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A Correspondence between Rigid Modules Over Path Algebras and Simple Curves on Riemann Surfaces

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ABSTRACT
We propose a conjectural correspondence between the set of rigid indecomposable modules over the path algebras of acyclic quivers and the set of certain non-self-intersecting curves on Riemann surfaces, and prove the correspondence for the two-complete rank 3 quivers.

KEYWORDS
quiver representation, real Schur root, Riemann surface

1. Introduction

In the study of the category of modules over a ring, geometric objects have often been used to describe the structures. In particular, the following problem has been considered fundamental. (For a small fraction of references, see [Apruzzese and Igusa; Fomin and Zelevinsky 02; Avramov, Buchweitz 00; Brüstle and Zhang 11; Canakci and Schroll 17; Musiker et al. 11; Zhang et al. 13].)

Problem 1. *Let R be a ring. Find a function f from a (sub)set of R-modules to a set of geometric objects so that the size of the (asymptotic) ext group between two modules M and N can be measured by the intersections of f(M) and f(N).*

The homological mirror symmetry (HMS), proposed by [Kontsevich 95], is one of the phenomena which answer this problem. The existence of such symmetry implies that there is a symplectic manifold S such that the number of intersections between two Lagrangians on S is closely related to the dimension of the ext group between the corresponding modules.

Pursuing this direction, in this article, we restrict ourselves to the following problem.

Problem 2. *Let R be an hereditary algebra. Find a function f from the set of indecomposable R-modules to a set of geometric objects so that the non-vanishing of the self-extension group of an indecomposable module M is precisely detected by the existence of the self-intersection of f(M).*

Every finite-dimensional hereditary algebra over an algebraically closed field is Morita equivalent to the path algebra of an acyclic quiver, i.e., a quiver without oriented cycles (See, e.g., [Assem et al. 06]). The number of vertices of a quiver is referred to as the rank of the quiver. The dimension vectors of indecomposable modules over a path algebra are called (positive) roots. A root α is real if the Euler inner product $\langle \alpha, \alpha \rangle$ is equal to 1, and imaginary if $\langle \alpha, \alpha \rangle \leq 0$.

We first consider the case that R is the path algebra of a two-complete quiver (i.e.,, an acyclic quiver with at least two arrows between every pair of vertices), and define a bijective function

$$f : \{\text{indecomposable modules corresponding to positive real roots}\} \to \{\text{admissible curves}\},$$

where admissible curves are certain paths on a Riemann surface (see Definition 2.1). Then we formulate the following conjecture:

Conjecture 1.1. *For an indecomposable R-module M, we have $Ext^1(M, M) = 0$ if and only if f(M) has no self-intersections.*

In this article, we prove this conjecture for two-complete rank 3 quivers. When $Ext^1(M, M) = 0$, the module M is called rigid, and the dimension vector of a rigid indecomposable module is called a real Schur root. To explain our result, we let

$$\mathcal{Z} := \left\{ (a, b, c) \in \mathbb{Z}^3 : \ \gcd(|b|, |c|) = 1 \right\}.$$

For each $z = (a, b, c) \in \mathcal{Z}$, define a curve η_z on the universal cover of a triangulated torus, consisting of

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two symmetric spirals and a line segment, so that a determines the number of times the spirals revolve and (b, c) determines the slope of the line segment. See Examples 2.2 (2). The curves η_z, $z \in \mathcal{Z}$, have no self-intersections. Now our result (Theorem 4.2) is the following.

Theorem 1.2. *Let R be the path algebra of a two-complete rank 3 quiver. Then there is a natural bijection between the set of rigid indecomposable modules and the sef of curves η_z, $z \in \mathcal{Z}$.*

This shows that real Schur roots are very special ones among all real roots in general. Our proof is achieved by expressing each real Schur root in terms of a sequence of simple reflections that corresponds to a non-self-intersecting path.[1] See Example 4.13.

For the general case, let R be the path algebra of any acyclic quiver. We still define an onto function

$$g : \{\text{admissible curves}\} \to \{\text{indecomposable modules corresponding to positive real roots}\},$$

and propose the following (See Conjecture 2.4):

Conjecture 1.3. *For an indecomposable R-module M, we have $\mathrm{Ext}^1(M,M) = 0$ if and only if $g^{-1}(M)$ contains a non-self-crossing curve.*

As tests for known cases, we prove this conjecture for equioriented quivers of types A and D, and for $A_2^{(1)}$ and all rank 2 quivers. We also consider the highest root of a quiver of type E_8 and provide such a path. If this conjecture holds true, then it gives an elementary geometric (and less recursive) criterion to distinguish real Schur roots among all positive real roots.

There have been a number of known criteria to tell whether a given real root is a real Schur root, some of which are in terms of sub-representations (due to Schofield [Schofield 92]), braid group actions (due to Crawley [Crawley-Boevey 92]), cluster variables (due to Caldero [Caldero and Keller 06]), or c-vectors (due to Chavez [Chavez 15]). Building on a result of Igusa–Schiffler [Igusa and Schiffler 10] and Baumeister–Dyer–Stump–Wegener [Baumeister et al. 14], Hubery and Krause [Hubery and Krause 16] characterized real Schur roots in terms of non-crossing partitions. There are also combinatorial descriptions for c-vectors in the same seed due to Speyer–Thomas [Speyer and Thomas 13] and Seven [Seven 15]. However none of these is of geometric nature, and most of them rely on heavy recursive procedures which are hard to apply in practice.

Also, a better description for real Schur roots is still needed to help understand a base step of the non-commutative HMS for path algebras. Note that the recent work of Shende–Treumann–Williams–Zaslow [Shende 15; Shende et al. 16; Treumann 18] suggests HMS for certain (not-necessarily acyclic) quivers including the ones coming from bicolored graphs on surfaces.

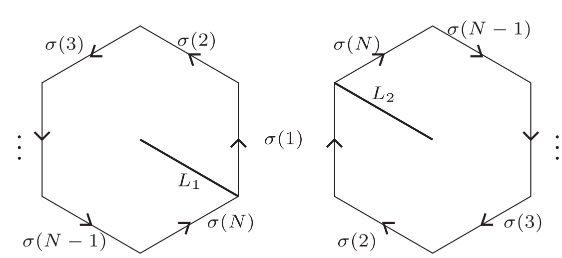
Our conjecture suggests the existence of the HMS phenomenon for the path algebra over an arbitrary acyclic quiver. In a subsequent project, we plan to investigate the HMS for path algebras over various quivers.

2. A conjectural correspondence

2.1. The statement of conjecture

Let $\mathcal{Q}$ be an acyclic (connected) quiver with N vertices labeled by $I := \{1, ..., N\}$. Denote by S_N the permutation group on I. Let $P_\mathcal{Q} \subset S_N$ be the set of all permutations σ such that there is no arrow from $\sigma(j)$ to $\sigma(i)$ for any $j > i$ on $\mathcal{Q}$. Note that if there exists an oriented path passing through all N vertices on $\mathcal{Q}$, in particular, if there is at least one arrow between every pair of vertices, then $P_\mathcal{Q}$ consists of a unique permutation.

For each $\sigma \in P_\mathcal{Q}$, we define a labeled Riemann surface Σ_σ[2] as follows. Let G_1 and G_2 be two identical copies of a regular N-gon. Label the edges of each of the two N-gons by $T_{\sigma(1)}, T_{\sigma(2)}, ..., T_{\sigma(N)}$ counter-clockwise. On G_i, let L_i be the line segment from the center of G_i to the common endpoint of $T_{\sigma(N)}$ and $T_{\sigma(1)}$. Fix the orientation of every edge of G_1 (resp. G_2) to be counter-clockwise (resp. clockwise) as in the following picture.



[1]After the first version of this paper was posted on the arXiv, Felikson and Tumarkin [Felikson and Tumarkin 17] proved Conjecture 1.1 for all 2-complete quivers. Moreover they characterized c-vectors in the same seed, using a collection of pairwise non-crossing admissible curves satisfying a certain word property.

[2]The punctured discs appeared in Bessis' work [Bessis 06]. For better visualization, here we prefer to use an alternative description using compact Riemann surfaces with one or two marked points.

Let Σ_σ be the (compact) Riemann surface of genus $\lfloor\frac{N-1}{2}\rfloor$ obtained by gluing together the two N-gons with all the edges of the same label identified according to their orientations. The edges of the N-gons become N different curves in Σ_σ. If N is odd, all the vertices of the two N-gons are identified to become one point in Σ_σ and the curves obtained from the edges are loops. If N is even, two distinct vertices are shared by all curves. Let $\mathcal{T}$ be the set of all curves, i.e., $\mathcal{T} = T_1 \cup \cdots \cup T_N \subset \Sigma_\sigma$, and V be the set of the vertex (or vertices) on $\mathcal{T}$.

Consider the Cartan matrix corresponding to $\mathcal{Q}$ and the root system of the associated Kac–Moody algebra. The simple root corresponding to $i \in I$ will be denoted by α_i. The simple reflections of the Weyl group W will be denoted by s_i, $i \in I$. More precisely, for each $\sigma \in P_{\mathcal{Q}}$,

$$W = \langle s_1, \ldots, s_N : s_1^2 = \cdots = s_N^2 = e \text{ and } (s_is_j)^{m_{ij}} = e \text{ for } i<j \rangle,$$

where e is the identity element and

$$m_{ij} = \begin{cases} 2, & \text{if there is no arrow from } \sigma(i) \text{ to } \sigma(j) \text{ on } \mathcal{Q}; \\ 3, & \text{if there is a single arrow from } \sigma(i) \text{ to } \sigma(j) \text{ on } \mathcal{Q}; \\ \infty, & \text{otherwise.} \end{cases}$$

Let $\mathfrak{W}$ be the set of words $\mathfrak{w} = i_1 i_2 \cdots i_k$ from the alphabet I such that no two consecutive letters i_p and i_{p+1} are the same. For each element $w \in W$, let $R_w \subset \mathfrak{W}$ be the set of words $i_1 i_2 \cdots i_k$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_k}$. Recall that the set of positive real roots and the set of reflections in W are in one-to-one correspondence. Also note that if $\mathcal{Q}$ is two-complete, i.e., $W = \langle s_1, \ldots, s_N : s_1^2 = \cdots = s_N^2 = e \rangle$, then there is a unique expression for every element $w \in W$ as a product of simple reflections (with no two consecutive simple reflections being the same), hence R_w contains a unique element. Define

$$R := \bigcup_{\substack{w \in W \\ w:\ \text{reflection}}} R_w \subset \mathfrak{W}.$$

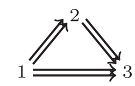
Definition 2.1. Let $\sigma \in P_{\mathcal{Q}}$. A *$\sigma$-admissible* curve is a continuous function $\eta : [0,1] \to \Sigma_\sigma$ such that

1. $\eta(x) \in V$ if and only if $x \in \{0,1\}$;
2. there exists $\epsilon>0$ such that $\eta([0,\epsilon]) \subset L_1$ and $\eta([1-\epsilon,1]) \subset L_2$;
3. if $\eta(x) \in \mathcal{T} \setminus V$ then $\eta([x-\epsilon, x+\epsilon])$ meets $\mathcal{T}$ transversally for sufficiently small $\epsilon>0$;
4. and $\upsilon(\eta) \in R$, where $\upsilon(\eta) := i_1 \cdots i_k$ is given by
$$\{x \in (0,1) : \eta(x) \in \mathcal{T}\} = \{x_1 < \cdots < x_k\}$$
$$\text{and} \quad \eta(x_\ell) \in T_{i_\ell} \text{ for } \ell \in \{1, \ldots, k\}.$$

If σ is clear from the context, a σ-admissible curve will be just called an admissible curve.

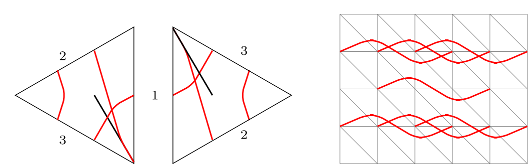
Note that for every $\mathfrak{w} \in R$, there is a σ-admissible curve η with $\upsilon(\eta) = \mathfrak{w}$. In particular, every positive real root can be represented by some admissible curve(s).

Example 2.2. Let $N=3$, and $\mathcal{Q}$ be the rank 3 acyclic quiver with double arrows between every pair of vertices as follows.



Let $\sigma \in S_3$ be the trivial permutation *id*. We have $P_{\mathcal{Q}} = \{id\}$.

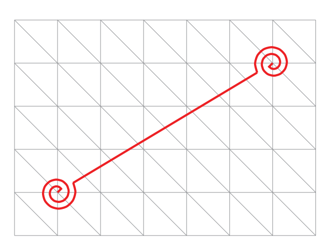
(1) First we consider a positive real root $\alpha_1 + 6\alpha_2 + 2\alpha_3 = s_2s_3\alpha_1$ and its corresponding reflection $w = s_2s_3s_1s_3s_2$. Then $R_w = \{23132\} \subset R$, and the following red curve becomes a σ-admissible curve η on Σ_σ, with $\upsilon(\eta) = 23132$. The picture on the right shows several copies of η on the universal cover of Σ_σ, where each horizontal line segment represents T_1, vertical T_3, and diagonal T_2. Clearly η has a self-intersection.



(2) Next we consider another positive real root

$$1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3$$
$$= (s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3\alpha_2, \qquad (2\text{–}1)$$

and its corresponding reflection $w = (s_3s_2s_1)^4 s_2s_3s_2s_1 s_2s_3s_2s_3s_2s_1s_2s_3s_2(s_1s_2s_3)^4$. Below we draw a copy of σ-admissible curve η with $\upsilon(\eta) = (321)^4 2321232\, 321232(123)^4$ on the universal cover of Σ_σ. Here η has no self-intersection.



Conjecture 2.4. *Let Γ_σ be the set of (isotopy classes of) σ-admissible curves η such that η has no self-intersections, i.e., $\eta(x_1) = \eta(x_2)$ implies $x_1 = x_2$ or $\{x_1, x_2\} = \{0, 1\}$. For each $\eta \in \Gamma_\sigma$, let $w \in W$ be the reflection such that $\upsilon(\eta) \in R_w$, and let $\beta(\eta)$ be the positive real root corresponding to w. Then $\{\beta(\eta) : \eta \in \cup_{\sigma \in P_{\mathcal{Q}}} \Gamma_\sigma\}$ is precisely the set of real Schur roots for $\mathcal{Q}$.*

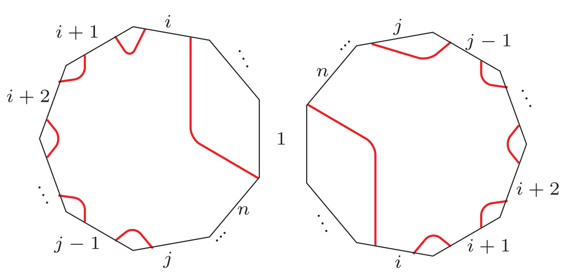
Remark 2.5. *The correspondence $\eta \mapsto \beta(\eta)$ is not one-to-one in general. If there are at least two arrows between every pair of vertices on $\mathcal{Q}$, then the conjecture predicts that it would be a bijection.*

2.2 Type A quivers

In this subsection we prove Conjecture 2.4 for equioriented quivers of type A. Let $\mathcal{Q}$ be the following quiver:

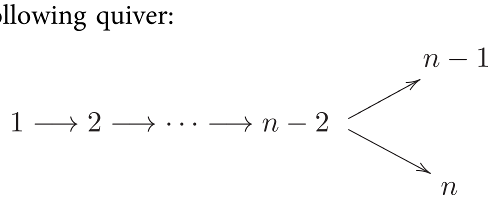
$$1 \to 2 \to \cdots \to n$$

Since all positive real roots are Schur, it is enough to show that every positive real root can be realized as $\beta(\eta)$ for some $\eta \in \Gamma_{id}$. Each positive real root is equal to $s_i s_{i+1} \cdots s_{j-1} \alpha_j$ for some $i \le j \in \{1, \ldots, n\}$, and the corresponding reflection is $w = s_i \cdots s_{j-1} s_j s_{j-1} \cdots s_i$. There exists an admissible curve η with no self-intersections and $\upsilon(\eta) = i \cdots (j-1) j (j-1) \cdots i \in R_w$ as depicted in the following picture, so we are done.



