Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

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COMMUNICATIONS IN ALGEBRA®
2017, VOL. 45, NO. 11, 4785–4800
http://dx.doi.org/10.1080/00927872.2017.1281935

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A combinatorial approach to root multiplicities of rank 2 hyperbolic Kac–Moody algebras

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ABSTRACT
In this paper we study root multiplicities of rank 2 hyperbolic Kac–Moody algebras using the combinatorics of Dyck paths.

ARTICLE HISTORY
Received 30 November 2016
Revised 19 December 2016
Communicated by K. Misra

KEYWORDS
Dyck path; hyperbolic Kac–Moody algebra; Root multiplicity

2010 MATHEMATICS SUBJECT CLASSIFICATION
Primary 17B67, 17B22; Secondary 05E15

1. Introduction

This paper takes a new approach to the study of root multiplicities for hyperbolic Kac–Moody algebras. Even though the root multiplicities are fundamental data in understanding the structures of Kac–Moody algebras, we have not seen much progress in this topic for the last 20 years. The method taken in this paper is totally new, though depending on the previous developments, and opens different perspectives that can bring new results on root multiplicities and make advancements, for example, toward *Frenkel's conjecture*. To begin with, let us first explain the background of the problem considered in this paper.

After introduced by Kac and Moody more than four decades ago, the Kac–Moody theory has become a standard generalization of the classical Lie theory. However, it is surprising how little is known beyond the affine case, even for certain hyperbolic algebras which were carefully studied by several authors.

The first difficulty in the hyperbolic case and other indefinite cases stems from wild behaviors of root multiplicities. To be precise, let $\mathfrak{g}$ be a Kac–Moody algebra with Cartan subalgebra $\mathfrak{h}$. For a root α, the root space $\mathfrak{g}_\alpha$ is given by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Then we have the root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha,$$

which is a decomposition of $\mathfrak{g}$ into finite-dimensional subspaces, where Δ^+ (resp. Δ^-) is the set of positive (resp. negative) roots. The dimension of the root space $\mathfrak{g}_\alpha$ is called the *multiplicity* of α. Obviously, root multiplicities are fundamental data to understand the structure of a Kac–Moody algebra $\mathfrak{g}$. However, the status of our knowledge shows a dichotomy according to types of $\mathfrak{g}$.

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Recall that the Weyl group W of $\mathfrak{g}$ acts on the set Δ of all roots, preserving root multiplicities. If α is a real root, α has an expression $\alpha = w\alpha_i$ for $w \in W$ where α_i is a simple root. It follows that $\dim(\mathfrak{g}_\alpha) = 1$. Since all roots in finite-dimensional Lie algebras are real, all root spaces in finite-dimensional Lie algebras are one dimensional. Let $\mathfrak{g}$ be an untwisted affine Kac–Moody algebra of rank $\ell+1$. Then the multiplicity of every imaginary root of $\mathfrak{g}$ is ℓ ([8, Corollary 7.4]). There is a similar formula for twisted affine Kac–Moody algebras as well ([8, Corollary 8.3]).

For hyperbolic and more general indefinite Kac–Moody algebras, the situation is vastly different, due to the exponential growth of the imaginary root spaces. Our knowledge of the dimensions of imaginary root spaces is far from being complete, though there are known formulas for root multiplicities.

The first formulas for root multiplicities of Kac–Moody algebras are a closed form formula by Berman and Moody [1] and a recursive formula by Peterson and Kac [8, 20]. Both formulas are based on the *denominator identity* for a Kac–Moody algebra $\mathfrak{g}$ and enable us to calculate the multiplicity of a given root (of a reasonable height). The paper by Feingold and Frenkel [5], where the hyperbolic Kac–Moody algebra $\mathfrak{F}$ of type $HA_1^{(1)}$ was studied, included the first results about hyperbolic root multiplicities giving an infinite number of them a combinatorial meaning as values of a partition function. Their method used the $\mathbb{Z}$-grading of $\mathfrak{F}$ by level with respect to the affine subalgebra $A_1^{(1)}$ and gave closed formulas for all the root multiplicities on levels ± 1 and ± 2. Using the same method, Kac et al. [9] calculated some root multiplicities for $HE_8^{(1)}(= E_{10})$.

These methods were further systematically developed and generalized by Kang [10, 13] for arbitrary Kac–Moody algebras and have been adopted in many works on root multiplicities of indefinite Kac–Moody algebras. In his construction, the first author adopted homological techniques and Kostant's formula [7] to devise a method that works for higher levels. For example, he applied his method to compute root multiplicities of the algebra $\mathfrak{F}$ of type $HA_1^{(1)}$ up to level 5 [11, 12].

Despite all these results, we still do not have any unified, efficient approach to computing all root multiplicities. Essentially these methods give answers to root multiplicities one at a time, with no general formulas or effective bounds on multiplicities. In particular, these formulas are given by certain *alternating* sums of *rational* numbers and make it difficult to control overall behavior of root multiplicities. Therefore it is already quite hard to find effective upper or lower bounds for root multiplicities for hyperbolic and other indefinite Kac–Moody algebras.

