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Rigid reflections and Kac-Moody algebras

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Abstract Given any Coxeter group, we define rigid reflections and rigid roots using non-self-intersecting curves on a Riemann surface with labeled curves. When the Coxeter group arises from an acyclic quiver, they are related to the rigid representations of the quiver. For a family of rank 3 Coxeter groups, we show that there is a surjective map from the set of reduced positive roots of a rank 2 Kac-Moody algebra onto the set of rigid reflections. We conjecture that this map is bijective.

Keywords Coxeter groups, rigid reflections, rigid roots, non-self-crossing curves, Kac-Moody algebras

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1 Introduction

Let Q be an acyclic quiver of rank n, i.e., a quiver with n vertices and without oriented cycles, and $\mathrm{mod}(Q)$ be the category of finite dimensional representations of Q, or equivalently, the category of finite dimensional modules over the path algebra of Q. Among the objects in $\mathrm{mod}(Q)$, the indecomposable representations M with $\mathrm{Ext}^1(M,M)=0$ are called *rigid* and their dimension vectors are called *real Schur roots*. They play prominent roles in understanding the category $\mathrm{mod}(Q)$. Moreover, the real Schur roots form an important subset of the set of positive real roots of the Kac-Moody algebra $\mathfrak{g}(Q)$ associated with Q.

In an attempt to establish a form of homological mirror symmetry [4], we proposed in a previous paper [5] a correspondence between rigid representations in $\mathrm{mod}(Q)$ and the set of certain non-self-intersecting curves on a Riemann surface Σ with n labeled curves. The conjecture is now proven by Felikson and Tumarkin [1] for 2-*complete* quivers Q where, by definition, every pair of vertices in Q is connected by more than two edges. However, it is wide open for general acyclic quivers.

The conjectural correspondence factors through a family of reflections in the Weyl group of $\mathfrak{g}(Q)$ to relate non-self-intersecting curves in Σ with real Schur roots. Since reflections make sense for any Coxeter

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groups, one can consider such a family of reflections in an arbitrary Coxeter group. Indeed, let W be a Coxeter group with n ordered generators:

$$W = \langle s_1, s_2, \dots, s_n : s_1^2 = \cdots = s_n^2 = e, (s_i s_j)^{m_{ij}} = e \ (i \neq j) \rangle,$$

where $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$. Then we define *rigid reflections* in W using non-self-intersecting curves on the Riemann surface Σ and *rigid roots* to be the associated positive roots to the rigid reflections in the set of positive roots of W (see Definitions 2.2 and 2.5). These definitions are in line with the conjectural correspondence. In particular, when $m_{ij} = \infty$ for all $i \neq j$, the set of rigid reflections is in bijection with the set of rigid representations of any 2-complete acyclic quiver Q of rank n.

This paper is concerned with an unexpected, surprising phenomenon that the rigid roots of W are parametrized by the positive roots of a seemingly unrelated Kac-Moody algebra $\mathcal{H}$. This phenomenon seems true for a wide range of Coxeter groups W, and we will show this for a family of rank 3 Coxeter groups in this paper.

To be precise, fix a positive integer $m \geqslant 2$ and consider the following Coxeter group:

$$W(m) = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^m = (s_2 s_3)^m = e \rangle.$$

Here, we put $m_{13} = m_{31} = \infty$ as the usual convention. Let $\mathcal{H}(m)$ be the rank 2 Kac-Moody algebra associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -m \\ -m & 2 \end{smallmatrix}\right)$. We denote an element of the root lattice of $\mathcal{H}(m)$ by $[a, b]$, $a, b \in \mathbb{Z}$, where $[1, 0]$ and $[0, 1]$ are the positive simple roots. A root $[a, b]$ of $\mathcal{H}(m)$ is called *reduced* if $\gcd(a, b) = 1$ and $ab \neq 0$. A reduced root determines a non-self-intersecting curve η on the torus Σ with triangulation by three labeled curves. Then we define a function, $[a, b] \mapsto s([a, b]) \in W$, by reading off the labels of the intersection points of η with the labeled curves on Σ, and make the following conjecture.

Conjecture 1.1. *For $m \geqslant 2$, the function, $[a, b] \mapsto s([a, b])$, is a bijection from the set of reduced positive roots of $\mathcal{H}(m)$ to the set of rigid reflections of $W(m)$.*

The case $m = 2$ will be verified in Example 2.9, and the case $m = 3$ will be established in a forthcoming paper[1)] where mutations of quivers and cluster variables will be exploited. As the main result of this paper, we prove the following theorem.

Theorem 1.2. *For $m \geqslant 2$, the function in Conjecture* 1.1 *is a surjection.*

Our proof of Theorem 1.2 shows that the Weyl group of $\mathcal{H}(m)$, which is isomorphic to the infinite dihedral group, governs the symmetries of the set of rigid reflections of $W(m)$, and utilizes these symmetries to make an induction argument work on the values of the square norm of $[a, b]$. It is intriguing that such a nice structure dwells in the set of rigid reflections.

The organization of this paper is as follows. In Section 2, we define rigid reflections and rigid roots and provide examples. After introducing notations for rank 2 Kac-Moody algebras $\mathcal{H}(m)$, we state the main theorem and illustrate with examples. In particular, the case $m = 2$ is completely described. Section 3 is devoted to a proof of the main theorem. We first establish several lemmas, and a main step is achieved in Proposition 3.9 whose proof is an inductive algorithm for, given $[a, b]$ in the positive root lattice, how to find a reduced positive root $[a_0, b_0]$ of $\mathcal{H}(m)$ with the same rigid reflections, i.e., $s([a_0, b_0]) = s([a, b])$. Then Lemma 3.11 shows that it is enough to consider the positive root lattice, which completes the proof.

2 Rigid reflections and the main theorem

As in the introduction, let

$$W = \langle s_1, s_2, \dots, s_n : s_1^2 = \cdots = s_n^2 = e, \ (s_i s_j)^{m_{ij}} = e \rangle$$

be a Coxeter group with $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$. In this section, after the rigid reflections in W and rigid roots in the root system of W are defined, the main theorem of this paper will be stated.

1) Lee K-H, Lee K. A correspondence between rigid modules over path algebras and simple curves on Riemann surfaces II. In preparation

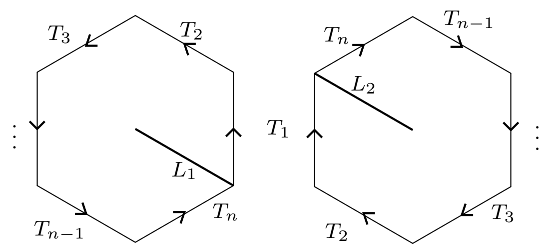

Figure 1 Two copies of a regular n-gon

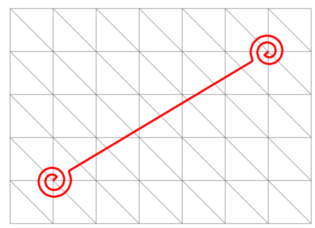

Figure 2 (Color online) Non-self-crossing admissible curve for $n = 3$

To begin with, we need a Riemann surface Σ equipped with n labeled curves as below. Let G_1 and G_2 be two identical copies of a regular n-gon. Label the edges of each of the two n-gons by $T_1, T_2, \ldots, T_n$ counter-clockwise. On G_i $(i = 1, 2)$, let L_i be the line segment from the center of G_i to the common endpoint of T_n and T_1. Later, these line segments will only be used to designate the end points of admissible curves and will not be used elsewhere. Fix the orientation of every edge of G_1 (resp. G_2) to be counter-clockwise (resp. clockwise) as in Figure 1.

