarkable connections between root systems of rank 2 (not necessarily symmetric) hyperbolic Kac–Moody algebras and quasi-regular cusps on Hilbert modular surfaces attached to certain quadratic fields. A. Feingold studied the algebra $\mathcal{H}(3)$ and described the root system of $\mathcal{H}(3)$ in terms of Fibonacci numbers [5]. Kang and Melville extended this result to $\mathcal{H}(a)$, $a \geqslant 3$, using generalized Fibonacci numbers [14]. Furthermore, in the same paper, they studied root multiplicities of $\mathcal{H}(a)$, making use of Kang's formula for root multiplicities of Kac–Moody algebras [13]. In 2004, Feingold and Nicolai showed that the algebras $\mathcal{H}(a)$ can be embedded into the rank 3 hyperbolic Kac–Moody algebra $\mathcal{F}$ associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{smallmatrix}\right)$.

The Lie algebra $\mathcal{F}$ was shown in Feingold–Frenkel [6] to have connections with the theory of genus 2 Siegel modular forms. Then Gritsenko–Nikulin [8] showed that $\mathcal{F}$ is contained in a generalized Kac–Moody algebra $\mathcal{G}$ whose denominator function is a particular Siegel modular form, and they called $\mathcal{G}$ an automorphic correction of $\mathcal{F}$. The notion of automorphic correction was originated from Borcherds' work [1] on Monster Lie algebras. See Section 3 for the precise definition of automorphic correction due to Gritsenko and Nikulin [10].

The purpose of this paper is to investigate connections of the hyperbolic Kac–Moody algebras $\mathcal{H}(a)$, $a \geqslant 3$, to Hilbert modular forms from the point of view of automorphic correction. For each odd prime p, let $F = \mathbb{Q}(\sqrt{p})$, and to each positive solution of the Pell equation $a^2 - ps^2 = 4$, we associate a family of $\mathcal{H}(a)$'s. In Section 5.1, we show that there exists a chain of embeddings in each family (Theorem 5.5).

In particular, we consider three infinite families of $\mathcal{H}(a)$'s attached to the quadratic fields $\mathbb{Q}(\sqrt{p})$, $p \in \{5, 13, 17\}$, respectively. These three primes are the only primes for which there exists the unique weakly holomorphic modular form $f_m \in A_0^+(p, \chi_p)$ (see Section 4.1 for the definition of the space $A_0^+(p, \chi_p)$) with the principal part $s(m)^{-1}q^{-m}$ for each $m \geqslant 1$, where $s(m) = \begin{cases} 1, & \text{if } p \nmid m, \\ 2, & \text{if } p \mid m. \end{cases}$

Consider the first $\mathcal{H}(a)$ in each family, namely, $\mathcal{H}(3)$, $\mathcal{H}(11)$, $\mathcal{H}(66)$. In Section 5.2, we show that there exists a generalized Kac–Moody superalgebra $\widetilde{\mathcal{H}}$ for each of these $\mathcal{H}(a)$'s, which contains the $\mathcal{H}(a)$ as a subalgebra, and whose denominator function is a Hilbert modular form $\Phi_1(z)$ for $\mathbb{Q}(\sqrt{p})$ (Theorem 5.16). Here the fact that $\Phi_1(z)$ is an infinite product, so-called Borcherds product, is crucial. Borcherds [2] studied certain lifts of weight 0 weakly holomorphic modular forms to modular forms on orthogonal groups $O(2,2)$. Bruinier and Bundschuh [4] made explicit the correspondence between modular forms on $O(2,2)$ and Hilbert modular forms for $\mathbb{Q}(\sqrt{p})$, $p \equiv 1 \pmod 4$. We use their explicit correspondence in showing that $\Phi_1(z)$ is indeed the automorphic correction of the denominator function of $\mathcal{H}(a)$. For $p = 13, 17$, we also use the explicit calculation of Mayer [21].

Let $f_m(z) = s(m)^{-1}q^{-m} + \sum_{n=0}^{\infty} a_m(n)q^n$. It is known [4] that $a_m(n)$ are rational numbers with bounded denominators. When $p = 5, 13$, more is true. Indeed we verify that $s(n)a_m(n)$ are integers for all n (Lemma 4.2). If $p = 17$, it is likely that they are integers, but we were not able to verify it. We assume that they are integers. It is necessary since they are root multiplicities of the generalized Kac–Moody superalgebra $\widetilde{\mathcal{H}}$.

In Section 6, we apply the method of Hardy–Ramanujan–Rademacher [19] to calculate the asymptotics of the Fourier coefficients $a_m(n)$ (Theorem 6.1). In that way, we obtain information on the root multiplicities of $\widetilde{\mathcal{H}}$.

It is expected that our method can be applied to more general rank 2 hyperbolic Kac–Moody algebras. We will consider these general cases in a subsequent paper [17].

1. Rank 2 symmetric hyperbolic Kac–Moody algebras

In this section we fix our notations for hyperbolic Kac–Moody algebras. A general theory of Kac–Moody algebras can be found in [12], and the rank 2 hyperbolic case was studied by Lepowsky and Moody [20], Feingold [5], and Kang and Melville [14].

Let $A = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ be a generalized Cartan matrix with $a \geqslant 3$, and $\mathcal{H}(a)$ be the hyperbolic Kac–Moody algebra associated with the matrix A. In this section, we write $\mathfrak{g} = \mathcal{H}(a)$ if there is no need to specify a. Let $\{h_1, h_2\}$ be the set of simple coroots in the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2 \subset \mathfrak{g}$. Let $\{\alpha_1, \alpha_2\} \subset \mathfrak{h}^*$ be the set of simple roots, and $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ be the root lattice, and define $\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2$ and $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\alpha_1 + \mathbb{R}\alpha_2$. The set of roots of $\mathfrak{g}$ will be denoted by Δ, and the set of positive (resp. negative) roots by Δ^+ (resp. by Δ^-), and the set of real (resp. imaginary) roots by Δ_{re} (resp. by Δ_{im}). We will use the notation Δ^+_{re} to denote the set of positive real roots. Similarly, we use Δ^+_{im}, Δ^-_{re} and Δ^-_{im}.

We assume that $a^2 - 4 = ps^2$ for some $s \in \mathbb{N}$ and an odd prime p, and let $F = \mathbb{Q}(\sqrt{p})$. We denote by $\bar{x}$ the conjugate of $x \in F$ and write N and tr for the norm and trace of F. We denote the ring of integers of F by $\mathcal{O}$. By considering the Pell equation

$$a^2 - ps^2 = 4, \tag{1.1}$$

we obtain infinitely many pairs (a, s) for each p. We set

$$\eta = \frac{a + \sqrt{a^2 - 4}}{2} = \frac{a + s\sqrt{p}}{2}.$$

Then we have $\bar{\eta} = \eta^{-1}$ and $1 + \eta^2 = a\eta$. If $p \equiv 1 \pmod 4$, we fix a fundamental unit ε_0 of F so that $\eta = \varepsilon_0^{2k}$ for some $k \in \mathbb{N}$. In this case $N(\varepsilon_0) = -1$ and $N(\eta) = 1$. If $p \equiv 3 \pmod 4$, we fix a fundamental unit ε_0 of F so that $\eta = \varepsilon_0^k$ for some $k \in \mathbb{N}$. In this case $N(\varepsilon_0) = 1$. For example, if $p = 5$ then the smallest positive solution of the Pell equation is $(a, s) = (3, 1)$, if $p = 13$ then $(a, s) = (11, 3)$, and if $p = 17$ then $(a, s) = (66, 16)$, and we choose a fundamental unit ε_0 of $\mathcal{O}$ as follows:

$$\varepsilon_0 = \frac{1 + \sqrt{5}}{2} \quad \text{for } p = 5; \qquad \varepsilon_0 = \frac{3 + \sqrt{13}}{2} \quad \text{for } p = 13; \qquad \varepsilon_0 = 4 + \sqrt{17} \quad \text{for } p = 17.$$

The simple reflection corresponding to α_i in the root system of $\mathfrak{g}$ is denoted by r_i $(i = 1, 2)$, and the Weyl group by W. The eigenvalues of $r_1 r_2$ as a linear transformation on $\mathfrak{h}^*$ are η^2 and η^{-2}. Let γ^+ be an eigenvector for η^2 and we set $\gamma^- = r_2\gamma^+$. Then γ^- is an eigenvector for $\bar{\eta}^2$. Specifically, we choose

$$\gamma^+ = \frac{\alpha_1 + \bar{\eta}\alpha_2}{s} \quad \text{and} \quad \gamma^- = \frac{\alpha_1 + \eta\alpha_2}{s}.$$

We define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ to be given by the Cartan matrix A with respect to $\{\alpha_1, \alpha_2\}$. Then we have $(\gamma^+, \gamma^+) = (\gamma^-, \gamma^-) = 0$ and $(\gamma^+, \gamma^-) = -p$.

We will use the column vector notation for the elements in $\mathfrak{h}^*$ with respect to the basis $\{\gamma^+, \gamma^-\}$, i.e. we write $\binom{x}{y}$ for $x\gamma^+ + y\gamma^-$. Then we have

$$\alpha_1 = \frac{1}{\sqrt{p}} \begin{pmatrix} \eta \\ -\bar{\eta} \end{pmatrix} \quad \text{and} \quad \alpha_2 = \frac{1}{\sqrt{p}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

It is now easy to see that $\mathfrak{h}^*_{\mathbb{Q}} = \{\binom{x}{\bar{x}} \mid x \in F\}$. A symmetric bilinear form $\langle\cdot,\cdot\rangle$ on F is defined by $\langle x, y\rangle = -p\,\mathrm{tr}(x\bar{y})$. We define a map $\psi : \mathfrak{h}^*_{\mathbb{Q}} \to F$ by $\binom{x}{\bar{x}} \mapsto x$. Then the map ψ is an isometry from $(\mathfrak{h}^*_{\mathbb{Q}}, (\cdot,\cdot))$ to $(F, \langle\cdot,\cdot\rangle)$. In particular, the root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is mapped onto a sublattice of $\mathcal{O}/\sqrt{p}$. When $p \equiv 1 \pmod 4$, the inverse different $\mathfrak{d}^{-1}$ is equal to $\mathcal{O}/\sqrt{p}$, and we have $s\mathfrak{d}^{-1} \subset \psi(Q) \subset \mathfrak{d}^{-1}$, and the dual lattice $(\mathfrak{d}^{-1})'$ of $\mathfrak{d}^{-1}$ is $\frac{1}{p}\mathcal{O}$.