For hyperbolic Kac–Moody algebras, in the setting of the 'no-ghost' theorem from String theory, I. Frenkel [6] proposed a bound on the root multiplicities of hyperbolic Kac–Moody algebras.

Frenkel's conjecture: *Let $\mathfrak{g}$ be a symmetric hyperbolic Kac–Moody algebra associated to a hyperbolic lattice of dimension d and equipped with invariant form $(\cdot \mid \cdot)$ such that $(\alpha_i|\alpha_i) = 2$ for simple roots α_i. Then we have:*

$$\dim(\mathfrak{g}_\alpha) \ \le \ p^{(d-2)}\left(1 - \frac{(\alpha|\alpha)}{2}\right),$$

where the function $p^{(\ell)}(n)$ is the multi-partition function with ℓ colors.

Frenkel's conjecture is known to be true for any symmetric Kac–Moody algebra associated to a hyperbolic lattice of dimension 26 [6], though Kac et al. [9] showed that the conjecture fails for E_{10}. The conjecture is still open for the rank 3 hyperbolic Kac–Moody algebra $\mathfrak{F}$ and proposes arguably the most tantalizing question about root multiplicities.

Open Problem: *Prove Frenkel's conjecture for the rank 3 hyperbolic Kac–Moody algebra $\mathfrak{F}$.*

As mentioned earlier, Feingold and Frenkel [5] and Kang [11, 12] studied root multiplicities of $\mathfrak{F}$. There is another approach to root multiplicities of $\mathfrak{F}$ and other hyperbolic Kac–Moody algebras, taken by Niemann [19], which follows Borcherds' idea in construction of the fake Monster Lie algebra [2]. This approach was further pursued by Kim and Lee [15]. A recent survey on root multiplicities can be found in [3].

In this paper, we adopt quite a different methodology and investigate root multiplicities of rank two symmetric hyperbolic Kac–Moody algebras $\mathcal{H}(a)$ ($a \geq 3$) through combinatorial objects. More precisely, we use lattice paths, known as *Dyck paths*, to describe root multiplicities.

Suppose that $\alpha = r\alpha_1 + s\alpha_2$ is an imaginary root of $\mathcal{H}(a)$ with r and s relatively prime, for simplicity. Then our first main theorem (Theorem 3.7) shows that

Theorem 1.1. *We have*

$$\text{mult}\,(\alpha) = \sum_{\substack{D:\, \text{Dyck path} \\ \text{wt}(D)=\alpha}} c(D).$$

Here $c(D)$ has values 1, 0, or -1 and is immediately determined by the shape of the Dyck path D. The result for general α involves considering cyclic equivalence of paths and a minor correction term coming from paths with weight $\alpha/2$. An important feature is that this formula only contains integers and has clear combinatorial interpretation, and makes it possible to prove properties of root multiplicities through combinatorial manipulations of Dyck paths. For example, in the symmetric rank two case, we can prove an analogue of Frenkel's conjecture through combinatorics of Dyck paths.

Proposition 1.2. *Let $\mathfrak{g} = \mathcal{H}(a)$. Then we have:*

$$\text{mult}\,(\alpha) \leq p_t\left(1 - \frac{(\alpha|\alpha)}{2}\right),$$

where $\alpha = r\alpha_1 + s\alpha_2$, $t = \max(r, s)$ and $p_t(n)$ is the number of partitions of n with at most t parts.

Even though this upper bound is in the form of Frenkel's conjecture, it is actually crude. More interestingly, Theorem 1.1 gives a natural upper bound by only counting paths with $c(D) = 1$. This upper bound can be significantly improved by considering cancellation with paths having $c(D) = -1$. Namely, we consider a function Φ from $\{D : c(D) = -1\}$ to $\{D : c(D) = 1\}$. Suppose that $\alpha = r\alpha_1 + s\alpha_2$ is an imaginary root of $\mathcal{H}(a)$ with r and s relatively prime, for simplicity. Then we obtain

Theorem 1.3.

$$\text{mult}\,(\alpha) \leq \#\{D : \textit{Dyck path},\ \text{wt}(D) = \alpha,\ c(D) = 1,\ D \textit{ is not an image under } \Phi\}.$$

This upper bound is quite sharp and gives exact root multiplicities for roots up to height 16 with a suitable choice of Φ. In Section 5, the function Φ will be carefully constructed. The resulting upper bound is satisfactorily accurate and enlightens combinatorics of Dyck paths related to root multiplicities.

Our approach clearly extends to higher rank Kac–Moody algebras by replacing Dyck paths with certain lattice paths. In a subsequent paper, we will consider higher rank cases; in particular, we will study the Feingold–Frenkel rank 3 algebra $\mathfrak{F}$. We hope that our approach may bring significant advancements toward Frenkel's conjecture for the algebra $\mathfrak{F}$.