Let Σ be the Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ obtained by gluing together the two n-gons with all the edges of the same label identified according to their orientations. The edges of the n-gons become n different curves in Σ. If n is odd, all the vertices of the two n-gons are identified to become one point in Σ and the curves obtained from the edges become loops. If n is even, two distinct vertices are shared by all curves. Let $\mathcal{T} = T_1 \cup \cdots \cup T_n \subset \Sigma$, and V be the set of the vertex (or vertices) on $\mathcal{T}$.

Let $\mathfrak{W}$ be the set of words from the alphabet $\{1, 2, \ldots, n\}$, and let $\mathfrak{R} \subset \mathfrak{W}$ be the subset of words $\mathfrak{w} = i_1 i_2 \cdots i_k$ such that k is an odd integer and $i_j = i_{k+1-j}$ for all $j \in \{1, \ldots, k\}$, in other words, $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reflection in W. For $\mathfrak{w} = i_1 i_2 \cdots i_k \in \mathfrak{W}$, denote $s_{i_1} \cdots s_{i_k} \in W$ by $s(\mathfrak{w})$.

Definition 2.1. An *admissible* curve is a continuous function $\eta : [0, 1] \to \Sigma$ such that

(1) $\eta(x) \in V$ if and only if $x \in \{0, 1\}$;

(2) η starts and ends at the common end point of T_1 and T_n. More precisely, there exists $\epsilon > 0$ such that $\eta([0, \epsilon]) \subset L_1$ and $\eta([1 - \epsilon, 1]) \subset L_2$;

(3) if $\eta(x) \in \mathcal{T} \setminus V$ then $\eta([x - \epsilon, x + \epsilon])$ meets $\mathcal{T}$ transversally for sufficiently small $\epsilon > 0$.

If η is admissible, then we obtain $\upsilon(\eta) := i_1 \cdots i_k \in \mathfrak{W}$ given by

$$\{x \in (0, 1) : \eta(x) \in \mathcal{T}\} = \{x_1 < \cdots < x_k\} \quad \text{and} \quad \eta(x_\ell) \in T_{i_\ell} \quad \text{for} \quad \ell \in \{1, \ldots, k\}.$$

Note that the word $\upsilon(\eta)$ only depends on the isotopy class of the admissible curve, and the group element $s(\upsilon(\eta))$ does not change even if we allow non-transversal intersections with $\mathcal{T}$ in the isotopy, because all that can happen in a generic one parameter family is a simple tangency, which inserts/removes $s_i s_i$ somewhere in a presentation of $s(\upsilon(\eta))$.

Conversely, note that for every $\mathfrak{w} \in \mathfrak{W}$, there is an admissible curve η with $\upsilon(\eta) = \mathfrak{w}$. Hence, every element in W can be represented by some admissible curve(s). For brevity, let $s(\eta) := s(\upsilon(\eta))$.

Definition 2.2. An element $s_{i_1} s_{i_2} \cdots s_{i_k}$ in W is called a *rigid reflection* if there exists a non-self-crossing admissible curve η with $\upsilon(\eta) = i_1 \cdots i_k \in \mathfrak{R}$.

Example 2.3. Let $n = 3$, and $W = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = e\rangle$, i.e., $m_{ij} = \infty$ for $i \neq j$. Consider the universal cover of Σ and a curve η as in Figure 2.

Here each horizontal line segment represents T_1, vertical T_3, and diagonal T_2. One sees that η has no self-intersection in Σ. Thus we obtain the corresponding rigid reflection

$$s(\eta) = (s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2(s_1s_2s_3)^4.$$

On the other hand, the reflection $s_2s_3s_1s_3s_2$ comes from the curve η' on the left of Figure 3, which has a self-intersection. The picture on the right of Figure 3 shows several copies of η' on the universal cover.

Consequently, the reflection $s(\eta') = s_2s_3s_1s_3s_2$ is *not* rigid.

Example 2.4. Let $n = 8$, and we have a rigid reflection

$$(s_8s_7\cdots s_2s_1)^5(s_8s_7\cdots s_2)s_1(s_2\cdots s_7s_8)(s_1s_2\cdots s_7s_8)^5,$$

which corresponds to non-self-intersecting curve on Σ which is shown in Figure 4.

Let Φ be the root system of W, realized in the real vector space $\mathbf{E}$ with basis $\{\alpha_1, \ldots, \alpha_n\}$ with the symmetric bilinear form B defined by

$$B(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij}) \quad \text{for} \quad 1 \leqslant i, j \leqslant n.$$

For each $i \in \{1, \ldots, n\}$, define the action of s_i on $\mathbf{E}$ by

$$s_i(\lambda) = \lambda - 2B(\lambda, \alpha_i)\alpha_i, \quad \lambda \in \mathbf{E},$$

and extend it to the action of W on $\mathbf{E}$. Then each root $\alpha \in \Phi$ determines a reflection $s_\alpha \in W$ (see [2] for more details).

Definition 2.5. A positive root $\alpha \in \Phi$ of W is called *rigid* if the corresponding reflection $s_\alpha \in W$ is rigid.

Example 2.6. In Example 2.3, we obtained the rigid reflection

$$(s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2(s_1s_2s_3)^4.$$

It give rises to a rigid root

$$1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3 = (s_3s_2s_1)^4 s_2s_3s_2s_1s_2s_3\alpha_2.$$

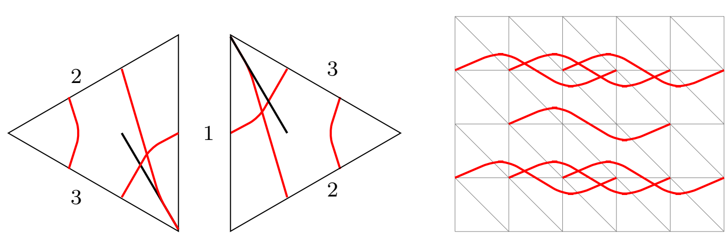

Figure 3 (Color online) Self-crossing admissible curve for $n = 3$

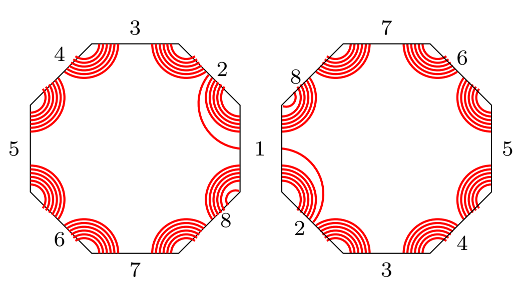

Figure 4 (Color online) Non-self-crossing admissible curve for $n = 8$

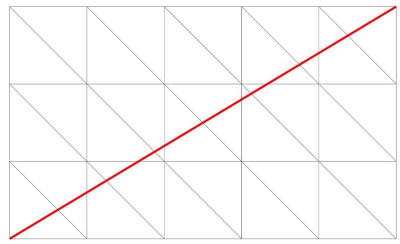

Figure 5 (Color online) Line segment $\eta([5,3])$

Fix a positive integer $m \geqslant 2$. As in Section 1, we set

$$W(m) = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^m = (s_2 s_3)^m = e \rangle.$$

Note that we put, in particular, $m_{13} = m_{31} = \infty$. Let $\mathcal{H}(m)$ be the rank 2 hyperbolic Kac-Moody algebra associated with the Cartan matrix $\left(\begin{smallmatrix} 2 & -m \\ -m & 2 \end{smallmatrix}\right)$. We denote an element of the root lattice of $\mathcal{H}(m)$ by $[a,b]$, $a, b \in \mathbb{Z}$, where $[1,0]$ and $[0,1]$ are the positive simple roots. A root $[a,b]$ of $\mathcal{H}(m)$ is called *reduced* if $\gcd(a,b) = 1$ and $ab \neq 0$. One can see that every non-simple real root is reduced.