Let ω_i $(i = 1, 2)$ be the fundamental weights of $\mathfrak{g}$. Then we have $\omega_1 = \frac{1}{4-a^2}(2\alpha_1 + a\alpha_2)$ and $\omega_2 = \frac{1}{4-a^2}(a\alpha_1 + 2\alpha_2)$. In the column vector notation,

$$\omega_1 = \frac{-1}{sp} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega_2 = \frac{-1}{sp} \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}.$$

We define

$$\rho := -(\omega_1 + \omega_2) = \frac{1}{sp} \begin{pmatrix} 1 + \eta \\ 1 + \bar{\eta} \end{pmatrix}.$$

The simple reflections have the matrix representations

$$r_1 = \begin{pmatrix} 0 & \eta^2 \\ \bar{\eta}^2 & 0 \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Weyl group W also acts on F by

$$r_1 x = \eta^2 \bar{x} \quad \text{and} \quad r_2 x = \bar{x} \quad \text{for } x \in F,$$

so that the isometry ψ is W-equivariant. Since $W = \{(r_1 r_2)^i, r_2(r_1 r_2)^i \mid i \in \mathbb{Z}\}$, we can calculate the set of positive real roots and obtain

$$\Delta_{\mathrm{re}}^+ = \left\{ \frac{1}{\sqrt{p}} \begin{pmatrix} \eta^j \\ -\bar{\eta}^j \end{pmatrix} \ (j > 0),\ \frac{1}{\sqrt{p}} \begin{pmatrix} -\bar{\eta}^j \\ \eta^j \end{pmatrix} \ (j \geqslant 0) \right\}.$$

The set of imaginary roots is described in [5,14]. We present it using our notations: First, we define the set

$$\Omega_k = \left\{ (m,n) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}\colon\ \sqrt{\frac{4k}{a^2-4}} \leqslant m \leqslant \sqrt{\frac{k}{a-2}},\ n = \frac{am - \sqrt{(a^2-4)m^2 - 4k}}{2} \right\}$$

for $k \geqslant 1$. Note that we have only to present $\psi(\Delta_{\mathrm{im}}^+)$. The set is given by

$$\psi\big(\Delta_{\mathrm{im}}^+\big) = \left\{ \frac{1}{\sqrt{p}} \eta^j (m\eta - n),\ \frac{1}{\sqrt{p}} \eta^j (n\eta - m),\ \frac{1}{\sqrt{p}} \bar{\eta}^j (n - m\bar{\eta}),\ \frac{1}{\sqrt{p}} \bar{\eta}^j (m - n\bar{\eta}) \right\}, \quad (1.2)$$

where $j \geqslant 0$ and $(m,n) \in \Omega_k$ for $k \geqslant 1$.

2. Modular forms on $O(2,2)$ as Hilbert modular forms

In this section, we review the result of [3] on the correspondence between Hilbert modular forms and modular forms on $O(2,2)$ in a special case after we consider the general case of modular forms on $O(n,2)$.

2.1. Modular forms on $O(n,2)$

Let (V,Q) be a non-degenerate quadratic space over $\mathbb{Q}$ of type $(n,2)$. Let $V(\mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Q}} V$ and $V(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Q}} V$, and $P(V(\mathbb{C})) = (V(\mathbb{C}) - \{0\})/\mathbb{C}^*$ be the corresponding projective space. We denote by $O_V(\mathbb{R})$ the orthogonal group of the space $V(\mathbb{R})$. Let $\mathcal{K}^+$ be a connected component of

$$\mathcal{K} = \big\{ [Z] \in P\big(V(\mathbb{C})\big)\colon\ (Z,Z) = 0,\ (Z,\bar{Z}) < 0 \big\}, \quad (2.1)$$

and let $O_V^+(\mathbb{R})$ be the subgroup of elements in $O_V(\mathbb{R})$ which preserve the components of $\mathcal{K}$.

For $Z \in V(\mathbb{C})$, write $Z = X + iY$ with $X, Y \in V(\mathbb{R})$. Given an even lattice $L \subset V$, we denote by O_L the orthogonal group of L and let $O_L^+ := O_L \cap O_V^+(\mathbb{R})$. Assume $\Gamma \subseteq O_L^+$ is a subgroup of finite index. Then Γ acts on $\mathcal{K}$ discontinuously. Let

$$\widetilde{\mathcal{K}}^+ = \{Z \in V(\mathbb{C}) - \{0\}:\ [Z] \in \mathcal{K}^+\}.$$

Let $k \in \frac{1}{2}\mathbb{Z}$, and χ be a multiplier system of Γ. Then a meromorphic function $\Phi : \widetilde{\mathcal{K}}^+ \to \mathbb{C}$ is called a *meromorphic modular form* of weight k and multiplier system χ for the group Γ, if

(1) Φ is homogeneous of degree $-k$, i.e., $\Phi(cZ) = c^{-k}\Phi(Z)$ for all $c \in \mathbb{C} - \{0\}$,
(2) Φ is invariant under Γ, i.e., $\Phi(\gamma Z) = \chi(\gamma)\Phi(Z)$ for all $\gamma \in \Gamma$.

This definition agrees with the one given in [10].

2.2. Hilbert modular forms on quadratic number fields

For a prime $p \equiv 1 \pmod 4$, let $F = \mathbb{Q}[\sqrt{p}]$, and let $\mathcal{O}_F$ and $\mathfrak{d}_F$ be the ring of integers and the different of F, respectively. We denote by $\bar{x}$ the conjugation of x in F. We set $\Gamma_F = SL_2(\mathcal{O}_F)$. Assume that $\Gamma \subseteq \Gamma_F$ is a subgroup of finite index. Let $\mathbb{H}$ be the upper-half plane and χ be a multiplier system of Γ. A meromorphic function $f : \mathbb{H}^2 \to \mathbb{C}$ is called a *Hilbert modular form* of weight k for Γ if

$$f(\gamma z) = \chi(\gamma) N(cz + d)^k f(z),$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and $N(cz + d) = (cz_1 + d)(\bar{c}z_2 + \bar{d})$ for $z = (z_1, z_2) \in \mathbb{H}^2$.

Consider the $\mathbb{Q}$-vector space $V = \mathbb{Q} \oplus \mathbb{Q} \oplus F$, and define a quadratic form Q on V by

$$Q(a, b, \nu) = -p(\nu\bar{\nu} + ab)$$

and a bilinear form B so that $B((a, b, \nu), (a, b, \nu)) = 2Q(a, b, \nu)$. Then (V, Q) is a quadratic space of type $(2, 2)$. We will consider the lattice $L = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{d}_F^{-1}$ in V.

Let

$$\widetilde{V} = \{X \in M_2(F):\ X^t = \bar{X}\} = \left\{\begin{pmatrix} a & \nu \\ \bar{\nu} & b \end{pmatrix}:\ a, b \in \mathbb{Q},\ \nu \in F\right\}.$$

Then $\widetilde{V}$ is a rational quadratic space with the quadratic form $\widetilde{Q}(X) = p\det(X)$. The corresponding bilinear form is $\widetilde{B}(X_1, X_2) = p\,\mathrm{tr}(X_1 X_2^*)$, where $X^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ for $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Here $SL_2(F)$ acts on $\widetilde{V}$ by $X \mapsto g.X = gX\bar{g}^t$ for $X \in \widetilde{V}$ and $g \in SL_2(F)$. Then $\widetilde{V}$ and V are isometric with the isometry given by

$$\widetilde{V} \to V, \quad \begin{pmatrix} a & \nu \\ \bar{\nu} & b \end{pmatrix} \mapsto (-a, b, \nu). \tag{2.2}$$

Under the isomorphism, we have

$$L = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{d}_F^{-1} \simeq \left\{ \begin{pmatrix} a & \nu \\ \bar{\nu} & b \end{pmatrix} \in \widetilde{V}\colon\ a, b \in \mathbb{Z},\ \nu \in \mathfrak{d}_F^{-1} \right\}. \tag{2.3}$$

Note that the dual lattice L' is given by

$$L' = \frac{1}{p}\mathbb{Z} \oplus \frac{1}{p}\mathbb{Z} \oplus \frac{1}{p}\mathcal{O}.$$

The two real embeddings $F \to \mathbb{R}^2$, $x \mapsto (x, \bar{x})$, induce an embedding $\widetilde{V} \mapsto M_2(\mathbb{R})$. Thus we have $\widetilde{V}(\mathbb{C}) = M_2(\mathbb{C})$, and let

$$\mathcal{K} = \bigl\{ [Z] \in P\bigl(M_2(\mathbb{C})\bigr)\colon\ \det(Z) = 0,\ \operatorname{tr}\bigl(Z\bar{Z}^*\bigr) < 0 \bigr\}.$$

We write $M(z) = \left(\begin{smallmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ for $z = (z_1, z_2) \in \mathbb{C}^2$. Note that $[M(z)] \in \mathcal{K}$ if and only if $\operatorname{Im}(z_1)\operatorname{Im}(z_2) > 0$. Let $\mathcal{K}^+$ be a connected component of $\mathcal{K}$. Then $\mathbb{H}^2 \to \mathcal{K}^+$, $z = (z_1, z_2) \mapsto [M(z)]$, is a biholomorphic map. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(F)$, we have

$$\gamma M(z) = N(cz + d) M(\gamma z).$$

One can easily see that $\Gamma_F \subset O_L^+$. Therefore, modular forms of weight k on $O(2,2)$ can be considered as Hilbert modular forms of weight k.