2. Rank two symmetric hyperbolic Kac–Moody algebras

In this section, we fix our notations for rank 2 hyperbolic Kac–Moody algebras. A general theory of Kac–Moody algebras can be found in [8], and the root systems of rank two hyperbolic Kac–Moody algebras were studied by Lepowsky and Moody [18] and Feingold [4]. Root multiplicities of these algebras were investigated by Kang and Melville [14].

Let $A = (a_{ij}) = \begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ be a generalized Cartan matrix with $a \geq 3$, and $\mathcal{H}(a)$ be the hyperbolic Kac–Moody algebra associated with the matrix A. In this section, we write $\mathfrak{g} = \mathcal{H}(a)$ if there is no need to specify a. Let $\{h_1, h_2\}$ be the set of simple coroots in the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_1 \oplus \mathbb{C}h_2 \subset \mathfrak{g}$.

Let $\{\alpha_1, \alpha_2\} \subset \mathfrak{h}^*$ be the set of simple roots, and $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ be the root lattice. The set of roots of $\mathfrak{g}$ will be denoted by Δ, and the set of positive (resp. negative) roots by Δ^+ (resp. by Δ^-), and the set of real (resp. imaginary) roots by $\Delta_{\rm re}$ (resp. by $\Delta_{\rm im}$). We will use the notation $\Delta^+_{\rm re}$ to denote the set of positive real roots. Similarly, we use $\Delta^+_{\rm im}$, $\Delta^-_{\rm re}$, and $\Delta^-_{\rm im}$. The Lie algebra $\mathfrak{g}$ has the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ and we define the *multiplicity* of α by $\operatorname{mult}\alpha := \dim \mathfrak{g}_\alpha$.

We define a symmetric bilinear form on $\mathfrak{h}^*$ by $(\alpha_i|\alpha_j) = a_{ij}$, where a_{ij} is the (i,j)-entry of the Cartan matrix A. The simple reflection corresponding to α_i in the root system of $\mathfrak{g}$ is denoted by r_i $(i = 1, 2)$, and the Weyl group W is given by $W = \{(r_1r_2)^i, r_2(r_1r_2)^i \mid i \in \mathbb{Z}\}$. Define a sequence $\{B_n\}$ by

$$B_0 = 0, \quad B_1 = 1, \quad B_{n+2} = aB_{n+1} - B_n \quad \text{for } n \geq 0.$$

It can be shown that

$$B_n = \frac{1-\gamma^{2n}}{\gamma^{n-1}(1-\gamma^2)} \qquad (n \geq 0),$$

where $\gamma = \frac{a+\sqrt{a^2-4}}{2}$. We will write $(A, B) = A\alpha_1 + B\alpha_2$. Then the set of positive real roots is given by

$$\Delta^+_{\rm re} = \{(B_n, B_{n+1}),\ (B_{n+1}, B_n) \mid n \geq 0\}.$$

See [14] for details. To describe the set of imaginary roots, we first define the set

$$\Omega_k = \left\{ (m,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : \sqrt{\frac{4k}{a^2-4}} \leq m \leq \sqrt{\frac{k}{a-2}},\ n = \frac{am - \sqrt{(a^2-4)m^2 - 4k}}{2} \right\}$$

for $k \geq 1$. Here, by definition, we do not include (m, n) in Ω_k unless n is an integer. For example, when $a = 3$ and $k = 100$, the values $m = 9, 10$ satisfy the inequalities in the condition. But the corresponding n is an integer for $m = 10$, precisely $n = 10$, and it is an irrational number for $m = 9$. Thus we have $(10, 10) \in \Omega_{100}$ and $(9, n) \neq \Omega_{100}$ for any $n \in \mathbb{Z}_{\geq 0}$.

Proposition 2.1 ([14]). *For $a \geq 3$, the set of positive imaginary roots α of $\mathcal{H}(a)$ with $(\alpha|\alpha) = -2k$ is*

$$\Delta^+_{{\rm im},k} = \left\{ \begin{array}{l} (m,n),\ (mB_{j+1} - nB_j, mB_{j+2} - nB_{j+1}), \\ (mB_{j+2} - nB_{j+1}, mB_{j+1} - nB_j) \end{array} \middle| (m,n) \in \Omega_k \text{ or } (n,m) \in \Omega_k,\ j \geq 0 \right\}.$$

The denominator identity is given by

$$\prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W} (-1)^{\ell(w)} e^{w\rho-\rho},$$

where $\ell(w)$ is the length of w and $\rho = (\alpha_1 + \alpha_2)/(2 - a)$.