Let $\mathcal{P}^+ = \{[a,b] : a, b \in \mathbb{Z}_{>0}, \gcd(a,b) = 1\}$. For every $[a,b] \in \mathcal{P}^+$, let $\eta([a,b])$ be the line segment from $(0,0)$ to (a,b) on the universal cover of the torus, which has no self-intersections. Write $s([a,b]) := s(\eta([a,b])) \in W(m)$ for the corresponding rigid reflection. For example, we have

$$s([5,3]) = s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3 s_2 s_1 s_2 s_3 s_2$$

as one can check in Figure 5.

With these definitions, we now state the main theorem of this paper.

Theorem 2.7. *The function, $[a,b] \mapsto s([a,b])$, is an onto function from the set of reduced positive roots of $\mathcal{H}(m)$ to the set of rigid reflections of $W(m)$.*

Equivalently, if we let $\beta([a,b])$ be the rigid root determined by the rigid reflection $s([a,b])$, then the above theorem asserts that the function, $[a,b] \mapsto \beta([a,b])$, is an onto function from the set of reduced positive roots of $\mathcal{H}(m)$ to the set of rigid roots of $W(m)$.

A proof of Theorem 2.7 will be given in the next section. In the rest of this section we will present some examples. Recall from [3] that

$$[a,b] \text{ is a root of } \mathcal{H}(m) \quad \text{if and only if} \quad a^2 + b^2 - mab \leqslant 1. \tag{2.1}$$

We will use this fact in the following example without further mentioning it.

Example 2.8. (1) Let $m = 3$. Consider the rigid reflection $s([4,1]) = s_2 s_3 s_2 s_3 s_2 s_3 s_2 = s_2$ and its rigid root $\beta([4,1]) = \alpha_2$. The point $[4,1]$ is not a root of $\mathcal{H}(3)$. However, these are covered by the root $[1,1]$ of $\mathcal{H}(3)$ since $s([1,1]) = s_2$ and $\beta([1,1]) = \alpha_2$.

One can also check $s([30,11]) = s_2 s_3 s_2 s_1 s_2 s_3 s_2 = s([3,2])$ and $\beta([30,11]) = \alpha_1 + 3\alpha_2 + 3\alpha_3 = \beta([3,2])$. Here, $[30,11]$ is not a root of $\mathcal{H}(3)$, whereas $[3,2]$ is.

(2) Now let $m = 4$. Then we have

$$s([5,2]) = s([13,2]) = s_2 s_3 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3 s_2 \quad \text{and} \quad \beta([5,2]) = \beta([13,2]) = \alpha_1 + 3\sqrt{2}\alpha_2 + 6\alpha_3.$$

Here, $[13,2]$ is not a root of $\mathcal{H}(4)$, but $[5,2]$ is a root.

(3) For a general m, let $x = 2\cos(\pi/m)$. Then $s([5,3]) = s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3 s_2 s_1 s_2 s_3 s_2$ and

$$\beta([5,3]) = (x^3 + x)\alpha_1 + (x^6 + 3x^4 + 2x^2 - 1)\alpha_2 + (x^5 + 3x^3 + 2x)\alpha_3.$$

Example 2.9. Assume that $m = 2$. Then the Kac-Moody algebra $\mathcal{H}(2)$ is the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ and its set of reduced positive roots is given by $\{[n, n+1], [n+1, n], [1,1] : n \geqslant 1\}$. On the other hand, since s_2 commutes with s_1 and s_3 in $W(2)$, we see that the set of rigid reflections in $W(2)$ is

$$\{s_1 (s_3 s_1)^{n-1},\ s_3 (s_1 s_3)^{n-1},\ s_2 : n \geqslant 1\},$$

and that the set of rigid roots of $W(2)$ is

$$\{n\alpha_1 + (n-1)\alpha_3,\ (n-1)\alpha_1 + n\alpha_3,\ \alpha_2 : n \geqslant 1\}.$$

Applying the maps $s(\cdot)$ and $\beta(\cdot)$ to the set of reduced positive roots, we obtain, for $n \geqslant 1$,

$$s([n,n+1]) = s_1(s_3s_1)^{n-1},\quad s([n+1,1]) = s_3(s_1s_3)^{n-1},\quad s([1,1]) = s_2,$$
$$\beta([n,n+1]) = n\alpha_1 + (n-1)\alpha_3,\quad \beta([n+1,1]) = (n-1)\alpha_1 + n\alpha_3,\quad \beta([1,1]) = \alpha_2.$$

Therefore, the maps are clearly bijections, and Conjecture 1.1 is verified in this case $m = 2$.

3 Proofs of the main results

In this section, we prove Theorem 2.7. The last lemma (Lemma 3.11) enables us to focus only on the positive root lattice $\mathcal{P}^+$. Lemma 3.6 shows that we can use a certain transformation to preserve rigid roots. Lemmas 3.2 and 3.8 guarantee this transformation to work inductively, and the inductive algorithm is given in Proposition 3.9. We explain the algorithm in Example 3.10.

Define a sequence F_n recursively by $F_0 = 0, F_1 = 1$, and $F_n = mF_{n-1} - F_{n-2}$. Define another sequence E_n by $E_0 = E_1 = 1$ and $E_n = mE_{n-1} - E_{n-2}$. Below we collect some general facts about these sequences, which will be frequently used for the rest of the paper.