3. Automorphic correction

In this section, we recall the theory of automorphic correction established by Gritsenko and Nikulin [8–10]. The original idea of automorphic correction can be traced back to Borcherds' work [1].

We assume that the following data (1)–(4) are given.

(1) We are given a lattice M with a non-degenerate integral symmetric bilinear form $(\cdot,\cdot)$ of signature $(n, 1)$ for some $n \in \mathbb{N}$.

(2) A nontrivial reflection group $W \subset O_M$ is given. The group W is generated by reflections in some roots of the lattice M. A vector $\alpha \in M$ is called a root if $(\alpha, \alpha) > 0$ and (α, α) divides $2(\alpha, \beta)$ for all $\beta \in M$.

(3) Consider the cone

$$V(M) = \bigl\{ \beta \in M \otimes \mathbb{R} \bigm| (\beta, \beta) < 0 \bigr\},$$

which is a union of two half cones. One of these half cones is denoted by $V^+(M)$. Choose a minimal set Π of roots which determines a fundamental chamber $\mathcal{M} \subset V^+(M)$ of W so that

$$\mathcal{M} = \{\beta \in \overline{V^+(M)} \mid (\beta, \alpha) \leqslant 0 \text{ for all } \alpha \in \Pi\}.$$

Moreover, we have a Weyl vector $\rho \in M \otimes \mathbb{Q}$ satisfying $(\rho, \alpha) = -(\alpha, \alpha)/2$ for each $\alpha \in \Pi$.

(4) Define the complexified cone $\Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M)$. For each $m \in \mathbb{N}$, we define a rank 2 lattice $P(m) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ with the symmetric integral bilinear form such that $(e_1, e_1) = (e_2, e_2) = 0$ and $(e_1, e_2) = -m$. Let $L = P(m) \oplus M$ be an extended lattice for some $m \in \mathbb{N}$. We consider the quadratic space $V = L \otimes \mathbb{Q}$ and obtain $\mathcal{K}^+$ as in (2.1). Define a map $\Omega(V^+(M)) \to \mathcal{K}$ by $z \mapsto [\frac{(z,z)}{2m} e_1 + e_2 + z]$. Then the space $\mathcal{K}^+$ is canonically identified with $\Omega(V^+(M))$. We are given a holomorphic automorphic form $\Phi(z)$ on $\Omega(V^+(M))$ with respect to a subgroup $\Gamma \subset O_L^+$ of finite index. The automorphic form Φ has a Fourier expansion of the form

$$\Phi(z) = \sum_{w \in W} \det(w) \Big(e\big(-(w(\rho), z)\big) - \sum_{a \in M \cap \mathcal{M}} m(a) e\big(-(w(\rho + a), z)\big) \Big),$$

where $e(x) = e^{2\pi i x}$ and $m(a) \in \mathbb{Z}$ for all $a \in M \cap \mathcal{M}$.

The matrix

$$A = \left(\frac{2(\alpha, \alpha')}{(\alpha, \alpha)} \right)_{\alpha, \alpha' \in \Pi}$$

defines a Kac–Moody algebra $\mathfrak{g}$. Moreover, the data (1)–(4) define a generalized Kac–Moody superalgebra $\mathcal{G}$ as in [10]. In particular, the function $\Phi(z)$ determines the set of imaginary simple roots of $\mathcal{G}$ in the following way: First, assume that $a \in M \cap \mathcal{M}$ and $(a, a) < 0$. If $m(a) > 0$ then a is an even imaginary simple root with multiplicity $m(a)$, and if $m(a) < 0$ then a is an odd imaginary simple root with multiplicity $-m(a)$. Next, assume that $a_0 \in M \cap \mathcal{M}$ is primitive (i.e. a_0 cannot be written as an integral multiple of another $b_0 \in M \cap \mathcal{M}$) and $(a_0, a_0) = 0$. Then we define $\mu(na_0) \in \mathbb{Z}$, $n \in \mathbb{N}$, by

$$1 - \sum_{k=1}^{\infty} m(ka_0) t^k = \prod_{n=1}^{\infty} (1 - t^n)^{\mu(na_0)},$$

where t is a formal variable. If $\mu(na_0) > 0$ then na_0 is an even imaginary simple root with multiplicity $\mu(na_0)$; if $\mu(na_0) < 0$ then na_0 is an odd imaginary simple root with multiplicity $-\mu(na_0)$.

We call $\mathcal{G}$ (or $\Phi(z)$) an *automorphic correction* of $\mathfrak{g}$. The automorphic form $\Phi(z)$ determines the set of simple imaginary roots of $\mathcal{G}$, and can be written, using the denominator identity for the generalized Kac–Moody superalgebra $\mathcal{G}$, as the product

$$\Phi(z) = e\big(-(\rho, z)\big) \prod_{\alpha \in \Delta(\mathcal{G})^+} \big(1 - e\big(-(\alpha, z)\big)\big)^{\text{mult}(\mathcal{G}, \alpha)},$$

where $\Delta(\mathcal{G})^+$ is the set of positive roots of $\mathcal{G}$ and $\mathrm{mult}(\mathcal{G},\alpha)$ is the root multiplicity of α in $\mathcal{G}$. In Section 5.2, we will construct automorphic corrections of the hyperbolic Kac–Moody algebras $\mathcal{H}(3)$, $\mathcal{H}(11)$, $\mathcal{H}(66)$ which are associated with the quadratic fields $\mathbb{Q}(\sqrt{p}\,)$, $p = 5, 13, 17$, respectively.

4. Hilbert modular forms as Borcherds products

In this section we summarize the results of [4] (cf. [21]).

4.1. Weakly holomorphic modular forms of weight 0

Let p be an odd prime and $A_k^+(p,\chi_p)$ (resp. $A_k^-(p,\chi_p)$) be the space of weakly holomorphic modular forms f of weight k for the group $\Gamma_0(p)$ with character χ_p such that $a(n) = 0$ if $\chi_p(n) = -1$ (resp. $\chi_p(n) = 1$), where $f = \sum_{n\geqslant n_0} a(n)q^n$ for some $n_0 \in \mathbb{Z}$ and $\chi_p(n) = (\frac{n}{p})$. Such an f is called a cusp form if it vanishes at all cusps: Since ∞, 0 are cusps in our case (see Section 6), $a(n) = 0$ for $n \leqslant 0$ and $f(-\frac{1}{z}) = \sum_{n\geqslant 1} b(n)q^{\frac{n}{p}}$. We denote by $S_k^+(p,\chi_p)$ (resp. $S_k^-(p,\chi_p)$) the subspace of cusp forms. For an integer n, define $s(n) = \begin{cases} 2 & \text{if } p \mid n, \\ 1 & \text{otherwise.}\end{cases}$

Theorem 4.1. *(See [4, Theorem 6].) There exists a weakly holomorphic modular form $f \in A_0^+(p,\chi_p)$ with prescribed principal part $\sum_{n<0} a(n)q^n$ if and only if $\sum_{n<0} s(n) \times a(n)b(-n) = 0$ for every cusp form $g = \sum_{m>0} b(m)q^m \in S_2^\delta(p,\chi_p)$, where $\delta = \chi_p(-1)$.*

In the rest of this section, we assume that $p \in \{5, 13, 17\}$. Then we have $S_2^+(p,\chi_p) = 0$. For a given positive integer m with $\chi_p(m) \neq -1$, we let

$$f_m = \sum_{n\geqslant -m} a_m(n)q^n = s(m)^{-1}q^{-m} + \sum_{n=0}^{\infty} a_m(n)q^n,$$

be the unique element of $A_0^+(p,\chi_p)$, whose principal part is $s(m)^{-1}q^{-m}$.

When $p = 5$,

$$\begin{aligned}
f_1 &= q^{-1} + 5 + 11q - 54q^4 + 55q^5 + 44q^6 - 395q^9 + 340q^{10} + \cdots, \\
f_4 &= q^{-4} + 15 - 216q + 4959q^4 + 22\,040q^5 - 90\,984q^6 + \cdots, \\
f_9 &= q^{-9} + 35 - 3555q + 922\,374q^4 + 7\,512\,885q^5 - 53\,113\,164q^6 + \cdots.
\end{aligned}$$

When $p = 13$,

$$\begin{aligned}
f_1 &= q^{-1} + 1 + q + 3q^3 - 2q^4 - q^9 - 4q^{10} + 10q^{12} + \cdots, \\
f_4 &= q^{-4} + 3 - 8q + 16q^3 + 29q^4 - 70q^9 - 2q^{10} - 32q^{11} + \cdots, \\
f_9 &= q^{-9} + 13 - 9q + 36q^3 - 198q^4 + \cdots.
\end{aligned}$$

When $p=17$,

$$f_1=q^{-1}+\frac{1}{2}-q+q^2+2q^4-q^8-2q^9+q^{13}-q^{15}+2q^{16}+\cdots,$$
$$f_4=q^{-4}+\frac{7}{2}+8q-2q^2+11q^4-5q^8+16q^9-56q^{13}+\cdots,$$
$$f_9=q^{-9}+\frac{7}{2}-18q-27q^2+36q^4+243q^8+41q^9-279q^{13}+\cdots.$$

If $p=5,13$, we can prove that f_1 has integer coefficients. This follows from the fact that

$$f_1(z)=\frac{E_2^{(p)}(z)}{H_2(z)},$$

where $E_2^{(p)}$ is the normalized Eisenstein series of weight 2 for $\Gamma_0(p)$ with the trivial character and H_2 is the Eisenstein series with the character χ_p corresponding to the cusp 0 (there is a typo in [21, p. 114]):

$$E_2^{(p)}(z)=1+\frac{24}{p-1}\sum_{n=1}^{\infty}\bigl(\sigma(n)-p\sigma(n/p)\bigr)q^n,$$
$$H_2(z)=\sum_{n=1}^{\infty}\Bigl(\sum_{d|n}d\chi_p(n/d)\Bigr)q^n=q+O\bigl(q^2\bigr).$$

Here we put the convention that if $p\nmid n$, $\sigma(n/p)=0$. Note that if $p=5$, $H_2(z)=H^{(q)}(z)=\frac{\eta(5z)^5}{\eta(z)}$. When $p=13$, $H_2(z)$ may have a zero in the upper half plane, and yet it is remarkable that the quotient $\frac{E_2^{(p)}(z)}{H_2(z)}$ does not have a pole in the upper half plane.