3. Contribution multiplicity

In this section, we fix a hyperbolic Kac–Moody algebra $\mathcal{H}(a)$, $a \geq 3$. First, we recall Kang and Melville's result [14] on root multiplicities of $\mathcal{H}(a)$. For $r, s \in \mathbb{Z}_{\geq 0}$, write $\alpha = r\alpha_1 + s\alpha_2$. As in [14], we define a sequence $\{A_n\}_{n\geq 0}$ as follows:

$$A_0 = 0, \quad A_1 = 1,$$
$$A_{n+2} = aA_{n+1} - A_n + 1 \text{ for } n \geq 0.$$

Let

$$\mathscr{C} = \left\{ \mathbf{c} = (c_0^0, c_0^1, c_1^0, c_1^1, \ldots) \mid c_i^j \text{ are non-negative integers}, j \in \{0,1\}, i \geq 0 \right\},$$

and let

$$\mathscr{C}(\alpha) = \left\{ \mathbf{c} \in \mathscr{C} \mid \sum_{i\geq 0}(c_i^0 A_{i+1} + c_i^1 A_i) = r,\ \sum_{i\geq 0}(c_i^0 A_i + c_i^1 A_{i+1}) = s \right\}. \tag{3.1}$$

For $\tau \in Q$, we write $\tau|\alpha$ if $\alpha = d\tau$ for some $d \in \mathbb{Z}_{>0}$, and set $\alpha/\tau = d$.

Proposition 3.2 ([14, Propositions 2.1, 2.2]). *We have*

$$\mathrm{mult}\,(\alpha) = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right)\frac{\tau}{\alpha} \sum_{\mathbf{c}\in\mathscr{C}(\tau)} (-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0 + c_i^1) - 1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}, \tag{3.3}$$

where μ is the Möbius function.

For $r, s \in \mathbb{Z}_{\geq 0}$, define a *Dyck path of size $r \times s$* to be a lattice path from $(0,0)$ to (r,s) that never goes above the main diagonal joining $(0,0)$ and (r,s). We identify a Dyck path with a word in alphabet $\{1,2\}$, where 1 represents a horizontal move and 2 a vertical move. Then a Dyck path has 12-corners (i.e., corners of shape ⌟) and 21-corners (i.e., corners of shape ⌜). We consider the end points $(0,0)$ and (r,s) as 21-corners. We define the *weight* of a Dyck path D of size $r \times s$ to be $\mathrm{wt}(D) := r\alpha_1 + s\alpha_2 \in Q$.

We say that two Dyck paths D_1 and D_2 are *equivalent* if D_1 is a cyclic permutation of D_2 (as words in the alphabet $\{1,2\}$). Then we obtain *equivalence classes* of Dyck paths. When no confusion arises, we will frequently identify an equivalence class $\mathcal{D}$ with any representative $D \in \mathcal{D}$. For an equivalence class $\mathcal{D}$, the *weight* $\mathrm{wt}(\mathcal{D})$ is well defined. The concatenation of Dyck paths $D_1, D_2, \ldots, D_r$ will be denoted by $D_1D_2\cdots D_r$. For a positive integer d and a Dyck path D, the concatenation D^d is defined in an obvious way. We distinguish a concatenation from its resulting path. The resulting path of a concatenation $D_1D_2\cdots D_r$ will be denoted by $\pi(D_1D_2\cdots D_r)$. A Dyck path D is said to be *primitive* if $D \neq \pi(D_0^d)$ for any subpath D_0 and $d \geq 2$. Likewise, an equivalence class $\mathcal{D}$ is said to be *primitive* if any element D of $\mathcal{D}$ is primitive.

Definition 3.4. For any positive integers u, v, denote by $L_{u\times v}$ the Dyck path of size $u\times v$, which consists of u horizontal edges followed by v vertical edges, and call it an *elementary* path. We say that the elementary Dyck path $L_{u\times v}$ is of

$$\begin{cases} \text{type } (-1), & \text{if } A_n \leq \min(u,v),\ \max(u,v) < A_{n+1},\ \text{and } n:\text{even} > 0; \\ \text{type } (1), & \text{if } A_n \leq \min(u,v),\ \max(u,v) < A_{n+1},\ \text{and } n:\text{odd}; \\ \text{type } (0), & \text{if } A_n \leq \min(u,v) < A_{n+1} \leq \max(u,v),\ \text{and } n > 0. \end{cases}$$

For a given Dyck path D, define $\mathscr{S}(D)$ to be the set of all concatenations of copies of $L_{A_{i+1}\times A_i}$ and copies of $L_{A_i\times A_{i+1}}$ for $i \geq 0$ in some order that realize D. For a concatenation $\mathbf{s}$ in $\mathscr{S}(D)$, the number of copies of $L_{A_{i+1}\times A_i}$ is denoted by $c_i^0(\mathbf{s})$ and the number of copies of $L_{A_i\times A_{i+1}}$ by $c_i^1(\mathbf{s})$. We define

$$\mathrm{seq}(\mathbf{s}) = (c_i^0(\mathbf{s}), c_i^1(\mathbf{s}))_{i\geq 0} \in \mathscr{C} \quad \text{and} \quad \mathrm{sgn}(\mathbf{s}) = (-1)^{\sum_{i:\mathrm{odd}}(c_i^0(\mathbf{s})+c_i^1(\mathbf{s}))}.$$

