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General Section

Rationality and p-adic properties of reduced forms of half-integral weight

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In this paper we study special bases of certain spaces of half-integral weight weakly holomorphic modular forms. We establish a criterion for the integrality of Fourier coefficients of such bases. By using recursive relations between Hecke operators, we derive relations of Fourier coefficients of each basis element and obtain congruences of the Fourier coefficients, which extend known congruences for traces of singular moduli.

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1. Introduction and statement of results

Let N be a positive integer. For an odd integer k, we denote by $M^{!+\cdots+}_{k/2}(N)$ the space of weakly holomorphic modular forms of weight $k/2$ on $\Gamma_0(4N)$ whose n-th Fourier coefficient vanishes unless $(-1)^{(k-1)/2}\, n$ is a square modulo $4N$. For the moment, we assume that N is contained in the set

$$\mathfrak{S} = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

Then the group $\Gamma_0^*(N)$, which is the group generated by $\Gamma_0(N)$ and all Atkin–Lehner involutions W_e for $e \parallel N$, has genus 0. From the correspondence between Jacobi forms and half-integral weight forms (cf. [10, Theorem 5.6]), we see that for any $D \in \mathbb{Z}_{>0}$ with $D \equiv \square \pmod{4N}$, there is a unique modular form $g_{D,N} \in M^{!+\cdots+}_{3/2}(N)$ having a Fourier expansion of the form

$$g_{D,N}(\tau) = q^{-D} + \sum_{d\geq 0} B^{(N)}(D,d)\, q^d \quad (q = e^{2\pi i\tau},\ \tau \in \mathbb{H}).$$

Here, $\mathbb{H}$ denotes the complex upper half plane.

Let ℓ be a prime with $\ell \nmid 4N$. Then the Hecke operator $T_{k/2,4N}(\ell^2)$, originally defined on the space of weakly holomorphic modular forms of weight $k/2$ on $\Gamma_0(4N)$, acts on $M^{!+\cdots+}_{k/2}(N)$. We define $T_{k/2,4N}(\ell^{2n})$ for $n \geq 2$ recursively by

$$T_{k/2,4N}(\ell^{2n}) := T_{k/2,4N}(\ell^{2n-2})T_{k/2,4N}(\ell^2) - \ell^{k-2}T_{k/2,N}(\ell^{2n-4}).$$

For any positive integer m with $\gcd(m, 4N) = 1$, define $T_{k/2,4N}(m^2)$ multiplicatively and set

$$g^{(m)}_{D,N} := g_{D,N} \mid T_{3/2,4N}(m^2).$$

We denote by $B^{(N)}_m(D,d)$ the d-th Fourier coefficient of $g^{(m)}_{D,N}(\tau)$:

$$g^{(m)}_{D,N}(\tau) = (\text{principal part}) + \sum_{d\geq 0} B^{(N)}_m(D,d)\, q^d.$$

By the works of Zagier [29] and Kim [14], the coefficients $B^{(N)}_m(D,d)$ can be interpreted as traces of CM values of certain modular functions (or traces of singular moduli). Remarkably, the coefficients $B^{(N)}_m(D,d)$ show many congruence properties, and many authors studied them. In 2005, Ahlgren and Ono [2] showed that if $p \nmid m$ is an odd prime and $\left(\frac{-d}{p}\right) = 1$, then

$$B^{(1)}_m(1, p^2 d) \equiv 0 \pmod{p}.$$

Edixhoven [9] used the p-adic geometry of modular curves to show that, for any m and any d with $\left(\frac{-d}{p}\right) = 1$, we have

$$B_m^{(1)}(1, p^{2n}d) \equiv 0 \pmod{p^n}.$$

When p is an odd prime, Jenkins [12] obtained a recursive formula for $B_1^{(1)}(D, p^{2n}d)$ in terms of $B_1^{(1)}(D, p^{2k}d)$ with $k < n$. As a corollary he proved that if $\left(\frac{-d}{p}\right) = \left(\frac{D}{p}\right) \neq 0$, then we have

$$B_1^{(1)}(D, p^{2n}d) = p^n B_1^{(1)}(p^{2n}D, d).$$

Guerzhoy [11] showed that if D and $-d$ are fundamental discriminants with $\left(\frac{-d}{p}\right) = \left(\frac{D}{p}\right)$, then, for any m, we have

$$B_m^{(1)}(D, p^{2n}d) = p^n B_m^{(1)}(p^{2n}D, d).$$

In 2012, Ahlgren [1] proved a general theorem which implies the above results as special cases. On the other hand, Osburn [21] proved that if d is a positive integer such that $-d$ is congruent to a square modulo $4N$ and if $p \neq N$ is an odd prime which splits in $\mathbb{Q}(\sqrt{-d})$, then

$$B_1^{(N)}(1, p^2 d) \equiv 0 \pmod{p}.$$

Jenkins [13] and Koo and Shin [18] obtained the following generalization of Osburn's result: for a positive integer d such that $-d \equiv \square \pmod{4N}$ and an odd prime $p \neq N$ which splits in $\mathbb{Q}(\sqrt{-d})$,

$$B_1^{(N)}(1, p^{2n}d) \equiv 0 \pmod{p^n}$$

for all $n \geq 1$.

The purpose of this paper is to generalize all these congruences to more general modular forms. To be precise, from now on, we assume that $N \geq 1$ is odd and square-free. For an even Dirichlet character χ modulo $4N$, we denote by $M^!_{k/2}(4N, \chi)$ the space of weakly holomorphic modular forms of weight $k/2$ on $\Gamma_0(4N)$ with Nebentypus χ. The subspace of holomorphic forms and that of cuspforms are denoted by $M_{k/2}(4N, \chi)$ and $S_{k/2}(4N, \chi)$ respectively.

Let $\mathcal{D}$ be a discriminant form of level $4N$ satisfying some additional conditions which will be given in Section 2.2. (For the basics on discriminant forms, see Section 2.1 below.) Then $\mathcal{D}$ determines an even Dirichlet character χ modulo $4N$ and a sign vector $\epsilon = (\epsilon_p)_p$ over $p = 2$ or $p \mid N$ with $\chi_p \neq 1$, where the character χ is decomposed into p-components: $\chi = \prod_p \chi_p$. Set $\chi' = \chi\left(\frac{4N}{\cdot}\right)$.

We define the associated modular form space $M^{!\epsilon}_{k/2}(N, \chi')$ to be the subspace of $M^{!}_{k/2}(4N, \chi')$ consisting of the forms $f \in M^{!}_{k/2}(4N, \chi')$ satisfying the so-called *ϵ-condition*, which will be defined in Section 2.2. We let

$$M^{\epsilon}_{k/2}(N, \chi') = M^{!\epsilon}_{k/2}(N, \chi') \cap M_{k/2}(4N, \chi') \quad \text{and}$$
$$S^{\epsilon}_{k/2}(N, \chi') = M^{!\epsilon}_{k/2}(N, \chi') \cap S_{k/2}(4N, \chi').$$

Let us give an example. Consider the following even lattice

$$L = \left\{ \begin{pmatrix} a & b/N \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}, \tag{1.1}$$

with $Q(\alpha) = -N \det(\alpha)$ and $(\alpha, \beta) = N\mathrm{tr}(\alpha\beta)$. We denote by L' the dual lattice of L. Then the space $M^{!\epsilon}_{k/2}(N, \chi')$ associated with the discriminant form L'/L is exactly the same as the space $M^{!+\cdots+}_{k/2}(N)$. Hence the ϵ-condition can be considered as a generalization of the Kohnen plus condition.

Now we further assume that $\chi_p \neq 1$ for each $p \mid N$, so $\chi' = 1$. In [31], Zhang defined a family of forms in $M^{!\epsilon}_{k/2}(N, 1)$, called *reduced forms*. (For the definition, see Section 2.2.) If a reduced form f_m exists for some $m \in \mathbb{Z}$, it must be unique and $\chi_p(m) \neq -\epsilon_p$ for each $p \mid N$. The set of reduced modular forms forms a basis for $M^{!\epsilon}_{k/2}(N, 1)$. When $k = 3$ and $N \in \mathfrak{S}$, the reduced form f_{-D} exists for each $D > 0$ which is a square modulo $4N$ (cf. Proposition 2.15 below). In fact, $s(-D)f_{-D} = g_{D,N}$ for every D where $s(-D)$ is a scaling constant. Thus the reduced forms are natural generalizations of the forms $g_{D,N}$.

In order to generalize the congruences mentioned above to reduced forms, we first need to check integrality of the Fourier coefficients of reduced forms. We establish the following proposition which allows us to check whether a fixed reduced form has integer Fourier coefficients.

Proposition 1.2. *Let k be an odd integer. Assume that $f = \sum_n a(n)q^n \in M^{!\epsilon}_{k/2}(N, \chi') \cap \mathbb{Q}((q))$ with bounded denominator, and that $a(n) \neq 0$ for some $n < 0$. Furthermore, let k' be the smallest positive integer which satisfies $k' \geq |\mathrm{ord}_\infty(f)|/4N$ and $k + 12k' > 0$. If $a(n) \in \mathbb{Z}$ for $n \leq \mathrm{ord}_\infty(f) + \frac{k+12k'}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]$, then $a(n) \in \mathbb{Z}$ for all n.*

Let $\mathcal{D}^*$ be the dual discriminant form of $\mathcal{D}$. It is known that the corresponding data to $\mathcal{D}^*$ is $(4N, \chi', \epsilon^*)$ with $\epsilon^*_p = \chi_p(-1)\epsilon_p$. Denote by $M^{\epsilon^*}_{k/2}(N, \chi')$ the space of modular forms associated to $\mathcal{D}^*$. We denote by $a(m, n)$ the n-th Fourier coefficient of the reduced form f_m. We prove the following theorem which turns the integrality problem for reduced forms into checking finitely many of them.