In general, it is known that $a_m(n)$ are rational numbers and have a bounded denominator. See Proposition 8 in [4]. But we can see that more is true in the case of $p=5,13$.

Lemma 4.2. *Let $p=5,13$. Then $s(n)a_m(n)$ are integers for all $n\geqslant 0$.*

Proof. Let $\widetilde{H}(z)=\frac{\eta(z)^k}{\eta(pz)^k}$, where $k=\frac{24}{gcd(24,p-1)}$. It belongs to $A_0(p,1)$. In [21, p. 112] we can see that $f_m(z)$, $m<p$, are generated by $\widetilde{H}(z)$ and $f_1(z)$ over $\mathbb{Z}$. Hence they have integer coefficients. On the other hand, f_p is obtained from $\frac{1}{2}E_0$ by subtracting suitable integer multiples of f_m, $m<p$. Hence it is enough to observe that if $E_0(z)=q^{-p}+\sum_{n=-p+1}^{\infty}b(n)q^n$, $b(n)$ is an even integer for $p\nmid n$. Recall that

$$E_0(z)=E_2^+(z)\frac{E_4E_6}{\Delta}(pz),\quad E_2^+(z)=1+\frac{2}{L(-1,\chi_p)}\sum_{n=1}^{\infty}\sum_{d|n}d\bigl(\chi_p(d)+\chi_p(n/d)\bigr)q^n.$$

Here $L(-1,\chi_5) = -\frac{2}{5}$, and $L(-1,\chi_{13}) = -2$, and if $p \nmid n$, then $p \nmid d$ for any $d \mid n$. So the possible values of $\chi_p(d)+\chi_p(n/d)$ are 0, ± 2. Therefore, we can write $E_2^+(z) = 1+2X+Y$, with $X = \sum_{n=1,\, p\nmid n}^{\infty} A(n)q^n$ and $Y = \sum_{n=1}^{\infty} B(n)q^{pn}$. On the other hand, $\frac{E_4E_6}{\Delta}(pz) = q^{-p}(1+Z)$, with $Z = \sum_{n=1}^{\infty} C(n)q^{pn}$, where $A(n)$, $B(n)$, $C(n)$ are integers. Hence our assertion is clear.

If $m > p$, $f_m(z)$ can be obtained from $j(pz)f_{m-p}(z)$ by subtracting suitable integer multiples of $f_{m'}$, $m' < m$. Hence by induction, we can see that for each m, $s(n)a_m(n)$ are integers. □

When $p = 17$, it is likely that $s(n)a(n)$ are integers for $f_1 = q^{-1} + \sum_{n\geqslant 0} a(n)q^n$. But we were not able to verify it. In Section 5.2, we will assume that $s(n)a(n)$ are integers for all $n \geqslant 1$.

4.2. Borcherds lifts

Let $p \in \{5, 13, 17\}$ and $F = \mathbb{Q}[\sqrt{p}]$. Denote the ring of integers of F by $\mathcal{O} = \mathbb{Z}[\frac{1+\sqrt{p}}{2}]$ and the different of F by $\mathfrak{d} = (\sqrt{p})$. We keep the fundamental units $\varepsilon_0 = \frac{1+\sqrt{5}}{2}$ for $p = 5$, $\varepsilon_0 = \frac{3+\sqrt{13}}{2}$ for $p = 13$, $\varepsilon_0 = 4+\sqrt{17}$ for $p = 17$ as in Section 1. Let (z_1, z_2) be a standard variable on $\mathbb{H}^2$ and write (y_1, y_2) for its imaginary part. The Hilbert modular group $\Gamma_F = SL_2(\mathcal{O})$ acts on $\mathbb{H}^2$ in the usual way. For a positive integer m with $\chi_p(m) \neq -1$, let

$$S(m) = \bigcup_{\substack{\lambda \in \mathfrak{d}^{-1} \\ N(\lambda) = -\frac{m}{p}}} \bigl\{(z_1, z_2) \in \mathbb{H}^2\colon\ \lambda y_1 + \bar{\lambda} y_2 = 0\bigr\}.$$

Let $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A_0^+(p, \chi_p)$ and assume that $s(n)a(n) \in \mathbb{Z}$ for all $n < 0$. Let $\mathcal{W} \subset \mathbb{H}^2$ be a Weyl chamber attached to f, i.e., a connected component of

$$\mathbb{H}^2 - \bigcup_{\substack{n<0 \\ a(n)\neq 0}} S(-n).$$

For $\lambda \in \mathfrak{d}^{-1}$, we write $(\lambda, \mathcal{W}) > 0$ if $\lambda y_1 + \bar{\lambda} y_2 > 0$ for all $(z_1, z_2) \in \mathcal{W}$. Put $N = \min\{n\colon\ a(n) \neq 0\}$. Then we have:

Theorem 4.3. *(See [2,4].) The Borcherds lift of f is given by*

$$\Psi(z_1, z_2) = e(\rho_{\mathcal{W}} z_1 + \overline{\rho_{\mathcal{W}}} z_2) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ (\nu, \mathcal{W}) > 0}} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})}.$$

Here $\Psi(z_1, z_2)$ is a Hilbert modular form of weight $a(0)$, and the product converges absolutely for all (z_1, z_2) with $y_1 y_2 > \frac{|N|}{p}$ outside the set of poles. (See below for the definition of $\rho_{\mathcal{W}}$.)

The Weyl vector $\rho_{\mathcal{W}}$ and its conjugate $\overline{\rho_{\mathcal{W}}}$ are contained in $(\mathrm{tr}(\varepsilon_0))^{-1}\mathfrak{d}^{-1}$, where $\varepsilon_0 > 0$ is the fundamental unit of F. More precisely, the vectors $\rho_{\mathcal{W}}$ and $\overline{\rho_{\mathcal{W}}}$ are given as follows: For a negative integer n with $a(n) \neq 0$, define

$$R(\mathcal{W}, n) = \left\{\lambda \in \mathfrak{d}^{-1}\colon\ \lambda > 0,\ N(\lambda) = \frac{n}{p},\ \lambda y_1 + \bar{\lambda} y_2 < 0,\ \varepsilon_0^2 \lambda y_1 + \bar{\varepsilon}_0^2 \bar{\lambda} y_2 > 0,\right.$$
$$\left.\text{for all } (z_1, z_2) \in \mathcal{W}\right\}.$$

Then

$$\rho_{\mathcal{W}} y_1 + \overline{\rho_{\mathcal{W}}} y_2 = \sum_{n<0} s(n)a(n) \frac{1}{\mathrm{tr}(\varepsilon_0)} \sum_{\lambda \in R(\mathcal{W},n)} (\varepsilon_0 \lambda y_1 + \bar{\varepsilon}_0 \bar{\lambda} y_2). \tag{4.4}$$

Let Ψ_m be the Borcherds lift of $f_m = s(m)^{-1}q^{-m} + \sum_{n=0}^{\infty} a(n)q^n$. Write $m = q_1^{k_1} \cdots q_r^{k_r}$ into the prime factorization with distinct primes q_i. First, assume that $(\frac{q_i}{p}) = -1$ with an odd k_i for some i. Then m cannot be the norm of an ideal in $\mathcal{O}$, and $S(m)$ is empty, and $\mathbb{H}^2$ is the only Weyl chamber for f_m. In this case, $\rho_{\mathcal{W}} = 0$ and Theorem 4.3 yields

$$\Psi_m(z_1, z_2) = \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})}. \tag{4.5}$$

Now we assume that $(\frac{q_i}{p}) \neq 1$ for all i and consider Ψ_{m^2} for $m \in \mathbb{N}$. For example, if $p = 17$, we consider Ψ_1 and Ψ_9; however we do not consider Ψ_4 or Ψ_{16} since $(\frac{2}{17}) = 1$. Then $S(m^2)$ is not empty. More importantly, since no q_i splits in F, the condition $N(\nu) = -m^2/p$ implies that $\nu = \pm \frac{m}{\sqrt{p}} \varepsilon_0^{2j}$ for some $j \in \mathbb{Z}$. Let $\mathcal{W}$ be the Weyl chamber attached to f_{m^2} that contains the point $(\sqrt{-1}, 2\sqrt{-1})$. Then $R(\mathcal{W}, -m^2) = \{\frac{m}{\sqrt{p}}\}$ and no other elements are included due to the condition on m. Hence, we obtain from (4.4)

$$\rho_{\mathcal{W}} = \frac{m\varepsilon_0}{\mathrm{tr}(\varepsilon_0)\sqrt{p}} \quad \text{for } m \in \mathbb{N}.$$