Lemma 3.1. (a) *The sequence $F_0, F_1, \ldots, F_n, \ldots$ is monotone, and the pairs (F_n, F_{n-1}) run through all the integer points on the quadric $x^2 + y^2 - mxy = 1$ subject to the condition $x > y \geqslant 0$.*

(b)

$$\det\begin{pmatrix} F_n & F_{n-1} \\ F_{n+1} & F_n \end{pmatrix} = 1.$$

In particular, we get

$$\frac{F_0}{F_1} < \frac{F_1}{F_2} < \cdots < \frac{F_n}{F_{n+1}} < \cdots.$$

(c) *For every $[a,b] \in \mathcal{P}^+$ such that $a > b$ and $a^2 + b^2 - mab > 1$, there exits a unique n such that*

$$\frac{F_{n-1}}{F_n} < \frac{b}{a} < \frac{F_n}{F_{n+1}}.$$

(d) *We have*

$$\frac{E_{n-1}}{E_n} > \frac{F_{n-1}}{F_n},\quad F_n = (m-1)F_{n-1} + E_{n-1},$$

and

$$\det\begin{pmatrix} E_{n-1} & E_n \\ F_{n-1} & F_n \end{pmatrix} = 1.$$

Proof. (a) Let

$$H = \{(x,y) \in \mathbb{Z}^2_{\geqslant 0} : x > y \text{ and } x^2 + y^2 - mxy = 1\}.$$

Suppose that $(x,y) \in H$. Then $y^2 + (my - x)^2 - my(my - x) = 1$. We also have $y > my - x$, because if $y \leqslant my - x$ then $y^2 - 1 = x(my - x) \geqslant xy > y^2$, which is absurd. Hence $(y, my - x) \in H$. Iterate this until we get $(1,0) \in H$. By backtracking, we get $(x,y) = (F_n, F_{n-1})$ for some n, and $F_0 < F_1 < \cdots < F_n \cdots$.

(b) See [7, Lemma 3.1].

(c) The sequence $0 = \frac{F_0}{F_1}, \frac{F_1}{F_2}, \ldots, \frac{F_n}{F_{n+1}}, \ldots$ converges to $\frac{m - \sqrt{m^2-4}}{2}$, because

$$\lim_{n\to\infty}\left(\left(\frac{F_n}{F_{n+1}}\right)^2 - m\left(\frac{F_n}{F_{n+1}}\right) + 1\right) = \lim_{n\to\infty}\frac{1}{F_{n+1}^2} = 0.$$

Since $a^2+b^2-mab>1$ implies $(\frac{b}{a})^2-m(\frac{b}{a})+1>0$, we get $\frac{b}{a}<\frac{m-\sqrt{m^2-4}}{2}$ or $\frac{b}{a}>\frac{m+\sqrt{m^2-4}}{2}$. The inequality $a>b$ implies $\frac{b}{a}<\frac{m-\sqrt{m^2-4}}{2}$. By (a), we have $(a,b)\notin H$, so $(a,b)\neq(F_n,F_{n-1})$ for any n. Actually $(a,b)\neq(gF_n,gF_{n-1})$ for any g and n, since $\gcd(a,b)=1$. Therefore there exits a unique n such that

$$\frac{F_{n-1}}{F_n}<\frac{b}{a}<\frac{F_n}{F_{n+1}}.$$

(d) It is easy to see $F_n=(m-1)F_{n-1}+E_{n-1}$ by induction on n. This implies

$$\det\begin{pmatrix}E_{n-1} & E_n\\ F_{n-1} & F_n\end{pmatrix}=1$$

and $\frac{E_{n-1}}{E_n}>\frac{F_{n-1}}{F_n}$. □

The following lemma will be used for the inductive algorithm in the proof of Proposition 3.9.

Lemma 3.2. *Assume that $[a,b]\in\mathcal{P}^+$ is not a root of $\mathcal{H}(m)$, and that*

$$\frac{F_{n-1}}{F_n}<\frac{b}{a}<\frac{F_n}{F_{n+1}}.$$

We have

$$\text{either}\quad [a,b]-m(-F_{n-1}a+F_nb)[F_n,F_{n-1}]\in\mathcal{P}^+,\quad \text{or}\quad [a,b]+m(-F_na+F_{n+1}b)[F_{n+1},F_n]\in\mathcal{P}^+.$$

Proof. The properties of the sequence $\{F_i\}$ imply that $[a,b]=\alpha[F_n,F_{n-1}]+\beta[F_{n+1},F_n]$, where α and β are positive integers. Moreover, we have

$$\beta=\det\begin{pmatrix}F_n & F_{n-1}\\ a & b\end{pmatrix}=-F_{n-1}a+F_nb$$

and

$$\alpha=\det\begin{pmatrix}a & b\\ F_{n+1} & F_n\end{pmatrix}=F_na-F_{n+1}b.$$

Therefore

$$\begin{aligned}[c,d]&:=[a,b]-m\beta[F_n,F_{n-1}]=\alpha[F_n,F_{n-1}]+\beta[F_{n+1},F_n]-m\beta[F_n,F_{n-1}]\\&=[\alpha F_n-\beta F_{n-1},\alpha F_{n-1}-\beta F_{n-2}],\end{aligned}$$

and

$$\begin{aligned}[e,f]&:=[a,b]-m\alpha[F_{n+1},F_n]=\alpha[F_n,F_{n-1}]+\beta[F_{n+1},F_n]-m\alpha[F_{n+1},F_n]\\&=[\beta F_{n+1}-\alpha F_{n+2},\beta F_n-\alpha F_{n+1}],\end{aligned}$$

where we used the recursive relation $F_n=mF_{n-1}-F_{n-2}$ four times. Now the property

$$\frac{F_{n-2}}{F_{n-1}}<\frac{F_{n-1}}{F_n}<\frac{F_n}{F_{n+1}}<\frac{F_{n+1}}{F_{n+2}}$$

implies that

$$\begin{aligned}&d=\alpha F_{n-1}-\beta F_{n-2}<0\\&\quad\Rightarrow c=\alpha F_n-\beta F_{n-1}<0\\&\quad\Rightarrow -f=\alpha F_{n+1}-\beta F_n<0\\&\quad\Rightarrow -e=\alpha F_{n+2}-\beta F_{n+1}<0.\end{aligned}$$

In particular, if at least one of the numbers c and d is negative, then both e and f are positive. It remains to show that $\gcd(c,d)=1$ and $\gcd(e,f)=1$.

Let $u = F_{n-1}, v = F_n$ and $w = F_{n+1}$ for convenience. Then we have $mv = u + w$, $v^2 = 1 + uw$ and

$$[c,d] = [(1+muv)a - mv^2b, mu^2a + (1-muv)b],$$
$$[e,f] = [(1-mvw)a + mw^2b, -mv^2a + (1+mvw)b].$$

Note that the matrices

$$\begin{pmatrix} 1+muv & -mv^2 \\ mu^2 & 1-muv \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1-mvw & mw^2 \\ -mv^2 & 1+mvw \end{pmatrix}$$

have determinant 1. Since $\gcd(a,b) = 1$, we have $\gcd(c,d) = 1$ and $\gcd(e,f) = 1$. □

Let (a_1, a_2) be a pair of nonnegative integers, not both zero, with $a_1 \geqslant a_2$. A *maximal Dyck path* of type $a_1 \times a_2$, denoted by $\mathcal{D}^{a_1\times a_2}$, is a lattice path from $(0,0)$ to (a_1, a_2) that is as close as possible to the diagonal joining $(0,0)$ and (a_1, a_2) without ever going above it. Assign $s_2s_3 \in W(m)$ to each horizontal edge of $\mathcal{D}^{a_1\times a_2}$, and $s_2s_1 \in W(m)$ to each vertical edge. Read these elements in the order of edges along $\mathcal{D}^{a_1\times a_2}$. Then we get a product of copies of s_2s_3 and s_2s_1. Denote the product by $s^{a_1\times a_2}$.