2.3. Type D quivers

In this subsection, we prove Conjecture 2.4 for the following quiver:



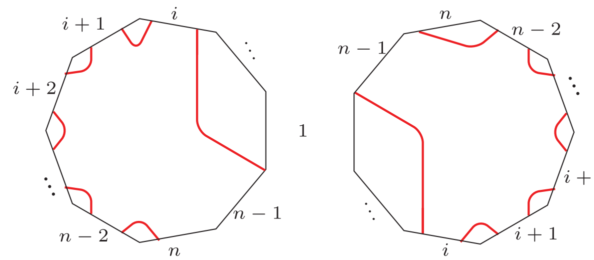
For the corresponding root system of type D_n, all positive real roots are Schur. Each positive real root is equal to one of the following:

$$s_i s_{i+1} \cdots s_{j-1} \alpha_j, \quad 1 \le i \le j \le n-1, \tag{2–2}$$

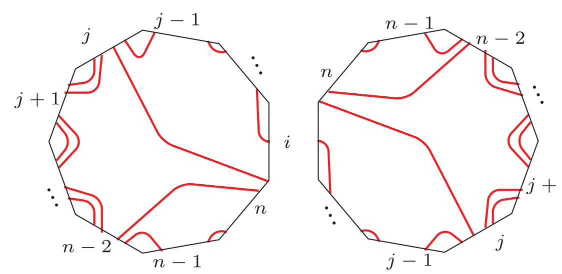
$$s_i s_{i+1} \cdots s_{n-2} \alpha_n, \quad 1 \le i \le n-1, \tag{2–3}$$

$$(s_j s_{j+1} \cdots s_{n-2})(s_n s_{n-1} s_{n-2} \cdots s_{i+1})\alpha_i, \quad 1 \le i < j \le n-1. \tag{2–4}$$

The roots in (2–2) are of the same form as in type A. For each of the corresponding reflections to the roots in (2–3), there exists an admissible curve η with no self-intersections on Σ_σ, where σ is either the permutation interchanging only $n-1$ and n (see the picture below) or the trivial permutation id.

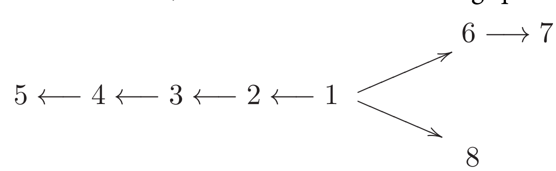


For each of the corresponding reflections to the roots in (2–4), there exists an admissible curve η with no self-intersections on Σ_{id}. Such curve is given below, where we omit drawing the edges $1, \ldots, i-1$ with which η does not intersect.



2.4. A quiver of type E_8

In this subsection, we consider the following quiver:



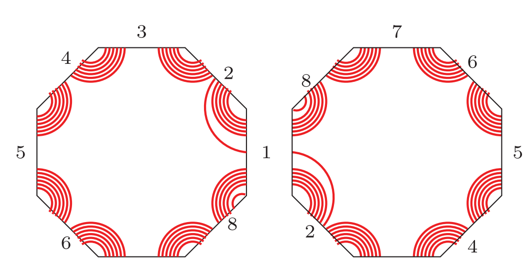
Again every positive real root is Schur. The highest positive real root

$$6\alpha_1 + 5\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

can be given by $(s_8 s_7 \cdots s_2 s_1)^5 (s_8 s_7 \cdots s_2)\alpha_1$, and one of non-reduced expressions for the corresponding reflection is

$$(s_8 s_7 \cdots s_2 s_1)^5 (s_8 s_7 \cdots s_2) s_1 (s_2 \cdots s_7 s_8)(s_1 s_2 \cdots s_7 s_8)^5,$$

which gives rise to the following non-self-intersecting curve on Σ_{id}.



On the actual Riemann surface Σ_{id}, this is just a spiral around one vertex followed by another spiral around the other vertex.

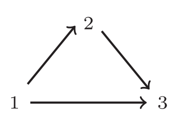
Remark 2.9. *It is not easy to give a unified proof of Conjecture 2.4 for all quivers of type ADE with arbitrary orietations.*

2.5. Rank 2 quivers

In this subsection, we prove Conjecture 2.4 for rank 2 quivers. Again in this case, all positive real roots are Schur. The reflection corresponding to each positive real root is of the form $s_is_js_i\cdots s_is_js_i$ with $\{i,j\}=\{1,2\}$. Clearly, there exists a spiral η (as an admissible curve) on the sphere Σ_σ such that η has no self-intersections and $\upsilon(\eta)=iji\cdots iji$, where σ is either id or the transposition (12).

2.6. Rank 3 tame quiver

Let Q be the rank 3 acyclic quiver of type $A_2^{(1)}$ as follows:



We have $P_Q=\{id\}$. The positive real Schur roots are, for $n\geq 0$,

$$(n+1)\alpha_1+n\alpha_2+n\alpha_3=\begin{cases}(s_1s_2s_3s_2)^{\frac{n}{2}}\alpha_1 & \text{if } n \text{ is even,}\\ (s_1s_2s_3s_2)^{\frac{n-1}{2}}s_1s_2\alpha_3 & \text{if } n \text{ is odd,}\end{cases} \tag{2–5}$$

$$(n+1)\alpha_1+(n+1)\alpha_2+n\alpha_3=(s_1s_2s_3s_2)^n s_1\alpha_2, \tag{2–6}$$

$$n\alpha_1+(n+1)\alpha_2+(n+1)\alpha_3 =\begin{cases}(s_2s_3s_2s_1)^{\frac{n}{2}}s_2\alpha_3 & \text{if } n \text{ is even,}\\ (s_2s_3s_2s_1)^{\frac{n-1}{2}}s_2s_3s_2\alpha_1 & \text{if } n \text{ is odd,}\end{cases} \tag{2–7}$$

$$n\alpha_1+n\alpha_2+(n+1)\alpha_3=(s_2s_3s_2s_1)^n s_2s_3\alpha_2, \tag{2–8}$$

$$\alpha_2 \quad \text{and} \quad \alpha_1+\alpha_3=s_1\alpha_3. \tag{2–9}$$

One can see that admissible curves corresponding to (2–5)–(2–8) are essentially determined by line segments with slopes $\frac{n}{n-1},\frac{n+2}{n},\frac{n+1}{n+2},\frac{n+1}{n+3}$, respectively, on the universal cover of the torus with triangulation as in Example 2.2. Curves corresponding to (2–7) and (2–8) are (isotopic to) line segments. When $n\geq 1$, curves corresponding to (2–5) and (2–6) revolve 180° around a vertex at the beginning, follow a line segment, and again revolve 180° around a vertex at the end. Clearly, such curves do not have self-intersections. The Schur roots α_2 and $\alpha_1+\alpha_3$ trivially correspond to non-self-intersecting curves.

Conversely, an admissible curve with no self-intersections on a torus becomes a curve on the universal cover isotopic to a union of two spirals around vertices, which are symmetric to each other, and a line segment in the middle of the two spirals. It can be checked that each of such curves gives rise to one of the real Schur roots listed in (2–5)–(2–9). Thus Conjecture 2.4 is verified in this case.

3. Preliminaries

3.1. Cluster variables

In this subsection, we review some notions from the theory of cluster algebras introduced by Fomin and Zelevinsky in [Fomin and Zelevinsky 02]. Our definition follows the exposition in [Fomin and Zelevinsky 07]. For our purpose, it is enough to define the coefficient-free cluster algebras of rank 3.

We consider a field $\mathcal{F}$ isomorphic to the field of rational functions in three independent variables.

Definition 3.1. A *labeled seed* in $\mathcal{F}$ is a pair $(\mathbf{x},B)$, where

- $\mathbf{x}=(x_1,x_2,x_3)$ is a triple from $\mathcal{F}$ forming a *free generating set* over $\mathbb{Q}$, and
- $B=(b_{ij})$ is a 3×3 integer *skew-symmetric* matrix.

That is, x_1,x_2,x_3 are algebraically independent over $\mathbb{Q}$, and $\mathcal{F}=\mathbb{Q}(x_1,x_2,x_3)$. We refer to $\mathbf{x}$ as the (labeled) *cluster* of a labeled seed $(\mathbf{x},B)$, and to the matrix B as the *exchange matrix*.

We use the notation$[x]_+=\max(x,0)$ and

$$\operatorname{sgn}(x)=\begin{cases}x/|x| & \text{if } x\neq 0;\\ 0 & \text{if } x=0.\end{cases}$$

Definition 3.2. Let $(\mathbf{x},B)$ be a labeled seed in $\mathcal{F}$, and let $k\in\{1,2,3\}$. The *seed mutation* μ_k in direction k transforms $(\mathbf{x},B)$ into the labeled seed $\mu_k(\mathbf{x},B)=(\mathbf{x}',B')$ defined as follows:

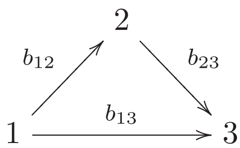
- The entries of $B'=(b'_{ij})$ are given by

$$b'_{ij}=\begin{cases}-b_{ij} & \text{if } i=k \text{ or } j=k;\\ b_{ij}+\operatorname{sgn}(b_{ik})\left[b_{ik}b_{kj}\right]_+ & \text{otherwise.}\end{cases} \tag{3–1}$$

- The cluster $\mathbf{x}'=(x'_1,x'_2,x'_3)$ is given by $x'_j=x_j$ for $j\neq k$, whereas $x'_k\in\mathcal{F}$ is determined by the *exchange relation*

$$x'_k=\frac{\prod x_i^{[b_{ik}]_+}+\prod x_i^{[-b_{ik}]_+}}{x_k}. \tag{3–2}$$

Let $B = (b_{ij})$ be a 3×3 skew-symmetric matrix, and $\mathcal{Q}$ be the rank 3 acyclic quiver corresponding to B, with the set $I = \{1, 2, 3\}$ of vertices, such that the quiver $\mathcal{Q}$ has b_{ij} arrows from i to j for $b_{ij} > 0$. We will write $i \xrightarrow{b_{ij}} j$ to represent the b_{ij} arrows. Assume that $|b_{ij}| \geq 2$ for $i \neq j$. Without loss of generality, we further assume that the vertex 1 of $\mathcal{Q}$ is a source, the vertex 2 a node, and the vertex 3 a sink:



The matrix resulting from the mutation of B at the vertex $i \in I$, will be denoted by $B(i)$, and the matrix resulting from the mutation of $B(i)$ at the vertex $j \in I$ by $B(ij)$. Then the matrix $B(i_1 i_2 \cdots i_k), i_p \in I$ is the result of k mutations. We will write $B(\mathfrak{w}) = (b_{ij}(\mathfrak{w}))$ for $\mathfrak{w} = i_1 \cdots i_k \in \mathfrak{W}$. When $\mu_{\mathfrak{w}}$ is a sequence $(\mu_{i_1}, \ldots, \mu_{i_k})$ of mutations performed from left to right, we write $\Xi(\mathfrak{w}) = \mu_{\mathfrak{w}}(\Xi) = \mu_{i_k} \circ \cdots \circ \mu_{i_1}(\Xi)$ for a labeled seed Ξ.

The *cluster variables* are the elements of clusters obtained by sequences of seed mutations from the initial seed $((x_1, x_2, x_3), B)$. The remarkable *Laurent phenomenon* [Fomin and Zelevinsky 02] and *Positivity* theorem [Davison 18; Davison et al. 15; Gross et al. 18; Lee and Schiffler 13; Lee and Schiffler 15] imply the following:

Theorem 3.5. *Each cluster variable is a Laurent polynomial over $\mathbb{Z}_{\geq 0}$ in the initial cluster variables x_1, x_2, x_3.*

Thanks to the Laurent Phenomenon, the denominator of every cluster variable is well defined when expressed in reduced form.

Example 3.6. Let $\Xi = \left((x_1, x_2, x_3), \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \right)$ be the initial seed. The mutation in direction three yields

$$\Xi(3) = \left(\left(x_1, x_2, \frac{x_1^2 x_2^2 + 1}{x_3} \right), \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & -2 \\ 2 & 2 & 0 \end{pmatrix} \right).$$

Applying the mutation in direction 2 to $\Xi(3)$, we obtain

$$\Xi(32) = \left(\left(x_1, \frac{x_1^2 (1 + x_1^2 x_2^2)^2 + x_3^2}{x_2 x_3^2}, \frac{x_1^2 x_2^2 + 1}{x_3} \right), \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix} \right).$$

More mutations produce

$$\Xi((321)^4) = \left(\left(\frac{P_1}{x_1^{4895} x_2^{12816} x_3^{33552}}, \frac{P_2}{x_1^{1870} x_2^{4895} x_3^{12816}}, \frac{P_3}{x_1^{714} x_2^{1870} x_3^{4895}} \right), \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \right),$$

where the corresponding quiver is acyclic. We apply the mutation in direction 2 to obtain

$$\Xi((321)^4 2) = \left(\left(\frac{P_1}{x_1^{4895} x_2^{12816} x_3^{33552}}, \frac{P_4}{x_1^{7920} x_2^{20737} x_3^{54288}}, \frac{P_3}{x_1^{714} x_2^{1870} x_3^{4895}} \right), \begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -2 \\ -6 & 2 & 0 \end{pmatrix} \right),$$

where the corresponding quiver is cyclic. We calculate three more mutations:

$$\Xi((321)^4 23) = \left(\left(\frac{P_1}{x_1^{4895} x_2^{12816} x_3^{33552}}, \frac{P_4}{x_1^{7920} x_2^{20737} x_3^{54288}}, \frac{P_5}{x_1^{28656} x_2^{75026} x_3^{196417}} \right), \begin{pmatrix} 0 & 10 & -6 \\ -10 & 0 & 2 \\ 6 & -2 & 0 \end{pmatrix} \right),$$

$$\Xi((321)^4 231) = \left(\left(\frac{P_6}{x_1^{167041} x_2^{437340} x_3^{1144950}}, \frac{P_4}{x_1^{7920} x_2^{20737} x_3^{54288}}, \frac{P_5}{x_1^{28656} x_2^{75026} x_3^{196417}} \right), \begin{pmatrix} 0 & -10 & 6 \\ 10 & 0 & -58 \\ -6 & 58 & 0 \end{pmatrix} \right),$$

$$\Xi((321)^4 2312) = \left(\left(\frac{P_6}{x_1^{167041} x_2^{437340} x_3^{1144950}}, \frac{P_7}{x_1^{1662490} x_2^{4352663} x_3^{11395212}}, \frac{P_5}{x_1^{28656} x_2^{75026} x_3^{196417}} \right), \begin{pmatrix} 0 & 10 & -574 \\ -10 & 0 & 58 \\ 574 & -58 & 0 \end{pmatrix} \right).$$

Here all P_i are polynomials in x_1, x_2, x_3 with no monomial factors. Compare (2–1) with the new cluster variable $P_7 / x_1^{1662490} x_2^{4352663} x_3^{11395212}$ in $\Xi((321)^4 2312)$.

3.2. Positive real roots

Let $A = (a_{ij})$ be the symmetric Cartan matrix corresponding to B and $\mathfrak{g}$ be the associated Kac–Moody algebra. The simple roots of $\mathfrak{g}$ will be denoted by α_i,

$i \in I$. Let $Q_+ = \oplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$ be the positive root lattice. We have the canonical bilinear form $(\cdot,\cdot)$ on Q_+ defined by $(\alpha_i, \alpha_j) = a_{ij}$ for $i, j \in I$. The simple reflections will be denoted by s_i, $i \in I$ and the Weyl group by W. As before, let $\mathfrak{W}$ be the set of words $\mathfrak{w} = i_1 i_2 \cdots i_k$ in the alphabet I such that no two consecutive letters i_p and i_{p+1} are the same. Since $W = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = e\rangle$, we regard $\mathfrak{W}$ as the set of reduced expressions of the elements of W. Assume that $\mathfrak{W}$ also has the empty word $\emptyset$.