By [21, p. 82] $(\nu, \mathcal{W}) > 0$ is equivalent to $(\nu, \tau) > 0$ for a point $\tau \in \mathcal{W}$. So in our case, it is equivalent to $\nu + 2\bar{\nu} > 0$. If $\nu \gg 0$ (and $\nu + 2\bar{\nu} > 0$) for $\nu \in \mathfrak{d}^{-1}$ then $N(\nu) > 0$. If $\nu \not\gg 0$ and $\nu + 2\bar{\nu} > 0$, then $N(\nu) < 0$ and $a(p\nu\bar{\nu}) \neq 0$ only for ν with $N(\nu) = -m^2/p$, in which case $s(p\nu\bar{\nu})a(p\nu\bar{\nu}) = 1$. Therefore,

$$\begin{aligned}
\Psi_{m^2}(z_1, z_2) &= e\left(\frac{m\varepsilon_0 z_1}{\mathrm{tr}(\varepsilon_0)\sqrt{p}} - \frac{m\bar{\varepsilon}_0 z_2}{\mathrm{tr}(\varepsilon_0)\sqrt{p}}\right) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu + 2\bar{\nu} > 0}} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})} \\
&= e\left(\frac{m\varepsilon_0 z_1}{\mathrm{tr}(\varepsilon_0)\sqrt{p}} - \frac{m\bar{\varepsilon}_0 z_2}{\mathrm{tr}(\varepsilon_0)\sqrt{p}}\right) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})} \\
&\quad \times \prod_{\substack{\nu \in \mathfrak{d}^{-1},\, \nu + 2\bar{\nu} > 0 \\ N(\nu) = -m^2/p}} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr).
\end{aligned} \tag{4.6}$$

5. Embedding of hyperbolic Kac–Moody algebras

In this section we associate a family of $\mathcal{H}(a)$'s to each odd prime p and prove that there exists a chain of embeddings among the algebras in each family. When $p = 5, 13$ or 17, we construct an automorphic correction of the first $\mathcal{H}(a)$ in each family, i.e. $\mathcal{H}(3)$, $\mathcal{H}(11)$, $\mathcal{H}(66)$ for $p = 5, 3, 17$, respectively. The automorphic correction will be given by the Hilbert modular form Ψ_1 considered in the previous section. We also consider other Ψ_m ($m \neq 1$) and see where the obstructions are for this to be an automorphic correction.

5.1. Embedding of $\mathcal{H}(a)$

We fix an odd prime p. We consider the Pell equation (1.1) again. Recall that we fixed a fundamental unit ε_0 of F. We enumerate the solutions (a_k, s_k) ($k = 1, 2, \ldots$) of the equation so that if $p \equiv 1 \pmod 4$,

$$\eta_k = \frac{a_k + s_k\sqrt{p}}{2} = \varepsilon_0^{2k}, \quad k = 1, 2, \ldots,$$

and if $p \equiv 3 \pmod 4$,

$$\eta_k = \frac{a_k + s_k\sqrt{p}}{2} = \varepsilon_0^{k}, \quad k = 1, 2, \ldots.$$

Note that we have

$$\eta_k^j = \eta_{kj} \quad \text{for } k, j \in \mathbb{N}.$$

In Section 1, we established an isometry of $\mathfrak{h}_{\mathbb{Q}}^*$ to F for each $\mathcal{H}(a_k)$ and obtained the set of positive real roots of $\mathcal{H}(a_k)$. Since the isometry depends on k, we denote it by ψ_k. Then we have

$$\psi_k\left(\Delta_{\mathrm{re}}^+\right) = \left\{ \frac{1}{\sqrt{p}}\eta_k^j \ (j > 0), \ -\frac{1}{\sqrt{p}}\bar{\eta}_k^j \ (j \geqslant 0)\right\}. \tag{5.1}$$

We call an element of $\psi_k(\Delta_{\mathrm{re}}^+)$ a positive real root by abusing the terminology. The set $\psi_k(\Delta_{\mathrm{im}}^+)$ is given in (1.2). We will apply the following proposition to establish embeddings of $\mathcal{H}(a_k)$.

Proposition 5.2. *(See [7].) Let Δ be the set of roots of a Kac–Moody algebra $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$, and let Δ_{re}^+ be the set of positive real roots of $\mathfrak{g}$. Let $\beta_1, \ldots, \beta_n \in \Delta_{\mathrm{re}}^+$ be chosen such that for all $1 \leqslant i \neq j \leqslant n$, we have $\beta_i - \beta_j \notin \Delta$. For $1 \leqslant i \leqslant n$, let E_i and F_i be nonzero root vectors in the root spaces corresponding to β_i and $-\beta_i$, respectively, and let $H_i = [E_i, F_i] \in \mathfrak{h}$. Then the Lie subalgebra of $\mathfrak{g}$ generated by $\{E_i, F_i, H_i \mid 1 \leqslant i \leqslant n\}$ is a Kac–Moody algebra with Cartan matrix $\left(\frac{2(\beta_i,\beta_j)}{(\beta_j,\beta_j)}\right)_{1\leqslant i,j\leqslant n}$.*

We fix k for the time being. Consider two positive real roots of $\mathcal{H}(a_k)$:

$$\beta_1 = \frac{1}{\sqrt{p}}\eta_k^j \quad \text{and} \quad \beta_2 = -\frac{1}{\sqrt{p}}\bar{\eta}_k^j \quad \text{for } j > 0. \tag{5.3}$$

Since $\beta_1 - \beta_2 = \frac{1}{\sqrt{p}}(\eta_k^j + \bar{\eta}_k^j) = \frac{1}{\sqrt{p}}(\eta_{kj} + \bar{\eta}_{kj}) = \frac{1}{\sqrt{p}}a_{kj}$, it is clear that $\beta_1 - \beta_2$ is not a root. We also see that $\langle \beta_i, \beta_i \rangle = 2$ $(i = 1, 2)$ and

$$\langle \beta_1, \beta_2 \rangle = -p\left(\frac{1}{\sqrt{p}}\eta_k^j \frac{1}{\sqrt{p}}\eta_k^j + \frac{1}{\sqrt{p}}\bar{\eta}_k^j \frac{1}{\sqrt{p}}\bar{\eta}_k^j\right) = -\left(\eta_k^{2j} + \bar{\eta}_k^{2j}\right) = -a_{2kj}.$$

Similarly, if we take

$$\beta_1 = \frac{1}{\sqrt{p}}\eta_k^j \quad \text{and} \quad \beta_2 = -\frac{1}{\sqrt{p}}\bar{\eta}_k^{j-1} \quad \text{for } j > 0, \tag{5.4}$$

then $\beta_1 - \beta_2 = \frac{1}{\sqrt{p}}(\eta_k^j + \bar{\eta}_k^{j-1}) = \frac{1}{\sqrt{p}}\bar{\eta}_k^{j-1}(1 + \eta_k^{2j-1})$. Comparing it with elements in $\psi_k(\Delta_{\mathrm{re}}^+)$ and $\psi_k(\Delta_{\mathrm{im}}^+)$, we see that $\beta_1 - \beta_2$ is not a root. We also have that $\langle \beta_i, \beta_i \rangle = 2$ $(i = 1, 2)$ and

$$\langle \beta_1, \beta_2 \rangle = -p\left(\frac{1}{\sqrt{p}}\eta_k^{j-1} \frac{1}{\sqrt{p}}\eta_k^j + \frac{1}{\sqrt{p}}\bar{\eta}_k^{j-1} \frac{1}{\sqrt{p}}\bar{\eta}_k^j\right) = -\left(\eta_k^{2j-1} + \bar{\eta}_k^{2j-1}\right) = -a_{k(2j-1)}.$$

Hence we obtain the following theorem.

Theorem 5.5. *Let k and l be positive integers, and assume that $k \,|\, l$. Then there exists an embedding of $\mathcal{H}(a_l)$ into $\mathcal{H}(a_k)$ as a Lie subalgebra. Moreover, the root space of β in $\mathcal{H}(a_l)$ is embedded into the root space of α in $\mathcal{H}(a_k)$ so that $\psi_l(\beta) = \eta_{kj}\psi_k(\alpha)$ if $l = 2kj$ for some $j \in \mathbb{N}$ and $\psi_l(\beta) = \eta_{k(j-1)}\psi_k(\alpha)$ if $l = k(2j-1)$ for some $j \in \mathbb{N}$.*

Proof. Applying Proposition 5.2 to the above computations, we obtain the first assertion. For the second assertion, we have only to investigate the simple roots. We notice that the simple roots of $\mathcal{H}(a_l)$ are $\frac{1}{\sqrt{p}}\eta_l$ and $-\frac{1}{\sqrt{p}}$. Assume that $l = 2kj$. If we multiply the simple roots of $\mathcal{H}(a_l)$ by $\bar{\eta}_{kj}$, we obtain $\frac{1}{\sqrt{p}}\eta_{kj}$ and $-\frac{1}{\sqrt{p}}\bar{\eta}_{kj}$, which are the roots in (5.3) and generate a copy of $\mathcal{H}(a_l)$ inside $\mathcal{H}(a_k)$. Now assume that $l = k(2j-1)$. Multiplying the simple roots of $\mathcal{H}(a_l)$ by $\bar{\eta}_{k(j-1)}$, we get $\frac{1}{\sqrt{p}}\eta_{kj}$ and $-\frac{1}{\sqrt{p}}\bar{\eta}_{k(j-1)}$, which are the roots in (5.4). This proves the theorem. □

We write $\mathrm{mult}(a_k, \alpha)$ for the multiplicity of α in $\mathcal{H}(a_k)$ for $k \in \mathbb{N}$.

Corollary 5.6. *Assume that we have either $\psi_l(\beta) = \eta_{kj}\psi_k(\alpha)$ and $l = 2kj$ for some $j \in \mathbb{N}$, or $\psi_l(\beta) = \eta_{k(j-1)}\psi_k(\alpha)$ and $l = k(2j-1)$ for some $j \in \mathbb{N}$. Then we have the inequalities:*

$$\mathrm{mult}(a_l, \beta) \leqslant \mathrm{mult}(a_k, \alpha).$$

Remark 5.7. An upper bound for $\operatorname{mult}(a_k, \alpha)$ is given by the homogeneous dimension of the corresponding free Lie algebra (see [14]). Since the depth (or height) of β is much smaller than that of α, the number $\operatorname{mult}(a_l, \beta)$ can be considered inductively as a lower bound for $\operatorname{mult}(a_k, \alpha)$.