Let σ_1 and σ_2 be the simple reflections of $\mathcal{H}(m)$ associated with the simple roots $[1,0]$ and $[0,1]$, respectively. Then they act on $[a,b] \in \mathbb{Z}^2$ in the usual way by

$$\sigma_1[a,b] = [-a+mb, b] \quad \text{and} \quad \sigma_2[a,b] = [a, -b+ma].$$

Lemma 3.3. *Assume that $[a,b] \in \mathcal{P}^+$ with $a \geqslant b$, and write $[c,d] = \sigma_1\sigma_2[a,b]$. Then we have*

(1) $s([a,b]) = s_3s_2s^{a\times b}s_1$;

(2) $s_1s_3s_2s^{a\times b}s_2s_3s_1 = s^{c\times d}$;

(3) $s_3s_2s_1s([a,b])s_1s_2s_3 = s([c,d])$.

Proof. (1) This is straightforward.

(2) Given $\mathcal{D}^{a\times b}$, we replace a horizontal step, which is followed by another horizontal step, with $\mathcal{D}^{(m^2-1)\times m}$ and a two-step path with horizontal step and an immediate vertical step with $\mathcal{D}^{(m^2-m-1)\times(m-1)}$. Then the resulting path is $\mathcal{D}^{c\times d}$. This transformation of Dyck paths can also be obtained from a sequence of Dyck path mutations considered in [6, Section 3].

For example, when $m = 3$, we have Figure 6, and obtain $\mathcal{D}^{13\times 5}$ from $\mathcal{D}^{2\times 1}$ through the transformation in Figure 7. We consider the associated Coxeter group elements

$$\begin{aligned} s^h &:= s^{(m^2-1)\times m} = s([m,1])^{m-1}s([m-1,1]) \\ &= ((s_2s_3)^m s_2s_1)^{m-1}(s_2s_3)^{m-1}s_2s_1 = (s_2s_1)^{m-1}s_3s_1 = s_1s_2s_3s_1, \\ s^{hv} &:= s^{(m^2-m-1)\times(m-1)} = s([m,1])^{m-2}s([m-1,1]) = s_1s_2s_1s_2s_3s_1, \end{aligned}$$

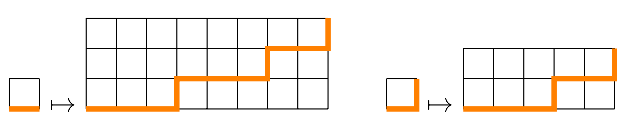

Figure 6 (Color online) Transformations of Dyck paths

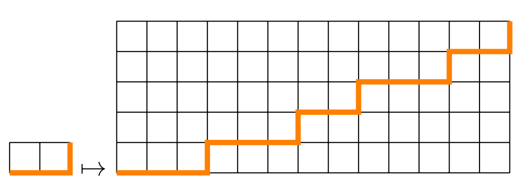

Figure 7 (Color online) The transformation of a Dyck path

and obtain

$$s_2s_3s_1s^h s_1s_3s_2 = (s_2s_3s_1)(s_1s_2s_3s_1)(s_1s_3s_2) = s_2s_3,$$
$$s_2s_3s_1s^{hv} s_1s_3s_2 = (s_2s_3s_1)(s_1s_2s_1s_2s_3s_1)(s_1s_3s_2) = (s_2s_3)(s_2s_1).$$

This proves the assertion.

(3) This is an immediate consequence of the parts (1) and (2). □

Lemma 3.4. *We have the following formulas*:

(a) $s^{F_2\times F_1} = s_2s_1$ *and* $s^{E_2\times E_1} = s_3s_1$. *Moreover,*

$$s^{F_n\times F_{n-1}} = \begin{cases} s_1(s_3s_2s_1)^{(n-3)/2}s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1, & \text{for } n \geqslant 3 \text{ odd,} \\ s_1(s_3s_2s_1)^{(n-4)/2}s_3s_1s_2s_3(s_1s_2s_3)^{(n-4)/2}s_1, & \text{for } n \geqslant 4 \text{ even,} \end{cases}$$

$$s^{E_n\times E_{n-1}} = \begin{cases} s_1(s_3s_2s_1)^{(n-3)/2}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1, & \text{for } n \geqslant 3 \text{ odd,} \\ s_1(s_3s_2s_1)^{(n-4)/2}s_3s_2s_3s_1s_2s_3(s_1s_2s_3)^{(n-4)/2}s_1, & \text{for } n \geqslant 4 \text{ even.} \end{cases}$$

(b) $(s_3s_2s_1s([F_n,F_{n-1}]))^m = (s_1s_2s_3s([F_n,F_{n-1}]))^m = e$ *for all* $n \geqslant 1$.

Proof. (a) Let $\mathcal{F}_n := \mathcal{D}^{F_n\times F_{n-1}}$ and $\mathcal{E}_n := \mathcal{D}^{E_n\times E_{n-1}}$. We use induction on n. It is easy to check the base cases. Suppose $n \geqslant 3$. Then the Dyck path $\mathcal{F}_n$ consists of $m-1$ copies of $\mathcal{F}_{n-1}$ followed by one copy of $\mathcal{E}_{n-1}$. This is because $((m-1)F_{n-1},(m-1)F_{n-2})$ is below the diagonal, and Pick's theorem implies that there is no integral point in the interior of the triangle formed by $(0,0),(F_n,F_{n-1})$, and $((m-1)F_{n-1},(m-1)F_{n-2})$. Similarly, the Dyck path $\mathcal{E}_n$ consists of $m-2$ copies of $\mathcal{F}_{n-1}$ followed by one copy of $\mathcal{E}_{n-1}$. It is straightforward to check the induction process as follows.

Suppose that n is odd. From the induction hypothesis, we have

$$s^{F_n\times F_{n-1}} = s_1(s_3s_2s_1)^{(n-3)/2}s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1,$$
$$s^{E_n\times E_{n-1}} = s_1(s_3s_2s_1)^{(n-3)/2}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1.$$

Then

$$\begin{aligned} s^{F_{n+1}\times F_n} &= (s_1(s_3s_2s_1)^{(n-3)/2}s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1)^{m-1}(s_1(s_3s_2s_1)^{(n-3)/2}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1) \\ &= s_1(s_3s_2s_1)^{(n-3)/2}(s_2s_3)^{m-1}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1 \\ &= s_1(s_3s_2s_1)^{(n-3)/2}s_3s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1, \end{aligned}$$

and

$$\begin{aligned} s^{E_{n+1}\times E_n} &= (s_1(s_3s_2s_1)^{(n-3)/2}s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1)^{m-2}(s_1(s_3s_2s_1)^{(n-3)/2}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1) \\ &= s_1(s_3s_2s_1)^{(n-3)/2}(s_2s_3)^{m-2}s_2s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1 \\ &= s_1(s_3s_2s_1)^{(n-3)/2}s_3s_2s_3s_1s_2s_3(s_1s_2s_3)^{(n-3)/2}s_1. \end{aligned}$$

The other case can be similarly proved.

(b) Each of $s_3s_2s_1s([F_n,F_{n-1}]) = s_3s_2s_1s_3s_2s^{F_n\times F_{n-1}}s_1$ and $s_1s_2s_3s([F_n,F_{n-1}]) = s_1s^{F_n\times F_{n-1}}s_1$ is a conjugate of one of s_1s_2, s_2s_1, s_2s_3, or s_3s_2, which implies the statement. □

Remark 3.5. Notice that $s_3s_2s_1$ can be considered as a curve going around below an integral point and $s_1s_2s_3$ going around above an integral point (see the illustrations in Figure 8).