Theorem 1.3. *Let $m_\epsilon = \max\{m : f^*_m \in M^{\epsilon^*}_{2-k/2}(N, \chi') \text{ exists}\}$. Assume that for all $n \in \mathbb{Z}$ and $m \geq -4N - m_\epsilon$, we have $s(m)a(m, n) \in \mathbb{Z}$. Then $s(m)a(m, n) \in \mathbb{Z}$ for all $m, n \in \mathbb{Z}$.*

Therefore, to check the integrality of reduced forms, it suffices to show the integrality of a finite number of Fourier coefficients satisfying the conditions of both Proposition 1.2 and Theorem 1.3. We give an example to illustrate this.

Example 1.4. We consider the space $M_{1/2}^{!+\cdots+}(7,1)$. Then we have $m_\epsilon = -1$. Define

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \qquad (2 < k \in 2\mathbb{Z})$$

to be the normalized Eisenstein series, and denote by $[\cdot,\cdot]_n$ $(n \geq 1)$ the n-th Rankin–Cohen bracket (cf. [6, pp. 53–58]). Set

$$\begin{aligned}
&\mathrm{RC}_1 = \tfrac{[\theta,E_{10}(28\tau)]_1}{\Delta(28\tau)}, && \mathrm{RC}_2 = \tfrac{[\theta,E_8(28\tau)]_2}{\Delta(28\tau)}, && \mathrm{RC}_3 = \tfrac{[\theta,E_6(28\tau)]_3}{\Delta(28\tau)}, && \mathrm{RC}_4 = \tfrac{[\theta,E_4(28\tau)]_4}{\Delta(28\tau)},\\
&\mathrm{RC}_5 = \tfrac{[\mathrm{RC}_1,E_{10}(28\tau)]_1}{\Delta(28\tau)}, && \mathrm{RC}_6 = \tfrac{[\mathrm{RC}_1,E_8(28\tau)]_2}{\Delta(28\tau)}, && \mathrm{RC}_7 = \tfrac{[\mathrm{RC}_1,E_6(28\tau)]_3}{\Delta(28\tau)}, && \mathrm{RC}_8 = \tfrac{[\mathrm{RC}_1,E_4(28\tau)]_4}{\Delta(28\tau)},\\
&\mathrm{RC}_9 = \tfrac{[\mathrm{RC}_2,E_{10}(28\tau)]_1}{\Delta(28\tau)}, && \mathrm{RC}_{10} = \tfrac{[\mathrm{RC}_2,E_8(28\tau)]_2}{\Delta(28\tau)}, && \mathrm{RC}_{11} = \tfrac{[\mathrm{RC}_2,E_6(28\tau)]_3}{\Delta(28\tau)}, && \mathrm{RC}_{12} = \tfrac{[\mathrm{RC}_2,E_4(28\tau)]_4}{\Delta(28\tau)},\\
&\mathrm{RC}_{13} = \tfrac{[\mathrm{RC}_1,E_{10}(28\tau)]_1}{\Delta(28\tau)}, && \mathrm{RC}_{14} = \tfrac{[\mathrm{RC}_3,E_8(28\tau)]_2}{\Delta(28\tau)}, && \mathrm{RC}_{15} = \tfrac{[\mathrm{RC}_3,E_6(28\tau)]_3}{\Delta(28\tau)}, && \mathrm{RC}_{16} = \tfrac{[\mathrm{RC}_3,E_4(28\tau)]_4}{\Delta(28\tau)}.
\end{aligned}$$

In addition, we set

$$f = \tfrac{1}{5600}\mathrm{RC}_1 + \tfrac{7}{103680}\mathrm{RC}_2 + \tfrac{1}{80640}\mathrm{RC}_3 + \tfrac{1}{705600}\mathrm{RC}_4 - \tfrac{41687}{1800}\theta,$$

and define

$$\mathrm{RC}_{17} = \tfrac{[f,E_4(28\tau)]_4}{\Delta(28\tau)}.$$

By taking linear combinations of these Rankin-Cohen brackets, we find

$$\begin{aligned}
s(0)f_0 &= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots,\\
s(-3)f_{-3} &= q^{-3} - 3q - 2q^4 + 6q^8 + 5q^9 - 10q^{16} + \cdots,\\
s(-7)f_{-7} &= q^{-7} - 10q + 4q^4 + 28q^8 - 24q^9 + 60q^{16} + \cdots,\\
s(-12)f_{-12} &= q^{-12} - 10q - 25q^4 - 6q^8 + 46q^9 + 152q^{16} + \cdots,\\
s(-19)f_{-19} &= q^{-19} - q - 50q^4 - 50q^8 - 153q^9 + 798q^{16} + \cdots,\\
s(-20)f_{-20} &= q^{-20} - 22q + 26q^4 - 180q^8 - 78q^9 - 338q^{16} + \cdots,\\
s(-24)f_{-24} &= q^{-24} - 2q - 28q^4 + 225q^8 - 450q^9 - 2976q^{16} + \cdots,\\
s(-27)f_{-27} &= q^{-27} + 12q + 52q^4 - 468q^8 + 156q^9 - 1300q^{16} + \cdots.
\end{aligned}$$

For example, we obtain

$$\begin{aligned}
f_{-3} = &-\tfrac{92368453}{1197504000}\,\mathrm{RC}_1 - \tfrac{1105849}{739031040}\,\mathrm{RC}_2 - \tfrac{7775323}{804722688000}\,\mathrm{RC}_3 + \tfrac{31109}{68584320000}\,\mathrm{RC}_4\\
&- \tfrac{1}{49268736000}\,\mathrm{RC}_7 + \tfrac{1}{862202880000}\,\mathrm{RC}_8 - \tfrac{1}{86910050304000}\,\mathrm{RC}_{12}
\end{aligned}$$

$$+\frac{1}{216309458534400}\,\mathrm{RC}_{15}+\frac{83841213721}{1026432000}\,\theta$$
$$=q^{-3}-3q-2q^4+6q^8+5q^9-10q^{16}+\cdots.$$

By Proposition 1.2, the forms $s(0)f_0,\ldots,s(-27)f_{-27}$ have integer Fourier coefficients. It follows from Theorem 1.3 that every reduced form in $M^{!+\cdots+}_{1/2}(7,1)$ has integer Fourier coefficients.

Now we assume that, for any reduced form

$$f_m=\sum_n a(m,n)\,q^n\in M^{!\epsilon}_{k/2}(N,1),$$

the form $s(m)f_m$ has integer Fourier coefficients. Furthermore, let $k\geq 3$ be an odd integer and set $\lambda=(k-1)/2$. Then the reduced form $f_m\in M^{!\epsilon}_{k/2}(N,1)$ exists for every $m\in\mathbb{Z}_{<0}$ with $\chi_p(m)\neq-\epsilon_p$ for all $p\mid N$. We write

$$F_m(\tau)=s(m)f_m(\tau)=q^m+\sum_{\substack{d\geq 0\\ \chi_p(d)\neq-\epsilon_p\text{ for all } p|N}}B^{(N)}(m,d)\,q^d.$$

Note that the Hecke operator $T_{k/2,4N}(\ell^2)$ acts on the space $M^{!\epsilon}_{k/2}(N,1)$ for each prime ℓ with $\gcd(\ell,4N)=1$. For any positive integer t with $\gcd(t,4N)=1$, define

$$F^{(t)}_m:=F_m\mid T_{k/2,4N}(t^2).$$

Then we obtain the coefficients $B^{(N)}_t(m,d)$ from the equation

$$F^{(t)}_m(\tau)=(\text{principal part})+\sum_{\substack{d\geq 0,\\ \chi_p(d)\neq-\epsilon_p\text{ for all } p|N}}B^{(N)}_t(m,d)\,q^d.$$

We state our main theorem which describes various relations among the coefficients $B^{(N)}_t(m,d)$.

Theorem 1.5. *We have the following:*

(i) $B^{(N)}_t(m,\ell^{2n+2}d)-\ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right)B^{(N)}_t(m,\ell^{2n}d)=\ell^{(k-2)n}\Big\{B^{(N)}_t(\ell^{2n}m,\ell^2 d)-B^{(N)}_t(\ell^{2n-2}m,d)\Big\}$.

(ii) *If $\ell\nmid d$, then*

$$\ell^{(k-2)n}B^{(N)}_t(\ell^{2n}m,d)=B^{(N)}_t(m,\ell^{2n}d)$$
$$+\left[\left(\frac{(-1)^\lambda d}{\ell}\right)-\left(\frac{(-1)^\lambda m}{\ell}\right)\right]\cdot\sum_{k=1}^{n}\ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1}B^{(N)}_t(m,\ell^{2n-2k}d).$$

(iii) *If* $\ell \parallel d$, *then*

$$\ell^{(k-2)n} B_t^{(N)}(\ell^{2n}m, d) = B_t^{(N)}(m, \ell^{2n}d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) \cdot B_t^{(N)}(m, \ell^{2n-2}d).$$

As a corollary, we obtain the following congruences:

Corollary 1.6. *Assume that* $S^{\epsilon}_{k/2}(N,1) = 0$.