If $\mathcal{D}$ is an equivalence class, we observe that $\mathscr{S}(D_1)$ is in one-to-one correspondence with $\mathscr{S}(D_2)$ through cyclic permutation for $D_1, D_2 \in \mathcal{D}$. For an equivalence class $\mathcal{D}$, we define the set $\mathscr{S}(\mathcal{D})$ to be equal to $\mathscr{S}(D)$ for a fixed Dyck path $D \in \mathcal{D}$. Now the *contribution multiplicity* $c(\mathcal{D})$ of $\mathcal{D}$ is defined by

$$c(\mathcal{D}) = \sum_{\mathbf{s}\in\mathscr{S}(\mathcal{D})} \mathrm{sgn}(\mathbf{s}).$$

For a Dyck path D, a subpath D_0 of D is called *framed* if the starting point and the ending point of D_0 are both 21-corners.

Lemma 3.5. *For any Dyck path D, we have*

$$c(D) = \begin{cases} 0, & \text{if } D \text{ contains a framed subpath of type } (0); \\ (-1)^{\#\text{ of framed subpaths of } D \text{ of type } (-1)}, & \text{otherwise.} \end{cases}$$

Proof. Assume that $D = \pi(D_1D_2)$. Then we have

$$\begin{aligned} c(D) &= \sum_{\mathbf{s}\in\mathscr{S}(D)} \operatorname{sgn}(\mathbf{s}) \\ &= \sum_{(\mathbf{s}_1,\mathbf{s}_2)\in\mathscr{S}(D_1)\times\mathscr{S}(D_2)} \operatorname{sgn}(\mathbf{s}_1)\operatorname{sgn}(\mathbf{s}_2) \\ &= \sum_{\mathbf{s}_1\in\mathscr{S}(D_1)} \operatorname{sgn}(\mathbf{s}_1) \sum_{\mathbf{s}_2\in\mathscr{S}(D_2)} \operatorname{sgn}(\mathbf{s}_2) \\ &= c(D_1)c(D_2). \end{aligned}$$

Thus it is enough to consider the case when D is an elementary path. In this case, we need to prove that $c(D)$ is equal to its type. We will use induction. Clearly, $c(L_{1\times 0}) = c(L_{0\times 1}) = 1$, and the assertion of the lemma is true. Suppose that the assertion is true for $L_{u\times v}$. We will prove the case $L_{(u+1)\times v}$. The other case $L_{u\times(v+1)}$ is obtained from the symmetry.

Write $L = L_{u\times v}$ and $L_1 = L_{(u+1)\times v}$ to ease the notations. Assume that $c(L)$ is equal to its type. If L and L_1 are of the same type, then we get all the elements of $\mathscr{S}(L_1)$ from those of $\mathscr{S}(L)$ by adding 1 to $c_0^0(\mathbf{s})$, $\mathbf{s} \in \mathscr{S}(L)$, and $c(L_1) = c(L)$.

If L is of type (1) and L_1 is of type (0), then the path L_1 newly contains $L_{A_{n+1}\times A_n}$ as a subpath for n odd, where $A_{n+1} = u + 1$. Consequently, $c(L_1) = c(L) - 1 = 0$ by induction. If L is of type (-1) and L_1 is of type (0), then the path L_1 newly contains $L_{A_{n+1}\times A_n}$ as a subpath for n even, where $A_{n+1} = u + 1$. Thus, again, we have $c(L_1) = c(L) + 1 = 0$.

Similarly, if L is of type (0) and L_1 is of type (-1) (respectively, if L is of type (0) and L_1 is of type (1)), then L_1 newly contains $L_{A_{n+1}\times A_n}$ as a subpath for n odd (respectively, for n even), where $A_{n+1} = u + 1$. Thus we have $c(L_1) = c(L) - 1 = -1$ (respectively, $c(L_1) = c(L) + 1 = 1$). Now, by induction, we are done. □

Remark 3.6. The above lemma enables us to compute $c(D)$ efficiently and combinatorially. In particular, $c(D) = 1$ if D contains no framed subpaths of type (0) and an even number of framed subpaths of type (-1).

The following theorem is a combinatorial realization of Kang and Melville's formula (3.3), which says that the root multiplicity of α is equal to the sum of contribution multiplicities $c(\mathcal{D})$ of primitive equivalence classes $\mathcal{D}$ of weight α plus some correction term.