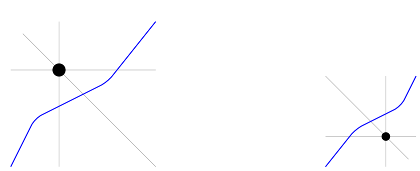

Figure 8 (Color online) Curves going around an integral point

This fact will be used frequently in the proof of the following lemma.

Lemma 3.6. *For a fixed positive integer n and $[a,b]\in\mathcal{P}^+$ with $a>b$, let $\kappa=-F_{n-1}a+F_nb$. Assume that*

$$a':=a-m|\kappa|F_n>0 \quad \textit{and} \quad b':=b-m|\kappa|F_{n-1}>0.$$

Then

$$s([a',b'])=s([a,b]).$$

Proof. If $\kappa=0$ then trivial. Here, we give a proof for the case of $\kappa>0$, as the $\kappa<0$ case is similar. We need to be able to locate the integral points inside the triangle, say T, formed by $(0,0)$, (a,b) and (a',b'). Pick's theorem implies that there are exactly $m\binom{\kappa}{2}$ integral points in the interior of T. For each $i\in\{1,\ldots,\kappa\}$, let $L_i\subset\mathbb{R}^2$ be the line segment from $(\frac{i}{\kappa}a',\frac{i}{\kappa}b')$ to $(\frac{i}{\kappa}a,\frac{i}{\kappa}b)$. For technical simplicity, assume that L_κ is open ended at (a,b) so that L_κ contains exactly $m\kappa$ integral points. Also note that $\kappa>0$ implies $\frac{b}{a}<\frac{b'}{a'}$.

First, observe that all integer points inside T have to be on the intervals L_i. Indeed, if (c,d) is an integer point inside T, consider the triangle with vertices (a,b), (c,d), and $(a-F_n,b-F_{n-1})$ (the integer point on the interval L_κ, next to (a,b)). Suppose that its area is $i/2$, where i is an integer. Note that the area of the triangle with vertices $(0,0)$, (a,b), and $(a-F_n,b-F_{n-1})$ equals

$$\frac{1}{2}\det\begin{pmatrix}F_n & F_{n-1}\\ a & b\end{pmatrix}=\frac{\kappa}{2}.$$

Therefore, (c,d) has to be $\frac{\kappa}{i}$ times closer to the interval L_κ, than $(0,0)$. So, (c,d) belongs to $L_{\kappa-i}$.

Note that the distances between consecutive integer points on all L_i's are the same and equal to $\frac{1}{m\kappa}|L_\kappa|$, and the lengths of the intervals are given by $|L_i|=\frac{i}{\kappa}|L_\kappa|$. Therefore, since for $0<i<\kappa$ the end points of L_i are not integer points, each L_i contains exactly mi integer points.

For $i\in\{1,\ldots,\kappa\}$, let M_i be an admissible curve which starts at $(0,0)$, goes below $P_{i,1},\ldots,P_{i,mi}$ but above $P_{i-1,1},\ldots,P_{i-1,m(i-1)}$, and ends at (a,b). It would be useful to give two different names to M_i by letting M_i^- (resp. M_i^+) be a curve (isotopic to M_i) sufficiently close to $P_{i-1,1},\ldots,P_{i-1,m(i-1)}$ (resp. $P_{i,1},\ldots,P_{i,mi}$). Note that

$$s(M_i)=s(M_i^+)=s(M_i^-).$$

For $i\in\{1,\ldots,\kappa-1\}$, let S_i be the line segment from $P_{i-1,1}$ to $P_{i,1}$ (where $P_{0,1}=(0,0)$), and let T_i be the line segment from $P_{i,mi+1}$ to $P_{i+1,m(i+1)+1}$, where $P_{i,mi+1}$ is the integral point that makes $P_{i,mi}$ become the midpoint between $P_{i,mi-1}$ and $P_{i,mi+1}$ (see Figure 9).

Then, by Lemma 3.4(b) and Remark 3.5, we have

$$\begin{aligned}s([a,b])&=s(M_1)=s(M_1^+)=s(S_1)(s_3s_2s_1s([F_n,F_{n-1}]))^m s_1s_2s_3s(T_1)\cdots s_1s_2s_3s(T_{\kappa-1})\\&=s(S_1)(s_1s_2s_3s([F_n,F_{n-1}]))^m s_1s_2s_3s(T_1)\cdots s_1s_2s_3s(T_{\kappa-1})=s(M_2^-)=s(M_2)\\&=s(M_2^+)=s(S_1)s_1s_2s_3s(S_2)(s_3s_2s_1s([F_n,F_{n-1}]))^{2m}s_1s_2s_3s(T_2)\cdots s_1s_2s_3s(T_{\kappa-1})\\&=\cdots=s(S_1)s_1s_2s_3s(S_2)\cdots s_1s_2s_3s(S_{\kappa-1})(s_1s_2s_3s([F_n,F_{n-1}]))^{(\kappa-1)m}s_1s_2s_3s(T_{\kappa-1})\\&=s(M_\kappa^-)=s(M_\kappa^+)=s([a',b'])(s_3s_2s_1s([F_n,F_{n-1}]))^{\kappa m}=s([a',b']).\end{aligned}$$

The proof is completed. □

When $n=0$ in the above lemma, we have $\kappa=b$ and obtain the following corollary.

Corollary 3.7. *Let $[a,b]\in\mathcal{P}^+$. Then, for $j\in\mathbb{Z}_{>0}$,*

$$s([a+jmb,b])=s([a,b]).$$

For $[a,b]\in\mathcal{P}^+$, define

$$Q([a,b])=a^2+b^2-mab.$$

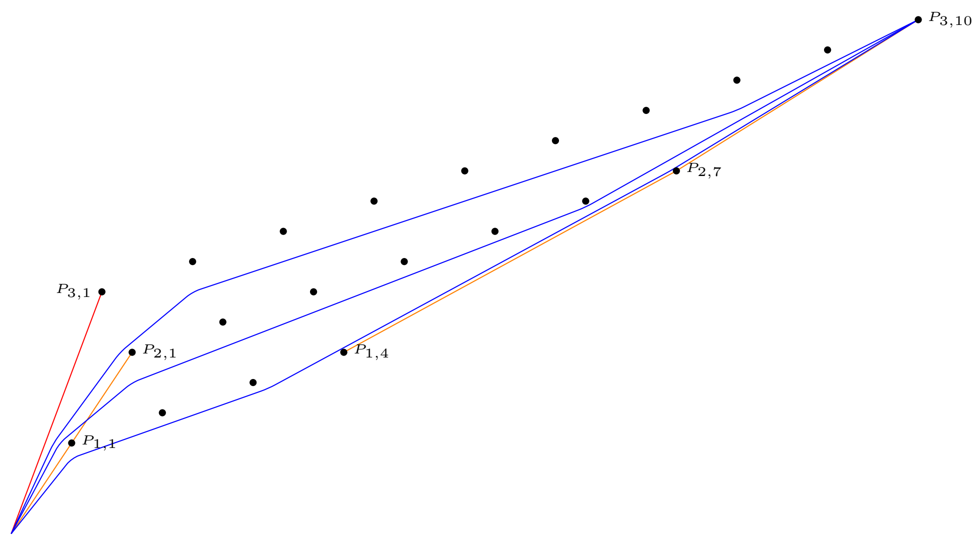

Figure 9 (Color online) A picture illustrating the case of $m = \kappa = 3$, $[a, b] = [48, 17]$ and $[a', b'] = [21, 8]$. The three blue-colored curves represent M_1, M_2 and M_3 (from the bottom). The four orange-colored line segments are S_1, S_2, T_1 and T_2.