(1) *If* $\left(\frac{-d}{\ell}\right) = \left(\frac{-m}{\ell}\right) \neq 0$, *or if* $\ell \parallel d$ *and* $\ell \parallel m$, *then for any positive integer* t *with* $(t, 4N) = 1$ *and* n, *we have*

$$B_t^{(N)}(m, \ell^{2n}d) = \ell^{(k-2)n}\, B_t^{(N)}(\ell^{2n}m, d) \equiv 0 \pmod{\ell^{(k-2)n}}.$$

(2) *If* $\chi_p(\ell d) \neq -\epsilon_p$ *for all* $p \mid N$, *then for any positive integer* t *with* $(t, 4N) = 1$ *and any* $n \geq 1$, *we get*

$$B_t^{(N)}(m, \ell^{2n+1}d) \equiv \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) B_t^{(N)}(m, \ell^{2n-1}d) \pmod{\ell^{(k-2)n}}.$$

As for the condition in the above corollary, we remark that if

$$N \in \{n \mid n \text{ is an odd square-free integer with } 1 \leq n < 37\} \cup \{39, 41, 47, 51, 55, 59, 69, 71, 87, 95, 105, 119\},$$

then $S^{+\cdots+}_{3/2}(N,1) = 0$. (See [5, Table 4].)

We organize this paper as follows. In Section 2, we present preliminaries on discriminant forms and modular forms of half-integral weight. Also, we recall the definitions of ϵ-condition and reduced forms. In Section 3, we prove Proposition 1.2 and Theorem 1.3, and in Section 4 we prove Theorem 1.5 and Corollary 1.6.

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2. Preliminaries

In this section, we review the basics on quadratic spaces, lattices and discriminant forms. Standard references for the theory of quadratic forms and lattices are [7], [8] and [16]. For the theory of discriminant forms, see [19], [20] and [26]. Moreover, for transitive discriminant forms, ϵ-conditions and reduced forms, we refer to [31].

2.1. Discriminant forms

Definition 2.1. Let $\mathcal{D}$ be a finite abelian group. A *quadratic form* Q on $\mathcal{D}$ is a map $Q : \mathcal{D} \to \mathbb{Q}/\mathbb{Z}$ such that

(1) $Q(nx) = n^2 Q(x)$ for all $n \in \mathbb{Z}$ and $x \in \mathcal{D}$,
(2) the map $\langle \cdot, \cdot \rangle : \mathcal{D} \times \mathcal{D} \to \mathbb{Q}/\mathbb{Z},\ (x, y) \mapsto \langle x, y \rangle := Q(x + y) - Q(x) - Q(y)$ is bilinear.

The quadratic form Q on $\mathcal{D}$ is called *nondegenerate* if

$$\langle x, y \rangle = 0 \text{ for all } y \in \mathcal{D} \text{ implies } x = 0.$$

In this case, we call the pair $(\mathcal{D}, Q)$ is a *finite quadratic module* or a *discriminant form*.

Let $(\mathcal{D}, Q)$ be a discriminant form. The *level* of $(\mathcal{D}, Q)$ is the smallest positive integer M such that $MQ(x) = 0$ for all $x \in \mathcal{D}$. Let $(\mathcal{D}', Q')$ be another discriminant form. A $\mathbb{Z}$-module homomorphism $\sigma : \mathcal{D} \to \mathcal{D}'$ is called *isometry* if σ is injective and $Q = Q' \circ \sigma$. If there is a bijective isometry between $(\mathcal{D}, Q)$ and $(\mathcal{D}', Q')$, then we say that $(\mathcal{D}, Q)$ and $(\mathcal{D}', Q')$ are *isometric* and we write $(\mathcal{D}, Q) \simeq (\mathcal{D}', Q')$. The set of all bijective isometries of $\mathcal{D}$ onto itself forms a group under composition; we denote it by $\mathrm{Aut}(\mathcal{D})$.

Many concepts and results in the theory of discriminant forms are closely related to lattices. So we recall some basic facts about lattices.

Definition 2.2. Let K be a field and let V be a finite-dimensional vector space over K. By a *quadratic form* on V, we mean a map $Q : V \to K$ such that

(1) $Q(cv) = c^2 Q(v)$ for all $c \in K$ and $v \in V$,
(2) the map $\langle \cdot, \cdot \rangle : V \times V \to K,\ (v, w) \mapsto \langle v, w \rangle := Q(v+w) - Q(v) - Q(w)$ is a symmetric bilinear form on V.

The pair (V, Q) is called a *quadratic space* over K.

Let (V, Q) be a quadratic space of dimension n and let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis for V. The matrix $G = G_{\mathcal{B}} := (\langle v_i, v_j \rangle)$ is called the *Gram matrix of* V *with respect to the basis* $\mathcal{B}$. If $\det(G_{\mathcal{B}}) \neq 0$, then the quadratic space (V, Q) is called *nondegenerate*. This definition is independent of the choice of basis. If (V, Q) is nondegenerate, then $V^{\perp} = \{0\}$, and vice versa.

Let (V', Q') be another quadratic space over K. We say that (V, Q) and (V', Q') are *isometric* if there exists a linear isomorphism $\psi : V \to V'$ such that $Q(v) = Q'(\psi(v))$ for all $v \in V$. By Sylvester's law of inertia, if (V, Q) is an n-dimensional nondegenerate quadratic space over $\mathbb{R}$, then there exists a unique pair (r, s) of nonnegative integers such that (V, Q) is isometric to $\mathbb{R}^{r,s} := (\mathbb{R}^{r+s}, x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2)$. The value $\mathrm{sign}(V) := r - s$ is called the *signature* of (V, Q).

Now let $R = \mathbb{Z}$ or $\mathbb{Z}_p$ and let K be the field of fractions of R.

Definition 2.3. By an *R-lattice* of rank n, we mean a free R-module L of rank n with a K-valued symmetric R-bilinear map $\langle \cdot, \cdot \rangle : L \times L \to K$. An R-lattice L is called *nondegenerate* if $\langle x, y \rangle = 0$ for all $y \in L$ implies $x = 0$.

Let L be an R-lattice. Then the lattice L is nondegenerate if and only if $(L \otimes_R K, Q_K)$ is nondegenerate. Let F be an extension field of K. Then the bilinear map $\langle \cdot, \cdot \rangle$ can be extended to a symmetric F-bilinear form $\langle \cdot, \cdot \rangle$ on $L \otimes_R F$. If $Q_F : L \otimes_R F \to F$ is the map defined by $Q_F(x) = \frac{1}{2}\langle x, x \rangle$, then the pair $(L \otimes_R F, Q_F)$ is a quadratic space over F.

If $\mathcal{B} = \{x_1, \ldots, x_n\}$ is a basis for L, then the *Gram matrix* with respect to $\mathcal{B}$ is the matrix $G = G_{\mathcal{B}} := (\langle x_i, x_j \rangle)$. Its determinant is determined up to multiplication by an element of $(R^*)^2$; it is called the *discriminant* of L and denoted by $\mathrm{disc}(L)$. If $\langle x, y \rangle \in R$ for all $x, y \in L$, then L is called *integral*. If $\langle x, x \rangle \in 2R$ for every $x \in L$, then L is called *even*. From the polarization identity,

$$\langle x, y \rangle = \frac{1}{2}(\langle x + y, x + y \rangle - \langle x, x \rangle - \langle y, y \rangle),$$

we see that even lattices are integral.

Definition 2.4. Let L be an R-lattice. We define the *dual lattice* L' of L by the set

$$L' := \{x \in L \otimes_R K : (x, y) \in R \text{ for all } y \in L\}.$$

If L is nondegenerate, then L' is also a lattice. Moreover, L is integral if and only if $L \subset L'$.

Definition 2.5. Let L be a nondegenerate $\mathbb{Z}$-lattice.

(1) The *level* of L is the smallest positive integer M such that $M \cdot Q_{\mathbb{Q}}(x) \in 2\mathbb{Z}$ for all $x \in L'$.
(2) The *signature* of L is defined as the signature of the real quadratic space $(L \otimes_{\mathbb{Z}} \mathbb{R}, Q_{\mathbb{R}})$.

Since $(\mathbb{Z}^*)^2 = \{1\}$, the discriminant $\mathrm{disc}(L)$ is an integer uniquely determined by L. If L is integral, then L'/L is a finite abelian group of order $|\mathrm{disc}(L)|$. Assume further that L is even. Then the map $Q_{L'/L} : L'/L \to \mathbb{Q}/\mathbb{Z}$, $x + L \mapsto Q_{\mathbb{Q}}(x) \pmod{\mathbb{Z}}$ is a well-defined quadratic form on L'/L. In other words, the pair $(L'/L, Q_{L'/L})$ is a discriminant form; we call it the *discriminant form of L*.

Proposition 2.6 *([19], Theorem 1.3.2). Let $(\mathcal{D}, Q)$ be a discriminant form. Then there exists a nondegenerate even $\mathbb{Z}$-lattice L such that $(\mathcal{D}, Q) \simeq (L'/L, Q_{L'/L})$.*

Now we introduce the Jordan decomposition of discriminant forms. Let $(\mathcal{D}, Q)$ and $(\mathcal{D}', Q')$ be two discriminant forms. Then the map $Q \oplus Q' : \mathcal{D} \oplus \mathcal{D}' \to \mathbb{Q}/\mathbb{Z},\ (x, y) \mapsto Q(x)+Q'(y)$ is a nondegenerate quadratic form on $\mathcal{D}\oplus\mathcal{D}'$, hence the pair $(\mathcal{D}\oplus\mathcal{D}', Q\oplus Q')$ is a discriminant form, called the *(orthogonal) direct sum* of $(\mathcal{D}, Q)$ and $(\mathcal{D}', Q')$. The level of $\mathcal{D}\oplus\mathcal{D}'$ is the least common multiple of the levels of $\mathcal{D}$ and $\mathcal{D}'$. For a discriminant form $(\mathcal{D}, Q)$, denote by $\mathcal{D}^n$ the orthogonal direct sum of n copies of $\mathcal{D}$. A discriminant form $(\mathcal{D}, Q)$ is called *indecomposable* if $(\mathcal{D}, Q)$ is not isometric to any orthogonal direct sum of two nonzero discriminant forms; otherwise, $(\mathcal{D}, Q)$ is called *decomposable*.