Theorem 3.7. *For $\alpha \in \Delta^+$, we have*

$$\operatorname{mult}(\alpha) = \sum_{\substack{\mathcal{D}:\ \textit{primitive} \\ \mathrm{wt}(\mathcal{D})=\alpha}} c(\mathcal{D}) + \sum_{\substack{\mathcal{D}:\ \textit{primitive} \\ \mathrm{wt}(\mathcal{D})=\alpha/2}} \left\lfloor \frac{1-c(\mathcal{D})}{2} \right\rfloor.$$

By Lemma 3.5, we see that the second sum (i.e., the correction term) is nothing but the number of primitive $\mathcal{D}$ such that $\mathrm{wt}(\mathcal{D}) = \alpha/2$ and $c(\mathcal{D}) = -1$.

Proof. Write $\alpha = r\alpha_1 + s\alpha_2$. Before we deal with the general case, we first consider a simpler case and assume that r and s are relatively prime. Then the correction term is 0, and each equivalence class of weight α has only one primitive Dyck path. Recall that we defined $\mathscr{C}(\alpha)$ in (3.1). We claim that, for each $\mathbf{c} = (c_i^0, c_i^1)_{i\geq 0} \in \mathscr{C}(\alpha)$, the number of concatenations $\mathbf{s}$ such that $\mathrm{seq}(\mathbf{s}) = \mathbf{c}$ is $\frac{(\sum_{i\geq 0}(c_i^0+c_i^1)-1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}$. Indeed, let $\mathbf{p}$ be a concatenation of the c_i^0 copies of $L_{A_{i+1}\times A_i}$ and c_i^1 copies of $L_{A_i\times A_{i+1}}$ in some order, and consider the concatenation $\mathbf{p}^N$ for N sufficiently large. Then we can find a unique line with slope s/r which intersects the path $\pi(\mathbf{p}^N)$ so that the path never goes above the line. Since r and s are relatively prime, two consecutive intersection points uniquely determine a concatenation which is a cyclic permutation of $\mathbf{p}$, and the number of cyclic permutations is $\sum_{i\geq 0}(c_i^0 + c_i^1)$. Now the claim follows.

From Proposition 3.2 and the claim above, we obtain

$$\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\alpha}} c(\mathcal{D}) = \sum_{\mathcal{D}:\,\mathrm{wt}(\mathcal{D})=\alpha}\ \sum_{\mathbf{s}\in\mathscr{S}(\mathcal{D})} (-1)^{\sum_{i:\mathrm{odd}}(c_i^0(\mathbf{s})+c_i^1(\mathbf{s}))}$$

$$= \sum_{\mathbf{c}\in\mathscr{C}(\alpha)} (-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0 + c_i^1) - 1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}$$

$$= \mathrm{mult}\,(\alpha).$$

Now we consider arbitrary $r, s \in \mathbb{Z}_{\geq 0}$. We will show

$$\sum_{\mathbf{c}\in\mathscr{C}(\alpha)} (-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0 + c_i^1) - 1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}$$

$$= \sum_{\tau|\alpha} \frac{\tau}{\alpha}\left[\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\tau}} c(\mathcal{D}) + \sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\tau/2}} \left\lfloor \frac{1-c(\mathcal{D})}{2}\right\rfloor\right]. \tag{3.8}$$

Let $\mathbf{c} \in \mathscr{C}(\alpha)$. As before, assume that $\mathbf{p}$ is a concatenation of the c_i^0 copies of $L_{A_{i+1}\times A_i}$ and c_i^1 copies of $L_{A_i\times A_{i+1}}$ in some order, and consider the concatenation $\mathbf{p}^N$ for N sufficiently large. Then we can find a unique line with slope s/r which intersects the path $\pi(\mathbf{p}^N)$ so that the path never goes above the line. Then we obtain an equivalence class of concatenations of size $r \times s$. We choose a concatenation from the equivalence class and denote it again by $\mathbf{p}$.

If $\mathbf{p} = \mathbf{p}_0^d$ for some concatenation $\mathbf{p}_0$ of weight τ such that $\alpha/\tau = d$ and d is maximal, then the number of cyclic permutations of $\mathbf{p}$ is $\sum_{i\geq 0}(c_i^0 + c_i^1)/d$. Define the *contribution* of the equivalence class of $\mathbf{p}$ to be $(-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)}/d$. Then the total sum of contributions of equivalence classes of concatenations $\mathbf{p}$ such that $\mathrm{seq}(\mathbf{p}) = \mathbf{c}$ is given by $(-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0+c_i^1)-1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}$. One can see this by observing that $\frac{(\sum_{i\geq 0} c_i^0+c_i^1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}$ counts the number of concatenations and that $\frac{(\sum_{i\geq 0}(c_i^0+c_i^1)-1)!}{\prod_{i\geq 0} c_i^0! c_i^1!}$ is the *weighted* number of cyclic equivalence classes of concatenations when we assign a weight $1/d$ to an equivalence class of $\sum_{i\geq 0}(c_i^0 + c_i^1)/d$ members.