For $\mathfrak{w} = i_1 \cdots i_k \in \mathfrak{W}$, define

$$s_{\mathfrak{w}} := s_{i_1} \cdots s_{i_k} \in W.$$

A root $\beta \in Q_+$ is called *real* if $\beta = w\alpha_i$ for some $w \in W$ and α_i, $i \in I$. For a positive real root β, the corresponding reflection will be denoted by $r_\beta \in W$.

4. Real Schur roots of rank 3 quivers

In this section we prove Conjecture 2.4 for rank 3 quivers with multiple arrows between every pair of vertices. We describe the set of real roots by the isotopy classes of certain curves on the universal cover of a triangulated torus and characterize the curves corresponding to real Schur roots.

4.1. Curves representing real roots

For easier visualization, we restate the set-up from Section 2.1 in terms of the universal cover. Consider the following set of lines on $\mathbb{R}^2$:

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3,$$

where $\mathcal{T}_1 = \{(x,y) : y \in \mathbb{Z}\}, \mathcal{T}_2 = \{(x,y) : x+y \in \mathbb{Z}\}$, and $\mathcal{T}_3 = \{(x,y) : x \in \mathbb{Z}\}$. Together with $\mathcal{T}$, the space $\mathbb{R}^2$ can be viewed as the universal cover of a triangulated torus.

We also define

$$\mathcal{L}_1 = \left\{(x,y) \ : \ x-y \in \mathbb{Z}, \quad x-\lfloor x\rfloor < \frac{1}{2}\right\}$$

$$\text{and} \quad \mathcal{L}_2 = \left\{(x,y) \ : \ x-y \in \mathbb{Z}, \quad x-\lfloor x\rfloor > \frac{1}{2}\right\}.$$

Definition 4.1. An *admissible* curve is a continuous function $\eta : [0,1] \to \mathbb{R}^2$ such that

1. $\eta(x) \in \mathbb{Z}^2$ if and only if $x \in \{0,1\}$;
2. there exists $\epsilon > 0$ such that $\eta([0,\epsilon]) \subset \mathcal{L}_1$ and $\eta([1-\epsilon, 1)) \subset \mathcal{L}_2$;
3. if $\eta(x) \in \mathcal{T} \setminus \mathbb{Z}^2$ then $\eta([x-\epsilon, x+\epsilon])$ meets $\mathcal{T}$ transversally for sufficiently small $\epsilon > 0$;
4. and $\upsilon(\eta) \in R$, where $\upsilon(\eta) := i_1 \cdots i_k$ is given by

$$\{x \in (0,1) : \eta(x) \in \mathcal{T}\} = \{x_1 < \cdots < x_k\}$$
$$\text{and} \quad \eta(x_\ell) \in \mathcal{T}_{i_\ell} \ \text{for} \ \ell \in \{1, \dots, k\}.$$

4.2. Curves representing real Schur roots

Let

$$\mathcal{Z} := \{(a,b,c) \in \mathbb{Z}^3 : \ \gcd(|b|, |c|) = 1\},$$

where $\gcd(0,0) = \infty$ and $\gcd(x, 0) = x$ for nonzero x. Fix $z = (a,b,c) \in \mathcal{Z}$ and let

$$\epsilon = \begin{cases} 1/2, & \text{if } \max(|b|,|c|) = 1; \\ 1/2\sqrt{b^2+c^2}, & \text{otherwise.} \end{cases}$$

Let $C_{z,1} \subset \mathbb{R}^2$ be the spiral that (i) crosses the positive x-axis $|a|$ times; (ii) starts with the line segment from (0, 0) to $(\epsilon/2, \epsilon/2)$, goes around (0, 0), and ends at $(\epsilon b, \epsilon c)$; and (iii) revolves clockwise if $a > 0$ (resp. counterclockwise if $a < 0$). Let $C_{z,2}$ be the line segment from $(\epsilon b, \epsilon c)$ to $(b-\epsilon b, c-\epsilon c)$, and $C_{z,3}$ be the spiral obtained by rotating $C_{z,1}$ by 180° around $(b/2, c/2)$. Let η_z be the union of $C_{z,1}, C_{z,2}$, and $C_{z,3}$. We are ready to state our main theorem as follows.

Let Γ be the set of (isotopy classes of) admissible curves η such that η has no self-intersections on the torus, i.e., $\eta(x_1) = \eta(x_2) (\text{mod } \mathbb{Z} \times \mathbb{Z})$ implies $x_1 = x_2$ or $\{x_1, x_2\} = \{0, 1\}$. It is not hard to see that $\Gamma = \{\eta_z : z \in \mathcal{Z}\}$ by using Dehn twists. Recall that $\beta(\eta)$ is defined in Conjecture 2.4 for $\eta \in \Gamma$.

Theorem 4.2. *The set $\{\beta(\eta_z) \ : \ z \in \mathcal{Z}\}$ is precisely the set of real Schur roots for Q.*

Clearly, the above theorem implies that Conjecture 2.4 holds for rank 3 quivers with multiple arrows between every pair of vertices. We will prove this theorem after we state Theorem 4.17 below in Section 4.4.

4.3. Mutations of vectors and the definition of $\psi(\mathfrak{w})$

We define the triple $V(\mathfrak{w})$ of vectors on $\mathbb{R}^2$ for $\mathfrak{w} \in \mathfrak{W} \setminus \{\emptyset\}$ as follows. First, we define

$$V(1) = (\langle -1, 2\rangle, \langle -1, 1\rangle, \langle 0, 1\rangle),$$
$$V(2) = (\langle 0, 1\rangle, \langle 1, 1\rangle, \langle 1, 0\rangle),$$
$$V(3) = (\langle 1, 0\rangle, \langle 1, -1\rangle, \langle 2, -1\rangle).$$

Then, we inductively define $V(i_1 \cdots i_q)$ for $q > 1$. Suppose that $V(i_1 \cdots i_{q-1}) = (\vec{v_1}, \vec{v_2}, \vec{v_3})$. Then $V(i_1 \cdots i_q)$ is defined by

$$V(i_1\cdots i_q) := \begin{cases} (\vec{v_2}+\vec{v_3}, \vec{v_2}, \vec{v_3}), & \text{if } i_q = 1; \\ (\vec{v_1}, \vec{v_1}+\vec{v_3}, \vec{v_3}), & \text{if } i_q = 2; \\ (\vec{v_1}, \vec{v_2}, \vec{v_1}+\vec{v_2}), & \text{if } i_q = 3. \end{cases} \quad (4\text{–}1)$$

Write $V(\mathfrak{w}) = (\vec{v_1}(\mathfrak{w}), \vec{v_2}(\mathfrak{w}), \vec{v_3}(\mathfrak{w}))$. Let $p \in \{1,2,3\}$ and suppose that $\vec{v_p}(\mathfrak{w}) = \langle b, c\rangle$. Let ξ be (the isotopic class of) the line segment from (0, 0) to (b, c). One can see that $\gcd(|b|,|c|) = 1$ and so $\xi \in \Gamma$. We write

$$\beta(\vec{v_p}(\mathfrak{w})) = \beta(\xi) \text{ and } \upsilon(\vec{v_p}(\mathfrak{w})) = \upsilon(\xi) \in R. \quad (4\text{–}2)$$

For each $\mathfrak{w} \in \mathfrak{W} \setminus \{\emptyset\}$, we define a positive real root $\phi(\mathfrak{w})$ by

$$\phi(\mathfrak{w}) := \beta(\vec{v_p}(\mathfrak{w})),$$

where p is the last letter of $\mathfrak{w} \in \mathfrak{W} \setminus \{\emptyset\}$.

Lemma 4.5. *For $\{i,j,k\} = \{1,2,3\}$, we have*

$$\phi((ij)^n) = (s_i s_j)^{n-1} s_i \alpha_j \quad \text{and} \quad \phi((ij)^n i) = (s_i s_j)^n \alpha_i; \quad (4\text{–}3)$$

$$\phi(i(ji)^n k) = s_i (s_j s_i)^{2n} \alpha_k \text{ and } \phi(i(ji)^n jk) = s_i (s_j s_i)^{2n+1} \alpha_k; \quad (4\text{–}4)$$

$$\phi(i(ji)^n ki) = s_i (s_j s_i)^{2n} s_k (s_i s_j)^n \alpha_i, \quad (4\text{–}5)$$

$$\phi(i(ji)^n kj) = s_i (s_j s_i)^{2n} s_k (s_i s_j)^n s_i \alpha_j, \quad (4\text{–}6)$$

$$\phi(i(ji)^n jki) = s_i (s_j s_i)^{2n+1} s_k (s_i s_j)^{n+1} \alpha_i, \quad (4\text{–}7)$$

$$\phi(i(ji)^n jkj) = s_i (s_j s_i)^{2n+1} s_k (s_i s_j)^n s_i \alpha_j. \quad (4\text{–}8)$$

Proof. It is straightforward to check these identities directly, so we omit the proof. Instead we illustrate the calculation for $\mathfrak{w} = 121212$.

By (4–1), we get

$$\begin{aligned} V(1) &= (\langle -1,2\rangle, \langle -1,1\rangle, \langle 0,1\rangle), \\ V(12) &= (\langle -1,2\rangle, \langle -1,3\rangle, \langle 0,1\rangle), \\ &\vdots \\ V(121212) &= (\langle -1,6\rangle, \langle -1,7\rangle, \langle 0,1\rangle). \end{aligned}$$

Let ξ be the line segment from (0, 0) to $(-1, 7)$. Then $\upsilon(\xi) = (12)^5 1$, hence the corresponding real root is

$$\phi(121212) = \beta(\xi) = \beta(\langle -1,7\rangle) = s_1 s_2 s_1 s_2 s_1 \alpha_2.$$

Let $\mathcal{C}_1 = \{1, 12, 123, 1231, 12312, 123123, \dots\}$ and $\mathcal{C}_3 = \{3, 32, 321, 3213, 32132, 321321, \dots\} \subset \mathfrak{W}$. Note that the quiver corresponding to $B(\mathfrak{w})$ is acyclic if and only if $\mathfrak{w} \in \mathcal{C}_1 \cup \mathcal{C}_3 \cup \{\emptyset\}$. For $\mathfrak{w} = i_1 \cdots i_k \in \mathfrak{W}$, let $\ell(\mathfrak{w}) = k$ and

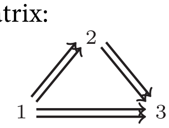
$$\rho(\mathfrak{w}) := \begin{cases} 0, & \text{if } \mathfrak{w} \text{ is the empty word } \emptyset, \text{ or } i_1 = 2; \\ \max\{p \ : \ i_1 \cdots i_p \in \mathcal{C}_1 \cup \mathcal{C}_3\}, & \text{otherwise.} \end{cases}$$

The following definition is important for the rest of the article.

Definition 4.12. Let $\hat{\mathfrak{w}} \in \mathfrak{W} \setminus \{\emptyset\}$, and write $\hat{\mathfrak{w}} = \mathfrak{w}\mathfrak{v} \in \mathfrak{W}$ with the word $\mathfrak{w}$ being the longest word such that $B(\mathfrak{w})$ is acyclic. Assume that $\mathfrak{w} = i_1 \dots i_k$. Then, we have $k = \rho(\hat{\mathfrak{w}})$. Define a positive real root $\psi(\hat{\mathfrak{w}})$ by

$$\psi(\hat{\mathfrak{w}}) := \begin{cases} s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} & \text{if } \mathfrak{v} = \emptyset; \\ s_{\mathfrak{w}} \phi(\mathfrak{v}) & \text{otherwise.} \end{cases}$$

Example 4.13. Let $\mathcal{Q}$ be the following rank 3 acyclic quiver and B be the corresponding skew-symmetric matrix:



$$, \qquad B = \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}.$$

Consider $\hat{\mathfrak{w}} = (321)^4 2132 \in \mathfrak{W}$. Then $\mathfrak{w} = (321)^4$ and $\mathfrak{v} = 2132$. One easily obtains

$$V(\mathfrak{v}) = V(2132) = (\langle 2,1\rangle, \langle 5,3\rangle, \langle 3,2\rangle).$$

Thus $\vec{v_2}(\mathfrak{v}) = \langle 5,3\rangle$. By recording the intersections of the line segment ξ from (0, 0) to (5, 3) with $\mathcal{T}$, we have $\upsilon(\xi) = 2321232321232$ and

$$\phi(\mathfrak{v}) = \beta(\xi) = s_2 s_3 s_2 s_1 s_2 s_3 \alpha_2.$$

Combining these, we obtain

$$\begin{aligned} \psi(\hat{\mathfrak{w}}) &= s_{\mathfrak{w}} \phi(\mathfrak{v}) = (s_3 s_2 s_1)^4 s_2 s_3 s_2 s_1 s_2 s_3 \alpha_2 \\ &= 1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3. \end{aligned}$$

This real root was considered in Example 2.2 (2). The word $\mathfrak{w}$ corresponds to the spirals and $\mathfrak{v}$ to the line segment ξ.

4.4. Denominators of cluster variables

Consider the cluster variables associated to the initial seed $\Xi = ((x_1, x_2, x_3), B)$. The denominator of a noninitial cluster variable will be identified with an element of the positive root lattice Q_+ through

$$x_1^{m_1} x_2^{m_2} x_3^{m_3} \mapsto m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3, \quad m_i \in \mathbb{Z}_{\geq 0}, \quad i \in I. \quad (4\text{–}9)$$

The denominators of the initial cluster variables x_1, x_2, x_3 correspond to $-\alpha_1, -\alpha_2, -\alpha_3$, respectively.

Theorem 4.15 [Caldero and Keller 06]. *The correspondence (4–9) is a bijection between the set of denominators of cluster variables, other than x_i, $i \in I$, and the set of positive real Schur roots of $\mathcal{Q}$.*

For any $\mathfrak{w} \in \mathfrak{W} \setminus \{\emptyset\}$, let $\Xi(\mathfrak{w})$ be the labeled seed obtained from the initial seed Ξ by the sequence $\mu_{\mathfrak{w}}$ of mutations. We denote by $(\beta_1(\mathfrak{w}), \beta_2(\mathfrak{w}), \beta_3(\mathfrak{w}))$ the triple of real Schur roots (or negative simple roots) obtained from the denominators of the cluster variables in the cluster of $\Xi(\mathfrak{w})$.

Example 4.16. In Example 3.6, we obtain the triple of real Schur roots from $\Xi\big((321)^4 2312\big)$:

$$\begin{aligned}
\beta_1((321)^4 2312) &= 167041\ \alpha_1 + 437340\ \alpha_2 + 1144950\ \alpha_3, \\
\beta_2((321)^4 2312) &= 1662490\ \alpha_1 + 4352663\ \alpha_2 + 11395212\ \alpha_3, \qquad \text{(cf: Example 4: 13)} \\
\beta_3((321)^4 2312) &= 28656\ \alpha_1 + 75026\ \alpha_2 + 196417\ \alpha_3.
\end{aligned}$$

Now, we state a description of the real Schur roots associated with the denominators of cluster variables, using sequences of simple reflections.

Theorem 4.17. *Let $\hat{\mathfrak{w}} \in \mathfrak{W} \setminus \{\emptyset\}$. If p is the last letter of $\hat{\mathfrak{w}}$, then we have*

$$\beta_p(\hat{\mathfrak{w}}) = \psi(\hat{\mathfrak{w}}). \qquad (4\text{–}10)$$

This theorem will be proved in Section 5. Assuming this theorem, we now prove Theorem 4.2.