The nontrivial indecomposable discriminant forms are as follows:

(1) Let q be a power of an odd prime p and let $\varepsilon \in \{\pm 1\}$. We denote by q^{ε} the discriminant form

$$\left(\mathbb{Z}/q\mathbb{Z}, x + q\mathbb{Z} \mapsto \frac{a}{q}x^2 + \mathbb{Z}\right),$$

where a is any integer satisfying $\left(\frac{2a}{p}\right) = \varepsilon$.

(2) Let q be a power of the prime 2. We define q_{II}^{+2} and q_{II}^{-2} to be

$$q_{II}^{+2} := \left(\mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}, (x_1 + q\mathbb{Z}, x_2 + q\mathbb{Z}) \mapsto \frac{1}{q}x_1x_2 + \mathbb{Z}\right),$$
$$q_{II}^{-2} := \left(\mathbb{Z}/q\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}, (x_1 + q\mathbb{Z}, x_2 + q\mathbb{Z}) \mapsto \frac{1}{q}(x_1^2 + x_1x_2 + x_2^2) + \mathbb{Z}\right),$$

respectively. For $\varepsilon \in \{\pm 1\}$ and $t \in \mathbb{Z}$ satisfying $(-1)^{(t^2-1)/8} = \varepsilon$, define q_t^{ε} by

$$q_t^{\varepsilon} := \left(\mathbb{Z}/q\mathbb{Z}, x + q\mathbb{Z} \mapsto \frac{t}{2q}x^2 + \mathbb{Z}\right).$$

Proposition 2.7 *([19]). Every discriminant form is isometric to an orthogonal direct sum of indecomposable discriminant forms.*

Now we define several invariants of discriminant forms.

Definition 2.8.

(1) Let p be an odd prime and let I be an indecomposable discriminant form. The *p-excess* of I^n is an element of $\mathbb{Z}/8\mathbb{Z}$ defined by

$$p\text{-excess}(I^n) := \begin{cases} n(q-1) \pmod 8 & \text{if } I = q^{+1} \text{ and } q \text{ is a power of } p, \\ n(q-1) \pmod 8 & \text{if } I = q^{-1} \text{ and } q = p^{2k} \text{ for some } k \in \mathbb{Z}_{>0}, \\ n(q-1)+4 \pmod 8 & \text{if } I = q^{-1} \text{ and } q = p^{2k-1} \text{ for some } k \in \mathbb{Z}_{>0}, \\ 0 \pmod 8 & \text{if } I \text{ is not of the form } (p^k)^{\varepsilon}. \end{cases}$$

Let $\mathcal{D}$ be an arbitrary discriminant form. If $\mathcal{D} = I_1^{n_1} \oplus \cdots \oplus I_k^{n_k}$ is a decomposition into indecomposables, then we define the p-excess of $\mathcal{D}$ by

$$p\text{-excess}(\mathcal{D}) := \sum_{j=1}^{k} p\text{-excess}(I_j^{n_j}).$$

(2) Let I be an indecomposable discriminant form. The *oddity* of I^n is defined by

$$\text{oddity}(I^n) := \begin{cases} 0 \pmod 8 & \text{if } I = q_{II}^{+2} \text{ and } q \text{ is a power of } 2, \\ 0 \pmod 8 & \text{if } I = q_{II}^{-2} \text{ and } q = 2^{2k} \text{ for some } k \in \mathbb{Z}_{>0}, \\ 4 \pmod 8 & \text{if } I = q_{II}^{-2} \text{ and } q = 2^{2k-1} \text{ for some } k \in \mathbb{Z}_{>0}, \\ t \pmod 8 & \text{if } I = q_t^{+1} \text{ and } q \text{ is a power of } 2, \\ t \pmod 8 & \text{if } I = q_t^{-1} \text{ and } q = 2^{2k} \text{ for some } k \in \mathbb{Z}_{>0}, \\ t+4 \pmod 8 & \text{if } I = q_t^{-1} \text{ and } q = 2^{2k-1} \text{ for some } k \in \mathbb{Z}_{>0}, \\ 0 \pmod 8 & \text{if } I = (p^k)^{\varepsilon} \text{ for some odd prime } p. \end{cases}$$

If $\mathcal{D}$ is an arbitrary discriminant form and $\mathcal{D} = I_1^{n_1} \oplus \cdots \oplus I_k^{n_k}$ is a decomposition into indecomposables, then we define the oddity of $\mathcal{D}$ as

$$\text{oddity}(\mathcal{D}) := \sum_{j=1}^{k} \text{oddity}(I_j^{n_j}).$$

(3) Let $\mathcal{D}$ be a discriminant form. Choose a nondegenerate even lattice L such that $\mathcal{D} \simeq L'/L$. Define the *signature* of $\mathcal{D}$ by

$$\text{signature}(\mathcal{D}) := \text{signature}(L) \pmod 8.$$

This value is in fact independent of the choice of L (cf. Theorem 1.1.1 and Theorem 1.3.1 of [19]).

The p-excess, oddity and signature satisfy the following relation.

Proposition 2.9 *(oddity formula). Let $\mathcal{D}$ be a discriminant form. Then*

$$\text{signature}(\mathcal{D}) + \sum_{p \geq 3} p\text{-excess}(\mathcal{D}) = \text{oddity}(\mathcal{D}).$$

2.2. ϵ-Conditions and reduced forms

The following definition is crucial for the rest of this section.

Definition 2.10. A discriminant form $\mathcal{D}$ is called *transitive* if for any $n \in \mathbb{Q}/\mathbb{Z}$ the group $\mathrm{Aut}(\mathcal{D})$ acts transitively on the set $\{x \in \mathcal{D} : Q(x) = n\}$.

Let $(\mathcal{D}, Q)$ be a discriminant form and denote the associated bilinear form by $\langle \cdot, \cdot \rangle$ as above. As an abelian group, $\mathcal{D} = \oplus_p \mathcal{D}_p$, where $\mathcal{D}_p = \{x \in \mathcal{D} : p^m x = 0 \text{ for some } m \in \mathbb{Z}_{>0}\}$. In fact the decomposition is orthogonal. Indeed, if p, q are two distinct primes, $x \in \mathcal{D}_p$ and $y \in \mathcal{D}_q$, then $p^m x = 0$, $q^n y = 0$ for some $m, n \in \mathbb{Z}_{>0}$. Choose $a, b \in \mathbb{Z}$ so that $ap^m + bq^n = 1$. Then $\langle x, y\rangle = (ap^m + bq^n)\langle x, y\rangle = \langle (ap^m)x, y\rangle + \langle x, (bq^n)y\rangle = 0$. If we let M_p the level of $\mathcal{D}_p$, then M_p is a power of p. Indeed, the order of $\mathcal{D}_p$ is a power of p. If p is an odd prime, then Jordan components of $\mathcal{D}_p$ are of the form $(p^k)^\epsilon$. Since, as mentioned above, the level of an orthogonal direct sum of a finite number of discriminant forms is the least common multiple of the levels of summands, M_p is a power of p. Similarly, M_2 is a power of 2. Since the M_p are pairwise relatively prime, $M = \prod_p M_p$.

The following results give us the classification of transitive discriminant forms.

Proposition 2.11 *([31], Proposition 2.1). A discriminant form $\mathcal{D}$ is transitive if and only if $\mathcal{D} = \oplus_p \mathcal{D}_p$, where*

(1) *if $p \equiv 1 \pmod 4$, then $\mathcal{D}_p$ is trivial or equal to $p^{\pm 1}$ or $(p^{+1})^2$,*
(2) *if $p \equiv -1 \pmod 4$, then $\mathcal{D}_p$ is trivial or equal to $p^{\pm 1}$ or $(p^{-1})^2$,*
(3) *$\mathcal{D}_2$ is trivial or equal to one of the following:*

$$(2^{\pm 1}_{\pm 3})^3, \quad (2^{\pm 1}_{\pm 2})^2, \quad 2^{\pm 1}_{\pm 1}, \quad 2^{-2}_{II}, \quad 4^{\pm 1}_t, \quad 4^{\pm 1}_t \oplus 2^{+1}_{+1}.$$

Let $\mathcal{D}$ be a transitive discriminant form of odd signature. Assume that $\mathcal{D}_2 = 2^{\pm 1}_{\pm 1}$. Then we associate the following data to $\mathcal{D}$:

(1) The level M of $\mathcal{D}$. By Proposition 2.11, $M = 4N$ for some odd squarefree integer N.
(2) An even Dirichlet character χ modulo N. By the Chinese remainder theorem, it suffices to define its p-component χ_p for each prime p. If $p \equiv 1 \pmod 4$ (resp. $p \equiv 3 \pmod 4$), then we set

$$\chi_p(d) := \begin{cases} 1 & \text{if } \mathcal{D}_p \text{ is trivial or } (p^{+1})^2 \text{ (resp. } (p^{-1})^2), \\ \left(\frac{d}{p}\right) & \text{if } \mathcal{D}_p = p^{\pm 1}. \end{cases}$$

On the other hand, χ_2 is defined by

$$\chi_2 := \begin{cases} 1 & \text{if } \left(\frac{-1}{|\mathcal{D}|}\right) = 1, \\ \left(\frac{-4}{d}\right) & \text{if } \left(\frac{-1}{|\mathcal{D}|}\right) = -1. \end{cases}$$

Also, let $\chi' = \chi\left(\frac{M}{\cdot}\right)$.