5.2. Automorphic correction of $\mathcal{H}(a)$

In the rest of this section, we will construct automorphic correction of $\mathcal{H}(a)$ for $a = 3, 11, 66$. Hence, $a = a_1$ for each prime $p \in \{5, 13, 17\}$ and we will write $\psi = \psi_1$ for convenience. Recall that we need to establish data (1)–(4) (Section 3). We already have data (1)–(3). More precisely, we put

$$M = \psi^{-1}(\mathfrak{d}^{-1}) \subset \mathfrak{h}^*_{\mathbb{Q}} \quad \text{for each } \mathbb{Q}(\sqrt{p}),\ p = 5, 13, 17,$$

and use the same bilinear form on $\mathfrak{h}^*_{\mathbb{Q}}$. Then M is of signature $(1,1)$. We take the same Weyl group W for the reflection group of M, and choose the cone

$$V^+(M) = \{x\gamma^+ + y\gamma^- \in \mathfrak{h}^*_{\mathbb{R}} \mid x > 0,\ y > 0\}. \tag{5.8}$$

We set $\Pi = \{\alpha_1, \alpha_2\}$ and obtain the Weyl chamber

$$\mathcal{M} = \{\beta \in V^+(M) \mid (\beta, \alpha_i) \leqslant 0,\ i = 1, 2\} = \mathbb{R}_{\leqslant 0}\omega_1 + \mathbb{R}_{\leqslant 0}\omega_2.$$

The Weyl vector is given by $\rho = -(\omega_1 + \omega_2)$. The Cartan matrix is the same A for $\mathcal{H}(a)$.

Now we consider the data (4). We have the complexified cone

$$\Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : \operatorname{Im}(z_1) > 0,\ \operatorname{Im}(z_2) > 0 \right\} \subset \mathfrak{h}^*$$

with respect to the basis $\{\gamma^+, \gamma^-\}$ and from our choice of $V^+(M)$ in (5.8). Then $\Omega(V^+(M))$ is naturally identified with $\mathbb{H}^2$. We choose the extended lattice $L = P(p) \oplus M$, which is essentially identical to L in (2.3). Then the space $\mathcal{K}^+$ is given by

$$\begin{aligned} \mathcal{K}^+ &= \left\{ \left[\frac{(z,z)}{2p} e_1 + e_2 + z \right] \in P(L(\mathbb{C})) : z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \Omega(V^+(M)) \right\} \\ &= \left\{ \left[-z_1 z_2 e_1 + e_2 + \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] \in P(L(\mathbb{C})) : (z_1, z_2) \in \mathbb{H}^2 \right\} \\ &\cong \left\{ \left[\begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix} \right] \in P(M_2(\mathbb{C})) : (z_1, z_2) \in \mathbb{H}^2 \right\}. \end{aligned}$$

The last identification follows from (2.2). The action of $SL_2(\mathcal{O})$ on $\mathbb{H}^2 \cong \Omega(V^+(M))$ is compatible with its action on $M_2(\mathbb{C})$, and we have $SL_2(\mathcal{O}) = \Gamma_F \subset O^+_L$. As we observed in Section 2.2, an automorphic form on $\Omega(V^+(M))$ is a Hilbert modular form. Hence an automorphic correction of $\mathcal{H}(a)$ is a Hilbert modular form which can be written as a product. We obtain natural examples from Section 4 where we considered the works of Bruinier and others on Hilbert modular forms as Borcherds products [2,4].

Actually, our automorphic correction will be a Hilbert modular form with respect to the congruence subgroup $\Gamma_0(p)$ defined by

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}):\ a, b, d \in \mathcal{O},\ c \in (p) \right\} \subset O_L^+,$$

where $(p) \subset \mathcal{O}$ is the principal ideal generated by p. The notation $\Gamma_0(p)$ is the same as the congruence subgroup $\Gamma_0(p)$ of $SL_2(\mathbb{Z})$. However, it will be clear from the context which group we mean.

We fix $p \in \{5, 13, 17\}$ and consider $\mathcal{H}(a_1)$. Recall that $a_1 = 3$ for $p = 5$, $a_1 = 11$ for $p = 13$ and $a_1 = 66$ for $p = 17$. We identify $\mathbb{H}^2$ with $\Omega(V^+(M)) \subset \mathfrak{h}^*$ as above. Then the Weyl group W acts on $\mathbb{H}^2$; in particular, we have

$$r_1(z_1, z_2) = \left(\eta_1^2 z_2, \bar{\eta}_1^2 z_1\right) \quad \text{and} \quad r_2(z_1, z_2) = (z_2, z_1).$$

Recall that the Weyl group W also acts on F by

$$r_1 \nu = \eta_1^2 \bar{\nu} \quad \text{and} \quad r_2 \nu = \bar{\nu} \quad \text{for } \nu \in F.$$

For $m \in \mathbb{N}$, we define a map $\psi^{(m)} : \mathfrak{h}_{\mathbb{Q}}^* \to F$ by $\binom{\nu}{\bar{\nu}} \mapsto m\nu$, i.e. $\psi^{(m)} = m\psi$. Then we obtain

$$\psi^{(m)}(\rho) = \frac{m}{sp}(1 + \eta_1) = \frac{m\varepsilon_0}{\operatorname{tr}(\varepsilon_0)\sqrt{p}}. \tag{5.9}$$

Lemma 5.10. *Assume that $m = q_1^{k_1} \cdots q_r^{k_r}$ is the prime factorization of m with distinct primes q_i and suppose that $(\frac{q_i}{p}) \neq 1$ for all i. Then we have $\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)$ if and only if $\nu \in \mathfrak{d}^{-1}$, $\nu + 2\bar{\nu} > 0$ and $N(\nu) = -m^2/p$.*

Proof. The "only if" part can be verified through straightforward computations. Consider the "if" part. Since no q_i splits in F, we obtain $\nu = \pm\frac{m}{\sqrt{p}}\varepsilon_0^{2j}$ for some $j \in \mathbb{Z}$ from the conditions $N(\nu) = -m^2/p$ and $\nu \in \mathfrak{d}^{-1}$. Recall that $\varepsilon_0^2 = \eta_1$. We obtain from the description of the positive real roots (5.1) that the additional condition $\nu + 2\bar{\nu} > 0$ makes $\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)$. □

Remark 5.11. It is important to notice that the same conditions as in the above lemma appear in the Borcherds lift Ψ_{m^2} in (4.6). In particular, we can write for such an m

$$\prod_{\substack{\nu \in \mathfrak{d}^{-1},\ \nu + 2\bar{\nu} > 0 \\ N(\nu) = -m^2/p}} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr) = \prod_{\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)} \bigl(1 - e(\nu z_1 + \bar{\nu} z_2)\bigr).$$

Proposition 5.12. *Let Ψ_m be the Borcherds lifts for $m \in \mathbb{N}$. Define $\overline{\Psi}_m(z_1, z_2) = \Psi_m(z_2, z_1)$ and write $m = q_1^{k_1} \cdots q_r^{k_r}$ into the prime factorization with distinct primes q_i.*

(1) *Assume that $(\frac{q_i}{p}) = -1$ with an odd k_i for some i. Then we have*

$$\overline{\Psi}_m(wz) = \overline{\Psi}_m(z) \quad \text{for } w \in W.$$

(2) *Assume that $(\frac{q_i}{p}) \neq 1$ for all i. Then we have*

$$\overline{\Psi}_{m^2}(wz) = \det(w)\overline{\Psi}_{m^2}(z) \quad \text{for } w \in W.$$

Proof. We have only to consider the simple reflections r_1 and r_2. First, consider the part (1). In this case, the Borcherds product Ψ_m is of the form (4.5). It is easy to see that

$$\nu \gg 0 \quad \Longleftrightarrow \quad \bar{\nu} \gg 0 \quad \Longleftrightarrow \quad \bar{\nu}\eta_1^2 \gg 0.$$

Then we obtain $\overline{\Psi}_m(r_1 z) = \overline{\Psi}_m(r_2 z) = \overline{\Psi}_m(z)$.

Now we consider the part (2). From (4.6) and Remark 5.11, we have

$$\begin{aligned}
\overline{\Psi}_{m^2}(z_1, z_2) &= \Psi_{m^2}(z_2, z_1) \\
&= e\left(\frac{m\varepsilon_0 z_2}{\operatorname{tr}(\varepsilon_0)\sqrt{p}} - \frac{m\bar{\varepsilon}_0 z_1}{\operatorname{tr}(\varepsilon_0)\sqrt{p}}\right) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(1 - e(\nu z_2 + \bar{\nu} z_1)\right)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})} \\
&\quad \times \prod_{\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)} \left(1 - e(\nu z_2 + \bar{\nu} z_1)\right).
\end{aligned}$$

The product over $\nu \gg 0$ is invariant under r_1 and r_2 as in the part (1). Write $\nu_i = \psi^{(m)}(\alpha_i)$, $i = 1, 2$. Each r_i sends ν_i to $-\nu_i$ and keeps the set $\psi^{(m)}(\Delta_{\mathrm{re}}^+ \setminus \{\alpha_i\})$ invariant. For $w = r_1$, we have

$$\begin{aligned}
&\overline{\Psi}_{m^2}\left(r_1(z_1, z_2)\right) \\
&\quad = \overline{\Psi}_{m^2}\left(\eta_1^2 z_2, \bar{\eta}_1^2 z_1\right) = \Psi_{m^2}\left(\bar{\eta}_1^2 z_1, \eta_1^2 z_2\right) \\
&\quad = A_1 \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(1 - e(\nu z_1 + \bar{\nu} z_2)\right)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})} \prod_{\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)} \left(1 - e\left(\nu\bar{\eta}_1^2 z_1 + \bar{\nu}\eta_1^2 z_2\right)\right) \\
&\quad = A_1 \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(1 - e(\nu z_2 + \bar{\nu} z_1)\right)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})} \prod_{\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)} \left(1 - e(r_1\nu z_2 + \overline{r_1\nu} z_1)\right) \\
&\quad = A_1 \frac{1 - e(-\nu_1 z_2 - \bar{\nu}_1 z_1)}{1 - e(\nu_1 z_2 + \bar{\nu}_1 z_1)} \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \left(1 - e(\nu z_2 + \bar{\nu} z_1)\right)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})}
\end{aligned}$$

$$\times \prod_{\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)} \big(1 - e(\nu z_2 + \bar{\nu} z_1)\big)$$

$$= -A_1 e(-\nu_1 z_2 - \bar{\nu}_1 z_1) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \big(1 - e(\nu z_2 + \bar{\nu} z_1)\big)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})}$$

$$\times \prod_{\nu \in \psi^{(m)}(\Delta_{\mathrm{re}}^+)} \big(1 - e(\nu z_2 + \bar{\nu} z_1)\big),$$

where we put

$$A_1 = e\left(\frac{m\bar{\varepsilon}_0 \bar{\eta}_1^2 z_1}{\mathrm{tr}(\varepsilon_0)\sqrt{p}} - \frac{m\bar{\varepsilon}_0 \eta^2 z_2}{\mathrm{tr}(\varepsilon_0)\sqrt{p}}\right).$$