Proof of Theorem 4.2 . By Theorems 4.15 and 4.17, we have only to prove that there exists a one-to-one correspondence $\hat{\mathfrak{w}} = \mathfrak{w}\mathfrak{v} \in \mathfrak{W} \setminus \{\emptyset\} \mapsto z = (a, b, c) \in \mathcal{Z}$ such that $\psi(\hat{\mathfrak{w}}) = \beta(\eta_z)$, where the word $\mathfrak{w}$ is the longest word such that $B(\mathfrak{w})$ is acyclic. By definition, we have $\mathfrak{w} \in \mathcal{C}_1 \cup \mathcal{C}_3$, and it determines the spiral C_1 (and C_3) and the number a. Next consider the vector $\vec{v_p}(\mathfrak{v}) = \langle b', c' \rangle$ and determine the sign for $\langle b, c \rangle = \pm\langle b', c' \rangle$ so that the line segment $C_2 \in \Gamma$ from $(\epsilon b, \epsilon c)$ to $(b - \epsilon b, c - \epsilon c)$ is connected to the spiral C_1 (and C_3) for sufficiently small $\epsilon > 0$. Then, we set $z = (a, b, c) \in \mathcal{Z}$ and define η_z to be the union of C_1, C_2 and C_3.

Conversely, given $z = (a, b, c) \in \mathcal{Z}$, we have the unique curve η_z consisting of $C_{z,1}, C_{z,2}$, and $C_{z,3}$ by definition. The spiral $C_{z,1}$ determines $\mathfrak{w} \in \mathfrak{W}$ by simply recording the consecutive intersections of $C_{z,1}$ with $\mathcal{T}_p$, $p = 1, 2, 3$. Since $\gcd(|b|, |c|) = 1$, the line segment $C_{z,2}$ or the vector $\langle b, c \rangle$ determines a unique $\mathfrak{v} \in \mathfrak{W}$ such that $\vec{v_p}(\mathfrak{v}) = \pm\langle b, c \rangle$ where p is the last letter of $\mathfrak{v}$. Namely, one can associate a Farey triple with $V(\mathfrak{u}) = (\vec{v_1}(\mathfrak{u}), \vec{v_2}(\mathfrak{u}), \vec{v_3}(\mathfrak{u}))$ by taking the ratio of two coordinates of each $\vec{v_i}(\mathfrak{u})$, $i = 1, 2, 3$, for each $\mathfrak{u} \in \mathfrak{W}$ and use the Farey tree (or the Stern–Brocot tree) to find $\mathfrak{v}$ (cf. [Aigner 13, pp. 52–53]). Then we set $\hat{\mathfrak{w}} = \mathfrak{w}\mathfrak{v}$. This establishes the inverse of the map $\hat{\mathfrak{w}} \in \mathfrak{W} \setminus \{\emptyset\} \mapsto z \in \mathcal{Z}$.

5. Proof of Theorem 4.17

This section is devoted to a proof of Theorem 4.17. Recall that $\ell(\hat{\mathfrak{w}}) = k$ and $\rho(\hat{\mathfrak{w}}) = \max\{p : i_1 \cdots i_p \in \mathcal{C}_1 \cup \mathcal{C}_3\}$ for $\hat{\mathfrak{w}} = i_1 \cdots i_k \in \mathfrak{W}$. Note that $\rho(\hat{\mathfrak{w}}) = \max\{p : B(i_1 \ldots i_p) \text{ is acyclic}\}$. It is easy to check (4–10) if $\ell(\hat{\mathfrak{w}}) = 1$, so we assume that $\ell(\hat{\mathfrak{w}}) \geq 2$. Let

$$\begin{aligned}
\delta(\hat{\mathfrak{w}}) := \max\{p : q + 1 \leq p \leq \ell(\hat{\mathfrak{w}}) \\
\text{and } i_q i_{q+1} \cdots i_p \text{ consists of two letters}\},
\end{aligned}$$

where $q = \max\big(1, \rho(\hat{\mathfrak{w}}) - 1\big)$. We also let $\mathfrak{w} = i_1 \cdots i_{\rho(\hat{\mathfrak{w}})}$.

We have $\ell(\hat{\mathfrak{w}}) \geq \delta(\hat{\mathfrak{w}}) \geq \rho(\hat{\mathfrak{w}})$ by definition. We plan to prove Theorem 4.17 by considering the following cases:

Case 1: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) = \rho(\hat{\mathfrak{w}})$,
Case 2: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) = \rho(\hat{\mathfrak{w}}) + 1$,
Case 3: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) \geq \rho(\hat{\mathfrak{w}}) + 2$,
Case 4: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) + 1$,
Case 5: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) + 2$,
Case 6: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) + 3$,
Case 7: $\ell(\hat{\mathfrak{w}}) \geq \delta(\hat{\mathfrak{w}}) + 4$.

In what follows, we always set $\{i, j, k\} = \{1, 2, 3\}$. Consider the natural partial order on Q_+, that is, $m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 \geq m'_1\alpha_1 + m'_2\alpha_2 + m'_3\alpha_3$ if and only if $m_i \geq m'_i$ for all $i \in I$. We set $c_{ij} = |b_{ij}|$ for $i \neq j$ and $c_{ij}(\mathfrak{v}) = |b_{ij}(\mathfrak{v})|$ for $i \neq j$ and $\mathfrak{v} \in \mathfrak{W}$.

5.1. Case 1: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) = \rho(\hat{\mathfrak{w}})$

If $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) = \rho(\hat{\mathfrak{w}})$ then $\hat{\mathfrak{w}} = \mathfrak{w} \in \mathcal{C}_1 \cup \mathcal{C}_3$, equivalently $B(\hat{\mathfrak{w}})$ is acyclic. Write $\mathfrak{w} = \mathfrak{u}jk$ for $\mathfrak{u} \in \mathfrak{W}$ and $j, k \in I$.

Lemma 5.1. *We have*

$$\beta_k(\mathfrak{u}jk) = s_{\mathfrak{u}} s_j(\alpha_k) = \psi(\mathfrak{u}jk). \qquad (5\text{–}1)$$

Proof. We have $\beta_u(u) = \alpha_u$ and $\beta_v(uv) = \alpha_v + c_{uv}\alpha_u = s_u(\alpha_v)$. Now, by induction, we have

$$\begin{aligned}\beta_j(\mathfrak{u}j) &= s_\mathfrak{u}(\alpha_j),\\ \beta_i(\mathfrak{u}j) &= \beta_i(\mathfrak{u}) = \beta_i(\mathfrak{u}'i) = s_{\mathfrak{u}'}(\alpha_i) = s_{\mathfrak{u}'}s_is_i(\alpha_i) = s_\mathfrak{u}s_i(\alpha_i),\\ \beta_k(\mathfrak{u}j) &= \beta_k(\mathfrak{u}''kij) = \beta_k(\mathfrak{u}''k) = s_{\mathfrak{u}''}(\alpha_k)\\ &= s_{\mathfrak{u}''}s_ks_is_is_k(\alpha_k) = s_\mathfrak{u}s_is_k(\alpha_k),\end{aligned}$$

where we write $\mathfrak{u} = \mathfrak{u}'i = \mathfrak{u}''ki$.

Then, we have

$$\begin{aligned}\beta_k(\mathfrak{u}jk) &= -\beta_k(\mathfrak{u}j) + c_{ik}\beta_i(\mathfrak{u}j) + c_{jk}\beta_j(\mathfrak{u}j)\\ &= -s_\mathfrak{u}s_is_k(\alpha_k) + c_{ik}s_\mathfrak{u}s_i(\alpha_i) + c_{jk}s_\mathfrak{u}(\alpha_j)\\ &= s_\mathfrak{u}[s_i(\alpha_k) - c_{ik}\alpha_i + c_{jk}\alpha_j] = s_\mathfrak{u}[\alpha_k + c_{ik}\alpha_i - c_{ik}\alpha_i + c_{jk}\alpha_j]\\ &= s_\mathfrak{u}[\alpha_k + c_{jk}\alpha_j] = s_\mathfrak{u}s_j(\alpha_k).\end{aligned}$$

Since $\psi(\mathfrak{u}jk) = s_\mathfrak{u}s_j(\alpha_k)$, we obtain

$$\beta_k(\mathfrak{w}) = \psi(\mathfrak{w}). \tag{5–2}$$

5.2. Case 2: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) = \rho(\hat{\mathfrak{w}}) + 1$

Suppose that $\mathfrak{w} = \mathfrak{u}kij$. Then $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) = \rho(\hat{\mathfrak{w}}) + 1$ implies that $\hat{\mathfrak{w}} = \mathfrak{w}i$.

Lemma 5.4. *Suppose that $\mathfrak{w} = \mathfrak{u}kij$ and $B(\mathfrak{w})$ is acyclic. Then, we have*

$$c_{ij}\beta_j(\mathfrak{w}) \geq c_{ik}\beta_k(\mathfrak{w}).$$

Proof. We use induction on the length of $\mathfrak{w}$. Base cases can be checked easily. We have

$$\beta_i(\mathfrak{u}ki) = -\beta_i(\mathfrak{u}k) + c_{ik}\beta_k(\mathfrak{u}k) + c_{ij}\beta_j(\mathfrak{u}k)$$

and

$$\begin{aligned}\beta_j(\mathfrak{w}) &= -\beta_j(\mathfrak{u}ki) + c_{jk}\beta_k(\mathfrak{u}ki) + c_{ij}\beta_i(\mathfrak{u}ki)\\ &= -\beta_j(\mathfrak{u}ki) + c_{jk}\beta_k(\mathfrak{u}ki)\\ &\quad + c_{ij}\left[-\beta_i(\mathfrak{u}k) + c_{ik}\beta_k(\mathfrak{u}k) + c_{ij}\beta_j(\mathfrak{u}k)\right]\\ &= (c_{jk} + c_{ij}c_{ik})\beta_k(\mathfrak{u}k) - c_{ij}\beta_i(\mathfrak{u}k) + \left(c_{ij}^2 - 1\right)\beta_j(\mathfrak{u}k).\end{aligned}$$

By induction, assume that $c_{jk}\beta_k(\mathfrak{u}k) \geq c_{ij}\beta_i(\mathfrak{u}k)$. Then, we have

$$\begin{aligned}c_{ij}\beta_j(\mathfrak{w}) &= c_{ij}(c_{jk} + c_{ij}c_{ik})\beta_k(\mathfrak{u}k) - b_{ij}^2\beta_i(\mathfrak{u}k)\\ &\quad + c_{ij}\left(c_{ij}^2 - 1\right)\beta_j(\mathfrak{u}k)\\ &\geq c_{ij}(c_{jk} + c_{ij}c_{ik})\beta_k(\mathfrak{u}k) - b_{ij}^2\beta_i(\mathfrak{u}k)\\ &\geq c_{ij}(c_{jk} + c_{ij}c_{ik})\beta_k(\mathfrak{u}k) - c_{ij}c_{jk}\beta_k(\mathfrak{u}k) = b_{ij}^2c_{ik}\beta_k(\mathfrak{u}k)\\ &\geq c_{ik}\beta_k(\mathfrak{u}k) = c_{ik}\beta_k(\mathfrak{w}).\end{aligned}$$

Suppose that $\mathfrak{v} \in \mathfrak{W}$ ends with j and consider $\mathfrak{v}i$. If, we have $c_{ij}(\mathfrak{v})\beta_j(\mathfrak{v}) \geq c_{ik}(\mathfrak{v})\beta_k(\mathfrak{v})$, we record this situation using $[j]$ below the i-arrow in the following diagram:

$$\xrightarrow{j} \mathfrak{v} \underset{[j]}{\xrightarrow{i}} \mathfrak{v}i.$$

Similarly, if

$$c_{ij}(\mathfrak{v})\beta_j(\mathfrak{v}) \leq c_{ik}(\mathfrak{v})\beta_k(\mathfrak{v}),$$

we write

$$\xrightarrow{j} \mathfrak{v} \underset{[k]}{\xrightarrow{i}} \mathfrak{v}i.$$

Remember that $B(\mathfrak{w})$ is acyclic and $B(\hat{\mathfrak{w}}) = B(\mathfrak{w}i)$ is cyclic. By definition, we have $\psi(\mathfrak{w}i) = s_\mathfrak{w}\alpha_i$. If $\mathfrak{w} = \emptyset$, then $\beta_i(\mathfrak{w}i) = \beta_i(i) = \alpha_i$; if $\mathfrak{w} = j$, then $\beta_i(\mathfrak{w}i) = \beta_i(ji) = s_j(\alpha_i)$; if $\mathfrak{w} = ij$, then $\beta_i(\mathfrak{w}i) = \beta_i(iji) = s_is_j(\alpha_i)$. In all these cases, we have (4–10).

Now suppose that $\mathfrak{w} = \mathfrak{u}kij$. By Lemma 5.4, we have $c_{ij}\beta_j(\mathfrak{w}) \geq c_{ik}\beta_k(\mathfrak{w})$. Thus we have

$$\xrightarrow{j} \mathfrak{w} \underset{[j]}{\xrightarrow{i}} \mathfrak{w}i.$$

By Case 1, we have

$$\begin{aligned}\beta_i(\mathfrak{w}i) &= -\beta_i(\mathfrak{w}) + c_{ij}\beta_j(\mathfrak{w}) = -s_\mathfrak{w}s_js_i(\alpha_i) + c_{ij}s_\mathfrak{w}s_j(\alpha_j)\\ &= s_\mathfrak{w}[s_j\alpha_i - c_{ij}\alpha_j] = s_\mathfrak{w}(\alpha_i).\end{aligned}$$

Thus we have

$$\beta_i(\mathfrak{w}i) = s_\mathfrak{w}(\alpha_i) = \psi(\mathfrak{w}i). \tag{5–3}$$

This proves (4–10) in this case.

5.3. Case 3: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) \geq \rho(\hat{\mathfrak{w}}) + 2$

Assume that $B(\mathfrak{w})$ is acyclic and $B(\mathfrak{w}i)$ is cyclic.

Lemma 5.6. *We have*

$$\xrightarrow{i} \mathfrak{w}i \underset{[i]}{\xrightarrow{j}} \mathfrak{w}ij \underset{[j]}{\xrightarrow{i}} \mathfrak{w}iji.$$

That is, we have

$$c_{ij}(\mathfrak{w}i)\beta_i(\mathfrak{w}i) \geq c_{jk}(\mathfrak{w}i)\beta_k(\mathfrak{w}i), \quad \beta_j(\mathfrak{w}ij) = s_\mathfrak{w}s_i(\alpha_j),$$
and $$c_{ij}(\mathfrak{w}ij)\beta_j(\mathfrak{w}ij) \geq c_{ik}(\mathfrak{w}ij)\beta_k(\mathfrak{w}ij), \quad \beta_i(\mathfrak{w}iji) = s_\mathfrak{w}s_is_j(\alpha_i).$$

Proof. If the length of $\mathfrak{w}$ is less than three, it can be checked directly. Otherwise, write $\mathfrak{w} = \mathfrak{u}kij$. Using (5–3), we have

$$\begin{aligned}c_{ij}(\mathfrak{w}i)\beta_i(\mathfrak{w}i) &= c_{ij}s_\mathfrak{w}(\alpha_i) = c_{ij}s_\mathfrak{u}s_ks_is_j(\alpha_i)\\ &= c_{ij}s_\mathfrak{u}\left[\left(c_{ij}^2 - 1\right)\alpha_i + c_{ij}\alpha_j + \left(c_{ij}^2c_{ik} - c_{ik} + c_{ij}c_{jk}\right)\alpha_k\right]\\ &= c_{ij}\left(c_{ij}^2 - 1\right)s_\mathfrak{u}(\alpha_i) + c_{ij}^2s_\mathfrak{u}(\alpha_j) + \left(c_{ij}^3c_{ik} - c_{ij}c_{ik} + c_{ij}^2c_{jk}\right)s_\mathfrak{u}(\alpha_k);\end{aligned}$$

on the other hand, using (5–1), we have

$$c_{jk}(\mathfrak{w}i)\beta_k(\mathfrak{w}i) = (c_{jk}+c_{ij}c_{ik})\beta_k(\mathfrak{u}k) = (c_{jk}+c_{ij}c_{ik})s_{\mathfrak{u}}(\alpha_k).$$

Since $\mathfrak{u}i$ and $\mathfrak{u}k$ are reduced expressions, we see that $s_{\mathfrak{u}}(\alpha_i)$ and $s_{\mathfrak{u}}(\alpha_k)$ are positive roots. We claim that $s_{\mathfrak{u}}(\alpha_i) \geq -s_{\mathfrak{u}}(\alpha_k)$. Indeed, writing $\mathfrak{u} = \mathfrak{u}'j$, we have

$$\begin{aligned} s_{\mathfrak{u}}(\alpha_k) + s_{\mathfrak{u}}(\alpha_j) &= s_{\mathfrak{u}'}s_j(\alpha_k) + s_{\mathfrak{u}'}s_j(\alpha_j) \\ &= s_{\mathfrak{u}'}(\alpha_k + c_{jk}\alpha_j) - s_{\mathfrak{u}'}(\alpha_j) \\ &= s_{\mathfrak{u}'}(\alpha_k) + (c_{jk}-1)s_{\mathfrak{u}'}(\alpha_j) \geq 0, \end{aligned} \tag{5–4}$$

since $s_{\mathfrak{u}'}(\alpha_k)$ is a positive root and $c_{jk} \geq 2$.