(3) The sign vector $\epsilon = (\epsilon_p)_p$. Here p ranges over 2 and prime factors of N such that χ_p is non-trivial. For such a prime, let

$$\epsilon_p := \begin{cases} \chi_p(2N/p)\delta_p & \text{if } p \text{ is odd and } \mathcal{D}_p = p^{\delta_p}, \\ t\left(\frac{-1}{M}\right) & \text{if } p = 2 \text{ and } \mathcal{D}_2 = 2_t^{+1}. \end{cases}$$

Conversely, given such a triple $(4N, \chi, \epsilon)$, we can recover $\mathcal{D}$. Indeed, if p is an odd prime with $\chi_p \neq 1$, then $\mathcal{D}_p = p^{\delta_p}$, where $\delta_p = \epsilon_p\chi_p(2N/p)$. If $p = 2$, then $\mathcal{D}_2 = 2_t^{+1}$, where $t = \epsilon_2\left(\frac{-1}{M}\right)$. For other prime factor p of N, we have $\mathcal{D}_p = p^{\pm 2}$ where the sign is determined uniquely by the transitivity.

Let k be an odd integer. Consider an element $(A, \phi) \in \mathrm{Mp}_2^+(\mathbb{R})$ of the metaplectic cover of $\mathrm{GL}_2^+(\mathbb{R})$, and let $f : \mathbb{H} \to \mathbb{C}$ be a function. The weight $k/2$ slash operator is defined by

$$(f|_{k/2}(A, \phi))(\tau) = \phi^{-k}(\tau)f(A\tau), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We define the operators τ_M, $W(m)$, $U(m)$, $Y(p)$ as follows.

(1) Let

$$\tau_M := \left(\begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, M^{\frac{1}{4}}(-i\tau)^{\frac{1}{2}}\right) \in \mathrm{Mp}_2^+(\mathbb{R}).$$

If $f \in M^!_{k/2}(M, \chi')$, then $f|_{k/2}\tau_M \in M^!_{k/2}(M, \chi)$ and $f|_{k/2}\tau_M^2 = f$ (cf. [25], Proposition 1.4).

(2) Let m be an odd divisor of M and choose $u, v \in \mathbb{Z}$ so that $vm + uM/m = 1$. Let

$$W(m) := \left(\begin{pmatrix} m & -1 \\ uM & vm \end{pmatrix}, \left(\frac{-4}{m}\right)^{-\frac{1}{2}} m^{-\frac{1}{4}}(uMz + mv)^{\frac{1}{2}}\right) \in \mathrm{Mp}_2^+(\mathbb{R}).$$

If $f \in M^!_{k/2}(M, \chi')$, then $f|_{k/2}W(m) \in M^!_{k/2}(M, \chi'\left(\frac{m}{\cdot}\right))$ (cf. [17], Proposition 2 or [28], (1.17)).

(3) For any divisor m of M and $f \in M^!_{k/2}(M, \chi')$, define $U(m)$ by

$$f|_{k/2}U(m) := \frac{1}{m}\sum_{j=1}^{m} f\left(\frac{\tau + j}{m}\right).$$

It is known that $f|_{k/2}U(m) \in M^!_{k/2}(M, \chi'\left(\frac{m}{\cdot}\right))$ (cf. [25], Proposition 1.5).

(4) For each odd prime divisor p of M, we define $Y(p)$ by

$$f|_{k/2}Y(p) := p^{1-\frac{k}{4}} f|_{k/2}U(p)W(p).$$

On the other hand, $Y(4)$ is defined by

$$f|_{k/2}Y(4) := 4^{1-\frac{k}{4}} f|_{k/4}U(4)W(N)\tau_M.$$

Then $f|_{k/2}Y(p), f|_{k/2}Y(4) \in M^!_{k/2}(M, \chi')$.

Definition 2.12. Let $\mathcal{D}$ be the discriminant form corresponding to $(4N, \chi', \epsilon)$, and k be an odd integer such that $k \equiv \text{signature}(\mathcal{D}) \pmod 4$. Then we define $M^{!\epsilon}_{k/2}(N, \chi')$ by the subspace of $M^!_{k/2}(4N, \chi')$ consisting of the functions $f = \sum_n a(n)q^n$ satisfying the following two conditions:

(1) $a(n) = 0$ if $n \equiv 2, -\epsilon_2 \pmod 4$ or $\left(\frac{n}{p}\right) = -\epsilon_p$ for some $p \mid N$ with $\chi_p \neq 1$,
(2) $f|_{k/2}Y(p) = -f$ for every $p \mid N$ with $\chi_p = 1$.

The ϵ-condition can be considered as a generalization of the Kohnen plus condition. Recall the even lattice L introduced in (1.1). Then its discriminant form $\mathcal{D} = L'/L \cong \mathbb{Z}/2N\mathbb{Z}$ is transitive with the 2-component $\mathcal{D}_2 = 2^{-1}_{\left(\frac{-1}{N}\right)}$ and the p-components $\mathcal{D}_p = p^{\left(\frac{2N/p}{p}\right)}$ for any odd prime p dividing N. We have $\epsilon_p = +1$ for all $p \mid N$ and $\chi' = \chi\left(\frac{4N}{\cdot}\right) = 1$. From the definition, we have $f = \sum_n a(n)q^n \in M^{!\epsilon}_{k/2}(N, 1)$ if and only if $a(n) = 0$ unless $(-1)^{\frac{k-1}{2}}n$ is a square modulo $4N$. Thus the space $M^{!\epsilon}_{k/2}(N, 1)$ is exactly the same as the space $M^{!+,\dots,+}_{k/2}(N)$. Eichler and Zagier denote the space $M^{\epsilon}_{k/2}(N, 1) = M^{!\epsilon}_{k/2}(N, 1) \cap M_{k/2}(4N, 1)$ by $M^{+,\dots,+}_{k/2}(N)$ in [10, p. 69].

Remark 2.13. It is known that the space $M^!_{k/2}(4N, \chi')$ decomposes into the direct sum of simultaneous eigenspace for the operators $Y(p)$, $p \mid N$ and $Y(4)$. The subspace $M^{!\epsilon}_{k/2}(N, \chi')$ is one of these simultaneous eigenspaces (cf. [31], p. 14).

Now we shall assume that $\chi_p \neq 1$ for each $p \mid N$, so $\chi' = 1$.

Definition 2.14. A form $f \in M^{!\epsilon}_{k/2}(N, \chi')$ is called *reduced* if $f = \frac{1}{s(m)}q^m + \sum_{\ell \geq m+1} a(\ell)q^\ell$ for some integer m and if for each $n > m$ with $a(n) \neq 0$, there does not exist $g \in M^{!\epsilon}_{k/2}(N, \chi')$ such that $g = q^n + O(q^{n+1})$. Here, $s(m) = \prod_{p \mid \gcd(N,m)} \left(1 + \frac{p}{|\mathcal{D}_p|}\right)$.

If a reduced form exists for some m, it is unique and $\chi_p(m) \neq -\epsilon_p$ for each $p \mid N$; we denote it by f_m. The set of reduced modular forms is a basis for $M^{!\epsilon}_{k/2}(N, \chi')$.

The following proposition determines $m < 0$ for which f_m exists. To state it, we need some notation. Let $\mathcal{D}^*$ be the dual discriminant form of $\mathcal{D}$ given by the same abelian group with the quadratic form $-Q$. It is known that $\mathcal{D}^*$ is also transitive and the corresponding data is $(4N, \chi', \epsilon^*)$ with $\epsilon^*_p = \chi_p(-1)\epsilon_p$.

Proposition 2.15 *([31], Proposition 6.1). Let* $B^* = \{m : f_m^* \in M^{\epsilon^*}_{2-k/2}(N, \chi') \text{ exists}\}$. *Then for any* $m < 0$ *with* $\chi_p(m) \neq -\epsilon_p$ *for all* $p \mid N$*, the reduced form* $f_m \in M^{!\epsilon}_{k/2}(N, \chi')$ *exists if and only if* $-m \notin B^*$.

3. Proofs of Proposition 1.2 and Theorem 1.3

In this section, we prove the rationality of Fourier coefficients of reduced forms following the lines in [4] and [30]. For $f = \sum_n a(n)q^n$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, define $f^\sigma = \sum_n \sigma(a(n))q^n$.

Lemma 3.1. *Let* χ *be a Dirichlet character modulo* N *with values in* $\mathbb{Q}$. *If* $f \in M^!_{k/2}(4N, \chi)$*, so is* f^σ.

Proof. It is known that $\mathrm{Aut}(\mathbb{C})$ acts on the space $M_{k/2}(\Gamma_1(4N), \chi)$, the space of holomorphic modular forms of weight $k/2$ on $\Gamma_1(4N)$ with Nebentypus χ, and $\sigma(M_{k/2}(4N, \chi)) = M_{k/2}(4N, \chi^\sigma)$. (See [23].) Since χ has values in $\mathbb{Q}$, $\mathrm{Aut}(\mathbb{C})$ acts on $M_{k/2}(4N, \chi)$.