Lemma 3.8. *Assume that*

$$\frac{F_{n-1}}{F_n} < \frac{b}{a} < \frac{F_n}{F_{n+1}}, \quad [a, b] \in \mathcal{P}^+.$$

Then, for any $j \in \mathbb{Z}_{>0}$, *we have*

$$Q([a, b] - mj(-F_{n-1}a + F_n b)[F_n, F_{n-1}]) < Q([a, b]),$$
$$Q([a, b] + mj(-F_n a + F_{n+1} b)[F_{n+1}, F_n]) < Q([a, b]).$$

Proof. Let $u = F_{n-1}, v = F_n$ and $w = F_{n+1}$ for convenience. First, set

$$x = a + mjauv - mjbv^2 \quad \text{and} \quad y = b + mjau^2 - mjbuv.$$

We want to prove

$$x^2 - mxy + y^2 < a^2 - mab + b^2.$$

We compute

$$\begin{aligned} x^2 - mxy + y^2 - (a^2 - mab + b^2) = {} & mj(mju^4 - m^2ju^mv + mju^2v^2 - mu^2 + 2uv)a^2 \\ & + mj(mju^2v^2 - m^2juv^m + mjv^4 - 2uv + mv^2)b^2 \\ & - 2mj(mju^mv - m^2ju^2v^2 + mjuv^m - u^2 + v^2)ab. \end{aligned} \tag{3.1}$$

Since $ua < vb$, we multiply the identity (3.1) by v/mj and then replace vb by ua to obtain the inequality

$$\begin{aligned} (3.1) \times (v/mj) < {} & (mju^4v - m^2ju^mv^2 + mju^2v^m - mu^2v + 2uv^2)a^2 \\ & + (mju^2v - m^2juv^2 + mjv^m - 2u + mv)u^2a^2 \\ & - 2(mju^mv - m^2ju^2v^2 + mjuv^m - u^2 + v^2)ua^2 = 0. \end{aligned}$$

Thus we have (3.1)< 0 as desired.

Now let

$$x_1 = a - mjavw + mjbw^2 \quad \text{and} \quad y_1 = b - mjav^2 + mjbvw,$$

and compute to obtain

$$\begin{aligned} x_1^2 - mx_1y_1 + {y_1}^2 - (a^2 - mab + b^2) = {} & mj(mjv^4 - m^2jv^mw + mjv^2w^2 + mv^2 - 2vw)a^2 \\ & + mj(mjv^2w^2 - m^2jvw^m + mjw^4 + 2vw - mw^2)b^2 \end{aligned}$$

$$-2mj(mjv^m w - m^2 jv^2 w^2 + mjvw^m + v^2 - w^2)ab. \quad (3.2)$$

Since $wb < va$, we multiply the identity (3.2) by v/mj and then replace va by wb to obtain the inequality

$$\begin{aligned}(3.2)\times(v/mj) < {} & (mjv^m - m^2jv^2w + mjvw^2 + mv - 2w)w^2b^2 \\ & + (mjv^m w^2 - m^2jv^2w^m + mjvw^4 + 2v^2w - mvw^2)b^2 \\ & - 2(mjv^m w - m^2jv^2w^2 + mjvw^m + v^2 - w^2)wb^2 = 0.\end{aligned}$$

This implies (3.2)<0 as desired. □

Combining the lemmas in this section, we obtain the following proposition, which is a main step toward the proof of Theorem 2.7.

Proposition 3.9. *Assume that* $[a,b] \in \mathcal{P}^+$. *Then there exists* $[a_0,b_0] \in \mathcal{P}^+$ *such that* $[a_0,b_0]$ *is a reduced root of* $\mathcal{H}(m)$ *and* $s([a_0,b_0]) = s([a,b])$ *or equivalently,* $\beta([a_0,b_0]) = \beta([a,b])$.

Proof. If $[a,b] \in \mathcal{P}^+$ is a root of $\mathcal{H}(m)$, it is already reduced from the definition of $\mathcal{P}^+$ and we simply take $[a_0,b_0]=[a,b]$. Assume that $[a,b]$ is not a root of $\mathcal{H}(m)$. Without loss of generality, we may further assume that $a > b$. Then, by Lemma 3.1 (c), we have $\frac{F_{n-1}}{F_n} < \frac{b}{a} < \frac{F_n}{F_{n+1}}$ for some $n \in \mathbb{Z}_{>0}$. By Lemma 3.2, we have either $[a,b] - m(-F_{n-1}a + F_nb)[F_n,F_{n-1}] \in \mathcal{P}^+$ or $[a,b] + m(-F_na + F_{n+1}b)[F_{n+1},F_n] \in \mathcal{P}^+$. Put $[a',b'] = [a,b] - m(-F_{n-1}a + F_nb)[F_n,F_{n-1}]$ or $[a',b'] = [a,b] + m(-F_na + F_{n+1}b)[F_{n+1},F_n]$, so that $[a',b'] \in \mathcal{P}^+$. Then we have $s([a',b']) = s([a,b])$ and $Q([a',b']) < Q([a,b])$ by Lemmas 3.6 and 3.8, respectively.

If $Q([a',b']) \leqslant 1$, then $[a',b']$ is a positive reduced root of $\mathcal{H}(m)$ from (2.1) and we take $[a_0,b_0] = [a',b']$. If $Q([a',b']) > 1$, then $[a',b']$ is not a root of $\mathcal{H}(m)$ and we repeat the process by putting a' and b' to be new a and b. Clearly, this process ends in a finite number of steps. □

Example 3.10. Assume that $m=3$, and consider $[487,186]$. Since $Q([487,186]) = 19$, it is not a root of $\mathcal{H}(3)$. Note that $\frac{21}{55} < \frac{186}{487} < \frac{55}{144}$. We compute $3\times(-55\times 487 + 144\times 186) = -3$ and

$$[487,186] - 3\times[144,55] = [55,21] \in \mathcal{P}^+.$$

Since $Q([55,21]) = 1$, $[55,21]$ is a real root of $\mathcal{H}(3)$ and the process ends here. Indeed, we have $s([487,186]) = s([55,21])$ and

$$\beta([487,186]) = \beta([55,21]) = 6\alpha_1 + 8\alpha_2 + 17\alpha_3.$$

Now consider $[1789,683]$ with $Q([1789,683]) = 1349$. Since $\frac{8}{21} < \frac{683}{1789} < \frac{21}{55}$, we get

$$[1789,683] + 3(-21\times 1789 + 55\times 683)[55,21] = [1129,431], \quad Q([1129,431]) = 605.$$

We continue to obtain

$$\begin{aligned}&[1129,431] + 3(-21\times 1129 + 55\times 431)[55,21] = [469,179], \quad Q([469,179]) = 149,\\ &[469,179] - 3(-8\times 469 + 21\times 179)[21,8] = [28,11], \quad Q([28,11]) = -19.\end{aligned}$$

Thus $[28,11]$ is an imaginary root of $\mathcal{H}(3)$, and we have

$$\beta([1789,683]) = \beta([28,11]) = 55\alpha_1 + 55\alpha_2 + 144\alpha_3.$$

Lemma 3.11. *Let* η *be any non-self-crossing admissible curve with* $\upsilon(\eta) \in \mathfrak{R}$. *Then there exists* $[a,b] \in \mathcal{P}^+$ *such that* $s(\eta) = s([a,b])$.