Now we have only to show that

$$-c_{ij}^2 + c_{ij}^3 c_{ik} - c_{ij}c_{ik} + c_{ij}^2 c_{jk} \geq c_{jk} + c_{ij}c_{ik},$$

which is equivalent to

$$c_{ij}^2 c_{jk} - c_{ij}^2 - c_{jk} + c_{ij}^3 c_{ik} - 2c_{ij}c_{ik} \geq 0.$$

We write the left-hand side of the inequality as

$$\left(c_{ij}^2 - 1\right)(c_{jk}-1) - 1 + \left(c_{ij}^2 - 2\right)c_{ij}c_{ik},$$

and we are done since $c_{ij}, c_{jk} \geq 2$.

Note that $c_{ij}(\mathfrak{w}i) = \left(\beta_i(\mathfrak{w}i), \beta_j(\mathfrak{w}i)\right)$. Indeed, since $\beta_i(\mathfrak{w}i) = s_{\mathfrak{w}}(\alpha_i)$ and $\beta_j(\mathfrak{w}i) = s_{\mathfrak{w}}s_j(\alpha_j)$, we have

$$\begin{aligned} \left(\beta_i(\mathfrak{w}i), \beta_j(\mathfrak{w}i)\right) &= \left(s_{\mathfrak{w}}(\alpha_i), s_{\mathfrak{w}}s_j(\alpha_j)\right) \\ &= -(\alpha_i, \alpha_j) = c_{ij} = c_{ij}(\mathfrak{w}i). \end{aligned}$$

Then, from $r_{\beta_i(\mathfrak{w}i)} = s_{\mathfrak{w}}s_i s_{\mathfrak{w}}^{-1}$, we obtain

$$\begin{aligned} \beta_j(\mathfrak{w}ij) &= -\beta_j(\mathfrak{w}i) + c_{ij}(\mathfrak{w}i)\beta_i(\mathfrak{w}i) \\ &= -\beta_j(\mathfrak{w}i) + \left(\beta_i(\mathfrak{w}i), \beta_j(\mathfrak{w}i)\right)\beta_i(\mathfrak{w}i) \\ &= -r_{\beta_i(\mathfrak{w}i)}\left(\beta_j(\mathfrak{w}i)\right) = -s_{\mathfrak{w}}s_i s_{\mathfrak{w}}^{-1}s_{\mathfrak{w}}s_j(\alpha_j) = s_{\mathfrak{w}}s_i(\alpha_j). \end{aligned} \tag{5–5}$$

A similar argument establishes $c_{ij}(\mathfrak{w}ij)\beta_j(\mathfrak{w}ij) \geq c_{ik}(\mathfrak{w}ij)\beta_k(\mathfrak{w}ij)$ and $\beta_i(\mathfrak{w}iji) = s_{\mathfrak{w}}s_i s_j(\alpha_i)$.

Lemma 5.9. *Let $\tilde{\mathfrak{w}} = \mathfrak{w}i(ji)^n$ for $n \in \mathbb{Z}_{\geq 0}$. Then, we have*

$$c_{ij}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}}) > c_{jk}(\tilde{\mathfrak{w}})\beta_k(\tilde{\mathfrak{w}}), \quad \beta_j(\tilde{\mathfrak{w}}j) = s_{\tilde{\mathfrak{w}}}(\alpha_j), \tag{5–6}$$

$$c_{ij}(\tilde{\mathfrak{w}}j)\beta_j(\tilde{\mathfrak{w}}j) > c_{ik}(\tilde{\mathfrak{w}}j)\beta_k(\tilde{\mathfrak{w}}j), \quad \beta_i(\tilde{\mathfrak{w}}ji) = s_{\tilde{\mathfrak{w}}}s_j(\alpha_i). \tag{5–7}$$

This means that we have

$$\mathfrak{w}i(ji)^n \underset{[i]}{\overset{j}{\to}} \mathfrak{w}i(ji)^n j \underset{[j]}{\overset{i}{\to}} \mathfrak{w}i(ji)^{n+1}$$

for each $n \in \mathbb{Z}_{\geq 0}$.

Proof. We have $\beta_k(\mathfrak{w}) = \beta_k\left(\mathfrak{w}i(ji)^n\right) = \beta_k\left(\mathfrak{w}i(ji)^n j\right)$. Similarly, $c_{ij} = c_{ij}(\mathfrak{w}i) = c_{ij}\left(\mathfrak{w}i(ji)^n\right) = c_{ij}\left(\mathfrak{w}i(ji)^n j\right)$. We write $\gamma = c_{ij}$ for simplicity. We use induction on n. The case $n=0$ is proven in Lemma 5.6. Thus we assume $n>0$. If we consider the vector $\left(\beta_i(\mathfrak{w}'), \beta_j(\mathfrak{w}')\right)$ for $\mathfrak{w}' = \mathfrak{w}i(ji)^m$ with $m<n$, the vector $\left(\beta_i(\mathfrak{w}'j), \beta_j(\mathfrak{w}'j)\right)$ after mutation j is given by the matrix $J_j := \begin{pmatrix} 1 & \gamma \\ 0 & -1 \end{pmatrix}$ through right multiplication, and the vector after mutation i for $\mathfrak{w}' = \mathfrak{w}i(ji)^m j$ is given by $J_i := \begin{pmatrix} -1 & 0 \\ \gamma & 1 \end{pmatrix}$. Similarly, if we consider the vector $\left(c_{jk}(\mathfrak{w}'), c_{ik}(\mathfrak{w}')\right)$, the matrices for mutation j and i are respectively given by the same matrices J_j and J_i.

Let $J = J_j J_i = \begin{pmatrix} \gamma^2-1 & \gamma \\ -\gamma & -1 \end{pmatrix}$. First, assume that $\gamma>2$. We denote two eigenvalues of J by λ_1 and λ_2 with $\lambda_1>\lambda_2$. Then we have $\lambda_1>1>\lambda_2>0$. A diagonalization $J = PDP^{-1}$ of J is given by $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $P = \begin{pmatrix} 1+\lambda_1 & 1+\lambda_2 \\ -\gamma & -\gamma \end{pmatrix}$. With $\lambda_1 + \lambda_2 = \gamma^2 - 2$ and $\lambda_1\lambda_2 = 1$, we compute to obtain

$$J^n = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^n(1+\lambda_1) - \lambda_2^n(1+\lambda_2) & \gamma\left(\lambda_1^n - \lambda_2^n\right) \\ -\gamma\left(\lambda_1^n - \lambda_2^n\right) & -\lambda_1^{n-1}(1+\lambda_1) + \lambda_2^{n-1}(1+\lambda_2) \end{pmatrix}.$$

We let

$$y_n := \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1^n - \lambda_2^n\right) = \lambda_1^{n-1} + \lambda_1^{n-3} + \cdots + \lambda_2^{n-3} + \lambda_2^{n-1},$$

$$x_n := \frac{1}{\lambda_1 - \lambda_2}\left(\lambda_1^n(1+\lambda_1) - \lambda_2^n(1+\lambda_2)\right) = y_{n+1} + y_n.$$

Then we have $J^n = \begin{pmatrix} x_n & \gamma y_n \\ -\gamma y_n & -x_{n-1} \end{pmatrix}$ for $n>0$.

Next, assume that $\gamma=2$, and let $y_n = n$ and $x_n = 2n+1$. Then, from direct computation, we have

$$J^n = \begin{pmatrix} 2n+1 & 2n \\ -2n & -2n+1 \end{pmatrix} = \begin{pmatrix} x_n & \gamma y_n \\ -\gamma y_n & -x_{n-1} \end{pmatrix} \quad \text{for } n>0,$$

and the same formula for J^n holds in this case as well.

We want to prove (5–6), which can be written as

$$\gamma\beta_i\left(\mathfrak{w}i(ji)^n\right) > c_{jk}\left(\mathfrak{w}i(ji)^n\right)\beta_k(\mathfrak{w}).$$

If the length of $\mathfrak{w}$ is less than 3, i.e. $\mathfrak{w} = \emptyset, j$ or ij, then $\beta_k(\mathfrak{w}) = 0$ and there is nothing to prove. Thus we assume $\mathfrak{w} = \mathfrak{u}kij$.

Using the matrices J^n, we rewrite the inequality as

$$\gamma\Big(x_n\beta_i(\mathfrak{w}i)-\gamma y_n\beta_j(\mathfrak{w}i)\Big)>\big(x_nc_{jk}(\mathfrak{w}i)-\gamma y_nc_{ik}(\mathfrak{w}i)\big)\beta_k(\mathfrak{w}), \tag{5–8}$$

which becomes

$$\gamma\big(x_ns_{\mathfrak{u}}s_ks_is_j(\alpha_i)-\gamma y_ns_{\mathfrak{u}}s_ks_i(\alpha_j)\big)>\big(x_n(c_{jk}+\gamma c_{ik})-\gamma y_nc_{ik}\big)s_{\mathfrak{u}}(\alpha_k). \tag{5–9}$$

We expand each side of (5–9).

$$\begin{aligned}\text{LHS} &= \gamma x s_{\mathfrak{u}}[(\gamma^2-1)\alpha_i+\gamma\alpha_j+(\gamma^2c_{ik}-c_{ik}+\gamma c_{jk})\alpha_k)]\\ &\quad -\gamma^2y_ns_{\mathfrak{u}}[\gamma\alpha_i+\alpha_j+(c_{jk}+\gamma c_{ik})\alpha_k]\\ &= [\gamma x_n(\gamma^2-1)-\gamma^3y_n]s_{\mathfrak{u}}(\alpha_i)+[\gamma^2x_n-\gamma^2y_n]s_{\mathfrak{u}}(\alpha_j)\\ &\quad +[\gamma x_n(\gamma^2c_{ik}-c_{ik}+\gamma c_{jk})-\gamma^2y_n(c_{jk}+\gamma c_{ik})]s_{\mathfrak{u}}(\alpha_k),\\ \text{RHS} &= [x_n(c_{jk}+\gamma c_{ik})-\gamma y_nc_{ik}]s_{\mathfrak{u}}(\alpha_k).\end{aligned}$$

We consider the coefficient of $s_{\mathfrak{u}}(\alpha_i)$ in LHS and find

$$\begin{aligned}x_n(\gamma^2-1)-\gamma^2y_n &= (y_{n+1}+y_n)(\gamma^2-1)-\gamma^2y_n\\ &= (\gamma^2-1)y_{n+1}-y_n\geq 0\end{aligned}$$

since $\gamma\geq 2$ and $y_{n+1}>y_n$.

Recall that we showed $s_{\mathfrak{u}}(\alpha_j)\geq -s_{\mathfrak{u}}(\alpha_k)$ in (5–4). We combine the coefficients of $s_{\mathfrak{u}}(\alpha_j)$ and $s_{\mathfrak{u}}(\alpha_k)$ in LHS and need to prove the following inequality.

$$\begin{aligned}&-\gamma^2x_n+\gamma^2y_n+\gamma^3x_nc_{ik}-\gamma x_nc_{ik}+\gamma^2x_nc_{jk}\\ &\quad -\gamma^2y_nc_{jk}-\gamma^3y_nc_{ik}\geq x_nc_{jk}+\gamma x_nc_{ik}-\gamma y_nc_{ik}.\end{aligned}$$

With $x_n=y_{n+1}+y_n$ substituted, the inequality is equivalent to

$$-\gamma^2y_{n+1}+\big(\gamma^3-2\gamma\big)y_{n+1}c_{ik}-\gamma y_nc_{ik}+\big(\gamma^2-1\big)y_{n+1}c_{jk}-y_nc_{jk}\geq 0. \tag{5–10}$$

Using $\gamma, c_{ik}, c_{jk}\geq 2$ and $y_{n+1}>y_n$, we see that

$$\begin{aligned}&-\gamma^2y_{n+1}+\big(\gamma^3-2\gamma\big)y_{n+1}c_{ik}-\gamma y_nc_{ik}\\ &\quad+\big(\gamma^2-1\big)y_{n+1}c_{jk}-y_nc_{jk}\\ &\geq -\gamma^2y_{n+1}+\big(\gamma^3-3\gamma\big)y_{n+1}c_{ik}+\big(\gamma^2-2\big)y_{n+1}c_{jk}\\ &\geq -\gamma^2y_{n+1}+2\big(\gamma^3-3\gamma\big)y_{n+1}+2\big(\gamma^2-2\big)y_{n+1}\\ &\quad=\gamma\big(2\gamma^2+\gamma-10\big)y_{n+1}\geq 0.\end{aligned}$$

Thus, the inequality (5–10) is proven, so is the inequality (5–6).

One can see that $c_{ij}(\tilde{\mathfrak{w}})=c_{ij}=\Big(\beta_i(\tilde{\mathfrak{w}}),\beta_j(\tilde{\mathfrak{w}})\Big)$ by induction. Then, a similar computation to (5.8) gives us $\beta_j(\tilde{\mathfrak{w}}j)=s_{\tilde{\mathfrak{w}}}(\alpha_j)$.

Now, we want to prove

$$\gamma\beta_j\big(\mathfrak{w}i(ji)^nj\big)>c_{ik}\big(\mathfrak{w}i(ji)^nj\big)\beta_k(\mathfrak{w}),$$

which is the same as (5–7). Since

$$J^nJ_j=\begin{pmatrix}x_n & \gamma y_n\\ -\gamma y_n & -x_{n-1}\end{pmatrix}\begin{pmatrix}1 & \gamma\\ 0 & -1\end{pmatrix}=\begin{pmatrix}x_n & \gamma y_{n+1}\\ -\gamma y_n & -x_n\end{pmatrix},$$

the inequality can be written as

$$\gamma\Big(\gamma y_{n+1}\beta_i(\mathfrak{w}i)-x_n\beta_j(\mathfrak{w}i)\Big)>\big(\gamma y_{n+1}c_{jk}(\mathfrak{w}i)-x_nc_{ik}(\mathfrak{w}i)\big)\beta_k(\mathfrak{w}),$$

and becomes

$$\begin{aligned}&\gamma(\gamma y_{n+1}s_{\mathfrak{u}}s_ks_is_j(\alpha_i)-x_ns_{\mathfrak{u}}s_ks_i(\alpha_j))>(\gamma y_{n+1}(c_{jk}+\gamma c_{ik})\\ &\quad -x_nc_{ik})s_{\mathfrak{u}}(\alpha_k).\end{aligned}$$

This can be proven in the same way as we did for (5–9). Similarly, we obtain $\beta_i(\tilde{\mathfrak{w}}ji)=s_{\tilde{\mathfrak{w}}}s_j(\alpha_i)$.