Note that $f\Delta^{k'} \in M_{k/2+12k'}(4N, \chi)$ for a sufficiently large positive integer k'. Here Δ is the unique normalized cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. The above observation shows that $(f\Delta^{k'})^\sigma \in M_{k/2+12k'}(4N, \chi)$. But Δ has integral Fourier coefficients, hence $f^\sigma\Delta^{k'} = (f\Delta^{k'})^\sigma \in M_{k/2+12k'}(4N, \chi)$ and $f^\sigma \in M^!_{k/2}(4N, \chi)$. □

Proposition 3.2. *Let* $k < 0$ *and let* $f = \sum_n a(n)q^n \in M^{!\epsilon}_{k/2}(N, \chi')$. *Suppose that* $a(n) \in \mathbb{Q}$ *for* $n < 0$. *Then all the coefficients* $a(n)$ *are rational with bounded denominators.*

Proof. Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. By Lemma 3.1, $f^\sigma \in M^!_{k/2}(4N, \chi')$. It is easy to check that the action of $\mathrm{Aut}(\mathbb{C})$ preserves the ϵ-condition. Since $a(n) \in \mathbb{Q}$ for $n < 0$, $h := f - f^\sigma$ is holomorphic at ∞. By [31, Corollary 5.5], $h \in M^{\epsilon}_{k/2}(N, \chi')$. But $k < 0$, so $h = 0$. It follows that f has rational coefficients.

We know that $\theta f\Delta^{k'} \in S_{(k+1)/2+12k'}(4N, \chi') \subset S_{(k+1)/2+12k'}(\Gamma_1(4N))$ for a sufficiently large positive integer k'. Shimura proved that $S_{(k+1)/2+12k'}(\Gamma_1(4N))$ has a basis $\mathcal{B}$ consisting of forms whose Fourier coefficients at ∞ are rational integers. (See [24, Theorem 3.52].) Let $S^{\mathbb{Q}}_{(k+1)/2+12k'}(\Gamma_1(4N))$ be the $\mathbb{Q}$-vector space of cusp forms in $S_{(k+1)/2+12k'}(\Gamma_1(4N))$ whose Fourier coefficients at ∞ are rational numbers. Then $\mathcal{B}$ is a $\mathbb{Q}$-basis of $S^{\mathbb{Q}}_{(k+1)/2+12k'}(\Gamma_1(4N))$ and $f\theta\Delta^{k'} \in S^{\mathbb{Q}}_{(k+1)/2+12k'}(\Gamma_1(4N))$. This implies that $f\theta\Delta^{k'}$ has coefficients with bounded denominators, and we conclude that the $a(n)$ are rational with bounded denominators. □

We are interested in integrality of Fourier coefficients. So we generalize Sturm's theorem to $M^{!\epsilon}_{k/2}(N, \chi')$. We begin with introducing the original Sturm's theorem.

Theorem 3.3 *([27]). Let* $\mathcal{O}_F$ *be the ring of integers of a number field* F*,* $\mathfrak{p}$ *any prime ideal,* N' *a positive integer and* k' *a positive integer. Assume* $f = \sum_n a(n)q^n \in M_{k'}(N', \chi) \cap \mathcal{O}_F[\![q]\!]$. *If* $a(n) \in \mathfrak{p}$ *for* $n \leq \frac{k'}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N')]$*, then* $a(n) \in \mathfrak{p}$ *for all* n.

Using Theorem 3.3, Kim, Lee and Zhang proved the following:

Corollary 3.4 *([15], Corollary 3.2). Let k' be a positive integer. Assume $f = \sum_n a(n)q^n \in M_{k'}(4N,\chi) \cap \mathbb{Q}[\![q]\!]$ with bounded denominator. If $a(n) \in \mathbb{Z}$ for $n \leq \frac{k'}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]$, then $a(n) \in \mathbb{Z}$ for all n.*

We extend this result to half-integral weight case. Let

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots, \quad (q = e^{2\pi i\tau},\ \tau \in \mathbb{H}).$$

Corollary 3.5. *Let $k > 0$ be an odd integer and assume that $f = \sum_n a(n)q^n \in M_{k/2}(4N,\chi) \cap \mathbb{Q}[\![q]\!]$ with bounded denominator. If $a(n) \in \mathbb{Z}$ for $n \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]$, then $a(n) \in \mathbb{Z}$ for all n.*

Proof. By multiplying θ^k, we have $f\theta^k \in M_k(4N,\chi)$. It suffices to show that all the coefficients of $f\theta^k$ are integers. Since $a(n) \in \mathbb{Z}$ for $n \leq \frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]$, the same property holds for the coefficients of $f\theta^k$. By Corollary 3.4, every coefficient of $f\theta^k$ is an integer. $\square$

Now we are ready to prove Proposition 1.2 and Theorem 1.3.

Proof of Proposition 1.2. Since $k' \geq |\mathrm{ord}_\infty(f)|/4N$, we see that $f(\tau)\Delta(4N\tau)^{k'} \in M^{\epsilon}_{k/2+12k'}(N,\chi)$ and that every coefficient of $f(\tau)\Delta(4N\tau)^{k'}$ less than or equal to $\frac{k+12k'}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4N)]$ is an integer. By Corollary 3.5, $f(\tau)\Delta(4N\tau)^{k'}$ has integer Fourier coefficients, hence so does f. $\square$

Proof of Theorem 1.3. Consider any reduced form $f_{m'}$ with $m' < -4N - m_\epsilon$. There exist integers $-4N - m_\epsilon \leq m'_0 < -m_\epsilon$ and $l \geq 1$ such that $m' = -4Nl + m'_0$. By maximality of m_ϵ, $f_{m'_0}$ exists. Consider now

$$g = j(4N\tau)^l f_{m'_0} = \sum_n b(n)q^n \in M^{!\epsilon}_{k/2}(N,\chi'),$$

where $j(\tau)$ denotes the classical j-function. It is known that j has integral Fourier coefficients. By the assumption on $f_{m'_0}$, we see that $b(n)s(m'_0) \in \mathbb{Z}$ for each n.

Now $s(m'_0)g$ and $s(m')f_{m'}$ share the same lowest power term, and we must have that

$$s(m')f_{m'} = s(m'_0)g - \sum_{m>m'} s(m'_0)b(m)s(m)f_m.$$

Hence $s(m')a_{m'}(n) = s(m'_0)b(n) - s(m)b(m)s(m'_0)a_m(n) \in \mathbb{Z}$ by the assumption and induction on m. $\square$

4. Proofs of Theorem 1.5 and Corollary 1.6

From now on, we shall assume that, for any reduced form

$$f_m = \sum_n a(m,n)q^n \in M^{!\epsilon}_{k/2}(N,\chi),$$

the modular form $s(m)f_m$ has integral Fourier coefficients. We remark that such integrality for each fixed reduced form can be verified by Proposition 1.2. Also, as we showed in Example 1.4, every reduced form in the space $M^{!+\cdots+}_{1/2}(7,1)$ satisfies the assumption.

We begin with a lemma.

Lemma 4.1. *Let $N \geq 1$ be a square-free integer. Then we have $M^{+\cdots+}_{3/2}(N,1) = S^{+\cdots+}_{3/2}(N,1)$.*

Proof. Let $f = \sum_{n\geq 0} a(n)q^n \in M^{+\cdots+}_{3/2}(N,1)$. By Borcherds' obstruction theorem (Theorem 3.1 of [3]), we get

$$s(0)a(0)b(0) = 0$$

for each $g = \sum_n b(n)q^n \in M^{\epsilon^*}_{1/2}(N,1)$. Since N is square-free,

$$M^{\epsilon^*}_{1/2}(N,1) = \mathbb{C}\theta$$

by [23, Theorem A]. Setting $g = \theta$, we obtain

$$s(0)a(0) = 0.$$

Since $s(0) \neq 0$, the form f vanishes at ∞. By [31, Proposition 5.3], the vector-valued form $\psi(f)$ is a cusp form. Here ψ is the map constructed in Chapter 5 of [31]. We conclude from [31, Corollary 5.5] that $f = \psi^{-1}(\psi(f)) \in S^{+\cdots+}_{3/2}(N,1)$. □

Remark 4.2. If $k \geq 3$ is an odd integer and $(k,\epsilon) \neq (3,+)$, then

$$M^{\epsilon^*}_{2-k/2}(N,1) = 0.$$

Let $N \geq 1$ be an odd square-free integer. Suppose that ℓ is a prime with $\ell \nmid 4N$. Consider a modular form

$$f(\tau) = \sum_n a(n)q^n \in M^{!\epsilon}_{k/2}(N,1),$$

where the sum is over n such that $\chi_p(n) \neq -\epsilon_p$ for all $p \mid N$. Then the action of the Hecke operator $T_{k/2,4N}(\ell^2)$ on f is given by

$$f(\tau) \mid T_{k/2,4N}(\ell^2) = \sum_n \left(a(\ell^2 n) + \ell^{\lambda-1} \left(\frac{(-1)^\lambda n}{\ell} \right) a(n) + \ell^{2\lambda-1} a(n/\ell^2) \right) q^n, \quad (4.3)$$

where $\lambda = (k-1)/2$ and the sum is over n such that $\chi_p(n) \neq -\epsilon_p$ for all $p \mid N$. Here we set $a(n/\ell^2) = 0$ if $\ell^2 \nmid n$. Note that $f(\tau) \mid T_{k/2,4N}(\ell^2) \in M^{!\epsilon}_{k/2}(N,1)$. We define $T_{k/2,4N}(\ell^{2n})$ for $n \geq 2$ recursively by

$$T_{k/2,4N}(\ell^{2n}) := T_{k/2,4N}(\ell^{2n-2}) T_{k/2,4N}(\ell^2) - \ell^{k-2} T_{k/2,4N}(\ell^{2n-4}).$$

Remark 4.4. For $n \geq 2$, our $T_{k/2,4N}(\ell^{2n})$ is different from the ℓ^{2n}-th Hecke operator given in [25]. See [22, p. 241] for details.