We obtain from (5.9) that

$$-\frac{m\bar{\varepsilon}_0 \eta^2}{\mathrm{tr}(\varepsilon_0)\sqrt{p}} - \nu_1 = \psi^{(m)}(r_1\rho) - \psi^{(m)}(\alpha_1) = \psi^{(m)}(\rho) = \frac{m\varepsilon_0}{\mathrm{tr}(\varepsilon_0)\sqrt{p}}.$$

Combining these computations, we see that $\overline{\Psi}_{m^2}(r_1(z_1, z_2)) = -\overline{\Psi}_{m^2}(z_1, z_2)$. Similarly, we can show that $\overline{\Psi}_{m^2}(r_2(z_1, z_2)) = -\overline{\Psi}_{m^2}(z_1, z_2)$. □

We define $\Phi_m(z) = \overline{\Psi}_m(pz)$. Then the function $\Phi_m(z)$ is a Hilbert modular form with respect to $\Gamma_0(p)$ thanks to the following lemma.

Lemma 5.13. *Let $g(z)$ be a Hilbert modular form for $\mathbb{Q}(\sqrt{p})$ with respect to $SL_2(\mathcal{O})$. Define $f(z) = \bar{g}(pz)$, where $\bar{g}(z_1, z_2) = g(z_2, z_1)$. Then the function $f(z)$ is a Hilbert modular form with respect to the congruence subgroup $\Gamma_0(p)$.*

Proof. By Theorem 4.2.2 in [21], the function $\bar{g}(z)$ is a Hilbert modular form with respect to $SL_2(\mathcal{O})$. Assume that μ is the multiplier system for $\bar{g}$. Then we define $\widetilde{\mu}$ on $\Gamma_0(p)$ by

$$\widetilde{\mu}\begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \mu\begin{pmatrix} a & bp \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in \Gamma_0(p).$$

For $\gamma \in \left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \in \Gamma_0(p)$, we have

$$f(\gamma z) = \bar{g}(p\gamma z) = \bar{g}\left(\frac{apz + bp}{cpz + d}\right)$$
$$= \mu\begin{pmatrix} a & bp \\ c & d \end{pmatrix} N(cpz + d)^k \bar{g}(pz) = \widetilde{\mu}\begin{pmatrix} a & b \\ pc & d \end{pmatrix} N(pcz + d)^k f(z).$$

Thus $f(z)$ is a Hilbert modular form with respect to $\Gamma_0(p)$ with the multiplier system $\widetilde{\mu}$. □

Since $F \cong \mathfrak{h}^*_{\mathbb{Q}} \subset \mathfrak{h}^*$, $\nu \mapsto \binom{\nu}{\bar\nu}$, and $\mathbb{H}^2 \cong \Omega(V^+(M)) \subset \mathfrak{h}^*$, $(z_1, z_2) \mapsto \binom{z_1}{z_2}$, the symmetric bilinear form on $\mathfrak{h}^*$ induces a pairing on $F \times \mathbb{H}^2$ given by

$$(\nu, z) = -p(\nu z_2 + \bar\nu z_1) \quad \text{for } \nu \in F \text{ and } z = (z_1, z_2) \in \mathbb{H}^2.$$

Write $m = q_1^{k_1} \cdots q_r^{k_r}$ as before. If $(\frac{q_i}{p}) = -1$ with an odd k_i for some i, then we can rewrite (4.5) and obtain

$$\begin{aligned}\Phi_m(z) &= \prod_{\substack{\nu \in \mathfrak{d}^{-1}\\ \nu \gg 0}} \bigl(1 - e\bigl(p(\nu z_2 + \bar\nu z_1)\bigr)\bigr)^{s(p\nu\bar\nu)a(p\nu\bar\nu)}\\ &= \prod_{\substack{\nu \in \mathfrak{d}^{-1}\\ \nu \gg 0}} \bigl(1 - e\bigl(-(\nu, z)\bigr)\bigr)^{s(p\nu\bar\nu)a(p\nu\bar\nu)}. \end{aligned} \tag{5.14}$$

We write $\rho_m = \psi^{(m)}(\rho)$. If $(\frac{q_i}{p}) \neq 1$ for all i, then we obtain from (4.6), (5.9) and Remark 5.11,

$$\begin{aligned}\Phi_{m^2}(z) &= e\bigl(p(\rho_m z_2 + \bar\rho_m z_1)\bigr) \prod_{\substack{\nu \in \mathfrak{d}^{-1}\\ \nu \gg 0}} \bigl(1 - e\bigl(p(\nu z_2 + \bar\nu z_1)\bigr)\bigr)^{s(p\nu\bar\nu)a(p\nu\bar\nu)}\\ &\quad \times \prod_{\nu \in \psi^{(m)}(\Delta^+_{\mathrm{re}})} \bigl(1 - e\bigl(p(\nu z_2 + \bar\nu z_1)\bigr)\bigr)\\ &= e\bigl(-(\rho_m, z)\bigr) \prod_{\substack{\nu \in \mathfrak{d}^{-1}\\ \nu \gg 0}} \bigl(1 - e\bigl(-(\nu, z)\bigr)\bigr)^{s(p\nu\bar\nu)a(p\nu\bar\nu)}\\ &\quad \times \prod_{\nu \in \psi^{(m)}(\Delta^+_{\mathrm{re}})} \bigl(1 - e\bigl(-(\nu, z)\bigr)\bigr). \end{aligned} \tag{5.15}$$

We write $f_1 = q^{-1} + \sum_{n \geqslant 0} a(n)q^n$. When $p = 17$, we assume that $s(n)a(n)$ are integers for all $n \geqslant 1$. This is necessary since $s(n)a(n)$ will be considered as root multiplicities in what follows. See the remark at the end of Section 4.1. It is known [21] that Ψ_1 is a cusp form of weight 5, 1, $\frac{1}{2}$ for $p = 5, 13, 17$, resp. and skew-symmetric, i.e., $\Psi_1(z_2, z_1) = -\Psi_1(z_1, z_2)$.

Now we state the main theorem of this paper.

Theorem 5.16. *Let $p \in \{5, 13, 17\}$. Then the Hilbert modular form Φ_1 provides an automorphic correction for the hyperbolic Kac–Moody algebra $\mathcal{H}(a_1)$, where $a_1 = 3$ for $p = 5$, $a_1 = 11$ for $p = 13$ and $a_1 = 66$ for $p = 17$. In particular, there exists a generalized Kac–Moody superalgebra $\widetilde{\mathcal{H}}$ whose denominator function is the Hilbert modular form Φ_1.*

Proof. From (5.15), we have

$$\Phi_1(z) = e\big(-(\rho, z)\big) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \big(1 - e\big(-(\nu, z)\big)\big)^{s(p\nu\bar{\nu})a(p\nu\bar{\nu})} \prod_{\nu \in \psi(\Delta_{\mathrm{re}}^+)} \big(1 - e\big(-(\nu, z)\big)\big).$$

We will drop ψ from the notation. We write $\Phi_1(z) = \sum_\mu b_\mu e(-(\mu, z))$. By Proposition 5.12, we obtain $\Phi_1(wz) = \det(w)\Phi_1(z)$ for $w \in W$. Since

$$\Phi_1(wz) = \sum_\mu b_{w\mu} e\big(-(\mu, z)\big),$$

we get

$$b_{w\mu} = \det(w) b_\mu. \tag{5.17}$$

One can easily see that $\rho + \nu \in V^+(M)$ for $\nu \in \mathfrak{d}^{-1}$, $\nu \gg 0$ and for $\nu \in \Delta_{\mathrm{re}}^+$. Hence the sum is over $\mu \in V^+(M)$ such that $\mu - \rho \in M$. Then we can write, using the fundamental chamber $\mathcal{M} \subset V^+(M)$,

$$\Phi_1(z) = \sum_{w \in W} \det(w)\Bigg(-\sum_{\substack{\rho+\nu \in \mathcal{M} \\ \nu \in M}} m(\nu) e\big(-\big(w(\rho + \nu), z\big)\big)\Bigg).$$

Assume that $\rho + \nu \in \mathcal{M}$. If $(\rho + \nu, \alpha_i) = 0$ for $i = 1, 2$, then $\rho + \nu$ is invariant under r_i, and $m(\nu) = 0$ from (5.17). Thus we may assume $(\rho + \nu, \alpha_i) < 0$. Then we have $(\nu, \alpha_i) \leqslant 0$ for $i = 1, 2$ and $\nu \in \mathcal{M}$ if $\nu \neq 0$. Since $m(0) = -1$, we have

$$\Phi_1(z) = \sum_{w \in W} \det(w)\Bigg(e\big(-\big(w(\rho), z\big)\big) - \sum_{\nu \in M \cap \mathcal{M}} m(\nu) e\big(-\big(w(\rho + \nu), z\big)\big)\Bigg).$$

This is exactly of the form required by the item (4) for an automorphic correction in Section 3. The data (1)–(3) have already been established at the beginning of Section 5.2. The existence of the corresponding generalized Kac–Moody superalgebra $\widetilde{\mathcal{H}}$ is a consequence of the theory of an automorphic correction as explained in Section 3. □

Remark 5.18. The automorphic correction Φ_1 is *reflective* in the definition of Gritsenko and Nikulin [10]. This means that the divisor of Φ_1 is a union of rational quadratic divisors which are orthogonal to some roots of the lattice $L = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{d}^{-1}$. More precisely, the divisor of Φ_1 is

$$\bigcup_{\substack{(a,b,\nu) \in L \\ ab - N(\nu) = 1/p}} \big\{(z_1, z_2) \in \mathbb{H}^2\colon\ a z_1 z_2 + \nu z_1 + \bar{\nu} z_2 + b = 0\big\}.$$

See [4,21] for more details.