Let $\tilde{\mathfrak{w}}=\mathfrak{w}i(ji)^n$ for $n\in\mathbb{Z}_{\geq 0}$ and $\mathfrak{v}=i(ji)^n$. By (4–3), we have

$$\phi(\mathfrak{v}j)=s_i(s_js_i)^n\alpha_j \text{ and } \phi(\mathfrak{v}ji)=s_i(s_js_i)^ns_j\alpha_i,$$

and obtain

$$\begin{aligned}\beta_j(\tilde{\mathfrak{w}}j) &= s_{\tilde{\mathfrak{w}}}(\alpha_j)=s_{\mathfrak{w}}\phi(\mathfrak{v}j)=\psi(\mathfrak{w}\mathfrak{v}j)\\ &=\psi(\tilde{\mathfrak{w}}j) \text{ and } \beta_i(\tilde{\mathfrak{w}}ji)=s_{\tilde{\mathfrak{w}}}s_j(\alpha_i)=\psi(\tilde{\mathfrak{w}}ji).\end{aligned}$$

Thus we have proven (4–10) in this case.

5.4. Case 4: $\ell(\hat{\mathfrak{w}})=\delta(\hat{\mathfrak{w}})+1$

Lemma 5.15. *Let* $\tilde{\mathfrak{w}}=\mathfrak{w}i(ji)^n$ *for* $n\in\mathbb{Z}_{\geq 0}$. *Then, we have*

$$c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}})>c_{ik}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}}),\quad \beta_k(\tilde{\mathfrak{w}}k)=s_{\tilde{\mathfrak{w}}}(s_js_i)^n(\alpha_k), \tag{5–11}$$

$$c_{ik}(\tilde{\mathfrak{w}}j)\beta_i(\tilde{\mathfrak{w}}j)>c_{jk}(\tilde{\mathfrak{w}}j)\beta_j(\tilde{\mathfrak{w}}j),\quad \beta_k(\tilde{\mathfrak{w}}jk)=s_{\tilde{\mathfrak{w}}}(s_js_i)^{n+1}(\alpha_k). \tag{5–12}$$

This means that we have

$$\xrightarrow[{[j]}]{i}\tilde{\mathfrak{w}}\xrightarrow[{[j]}]{k}\tilde{\mathfrak{w}}k \quad\text{and}\quad \xrightarrow[{[i]}]{j}\tilde{\mathfrak{w}}j\xrightarrow[{[i]}]{k}\tilde{\mathfrak{w}}jk.$$

Proof. First, we consider the case $n=0$ and see

$$\begin{aligned}c_{jk}(\mathfrak{w}i)\beta_j(\mathfrak{w}i) &= (c_{jk}+c_{ij}c_{ik})\beta_j(\mathfrak{w})\\ &= c_{jk}\beta_j(\mathfrak{w})+c_{ij}c_{ik}\beta_j(\mathfrak{w}),\\ c_{ik}(\mathfrak{w}i)\beta_i(\mathfrak{w}i) &= c_{ik}\big(-\beta_i(\mathfrak{w})+c_{ij}\beta_j(\mathfrak{w})\big)\\ &= -c_{ik}\beta_i(\mathfrak{w})+c_{ik}c_{ij}\beta_j(\mathfrak{w}).\end{aligned}$$

Thus, we have

$$c_{jk}(\mathfrak{w}i)\beta_j(\mathfrak{w}i)>c_{ik}(\mathfrak{w}i)\beta_i(\mathfrak{w}i). \tag{5–13}$$

We claim that $c_{jk}(\mathfrak{w}i)=\Big(\beta_k(\mathfrak{w}i),\beta_j(\mathfrak{w}i)\Big)$. Indeed, if the length of $\mathfrak{w}$ is greater than 3, we have

$$\begin{aligned}\Big(\beta_k(\mathfrak{w}i),\beta_j(\mathfrak{w}i)\Big) &= \big(s_{\mathfrak{w}}s_js_is_k(\alpha_k),s_{\mathfrak{w}}s_j(\alpha_j)\big)=\big(s_is_k(\alpha_k),\alpha_j\big)\\ &= -(\alpha_k+c_{ik}\alpha_i,\alpha_j)=c_{jk}+c_{ij}c_{ik}=c_{jk}(\mathfrak{w}i).\end{aligned}$$

Otherwise, it can be checked easily. Then, we obtain

$$\begin{aligned}\beta_k(\mathfrak{w}ik) &= -\beta_k(\mathfrak{w}i) + c_{jk}(\mathfrak{w}i)\beta_j(\mathfrak{w}i)\\ &= -\beta_k(\mathfrak{w}i) + \Big(\beta_k(\mathfrak{w}i), \beta_j(\mathfrak{w}i)\Big)\beta_j(\mathfrak{w}i)\\ &= -r_{\beta_j(\mathfrak{w}i)}\big(\beta_k(\mathfrak{w}i)\big) = -s_{\mathfrak{w}} s_j s_{\mathfrak{w}}^{-1}\ s_{\mathfrak{w}} s_j s_i s_k(\alpha_k)\\ &= s_{\mathfrak{w}} s_i(\alpha_k).\end{aligned}$$

Now assume $n > 0$. Using the matrix $J^n = \begin{pmatrix} x_n & \gamma y_n \\ -\gamma y_n & -x_{n-1}\end{pmatrix}$ defined in the proof of Lemma 5.9, the inequality $c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}}) > c_{ik}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}})$ can be written as

$$\begin{aligned}&\big(x_n c_{jk}(\mathfrak{w}i) - \gamma y_n c_{ik}(\mathfrak{w}i)\big)\Big(\gamma y_n \beta_i(\mathfrak{w}i) - x_{n-1}\beta_j(\mathfrak{w}i)\Big)\\ &> \big(\gamma y_n c_{jk}(\mathfrak{w}i) - x_{n-1} c_{ik}(\mathfrak{w}i)\big)\Big(x_n \beta_i(\mathfrak{w}i) - \gamma y_n \beta_j(\mathfrak{w}i)\Big),\end{aligned}$$

which is equivalent to

$$(\gamma^2 y_n^2 - x_n x_{n-1}) c_{jk}(\mathfrak{w}i)\beta_j(\mathfrak{w}i) > (\gamma^2 y_n^2 - x_n x_{n-1}) c_{ik}(\mathfrak{w}i)\beta_i(\mathfrak{w}i).$$

Since $\gamma^2 y_n^2 - x_n x_{n-1} = \det J^n = 1$, this inequality is the same as (5–13) and we are done.

We claim that $c_{jk}(\tilde{\mathfrak{w}}) = \Big(\beta_j(\tilde{\mathfrak{w}}), \beta_k(\tilde{\mathfrak{w}})\Big)$. Indeed, we have

$$\begin{aligned}&c_{jk}(\tilde{\mathfrak{w}}) = x_n c_{jk}(\mathfrak{w}i) - \gamma y_n c_{ik}(\mathfrak{w}i) \text{ and}\\ &\Big(\beta_j(\tilde{\mathfrak{w}}), \beta_k(\tilde{\mathfrak{w}})\Big) = \Big(\gamma y_n \beta_i(\mathfrak{w}i) - x_{n-1}\beta_j(\mathfrak{w}i), \beta_k(\mathfrak{w}i)\Big)\\ &\qquad = \gamma y_n\big(\beta_i(\mathfrak{w}i), \beta_k(\mathfrak{w}i)\big) - x_{n-1} c_{jk}(\mathfrak{w}i).\end{aligned}$$

Since $x_n + x_{n-1} = \gamma^2 y_n$, $c_{jk}(\mathfrak{w}i) = c_{jk} + \gamma c_{ik}$ and $\big(\beta_i(\mathfrak{w}i), \beta_k(\mathfrak{w}i)\big) = -c_{ik} + \gamma c_{jk} + \gamma^2 c_{ik}$, one sees that the claim holds. Then we obtain

$$\begin{aligned}\beta_k(\tilde{\mathfrak{w}}k) &= -\beta_k(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}})\\ &= -\beta_k(\tilde{\mathfrak{w}}) + \Big(\beta_j(\tilde{\mathfrak{w}}), \beta_k(\tilde{\mathfrak{w}})\Big)\beta_j(\tilde{\mathfrak{w}}) = -r_{\beta_j(\tilde{\mathfrak{w}})}\big(\beta_k(\tilde{\mathfrak{w}})\big)\\ &= -s_{\mathfrak{w}} s_i (s_j s_i)^{n-1} s_j (s_i s_j)^{n-1} s_i s_{\mathfrak{w}}^{-1} s_{\mathfrak{u}}(\alpha_k)\\ &= -s_{\mathfrak{w}} s_i (s_j s_i)^{n-1} s_j (s_i s_j)^{n-1} s_i s_j s_i s_k s_{\mathfrak{u}}^{-1} s_{\mathfrak{u}}(\alpha_k)\\ &= s_{\mathfrak{w}} s_i (s_j s_i)^{n-1} s_j (s_i s_j)^{n-1} s_i s_j s_i(\alpha_k) = s_{\tilde{\mathfrak{w}}}(s_j s_i)^n(\alpha_k),\end{aligned}$$

where we write $\mathfrak{w} = \mathfrak{u}kij$ as before.

Similarly, the inequality $c_{ik}(\tilde{\mathfrak{w}}j)\beta_i(\tilde{\mathfrak{w}}j) > c_{jk}(\tilde{\mathfrak{w}}j)\beta_j(\tilde{\mathfrak{w}}j)$ can be proven in the same way, using the matrix $J^n J_j = \begin{pmatrix} x_n & \gamma y_{n+1} \\ -\gamma y_n & -x_n\end{pmatrix}$ and $\det(J^n J_j) = -1$.

Furthermore, we see that

$$c_{ik}(\tilde{\mathfrak{w}}j) = (\beta_i(\tilde{\mathfrak{w}}j), \beta_k(\tilde{\mathfrak{w}}j))$$

and obtain

$$\begin{aligned}\beta_k(\tilde{\mathfrak{w}}jk) &= -r_{\beta_i(\tilde{\mathfrak{w}}j)}\big(\beta_k(\tilde{\mathfrak{w}})j\big)\\ &= -s_{\mathfrak{w}} s_i (s_j s_i)^{n-1} s_j s_i s_j (s_i s_j)^{n-1} s_i s_{\mathfrak{w}}^{-1} s_{\mathfrak{u}}(\alpha_k)\\ &= s_{\mathfrak{w}} s_i (s_j s_i)^{n-1} s_j s_i s_j (s_i s_j)^{n-1} s_i s_j s_i(\alpha_k) = s_{\tilde{\mathfrak{w}}}(s_j s_i)^{n+1}(\alpha_k).\end{aligned}$$

Let $\tilde{\mathfrak{w}} = \mathfrak{w}i(ji)^n$ for $n \in \mathbb{Z}_{\geq 0}$ and $\mathfrak{v} = i(ji)^n$. By (4–4), we have

$$\phi(\mathfrak{v}k) = s_i(s_j s_i)^{2n}\alpha_k \text{ and } \phi(\mathfrak{v}jk) = s_i(s_j s_i)^{2n+1}\alpha_k,$$

and obtain

$$\begin{aligned}&\beta_k(\tilde{\mathfrak{w}}k) = s_{\tilde{\mathfrak{w}}}(s_j s_i)^n(\alpha_k) = s_{\mathfrak{w}}\phi(\mathfrak{v}k)\\ &\quad = \psi(\tilde{\mathfrak{w}}k) \text{ and } \beta_k(\tilde{\mathfrak{w}}jk) = s_{\tilde{\mathfrak{w}}}(s_j s_i)^{n+1}(\alpha_k) = \psi(\tilde{\mathfrak{w}}jk).\end{aligned}$$

Thus, we have proven (4–10) in this case.

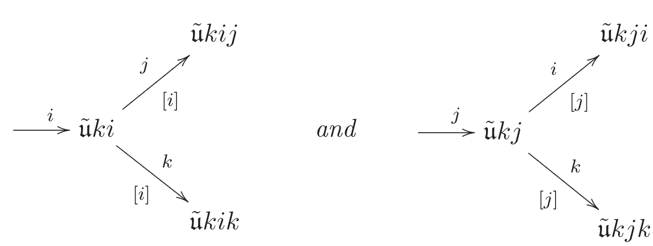
Before we go to the next case, we list the values of the bilinear form for various roots. Some of them have already been proved in the proof of Lemma 5.15. As the others can be easily checked, we omit the details.

Corollary 5.19. *We have*

$$\begin{aligned}&\Big(\beta_i(\tilde{\mathfrak{w}}), \beta_j(\tilde{\mathfrak{w}})\Big) = c_{ij}(\tilde{\mathfrak{w}}), \qquad (\beta_i(\tilde{\mathfrak{w}}j), \beta_j(\tilde{\mathfrak{w}}j)) = c_{ij}(\tilde{\mathfrak{w}}j),\\ &(\beta_i(\tilde{\mathfrak{w}}), \beta_k(\tilde{\mathfrak{w}})) = -c_{ik}(\tilde{\mathfrak{w}}) + c_{ij}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}}), (\beta_i(\tilde{\mathfrak{w}}j), \beta_k(\tilde{\mathfrak{w}}j)) = c_{ik}(\tilde{\mathfrak{w}}j),\\ &\Big(\beta_j(\tilde{\mathfrak{w}}), \beta_k(\tilde{\mathfrak{w}})\Big) = c_{jk}(\tilde{\mathfrak{w}}), (\beta_j(\tilde{\mathfrak{w}}j), \beta_k(\tilde{\mathfrak{w}}j)) = -c_{jk}(\tilde{\mathfrak{w}}j) + c_{ij}(\tilde{\mathfrak{w}}j)c_{ik}(\tilde{\mathfrak{w}}j).\end{aligned}$$

5.5. Case 5: $\ell(\hat{\mathfrak{w}}) = \delta(\hat{\mathfrak{w}}) + 2$

Lemma 5.20. *We have*



Proof. First, we prove

$$c_{ik}(\tilde{\mathfrak{w}}k)\beta_k(\tilde{\mathfrak{w}}k) \geq c_{ij}(\tilde{\mathfrak{w}}k)\beta_j(\tilde{\mathfrak{w}}k). \qquad (5\text{–}14)$$

We compute

$$\begin{aligned}c_{ik}(\tilde{\mathfrak{w}}k)\beta_k(\tilde{\mathfrak{w}}k) &= c_{ik}(\tilde{\mathfrak{w}})\Big(-\beta_k(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}})\Big)\\ &= -c_{ik}(\tilde{\mathfrak{w}})\beta_k(\tilde{\mathfrak{w}}) + c_{ik}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}}),\\ c_{ij}(\tilde{\mathfrak{w}}k)\beta_j(\tilde{\mathfrak{w}}j) &= \Big(-c_{ij}(\tilde{\mathfrak{w}}) + c_{ik}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}})\Big)\beta_j(\tilde{\mathfrak{w}})\\ &= -c_{ij}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}}) + c_{ik}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}}).\end{aligned}$$

Since, we have

$$\begin{aligned}c_{ik}(\tilde{\mathfrak{w}})\beta_k(\tilde{\mathfrak{w}}) &= c_{ik}\big(\mathfrak{w}i(ji)^{n-1}j\big)\beta_k\big(\mathfrak{w}i(ji)^{n-1}j\big)\\ &\le c_{ij}\big(\mathfrak{w}i(ji)^{n-1}j\big)\beta_j\big(\mathfrak{w}i(ji)^{n-1}j\big) = c_{ij}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}}),\end{aligned}$$

we see that the inequality (5–14) holds.