By Proposition 2.15, the reduced form $f_m \in M^{!\epsilon}_{k/2}(N,1)$ exists for every $m < 0$ with $\chi_p(m) \neq -\epsilon_p$ for all $p \mid N$. We write

$$F_m(\tau) = s(m) f_m(\tau) = q^m + \sum_{\substack{d \geq 0 \\ \chi_p(d) \neq -\epsilon_p \text{ for all } p \mid N}} B(m,d) q^d.$$

For any positive integer t with $\gcd(t, 4N) = 1$, define

$$F_m^{(t)} := F_m \mid T(t^2).$$

Then we obtain the coefficients $B_t(m,d)$ from the equation

$$F_m^{(t)}(\tau) = (\text{principal part}) + \sum_{\substack{d \geq 0, \\ \chi_p(d) \neq -\epsilon_p \text{ for all } p \mid N}} B_t(m,d) q^d.$$

For the rest of this section, let $k \geq 3$ be an odd integer and set $\lambda = (k-1)/2$, and assume

(1) $m \in \mathbb{Z}_{<0}$ such that $\chi_p(m) \neq -\epsilon_p$ for all $p \mid N$,
(2) ℓ is a prime with $\ell \nmid 4N$ and $\ell^2 \nmid m$.

Proposition 4.5. *Assume that $S^\epsilon_{k/2}(N,1) = 0$. Then, for any positive integer t with $(t, 4N) = 1$ and any positive integer n, we have*

$$F_m^{(t)} | T_{k/2,4N}(\ell^{2n}) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) F_m^{(t)} | T_{k/2,4N}(\ell^{2n-2}) = \ell^{(k-2)n} F^{(t)}_{\ell^{2n} m}.$$

Proof. For convenience, define $G_0^{(t)} := F_m^{(t)}$, and, for each $n \geq 1$,

$$G_n^{(t)} := F_m^{(t)} \mid T_{k/2,4N}(\ell^{2n}) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) F_m^{(t)} \mid T_{k/2,4N}(\ell^{2n-2}). \quad (4.6)$$

We need to show $G_n^{(t)} = \ell^{(k-2)n} F_{\ell^{2n}m}^{(t)}$. Since the Hecke operators commute, it suffices to prove the proposition in the case $t = 1$, which we now assume.

We claim that

$$G_n^{(1)} = G_{n-1}^{(1)} \mid T_{k/2,4N}(\ell^2) - \ell^{k-2} \cdot G_{n-2}^{(1)} \quad \text{for} \quad n \geq 2. \tag{4.7}$$

Indeed, if $n = 2$, then

$$\begin{aligned}
G_2^{(1)} - &G_1^{(1)} \mid T_{k/2,4N}(\ell^2) + \ell^{k-2} \cdot G_0^{(1)} \\
&= \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^4) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) F_m^{(1)} \mid T_{k/2,4N}(\ell^2) \right) \\
&\quad - \left(\left(F_m^{(1)} \mid T_{k/2,4N}(\ell^2) \right) \mid T_{k/2,4N}(\ell^2) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) F_m^{(1)} \mid T_{k/2,4N}(\ell^2) \right) \\
&\quad + \ell^{k-2} \cdot F_m^{(1)} \\
&= F_m^{(1)} \mid T_{k/2,4N}(\ell^4) - \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^2) \right) \mid T_{k/2,4N}(\ell^2) + \ell^{k-2} \cdot F_m^{(1)} = 0.
\end{aligned}$$

For $n \geq 3$,

$$\begin{aligned}
G_{n-1}^{(1)} &\mid T_{k/2,4N}(\ell^2) - \ell^{k-2} \cdot G_{n-2}^{(1)} \\
&= \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-2}) \right) \mid T_{k/2,4N}(\ell^2) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) \\
&\quad \cdot \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-4}) \right) \mid T_{k/2,4N}(\ell^2) \\
&\quad - \ell^{k-2} \cdot \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-4}) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-6}) \right) \\
&= \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-2}) T(\ell^2) - \ell^{k-2} \cdot F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-4}) \right) \\
&\quad - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) \\
&\quad \cdot \left(F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-4}) T(\ell^2) - \ell^{k-2} \cdot F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n-6}) \right) \\
&= F_m^{(1)} \mid T_{k/2,4N}(\ell^{2n}) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell} \right) \cdot F_m^{(t)} \mid T_{k/2,4N}(\ell^{2n-2}) = G_n^{(1)}.
\end{aligned}$$

Since $G_0^{(1)} = F_m^{(1)} = F_m$, the principal part of $G_0^{(1)}$ is q^m. By (4.3), the principal part of $G_1^{(1)}$ is $\ell^{k-2} q^{m\ell^2}$. Moreover, we see from (4.7) that, for all $n \geq 0$, the principal part of $G_n^{(1)}$ is equal to $\ell^{(k-2)n} q^{m\ell^{2n}}$. Since $F_{m\ell^{2n}}^{(1)} = F_{m\ell^{2n}}$ has principal part $q^{m\ell^{2n}}$, $G_n^{(1)} - \ell^{(k-2)n} F_{\ell^{2n}m}$ is holomorphic at the cusp ∞. Arguing as in the proof of Lemma 4.1, we have

$$G_n^{(1)} - \ell^{(k-2)n} F_{\ell^{2n}m} \in M_{k/2}^{\epsilon}(N, 1).$$

If $(k,\epsilon)=(3,+)$, then it follows from Lemma 4.1 that

$$G_n^{(1)} - \ell^{(k-2)n} F_{\ell^{2n}m} \in S_{k/2}^{\epsilon}(N,1).$$

Since $S_{k/2}^{\epsilon}(N,1)=\{0\}$ by assumption, we have $G_n^{(1)} = \ell^{(k-2)n} F_{\ell^{2n}m}$.

If $(k,\epsilon)\neq(3,+)$, then $M_{2-k/2}^{\epsilon^*}(N,1)=0$ (Remark 4.2). By Borcherds' obstruction theorem, there exists a reduced form g such that $g=1+O(q)$. We see from the definition of reduced forms that $B(m,0)=B(\ell^{2n}m,0)=0$. Hence the constant term of $G_n^{(1)} - \ell^{(k-2)n} F_{\ell^{2n}m}$ is zero, and thus

$$G_n^{(1)} - \ell^{(k-2)n} F_{\ell^{2n}m} \in S_{k/2}^{\epsilon}(N,1) = \{0\}.$$

Therefore we have $G_n^{(1)} = \ell^{(k-2)n} F_{\ell^{2n}m}$ in this case too. $\square$

Write

$$G_n^{(t)} = (\text{principal part}) + \sum_{\substack{d\geq 0,\\ \chi_p(d)\neq -\epsilon_p \text{ for all } p\mid N}} C_n(d) q^d.$$

Proposition 4.5 implies that, for all n and d,

$$C_n(d) = \ell^{(k-2)n} B_t(\ell^{2n}m, d). \tag{4.8}$$

Lemma 4.9. *The following are true:*

(i) *For any $d\geq 0$ with $\chi_p(d)\neq -\epsilon_p$ for all $p\mid N$, we have*

$$C_n(\ell^2 d) - \ell^{k-2}\cdot C_{n-1}(d) = C_0(\ell^{2n+2}d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n}d).$$

(ii) *If $\chi_p\neq -\epsilon_p$ for all $p\mid N$ and $\ell \parallel d$, then*

$$C_n(d) = C_0(\ell^{2n}d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n-2}d).$$

(iii) *If $\chi_p\neq -\epsilon_p$ for all $p\mid N$ and $\ell\nmid d$, then*

$$C_n(d) = C_0(\ell^{2n}d) + \left[\left(\frac{(-1)^\lambda d}{\ell}\right) - \left(\frac{(-1)^\lambda m}{\ell}\right)\right] \cdot \sum_{k=1}^{n} \ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1} C_0(\ell^{2n-2k}d).$$

Proof. We first prove (i). Note that

$$\begin{aligned}
&\ell^2 d\text{-th coefficient of } G_1^{(m)} = C_1(\ell^2 d),\\
&\ell^2 d\text{-th coefficient of } F_m^{(t)} \mid T_{k/2,4N}(\ell^2) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) F_m^{(t)}\\
&\quad = B_t(m,\ell^4 d) + \ell^{\lambda-1}\left(\frac{(-1)^\lambda \ell^2 d}{\ell}\right) B_t(m,\ell^2 d) + \ell^{k-2}\cdot B_t(m,d)\\
&\qquad - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m,\ell^2 d)\\
&\quad = B_t(m,\ell^4 d) + \ell^{k-2}\cdot B_t(m,d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m,\ell^2 d)\\
&\quad = C_0(\ell^4 d) + \ell^{k-2}\cdot C_0(d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^2 d).
\end{aligned}$$

By (4.6), we have

$$C_1(\ell^2 d) = C_0(\ell^4 d) + \ell^{k-2}\cdot C_0(d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^2 d).$$