We group the equivalence classes of concatenations $\mathbf{p}$ according to the resulting equivalence classes $\mathcal{D}$ of Dyck paths so that $\pi(\mathbf{p}) \in \mathcal{D}$, and define $\mathcal{T}_{\mathcal{D}}$ to be the total sum of contributions of the equivalence classes of $\mathbf{p}$ such that $\pi(\mathbf{p}) \in \mathcal{D}$. Then we have

$$\sum_{\mathbf{c}\in\mathscr{C}(\alpha)} (-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0 + c_i^1) - 1)!}{\prod_{i\geq 0} c_i^0! c_i^1!} = \sum_{\mathcal{D}:\mathrm{wt}(\mathcal{D})=\alpha} \mathcal{T}_{\mathcal{D}}. \tag{3.9}$$

We consider an equivalence class $\mathcal{D}$ of Dyck paths of weight α and choose a representative D. Let D_0 be a primitive subpath of D such that $D=\pi(D_0^d)$, and let $\mathscr{S}(D_0)=\{\mathbf{s}_1,\ldots,\mathbf{s}_k\}$. If $\mathbf{p}$ is a concatenation such that $\pi(\mathbf{p})=D$ then $\mathbf{p}$ is equal to a concatenation of d choices of $\mathbf{s}_i$ from $\mathscr{S}(D_0)$ with repetition allowed. Thus the total sum $\mathcal{T}_\mathcal{D}$ of contributions is equal to

$$\mathcal{T}_\mathcal{D}=\frac{1}{d}(\mathrm{sgn}(\mathbf{s}_1)+\cdots+\mathrm{sgn}(\mathbf{s}_k))^d=\frac{1}{d}c(D_0)^d.$$

By Lemma 3.5, we know that $c(D_0)=-1,0$ or 1. Unless $c(D_0)=-1$ and d is even, we have $\mathcal{T}_\mathcal{D}=\frac{1}{d}c(D_0)$. If $c(D_0)=-1$ and d is even, then we have $\mathcal{T}_\mathcal{D}=\frac{1}{d}=\frac{1}{d}c(D_0)+\frac{2}{d}$.

Now we obtain

$$\begin{aligned}\sum_{\mathcal{D}:\mathrm{wt}(\mathcal{D})=\alpha}\mathcal{T}_\mathcal{D}&=\sum_{\tau|\alpha}\frac{\tau}{\alpha}\sum_{\substack{\mathcal{D}_0:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D}_0)=\tau}}c(\mathcal{D}_0)+\sum_{2\tau|\alpha}\frac{2\tau}{\alpha}\sum_{\substack{\mathcal{D}_0:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D}_0)=\tau}}\delta_{c(\mathcal{D}_0)+1,0}\\ &=\sum_{\tau|\alpha}\frac{\tau}{\alpha}\left[\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\tau}}c(\mathcal{D})+\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\tau/2}}\left\lfloor\frac{1-c(\mathcal{D})}{2}\right\rfloor\right],\end{aligned}$$

where δ is the Kronecker delta. Combined with (3.9), this establishes the desired identity (3.8).

Finally, let $\beta=r_0\alpha_1+s_0\alpha_2$ be such that r_0 and s_0 are relatively prime and $\beta|\alpha$. Multiplying both sides of (3.8) by α/β, we obtain

$$\begin{aligned}&\frac{\alpha}{\beta}\sum_{\mathbf{c}\in\mathscr{C}(\alpha)}(-1)^{\sum_{i:\mathrm{odd}}(c_i^0+c_i^1)}\frac{(\sum_{i\geq 0}(c_i^0+c_i^1)-1)!}{\prod_{i\geq 0}c_i^0!c_i^1!}\\ &=\sum_{\tau|\alpha}\frac{\tau}{\beta}\left[\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\tau}}c(\mathcal{D})+\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\tau/2}}\left\lfloor\frac{1-c(\mathcal{D})}{2}\right\rfloor\right].\end{aligned}\tag{3.10}$$

It follows from the Möbius inversion and Proposition 3.2 that

$$\mathrm{mult}\,(\alpha)=\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\alpha}}c(\mathcal{D})+\sum_{\substack{\mathcal{D}:\,\text{primitive}\\ \mathrm{wt}(\mathcal{D})=\alpha/2}}\left\lfloor\frac{1-c(\mathcal{D})}{2}\right\rfloor.$$

This completes the proof. □

As a corollary, we can prove an analogue of Frenkel's conjecture.

Corollary 3.11. *We have*

$$\mathrm{mult}\,(\alpha)\ \leq\ p_t\left(1-\frac{(\alpha|\alpha)}{2}\right),$$

where $\alpha=r\alpha_1+s\alpha_2$, $t=\max(r,s)$ and $p_t(n)$ is the number of partitions of n with at most t parts.