Next, we prove

$$c_{jk}(\tilde{\mathfrak{w}}k)\beta_k(\tilde{\mathfrak{w}}k) \ge c_{ij}(\tilde{\mathfrak{w}}k)\beta_i(\tilde{\mathfrak{w}}k). \tag{5–15}$$

We compute

$$\begin{aligned}c_{jk}(\tilde{\mathfrak{w}}k)\beta_k(\tilde{\mathfrak{w}}k) &= -c_{jk}(\tilde{\mathfrak{w}})\beta_k(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})^2\beta_j(\tilde{\mathfrak{w}}),\\ c_{ij}(\tilde{\mathfrak{w}}k)\beta_i(\tilde{\mathfrak{w}}j) &= -c_{ij}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}}) + c_{ik}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}}).\end{aligned}$$

Since have $c_{jk}(\tilde{\mathfrak{w}})\beta_j(\tilde{\mathfrak{w}}) \ge c_{ik}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}})$ by Lemma 5.15 and $c_{jk}(\tilde{\mathfrak{w}})\beta_k(\tilde{\mathfrak{w}}) \le c_{ij}(\tilde{\mathfrak{w}})\beta_i(\tilde{\mathfrak{w}})$ by Lemma 5.9, the inequality (5–15) is proven.

In a similar way, one can prove

$$\begin{aligned}c_{ik}(\tilde{\mathfrak{w}}jk)\beta_k(\tilde{\mathfrak{w}}jk) &\ge c_{ij}(\tilde{\mathfrak{w}}jk)\beta_j(\tilde{\mathfrak{w}}jk),\\ c_{jk}(\tilde{\mathfrak{w}}jk)\beta_k(\tilde{\mathfrak{w}}jk) &\ge c_{ij}(\tilde{\mathfrak{w}}jk)\beta_i(\tilde{\mathfrak{w}}jk),\end{aligned}$$

establishing the diagram.

Corollary 5.23. *We have*

$$\begin{aligned}\beta_i(\tilde{\mathfrak{w}}ki) &= s_{\tilde{\mathfrak{w}}}(s_js_i)^n s_k(s_is_j)^n(\alpha_i),\ \beta_j(\tilde{\mathfrak{w}}kj)\\ &= s_{\tilde{\mathfrak{w}}}(s_js_i)^n s_k(s_is_j)^n s_i(\alpha_j),\\ \beta_i(\tilde{\mathfrak{w}}jki) &= s_{\tilde{\mathfrak{w}}}(s_js_i)^{n+1} s_k(s_is_j)^{n+1}(\alpha_i),\ \beta_j(\tilde{\mathfrak{w}}jkj)\\ &= s_{\tilde{\mathfrak{w}}}(s_js_i)^{n+1} s_k(s_is_j)^n s_i(\alpha_j).\end{aligned}$$

Let $\tilde{\mathfrak{w}} = \mathfrak{w}i(ji)^n$ for $n \in \mathbb{Z}_{\ge 0}$ and $\mathfrak{v} = i(ji)^n$. By (4–5) – (4–8), we have

$$\begin{aligned}\phi(\mathfrak{v}ki) &= s_i(s_js_i)^{2n} s_k(s_is_j)^n\alpha_i,\ \phi(\mathfrak{v}kj)\\ &= s_i(s_js_i)^{2n} s_k(s_is_j)^n s_i\alpha_j,\\ \phi(\mathfrak{v}jki) &= s_i(s_js_i)^{2n+1} s_k(s_is_j)^{n+1}\alpha_i,\ \phi(\mathfrak{v}jkj)\\ &= s_i(s_js_i)^{2n+1} s_k(s_is_j)^n s_i\alpha_j,\end{aligned}$$

and obtain

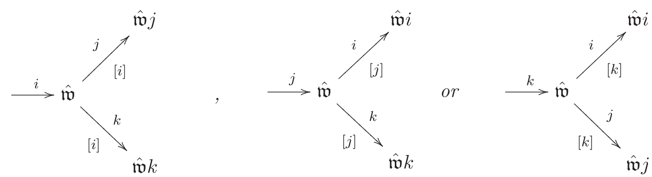
$$\begin{aligned}\beta_i(\tilde{\mathfrak{w}}ki) &= s_{\mathfrak{w}}\phi(\mathfrak{v}ki) = \psi(\mathfrak{w}\mathfrak{v}ki) = \psi(\tilde{\mathfrak{w}}ki),\\ \beta_j(\tilde{\mathfrak{w}}kj) &= s_{\mathfrak{w}}\phi(\mathfrak{v}kj) = \psi(\tilde{\mathfrak{w}}kj),\\ \beta_i(\tilde{\mathfrak{w}}jki) &= s_{\mathfrak{w}}\phi(\mathfrak{v}jki) = \psi(\tilde{\mathfrak{w}}jki),\\ \beta_j(\tilde{\mathfrak{w}}jkj) &= s_{\mathfrak{w}}\phi(\mathfrak{v}jkj) = \psi(\tilde{\mathfrak{w}}jkj).\end{aligned}$$

Thus, we have proven (4–10) in this case.

5.6. The case of $\ell(\mathfrak{w}) = \delta(\mathfrak{w}) + 3$

Write $\tilde{\mathfrak{u}} = \tilde{\mathfrak{w}}$ and $\tilde{\mathfrak{v}} = \mathfrak{v}$, or $\tilde{\mathfrak{u}} = \tilde{\mathfrak{w}}j$ and $\tilde{\mathfrak{v}} = \mathfrak{v}j$, so that $\tilde{\mathfrak{u}} = \mathfrak{w}\tilde{\mathfrak{v}}$.

Lemma 5.24. *We have*



Proof. First, we prove $c_{ij}(\tilde{\mathfrak{u}}ki)\beta_i(\tilde{\mathfrak{u}}ki) \ge c_{jk}(\tilde{\mathfrak{u}}ki)\beta_k(\tilde{\mathfrak{u}}ki)$. We have

$$\begin{aligned}c_{ij}(\tilde{\mathfrak{u}}ki)\beta_i(\tilde{\mathfrak{u}}ki) &= -c_{ij}(\tilde{\mathfrak{u}}k)\beta_i(\tilde{\mathfrak{u}}k) + c_{ij}(\tilde{\mathfrak{u}}k)c_{ik}(\tilde{\mathfrak{u}}k)\beta_k(\tilde{\mathfrak{u}}k),\\ c_{jk}(\tilde{\mathfrak{u}}ki)\beta_k(\tilde{\mathfrak{u}}ki) &= -c_{jk}(\tilde{\mathfrak{u}}k)\beta_k(\tilde{\mathfrak{u}}k) + c_{ik}(\tilde{\mathfrak{u}}k)c_{ij}(\tilde{\mathfrak{u}}k)\beta_k(\tilde{\mathfrak{u}}k).\end{aligned}$$

Since $c_{jk}(\tilde{\mathfrak{u}}k)\beta_k(\tilde{\mathfrak{u}}k) \ge c_{ij}(\tilde{\mathfrak{u}}k)\beta_i(\tilde{\mathfrak{u}}k)$ by Lemma 5.20, we are done.

Now we prove $c_{ik}(\tilde{\mathfrak{u}}ki)\beta_i(\tilde{\mathfrak{u}}ki) \ge c_{jk}(\tilde{\mathfrak{u}}ki)\beta_j(\tilde{\mathfrak{u}}ki)$. The left-hand side is

$$\begin{aligned}\mathrm{LHS} &= -c_{ik}(\tilde{\mathfrak{u}}k)\beta_i(\tilde{\mathfrak{u}}k) + c_{ik}(\tilde{\mathfrak{u}}k)^2\beta_k(\tilde{\mathfrak{u}}k)\\ &= -c_{ik}(\tilde{\mathfrak{u}})\beta_i(\tilde{\mathfrak{u}}) + c_{ik}(\tilde{\mathfrak{u}}k)^2\beta_k(\tilde{\mathfrak{u}}k),\end{aligned}$$

and the right-hand side is

$$\begin{aligned}\mathrm{RHS} &= -c_{jk}(\tilde{\mathfrak{u}}k)\beta_j(\tilde{\mathfrak{u}}k) + c_{ik}(\tilde{\mathfrak{u}}k)c_{ij}(\tilde{\mathfrak{u}}k)\beta_j(\tilde{\mathfrak{u}}k)\\ &= -c_{jk}(\tilde{\mathfrak{u}})\beta_j(\tilde{\mathfrak{u}}) + c_{ik}(\tilde{\mathfrak{u}}k)c_{ij}(\tilde{\mathfrak{u}}k)\beta_j(\tilde{\mathfrak{u}}k).\end{aligned}$$

We have $c_{ik}(\tilde{\mathfrak{u}}k)\beta_k(\tilde{\mathfrak{u}}k) \ge c_{ij}(\tilde{\mathfrak{u}}k)\beta_j(\tilde{\mathfrak{u}}k)$ by Lemma 5.20. If $\tilde{\mathfrak{u}} = \tilde{\mathfrak{w}}$, then we have $c_{jk}(\tilde{\mathfrak{u}})\beta_j(\tilde{\mathfrak{u}}) \ge c_{ik}(\tilde{\mathfrak{u}})\beta_i(\tilde{\mathfrak{u}})$ by Lemma 5.15 and we have LHS $\ge$ RHS. If $\tilde{\mathfrak{u}} = \tilde{\mathfrak{w}}j$, then we compute further and obtain

$$\begin{aligned}\mathrm{LHS} &= -c_{ik}(\tilde{\mathfrak{u}})\beta_i(\tilde{\mathfrak{u}}) + c_{ik}(\tilde{\mathfrak{u}})^2\big(-\beta_k(\tilde{\mathfrak{u}}) + c_{ik}(\tilde{\mathfrak{u}})\beta_i(\tilde{\mathfrak{u}})\big)\\ &= c_{ik}(\tilde{\mathfrak{u}})\Big(c_{ik}(\tilde{\mathfrak{u}})^2 - 1\Big)\beta_i(\tilde{\mathfrak{u}}) - c_{ik}(\tilde{\mathfrak{u}})^2\beta_k(\tilde{\mathfrak{u}}),\\ \mathrm{RHS} &= -c_{jk}(\tilde{\mathfrak{u}})\beta_j(\tilde{\mathfrak{u}}) + c_{ik}(\tilde{\mathfrak{u}})\Big(-c_{ij}(\tilde{\mathfrak{u}}) + c_{ik}(\tilde{\mathfrak{u}})c_{jk}(\tilde{\mathfrak{u}})\Big)\beta_j(\tilde{\mathfrak{u}})\\ &= c_{jk}(\tilde{\mathfrak{u}})\Big(c_{ik}(\tilde{\mathfrak{u}})^2 - 1\Big)\beta_j(\tilde{\mathfrak{u}}) - c_{ik}(\tilde{\mathfrak{u}})c_{ij}(\tilde{\mathfrak{u}})\beta_j(\tilde{\mathfrak{u}}).\end{aligned}$$

Since, we have

$$c_{ij}(\tilde{\mathfrak{u}})\beta_j(\tilde{\mathfrak{u}}) \ge c_{ik}(\tilde{\mathfrak{u}})\beta_k(\tilde{\mathfrak{u}}) \quad \text{and} \quad c_{ik}(\tilde{\mathfrak{u}})\beta_i(\tilde{\mathfrak{u}}) \ge c_{jk}(\tilde{\mathfrak{u}})\beta_j(\tilde{\mathfrak{u}})$$

by Lemma 5.9 and 5.15, we see that LHS $\ge$ RHS.

The inequalities for the second diagram can be proven similarly.

We need another lemma to complete our proof for this case. Consider two vectors (or line segments) $\vec{v_1}$ and $\vec{v_2}$, and define $\vec{v_1} * \vec{v_2}$ to be the piecewise linear curve resulting from moving $\vec{v_2}$ to a parallel position to concatenate $\vec{v_1}$ and $\vec{v_2}$ so that the end point of $\vec{v_1}$

and the starting point of $\vec{v_2}$ coincide. Assume that $\vec{v_1} * \vec{v_2}$ starts at (0, 0) and ends at a lattice point. We define

$$\mathfrak{v}(\vec{v_1} * \vec{v_2}) := p_1 \cdots p_\ell \in \mathfrak{W}$$

which records the consecutive intersections of $\vec{v_1} * \vec{v_2}$ with the sets $\mathcal{T}_{p_t}, t = 1, \ldots, \ell$ except the starting point and the ending point. This definition is compatible with (4–2) if we let $\vec{v_2} = \vec{0}$.

In the rest of the article, we will simply write $\mathfrak{v}(\vec{v})$ for $s_{\mathfrak{v}(\vec{v})} \in W$ to ease the notation.

Lemma 5.25. *Let $\mathfrak{v}pq \in \mathfrak{W}$. Then we have*

$$\mathfrak{v}(\vec{v_p}(\mathfrak{v}p))\beta(\vec{v_q}(\mathfrak{v}p)) = -\beta(\vec{v_q}(\mathfrak{v}pq)).$$

Proof. Let $\{p, q, r\} = \{1, 2, 3\}$. Clearly, we have

$$\begin{aligned}\vec{v_p}(\mathfrak{v}p) &= \frac{1}{2}\vec{v_p}(\mathfrak{v}p) + \frac{1}{2}\vec{v_r}(\mathfrak{v}p) + \frac{1}{2}\vec{v_q}(\mathfrak{v}p) \\ &= \frac{1}{2}\vec{v_q}(\mathfrak{v}pq) + \frac{1}{2}\vec{v_q}(\mathfrak{v}p).\end{aligned}$$

The curves $\vec{v_p}(\mathfrak{v}p)$ and $\frac{1}{2}\vec{v_q}(\mathfrak{v}pq) * \frac{1}{2}\vec{v_q}(\mathfrak{v}p)$ make a triangle with area $\frac{1}{4}$. Thus, there is no lattice point in the interior of the triangle. Consequently, we have $\mathfrak{v}(\vec{v_p}(\mathfrak{v}p)) = \mathfrak{v}(\frac{1}{2}\vec{v_q}(\mathfrak{v}pq) * \frac{1}{2}\vec{v_q}(\mathfrak{v}p))$.