Proof. Each non-self-crossing closed curve on the torus is a torus knot. As an admissible curve has a distinguished marked point, which is the origin, we allow Dehn twists around the origin. Hence (the lift of) the curve η (to the universal cover) is isotopic to a spiral (around the origin) followed by a line segment which is then followed by the opposite spiral (around the end point of η). Without loss of generality, we may assume that the first spiral goes around counterclockwise. Then $s(\eta)$ can be written in one of the forms

$$(s_3s_2s_1)^n s(\nu)(s_1s_2s_3)^n, \quad (s_3s_2s_1)^n s_3 s(\nu) s_3 (s_1s_2s_3)^n \quad \text{and} \quad (s_3s_2s_1)^n s_3s_2 s(\nu) s_2s_3 (s_1s_2s_3)^n$$

for some $n \geqslant 0$ and a line segment ν. We will consider the case when $n = 0$ and show that each of the forms is equal to $s([a,b])$ with $[a,b] \in \mathcal{P}^+$ and $a \geqslant b$. Then we immediately obtain the statement for any $n > 0$ by Lemma 3.3(3) and an induction argument. Thus we only need to consider each of the reflections $s(\nu)$, $s_3s(\nu)s_3$ and $s_3s_2s(\nu)s_2s_3$.

First, if $s(\eta) = s(\nu)$ with ν a line segment, then we have $s(\eta) = s([a,b])$ with $a \geqslant b$, if necessary, by applying Corollary 3.7.

Next, if $s(\eta) = s_3s(\nu)s_3$, then $s(\nu)$ starts with the letter 1; if $s(\nu)$ starts with the letter 2, then $s(\eta)$ would be of the form $s_3s_2s(\nu)s_2s_3$, which would fall into the next case. Let $D(\eta)$ be the lattice path from $(0,0)$ to $(0,1)$ then to the end point of η that goes North and West, and is closest to (but never crosses) the line segment ν.

Assign the element $s_2s_1 \in W(m)$ to each vertical edge of $D(\eta)$, and $s_1s_2s_3s_1$ to each horizontal edge. Let $w(\eta) \in W(m)$ be the product of these elements obtained by reading them while traveling along $D(\eta)$. Then $s(\eta) = s_3s_2w(\eta)s_1$.

Let (c,d) be the end point of η. The maximal Dyck path $\mathcal{D}^{(c+md)\times d}$ consists of (copies of) m consecutive horizontal edges followed by a vertical edge, and (copies of) $m-1$ consecutive horizontal edges (which is preceded by a vertical edge) followed by a vertical edge. Remember $s([c+md,d]) = s_3s_2s^{(c+md)\times d}s_1$ from Lemma 3.3(1).

A vertical edge of $D(\eta)$, which is followed by another vertical edge, corresponds to m consecutive horizontal edges followed by a vertical edge in $\mathcal{D}^{(c+md)\times d}$. A vertical edge followed by a horizontal edge in $D(\eta)$ corresponds to $m-1$ consecutive horizontal edges followed by a vertical edge in $\mathcal{D}^{(c+md)\times d}$. Moreover, the element s_2s_1 of a vertical edge of $D(\eta)$, which is followed by another vertical edge, is equal to $(s_2s_3)^m s_2s_1$ that is associated with m consecutive horizontal edges followed by a vertical edge in $\mathcal{D}^{(c+md)\times d}$. The combined element $(s_2s_1)(s_1s_2s_3s_1) = s_3s_1$ of a vertical edge followed by a horizontal edge in $D(\eta)$, is equal to $(s_2s_3)^{m-1}s_2s_1$ that is associated with $m-1$ consecutive horizontal edges followed by a vertical edge in $\mathcal{D}^{(c+md)\times d}$.

Hence $w(\eta) = s^{(c+md)\times d}$, so $s(\eta) = s([c+md,d])$. Since $c+d > 0$, we have $c+md > d$ and $[c+md,d] \in \mathcal{P}^+$. For example, Figure 10 illustrates the case $m = 3$ and $(c,d) = (-2,3)$. Here, we have

$$\begin{aligned} s(\eta) &= s_3s_2(s_2s_1)^2(s_1s_2s_3s_1)(s_2s_1)(s_1s_2s_3s_1)s_1 = s_3s_1s_3s_1s_3 \\ &= s_2s_3s_2s_3s_2s_1s_2s_3s_2s_3s_2s_1s_2s_3s_2s_3s_2 = s([7,3]). \end{aligned}$$

Lastly, suppose that $s(\eta) = s_3s_2s(\nu)s_2s_3$. Let (c,d) be the end point of η. If $[c+md,d] \in \mathcal{P}^+$, then a similar argument to the previous case shows that $s(\eta) = s([c+md,d])$. Otherwise, we have $c+md < 0$ and take a curve η' such that

$$s(\eta') = s_3s_2s(\nu')s_2s_3,$$

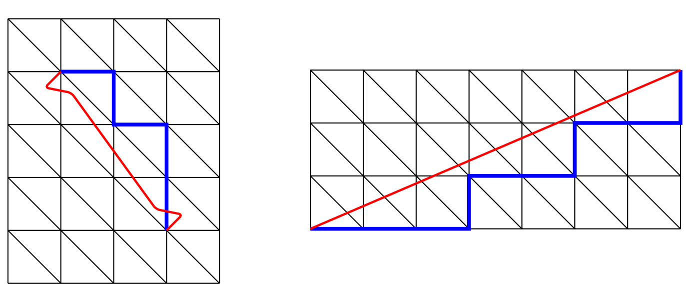

Figure 10 (Color online) Admissible curves and Dyck paths for $m = 3$ and $(c,d) = (-2,3)$

where ν' is a line segment between $(c + md + \epsilon, d - \epsilon)$ and $(-\epsilon, \epsilon)$ for some small $\epsilon > 0$. Then one can see that $s(\eta) = s(\eta')$. Repeating this process, we obtain a curve η'' whose end point is (c'', d) with $c'' + md > 0$ and we are done by Corollary 3.7 as $s(\eta) = s([c'' + md, d])$. $\square$

Proof of Theorem 2.7. Assume that $s(\eta)$ is a rigid reflection of $W(m)$ given by a non-self-intersecting admissible curve η. By Lemma 3.11, there exists $[a, b] \in \mathcal{P}^+$ such that $s(\eta) = s([a, b])$. Then by Proposition 3.9 there exists a reduced positive root $[a_0, b_0]$ of $\mathcal{H}(m)$ such that

$$s(\eta) = s([a, b]) = s([a_0, b_0]).$$

This completes the proof. $\square$

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