Hence,

$$C_1(\ell^2 d) - \ell^{k-2}\cdot C_0(d) = C_0(\ell^4 d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^2 d).$$

When $n \geq 2$, we use (4.7) to find that

$$\begin{aligned}
C_n(\ell^2 d) - \ell^{k-2}\cdot C_{n-1}(d) &= C_{n-1}(\ell^4 d) - \ell^{k-2}\cdot C_{n-2}(\ell^2 d) = \cdots\\
&= C_1(\ell^{2n} d) - \ell^{k-2}\cdot C_0(\ell^{2n-2} d).
\end{aligned}$$

From (4.6), we see that

$$C_1(\ell^{2n} d) = C_0(\ell^{2n+2} d) + \ell^{k-2}\cdot C_0(\ell^{2n-2} d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n} d).$$

Thus we obtain

$$C_n(\ell^2 d) - \ell^{k-2}\cdot C_{n-1}(d) = C_0(\ell^{2n+2} d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n} d).$$

We now prove (ii) and (iii). Observe that

$$d\text{-th coefficient of } G_1^{(t)} = C_1(d),$$

$$\begin{aligned}
&d\text{-th coefficient of } F_m^{(t)} \mid T_{k/2,4N}(\ell^2) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) F_m^{(t)} \\
&= B_t(m,\ell^2 d) + \ell^{\lambda-1}\left(\frac{(-1)^\lambda d}{\ell}\right) B_t(m,d) + \ell^{k-2}\cdot B_t(m,d/\ell^2) \\
&\quad - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m,d) \\
&= \begin{cases} B_t(m,\ell^2 d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m,d) & \text{if } \ \ell \parallel d, \\ B_t(m,\ell^2 d) + \ell^{\lambda-1}\left[\left(\frac{(-1)^\lambda d}{\ell}\right) - \left(\frac{(-1)^\lambda m}{\ell}\right)\right] B_t(m,d) & \text{if } \ \ell \nmid d, \end{cases} \\
&= \begin{cases} C_0(\ell^2 d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(d) & \text{if } \ \ell \parallel d, \\ C_0(\ell^2 d) + \ell^{\lambda-1}\left[\left(\frac{(-1)^\lambda d}{\ell}\right) - \left(\frac{(-1)^\lambda m}{\ell}\right)\right] C_0(d) & \text{if } \ \ell \nmid d. \end{cases}
\end{aligned}$$

By (4.6), we have

$$C_1(d) = \begin{cases} C_0(\ell^2 d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(d) & \text{if } \ \ell \parallel d, \\ C_0(\ell^2 d) + \ell^{\lambda-1}\left[\left(\frac{(-1)^\lambda d}{\ell}\right) - \left(\frac{(-1)^\lambda m}{\ell}\right)\right] C_0(d) & \text{if } \ \ell \nmid d. \end{cases}$$

On the other hand, it follows from (4.7) that

$$C_n(d) = C_{n-1}(\ell^2 d) - \ell^{k-2}\, C_{n-2}(d) + \ell^{\lambda-1}\left(\frac{(-1)^\lambda d}{\ell}\right) C_{n-1}(d)$$

for $n \geq 2$. Applying part (i) to $C_{n-1}(\ell^2 d) - \ell^{k-2}\, C_{n-2}(d)$, we obtain

$$C_n(d) = C_0(\ell^{2n} d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n-2} d) + \ell^{\lambda-1}\left(\frac{(-1)^\lambda d}{\ell}\right) C_{n-1}(d).$$

If $\ell \parallel d$, then we immediately obtain part (ii). Now assume that $\ell \nmid d$. Then by induction we have

$$\begin{aligned}
C_n(d) &= C_0(\ell^{2n} d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n-2} d) + \ell^{\lambda-1}\left(\frac{(-1)^\lambda d}{\ell}\right) C_{n-1}(d) \\
&= C_0(\ell^{2n} d) - \ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right) C_0(\ell^{2n-2} d) + \ell^{\lambda-1}\left(\frac{(-1)^\lambda d}{\ell}\right) C_0(\ell^{2n-2} d) \\
&\quad + \ell^{\lambda-1}\left(\frac{(-1)^\lambda d}{\ell}\right)\cdot\left[\left(\frac{(-1)^\lambda d}{\ell}\right) - \left(\frac{(-1)^\lambda m}{\ell}\right)\right] \\
&\quad \cdot \sum_{k=1}^{n-1} \ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1} C_0(\ell^{2n-2k-2} d) \\
&= C_0(\ell^{2n} d) + \ell^{\lambda-1}\left[\left(\frac{(-1)^\lambda d}{\ell}\right) - \left(\frac{(-1)^\lambda m}{\ell}\right)\right]\cdot C_0(\ell^{2n-2} d)
\end{aligned}$$

$$
\begin{aligned}
&+\left[\left(\frac{(-1)^\lambda d}{\ell}\right)-\left(\frac{(-1)^\lambda m}{\ell}\right)\right]\cdot\sum_{k=2}^{n}\ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1}C_0(\ell^{2n-2k}d)\\
&=C_0(\ell^{2n}d)+\left[\left(\frac{(-1)^\lambda d}{\ell}\right)-\left(\frac{(-1)^\lambda m}{\ell}\right)\right]\\
&\quad\cdot\sum_{k=1}^{n}\ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1}C_0(\ell^{2n-2k}d).
\end{aligned}
$$

This proves the identity in part (iii). □

We now prove Theorem 1.5 and Corollary 1.6.

Proof of Theorem 1.5. (i) By (4.8) and Lemma 4.9 (i), we have

$$
\begin{aligned}
&B_t(m,\ell^{2n+2}d)-\ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right)B_t(m,\ell^{2n}d)\\
=&C_0(\ell^{2n+2}d)-\ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right)C_0(\ell^{2n}d)=C_n(\ell^2 d)-\ell^{k-2}C_{n-1}(d)\\
=&\ell^{(k-2)n}B_t(\ell^{2n}m,\ell^2 d)-\ell^{k-2}\cdot\ell^{(k-2)(n-1)}B_t(\ell^{2n-2}m,d)\\
=&\ell^{(k-2)n}\left\{B_t(\ell^{2n}m,\ell^2 d)-B_t(\ell^{2n-2}m,d)\right\}.
\end{aligned}
$$

(ii) Using (4.8) and Lemma 4.9 (iii), we obtain

$$
\begin{aligned}
\ell^{(k-2)n}B_t(\ell^{2n}m,d)=C_n(d)&\\
&=C_0(\ell^{2n}d)+\left[\left(\frac{(-1)^\lambda d}{\ell}\right)-\left(\frac{(-1)^\lambda m}{\ell}\right)\right]\\
&\quad\cdot\sum_{k=1}^{n}\ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1}C_0(\ell^{2n-2k}d)\\
&=B_t(m,\ell^{2n}d)+\left[\left(\frac{(-1)^\lambda d}{\ell}\right)-\left(\frac{(-1)^\lambda m}{\ell}\right)\right]\\
&\quad\cdot\sum_{k=1}^{n}\ell^{(\lambda-1)k}\left(\frac{(-1)^\lambda d}{\ell}\right)^{k-1}B_t(m,\ell^{2n-2k}d).
\end{aligned}
$$

(iii) By (4.8) and Lemma 4.9 (ii),

$$
\begin{aligned}
\ell^{(k-2)n}B_t(\ell^{2n}m,d)=C_n(d)&=C_0(\ell^{2n}d)-\ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right)C_0(\ell^{2n-2}d)\\
&=B_t(m,\ell^{2n}d)-\ell^{\lambda-1}\left(\frac{(-1)^\lambda m}{\ell}\right)B_t(m,\ell^{2n-2}d).\qquad\square
\end{aligned}
$$

Proof of Corollary 1.6. (1) First, suppose that $\left(\frac{-d}{\ell}\right) = \left(\frac{-m}{\ell}\right) \neq 0$. Then $\ell \nmid d$. By Theorem 1.5 (ii),

$$\ell^{(k-2)n} B_t(\ell^{2n} m, d) = B_t(m, \ell^{2n} d).$$

Now assume that $\ell \parallel d$ and $\ell \parallel m$. Then by Theorem 1.5 (iii),

$$\ell^{(k-2)n} B_t(\ell^{2n} m, d) = B_t(m, \ell^{2n} d).$$

(2) If $\ell \nmid d$, then $\ell \parallel \ell d$. By Theorem 1.5 (iii),

$$\begin{aligned}
&B_t(m, \ell^{2n+1} d) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m, \ell^{2n-1} d) \\
=&B_t(m, \ell^{2n}(\ell d)) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m, \ell^{2n-2}(\ell d)) \\
=&\ell^{(k-2)n} B_t(\ell^{2n} m, \ell d) \equiv 0 \pmod{\ell^{(k-2)n}}.
\end{aligned}$$

If $\ell \mid d$, then by Theorem 1.5 (i), we obtain

$$\begin{aligned}
&B_t(m, \ell^{2n+1} d) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m, \ell^{2n-1} d) \\
&= B_t(m, \ell^{2n+2}(d/\ell)) - \ell^{\lambda-1} \left(\frac{(-1)^\lambda m}{\ell}\right) B_t(m, \ell^{2n}(d/\ell)) \\
&= \ell^{(k-2)n} \left(B_t(\ell^{2n} m, \ell d) - B_t(\ell^{2n-2} m, d/\ell)\right) \equiv 0 \pmod{\ell^{(k-2)n}}. \qquad \square
\end{aligned}$$

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