Remark 5.19. There are some obstructions when we try to interpret Φ_m ($m \neq 1$) as an automorphic correction. When we have Φ_m of the form (5.14), we do not have the product corresponding to the real roots. When the function Φ_{m^2} is of the form (5.15), we have both parts corresponding to the real roots and to the imaginary roots. However, the map $\psi^{(m)}$ is not an isometry. If we make $\psi^{(m)}$ an isometry by adjusting the bilinear form on F, the natural lattice would be $m\mathfrak{d}^{-1}$. Then Φ_{m^2} is not an automorphic correction of $\mathcal{H}(a_1)$.

6. Asymptotics for root multiplicities

In this section, we obtain asymptotics of Fourier coefficients of the modular forms f_m defined in Section 4.1. Note that the Fourier coefficients of f_1 are root multiplicities of the generalized Kac–Moody superalgebra $\widetilde{\mathcal{H}}$ with some modification. For the rank 3 hyperbolic Kac–Moody algebra $\mathcal{F}$ mentioned in the introduction, asymptotic upper bounds for root multiplicities were obtained in [15]. Other related works can be found in [16,17].

We apply the result of J. Lehner [19] on Fourier coefficients of modular forms using the method of Hardy–Ramanujan–Rademacher to our special case. We refer to [19] for unexplained notations: $\Gamma_0(p)$ has two cusps: $p_0 = \infty$, $p_1 = 0$ [18, p. 108].

For $f_m \in A_0^+(p, \chi_p)$ for $\chi_p(m) \neq -1$, let

$$f_m = s(m)^{-1} q^{-m} + \sum_{n=0}^{\infty} a_m(n) q^n.$$

Then as in [19, p. 314] we need to compute

$$A(p,n,m) = \sum_{d \in D_p} v^{-1}(M) e\left(\frac{nd - ma}{p}\right) = \sum_{d=1}^{p-1} \chi_p(d) e\left(\frac{nd - ma}{p}\right),$$

where $ad \equiv 1 \pmod p$. This is the Salié sum $T(n, -m; p)$ [11, p. 323]:

If $p \nmid m$, then

$$A(p,n,m) = T(n,-m;p) = \sqrt{p}\left(\frac{-m}{p}\right) \sum_{v^2 \equiv -mn \pmod p} e\left(\frac{2v}{p}\right).$$

Since $\chi_p(m) = 1$, we have $A(p,n,m) = \sqrt{p} \sum_{v^2 \equiv -mn \pmod p} e(\frac{2v}{p})$. Note that if $p \mid n$, then we obtain $A(p,n,m) = \sqrt{p}$.

If $p \mid m$,

$$A(p,n,m) = \sum_{d \in D_p} \chi_p(d) e\left(\frac{nd}{p}\right) = \begin{cases} \chi_p(n)\sqrt{p}, & \text{if } p \nmid n; \\ 0, & \text{if } p \mid n. \end{cases}$$

Next we need the Fourier expansion of f_m at 0: By [4, p. 54]

$$f_m\left(-\frac{1}{pz}\right) = f_m \mid W_p(z) = \frac{1}{\sqrt{p}} f \mid U_p(z) = \sqrt{p} \sum_{n \in \mathbb{Z}} a_m(pn) q^n.$$

Hence

$$f_m\left(-\frac{1}{z}\right) = \sqrt{p}\sum_{n\in\mathbb{Z}} a_m(pn)q^{\frac{n}{p}} = \begin{cases} \sqrt{p}\sum_{n=0}^{\infty} a_m(pn)q^{\frac{n}{p}}, & \text{if } p\nmid m, \\ \frac{\sqrt{p}}{s(m)}q^{-\frac{m}{p^2}} + \sqrt{p}\sum_{n=0}^{\infty} a_m(pn)q^{\frac{n}{p}}, & \text{if } p\,|\,m. \end{cases}$$

So by [19, p. 314] we have, if $p\nmid m$,

$$\begin{aligned} a_m(n) &= 2\pi\sum_{p|c}\frac{A(c,n,m)}{c}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{c}\right) \\ &= \frac{2\pi T(n,-m;p)}{p}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{p}\right) + \text{error term.} \end{aligned}$$

If $p\,|\,m$,

$$\begin{aligned} a_m(n) &= \pi\sum_{p|c}\frac{A(c,n,m)}{c}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{c}\right) \\ &\quad + \frac{\pi}{\sqrt{p}}\sum_{c=1}^{\infty}\frac{A(c,n,\frac{m}{p^2})}{c}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{pc}\right) \\ &= \frac{\pi}{\sqrt{p}}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{p}\right)\bigl(\chi_p(n)+1\bigr) + \text{error term.} \end{aligned}$$

Note that by definition, if $\chi_p(n) = -1$, $a_m(n) = 0$.

Now we show that the error term is smaller than the main term. In the case of $p\,|\,m$, the second term is similar to the first term. So it is enough to handle the case $p\nmid m$. By Weil's bound,

$$\bigl|A(c,n,m)\bigr| \leqslant (c,n,m)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c) \leqslant (n,m)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c) \leqslant (mn)^{\frac{1}{2}}c^{\frac{1}{2}}\tau(c),$$

where $\tau(c)$ is the number of positive divisors of c. We divide the error term into two regions: $p < c \leqslant 4\pi\sqrt{mn}$ and $c > 4\pi\sqrt{mn}$. Here

$$\begin{aligned} 2\pi\sum_{p<c\leqslant 4\pi\sqrt{mn}}\frac{A(c,n,m)}{c}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{c}\right) &\leqslant 2\pi m I_1\left(\frac{2\pi\sqrt{nm}}{p}\right)\sum_{p<c\leqslant 4\pi\sqrt{mn}}\frac{\tau(c)}{\sqrt{c}} \\ &\leqslant 8\pi^{\frac{3}{2}}m^{\frac{5}{4}}n^{\frac{1}{4}}(\log 4\pi\sqrt{mn}\,)\, I_1\left(\frac{2\pi\sqrt{nm}}{p}\right). \end{aligned}$$

Here we used the fact that $\sum_{c\leqslant x}\frac{\tau(c)}{\sqrt{c}} \leqslant 2\sqrt{x}\log x$.

On the other hand, since $I_1(z) \leqslant z$ for $0 < z < 1$,

$$2\pi \sum_{c>4\pi\sqrt{mn}} \frac{A(c,n,m)}{c}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{nm}}{c}\right) \leqslant 8\pi^2 m^{\frac{3}{2}} n^{\frac{1}{2}} \sum_{c>4\pi\sqrt{mn}} \frac{\tau(c)}{c^{\frac{3}{2}}}$$
$$\leqslant 48\pi^{\frac{3}{2}} m^{\frac{5}{4}} n^{\frac{1}{4}} (\log 4\pi\sqrt{mn}\,).$$

Here we used the fact that $\sum_{c>x} \frac{\tau(c)}{c^{\frac{3}{2}}} \leqslant 12x^{-\frac{1}{2}} \log x$.

Combining the above computations, we have proved the following theorem:

Theorem 6.1. *For a positive integer m with $\chi_p(m) \neq -1$, let $f_m = s(m)^{-1}q^{-m} + \sum_{n=0}^{\infty} a_m(n)q^n \in A_0^+(p,\chi_p)$. Then for any m, $a_m(n) > 0$ for all n, $p \mid n$.*

If $p \mid m$, we have $a_m(n) \geqslant 0$ for all n, and

$$a_m(n) = \frac{\pi}{\sqrt{p}}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{mn}}{p}\right)\left(\chi_p(n)+1\right) + O\left(m^{\frac{5}{4}} n^{\frac{1}{4}} \log 4\pi\sqrt{mn} I_1\left(\frac{2\pi\sqrt{nm}}{p}\right)\right).$$

If $p \nmid m$, we obtain

$$a_m(n) = \frac{2\pi}{\sqrt{p}}\left(\frac{m}{n}\right)^{\frac{1}{2}} I_1\left(\frac{4\pi\sqrt{mn}}{p}\right)\left(\sum_{v^2\equiv -mn \ (\mathrm{mod}\ p)} e\left(\frac{2v}{p}\right)\right)$$
$$+ O\left(m^{\frac{5}{4}} n^{\frac{1}{4}} (\log 4\pi\sqrt{mn}\,) I_1\left(\frac{2\pi\sqrt{nm}}{p}\right)\right).$$

For example, let $p = 5$, $m = 6$, $n = 9$. In this case, $\sum_{v^2\equiv -mn \ (\mathrm{mod}\ p)} e(\frac{2v}{5}) = 2\cos\frac{4\pi}{5}$. So $a_6(9) \sim \frac{2\pi}{\sqrt{5}}(6/9)^{\frac{1}{2}} I_1(\frac{4\pi\sqrt{54}}{5}) 2\cos\frac{4\pi}{5} = -35\,409\,600$. The exact value is $-35\,408\,776$. Let $p = 5$, $m = 10$, $n = 9$. In this case, $a_{10}(9) \sim \frac{2\pi}{\sqrt{5}}(10/9)^{1/2} I_1(\frac{4\pi\sqrt{90}}{5}) = 5\,391\,530\,000$. The exact value is $5\,391\,558\,200$.

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