Since the ending point of $\frac{1}{2}\vec{v_q}(\mathfrak{v}pq)$ is in $\mathcal{T}_q$, we may write

$$\mathfrak{v}\left(\frac{1}{2}\vec{v_q}(\mathfrak{v}pq) * \frac{1}{2}\vec{v_q}(\mathfrak{v}p)\right) = s_{i_1} \cdots s_{i_{k-1}} s_q s_{j_{\ell-1}} s_{j_{\ell-2}} \cdots s_{j_1},$$

where we have $\beta(\vec{v_q}(\mathfrak{v}pq)) = s_{i_1} \cdots s_{i_{k-1}} \alpha_q$ and $\beta(\vec{v_q}(\mathfrak{v}p)) = s_{j_1} \cdots s_{j_{\ell-1}} \alpha_q$. Now, we have

$$\begin{aligned}\mathfrak{v}(\vec{v_p}(\mathfrak{v}p))\beta(\vec{v_q}(\mathfrak{v}p)) &= \mathfrak{v}\left(\frac{1}{2}\vec{v_q}(\mathfrak{v}pq) * \frac{1}{2}\vec{v_q}(\mathfrak{v}p)\right)\beta(\vec{v_q}(\mathfrak{v}p)) \\ &= (s_{i_1} \cdots s_{i_{k-1}} s_q s_{j_{\ell-1}} s_{j_{\ell-2}} \cdots s_{j_1}) s_{j_1} \cdots s_{j_{\ell-1}} \alpha_q \\ &= s_{i_1} \cdots s_{i_{k-1}} s_q \alpha_q = -\beta(\vec{v_q}(\mathfrak{v}pq)).\end{aligned}$$

Corollary 5.26. *For $(p, q) = (i, j)$ or (j, i), we obtain*

$$\beta_q(\tilde{\mathfrak{u}}kpq) = s_{\mathfrak{w}}\phi(\tilde{\mathfrak{v}}kpq) = \psi(\tilde{\mathfrak{u}}kpq) \quad \text{and}$$
$$\beta_k(\tilde{\mathfrak{u}}kpk) = s_{\mathfrak{w}}\phi(\tilde{\mathfrak{v}}kpk) = \psi(\tilde{\mathfrak{u}}kpk).$$

Proof . We first show

$$c_{pq}(\tilde{\mathfrak{u}}kp) = (\beta_p(\tilde{\mathfrak{u}}kp), \beta_q(\tilde{\mathfrak{u}}kp)). \tag{5–16}$$

As the other cases are all similar, we only consider the case $\tilde{\mathfrak{u}} = \tilde{\mathfrak{w}}$ and $p = j, q = i$. We have

$$c_{pq}(\tilde{\mathfrak{u}}kp) = c_{ij}(\tilde{\mathfrak{w}}k) = -c_{ij}(\tilde{\mathfrak{w}}) + c_{ik}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}}).$$

On the other hand, since $(\beta_i(\tilde{\mathfrak{w}}), \beta_k(\tilde{\mathfrak{w}})) = -c_{ik}(\tilde{\mathfrak{w}}) + c_{ij}(\tilde{\mathfrak{w}})c_{jk}(\tilde{\mathfrak{w}})$ by Corollary 5.19, we get

$$\begin{aligned}(\beta_p(\tilde{\mathfrak{u}}kp), \beta_q(\tilde{\mathfrak{u}}kp)) &= \left(-\beta_j(\tilde{\mathfrak{w}}k) + c_{jk}(\tilde{\mathfrak{w}}k)\beta_k(\tilde{\mathfrak{w}}k), \beta_i(\tilde{\mathfrak{w}})\right) \\ &= \left(-\beta_j(\tilde{\mathfrak{w}}) - c_{jk}(\tilde{\mathfrak{w}})\beta_k(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})^2\beta_j(\tilde{\mathfrak{w}}), \beta_i(\tilde{\mathfrak{w}})\right) \\ &= -c_{ij}(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})c_{ik}(\tilde{\mathfrak{w}}) - c_{jk}(\tilde{\mathfrak{w}})^2 c_{ij}(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})^2 c_{ij}(\tilde{\mathfrak{w}}) \\ &= -c_{ij}(\tilde{\mathfrak{w}}) + c_{jk}(\tilde{\mathfrak{w}})c_{ik}(\tilde{\mathfrak{w}}).\end{aligned}$$

Thus, we have proven (5–16) in this case.

Since, we have

$$\beta_p(\tilde{\mathfrak{u}}kp) = s_{\mathfrak{w}}\beta(\vec{v_p}(\tilde{\mathfrak{v}}kp)) \quad \text{and}$$
$$\beta_q(\tilde{\mathfrak{u}}kp) = s_{\mathfrak{w}}\beta(\vec{v_q}(\tilde{\mathfrak{v}}kp)),$$

we obtain from (5–16)

$$\begin{aligned}\beta_q(\tilde{\mathfrak{u}}kpq) &= -\beta_q(\tilde{\mathfrak{u}}kp) + c_{pq}(\tilde{\mathfrak{u}}kp)\beta_p(\tilde{\mathfrak{u}}kp) \\ &= -r_{\beta_p(\tilde{\mathfrak{u}}kp)}\beta_q(\tilde{\mathfrak{u}}kp) \\ &= -s_{\mathfrak{w}}\mathfrak{v}(\vec{v_p}(\tilde{\mathfrak{v}}kp))s_{\mathfrak{w}}^{-1}s_{\mathfrak{w}}\beta(\vec{v_q}(\tilde{\mathfrak{v}}kp)) \\ &= -s_{\mathfrak{w}}\mathfrak{v}(\vec{v_p}(\tilde{\mathfrak{v}}kp))\beta(\vec{v_q}(\tilde{\mathfrak{v}}kp)).\end{aligned}$$

Now it follows from Lemma 5.25 that

$$\begin{aligned}\beta_q(\tilde{\mathfrak{u}}kpq) &= -s_{\mathfrak{w}}\mathfrak{v}(\vec{v_p}(\tilde{\mathfrak{v}}kp))\beta(\vec{v_q}(\tilde{\mathfrak{v}}kp)) \\ &= s_{\mathfrak{w}}\beta(\vec{v_q}(\tilde{\mathfrak{v}}kpq)) = s_{\mathfrak{w}}\phi(\tilde{\mathfrak{v}}kpq) = \psi(\tilde{\mathfrak{u}}kpq).\end{aligned}$$

Similarly, we have

$$c_{pk}(\tilde{\mathfrak{u}}kp) = (\beta_p(\tilde{\mathfrak{u}}kp), \beta_k(\tilde{\mathfrak{u}}kp)) \tag{5–17}$$

and compute

$$\begin{aligned}\beta_k(\tilde{\mathfrak{u}}kpk) &= -r_{\beta_p(\tilde{\mathfrak{u}}kp)}\beta_k(\tilde{\mathfrak{u}}kp) \\ &= -s_{\mathfrak{w}}\mathfrak{v}(\vec{v_p}(\tilde{\mathfrak{v}}kp))s_{\mathfrak{w}}^{-1}s_{\mathfrak{w}}\beta(\vec{v_k}(\tilde{\mathfrak{v}}kp)) \\ &= -s_{\mathfrak{w}}\mathfrak{v}(\vec{v_p}(\tilde{\mathfrak{v}}kp))\beta(\vec{v_k}(\tilde{\mathfrak{v}}kp)) \\ &= s_{\mathfrak{w}}\beta(\vec{v_k}(\tilde{\mathfrak{v}}kpk)) = s_{\mathfrak{w}}\phi(\tilde{\mathfrak{v}}kpk) = \psi(\tilde{\mathfrak{u}}kpk).\end{aligned}$$

5.7. The case of $\ell(\mathfrak{w}) \geq \delta(\mathfrak{w}) + 4$

Consider $\hat{\mathfrak{w}} \in \mathfrak{W}$ and write

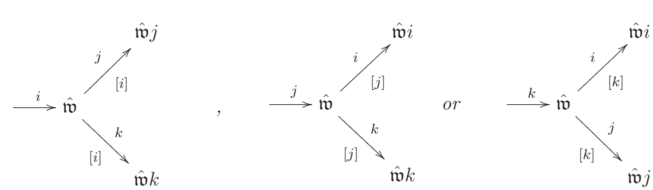
$$\hat{\mathfrak{w}} = \tilde{\mathfrak{u}}k\mathfrak{u}_1 k\mathfrak{u}_2 \ldots k\mathfrak{u}_\ell,$$

where we let

$$\mathfrak{u}_t = (ij)^{n_t}, (ji)^{n_t}, (ij)^{n_t}i \text{ or } (ji)^{n_t}j \text{ for some } n_t \geq 0$$

for $t = 1, 2, \ldots, \ell$.

Lemma 5.29. *We have*



Proof. Since the other cases are similar, we only consider the case $\mathfrak{u}_\ell = (ij)^n i$ for $n \geq 0$. We use induction on ℓ and n. When $\ell = 1$ and $n = 0$, the assertion follows from Lemma 5.24. Assume that $\ell \geq 1$ and $n \geq 1$, and suppose that $\mathfrak{u}_\ell = (ij)^n i$. First, we want to prove

$$c_{ij}(\hat{\mathfrak{w}})\beta_i(\hat{\mathfrak{w}}) \geq c_{jk}(\hat{\mathfrak{w}})\beta_k(\hat{\mathfrak{w}}). \tag{5–18}$$

Write $\mathfrak{w}' = \tilde{\mathfrak{u}}k\mathfrak{u}_1k\mathfrak{u}_2...\mathfrak{u}_{\ell-1}$. Let $\gamma = c_{ij}(\mathfrak{w}'k)$. Then, using the matrix J_n in the proof of Lemma 5.9 with new γ, we have

$$\begin{aligned}
c_{ij}(\hat{\mathfrak{w}})\beta_i(\hat{\mathfrak{w}}) &= \gamma\Big(x_n\beta_i(\mathfrak{w}'ki) - \gamma y_n\beta_j(\mathfrak{w}'ki)\Big)\\
&= \gamma x_n(-\beta_i(\mathfrak{w}'k) + c_{ik}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k)) - \gamma^2 y_n\beta_j(\mathfrak{w}'k)\\
&= -\gamma x_n\beta_i(\mathfrak{w}'k) + \gamma x_n c_{ik}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k) - \gamma^2 y_n\beta_j(\mathfrak{w}'k),\\
c_{jk}(\hat{\mathfrak{w}})\beta_k(\hat{\mathfrak{w}}) &= \Big(x_n c_{jk}(\mathfrak{w}'ki) - \gamma y_n c_{ik}(\mathfrak{w}'ki)\Big)\beta_k(\mathfrak{w}'k)\\
&= x_n\Big(-c_{jk}(\mathfrak{w}'k) + \gamma c_{ik}(\mathfrak{w}'k)\Big)\beta_k(\mathfrak{w}'k)\\
&\quad -\gamma y_n c_{ik}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k)\\
&= -x_n c_{jk}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k) + \gamma x_n c_{ik}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k)\\
&\quad -\gamma y_n c_{ik}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k).
\end{aligned}$$

Since we have, by induction,

$$c_{ik}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k) \geq \gamma\beta_j(\mathfrak{w}'k) \text{ and } c_{jk}(\mathfrak{w}'k)\beta_k(\mathfrak{w}'k) \geq \gamma\beta_i(\mathfrak{w}'k),$$

the inequality (5–17) follows.

Next we prove

$$c_{ik}(\hat{\mathfrak{w}})\beta_i(\hat{\mathfrak{w}}) \geq c_{jk}(\hat{\mathfrak{w}})\beta_j(\hat{\mathfrak{w}}). \tag{5–19}$$

Write $\mathfrak{w}'' = \tilde{\mathfrak{u}}k\mathfrak{u}_1k\mathfrak{u}_2...\mathfrak{u}_{\ell-1}ki(ji)^{n-1}j$. By induction, we have

$$\begin{aligned}
c_{ik}(\hat{\mathfrak{w}})\beta_i(\hat{\mathfrak{w}}) &= c_{ik}(\mathfrak{w}'')\Big(-\beta_i(\mathfrak{w}'') + c_{ij}(\mathfrak{w}'')\beta_j(\mathfrak{w}'')\Big)\\
&= -c_{ik}(\mathfrak{w}'')\beta_i(\mathfrak{w}'') + c_{ik}(\mathfrak{w}'')c_{ij}(\mathfrak{w}'')\beta_j(\mathfrak{w}''),\\
c_{jk}(\hat{\mathfrak{w}})\beta_j(\hat{\mathfrak{w}}) &= \Big(-c_{jk}(\mathfrak{w}'') + c_{ij}(\mathfrak{w}'')c_{ik}(\mathfrak{w}'')\Big)\beta_j(\mathfrak{w}'')\\
&= -c_{jk}(\mathfrak{w}'')\beta_j(\mathfrak{w}'') + c_{ik}(\mathfrak{w}'')c_{ij}(\mathfrak{w}'')\beta_j(\mathfrak{w}'').
\end{aligned}$$

Since $c_{jk}(\mathfrak{w}'')\beta_j(\mathfrak{w}'') \geq c_{ik}(\mathfrak{w}'')\beta_i(\mathfrak{w}'')$ by induction, we see that the inequality (5–19) holds.

We need another lemma.

Lemma 5.32. *Assume $\hat{\mathfrak{w}}$ ends with q. Then we have, for $p \neq q$,*

$$c_{pq}(\hat{\mathfrak{w}}) = \Big(\beta_p(\hat{\mathfrak{w}}), \beta_q(\hat{\mathfrak{w}})\Big).$$

Proof. We use induction. If $\hat{\mathfrak{w}} = \tilde{\mathfrak{u}}kq$, the assertion follows from (5–16) to (5–17). Now assume $\hat{\mathfrak{w}} = \check{\mathfrak{w}}rq$ for some $r \neq q$. Then, we have

$$\Big(\beta_p(\hat{\mathfrak{w}}), \beta_q(\hat{\mathfrak{w}})\Big) = \Big(\beta_p(\check{\mathfrak{w}}r), -\beta_q(\check{\mathfrak{w}}r) + c_{qr}(\check{\mathfrak{w}}r)\beta_r(\check{\mathfrak{w}}r)\Big).$$

If $p = r$, then, we have by induction

$$\begin{aligned}
\Big(\beta_p(\hat{\mathfrak{w}}), \beta_q(\hat{\mathfrak{w}})\Big) &= -c_{pq}(\check{\mathfrak{w}}r) + 2c_{pq}(\check{\mathfrak{w}}r)\\
&= c_{pq}(\check{\mathfrak{w}}r) = c_{pq}(\hat{\mathfrak{w}}),
\end{aligned}$$

and we are done.

If $p \neq r$, then, we have

$$c_{pq}(\hat{\mathfrak{w}}) = c_{pq}(\check{\mathfrak{w}}r) = -c_{pq}(\check{\mathfrak{w}}) + c_{pr}(\check{\mathfrak{w}})c_{qr}(\check{\mathfrak{w}}),$$

and obtain by induction

$$\begin{aligned}
\Big(\beta_p(\hat{\mathfrak{w}}), \beta_q(\hat{\mathfrak{w}})\Big) &= -\Big(\beta_p(\check{\mathfrak{w}}r), \beta_q(\check{\mathfrak{w}}r)\Big)\\
&\quad + c_{qr}(\check{\mathfrak{w}}r)\Big(\beta_p(\check{\mathfrak{w}}r), \beta_r(\check{\mathfrak{w}}r)\Big)\\
&= -\Big(\beta_p(\check{\mathfrak{w}}), \beta_q(\check{\mathfrak{w}})\Big) + c_{qr}(\check{\mathfrak{w}})c_{pr}(\check{\mathfrak{w}}r)\\
&= -c_{pq}(\check{\mathfrak{w}}) + c_{qr}(\check{\mathfrak{w}})c_{pr}(\check{\mathfrak{w}}).
\end{aligned}$$

This proves the desired identity.

Corollary 5.33. *Assume that $\hat{\mathfrak{w}} = \tilde{\mathfrak{u}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell \in \mathfrak{W}$ where $\mathfrak{u}_t = (ij)^{n_t}, (ji)^{n_t}, (ij)^{n_t}i$ or $(ji)^{n_t}j$ for some $n_t \geq 0$ for $t = 1, 2, ..., \ell$. Suppose that $\hat{\mathfrak{w}}$ does not end with p for $p = i, j$ or k. Then, we have*

$$\beta_p(\hat{\mathfrak{w}}p) = s_{\mathfrak{w}}\phi(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell p) = \psi(\hat{\mathfrak{w}}p). \tag{5–20}$$

Proof. With Lemma 5.29 established, the proof is very similar to that of Corollary 5.26. Suppose that $\hat{\mathfrak{w}}$ ends with q. By Lemma 5.32 and Lemma 5.25, we have

$$\begin{aligned}
\beta_p(\hat{\mathfrak{w}}p) &= -\beta_p(\hat{\mathfrak{w}}) + c_{pq}(\hat{\mathfrak{w}})\beta_q(\hat{\mathfrak{w}}) = -r_{\beta_q(\hat{\mathfrak{w}})}\beta_p(\hat{\mathfrak{w}})\\
&= -s_{\mathfrak{w}}\upsilon(\vec{v_q}(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell))s_{\mathfrak{w}}^{-1}s_{\mathfrak{w}}\beta(\vec{v_p}(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell))\\
&= -s_{\mathfrak{w}}\upsilon(\vec{v_q}(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell))\beta(\vec{v_p}(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell))\\
&= s_{\mathfrak{w}}\beta(\vec{v_p}(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell p))\\
&= s_{\mathfrak{w}}\phi(\tilde{\mathfrak{v}}k\mathfrak{u}_1k\mathfrak{u}_2...k\mathfrak{u}_\ell p) = \psi(\hat{\mathfrak{w}}p).
\end{aligned}$$

This completes the proof of Theorem 4.17.

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