Proof. Let $\alpha=r\alpha_1+s\alpha_2$. We assume by symmetry that $r\leq s$. Let $n=r-1+$(the number of unit boxes below the diagonal). We define a one-to-one function from the set of Dyck paths to the set of partitions of n by $D\mapsto(\gamma_0,\gamma_1,\ldots,\gamma_{s-1})$, where $\gamma_k=$ the number of unit boxes in the k-th row and below D for $k\in\{1,\ldots,s-1\}$ and $\gamma_0=n-\sum_{k=1}^{s-1}\gamma_k$. It is straightforward to see that $\gamma_0\geq\gamma_1\geq\cdots\geq\gamma_{s-1}$. Hence we have $\mathrm{mult}\,(\alpha)\leq p_s(n)$, so it suffices to show that $n\leq 1-\frac{(\alpha|\alpha)}{2}$.

Since mult (α) is invariant under the Weyl group action, we can further assume that $r \leq s \leq \frac{a}{2}r$. Then

$$1 - \frac{(\alpha|\alpha)}{2} = 1 + ars - r^2 - s^2 \geq 1 + \frac{a^2-4}{2a}rs \geq 1 + \frac{5}{6}rs \geq r - 1 + \frac{1}{2}rs \geq n. \qquad \square$$

As another corollary, we obtain combinatorial upper and lower bounds for root multiplicities:

Corollary 3.12. *We have*

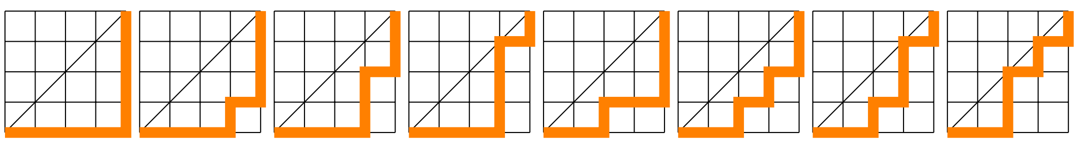
$$\sum_{\substack{\mathcal{D}:\,primitive\\ \mathrm{wt}(\mathcal{D})=\alpha}} c(\mathcal{D}) \leq \mathrm{mult}\,(\alpha) \leq \#\{\mathcal{D}:\, primitive,\ \mathrm{wt}(\mathcal{D}) = \alpha,\ c(\mathcal{D}) = 1\}.$$

Proof. The inequality for the lower bound is clear. For the upper bound, we need only to prove that

$$\#\{\mathcal{D}:\ \text{primitive},\ \mathrm{wt}(\mathcal{D}) = \alpha,\ c(\mathcal{D}) = -1\} \geq \#\{\mathcal{D}_0:\ \text{primitive},\ \mathrm{wt}(\mathcal{D}_0) = \alpha/2,\ c(\mathcal{D}_0) = -1\}.$$

Suppose that $\mathcal{D}_0$ is primitive with $\mathrm{wt}(\mathcal{D}_0) = \alpha/2$ and $c(\mathcal{D}_0) = -1$. We choose $D_0 \in \mathcal{D}_0$. Since $\mathrm{mult}(\alpha/2) \geq 0$, there exists a primitive D_1 with $\mathrm{wt}(D_1) = \alpha/2$ and $c(D_1) = 1$. Then $D := \pi(D_0D_1)$ is primitive, and we have $\mathrm{wt}(D) = \alpha$ and $c(D) = -1$. If we fix D_1, then the map $D_0 \mapsto D$ is injective. $\square$

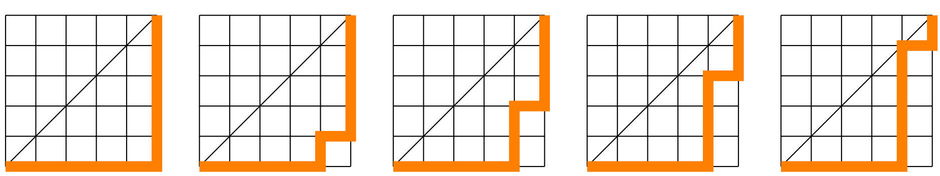
Example 3.13. Consider $\alpha = 4\alpha_1 + 4\alpha_2$ of $\mathcal{H}(3)$. Then we have the following representatives of equivalence classes of primitive Dyck paths and the corresponding contribution multiplicities.



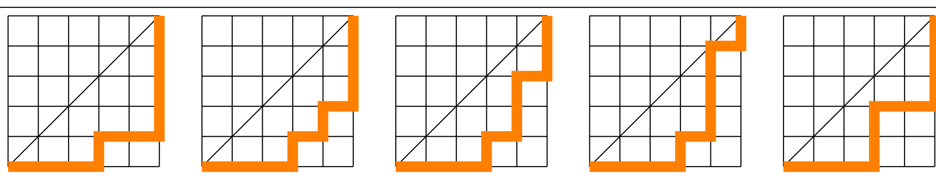
-1	1	1	1	1	1	1	1

Since $A_2 = 4$ for $\mathcal{H}(3)$, the weight $\alpha/2 = 2\alpha_1 + 2\alpha_2$ does not have any path D with $c(D) = -1$, and the correction term is 0. Thus we have mult $(\alpha) = 6$.

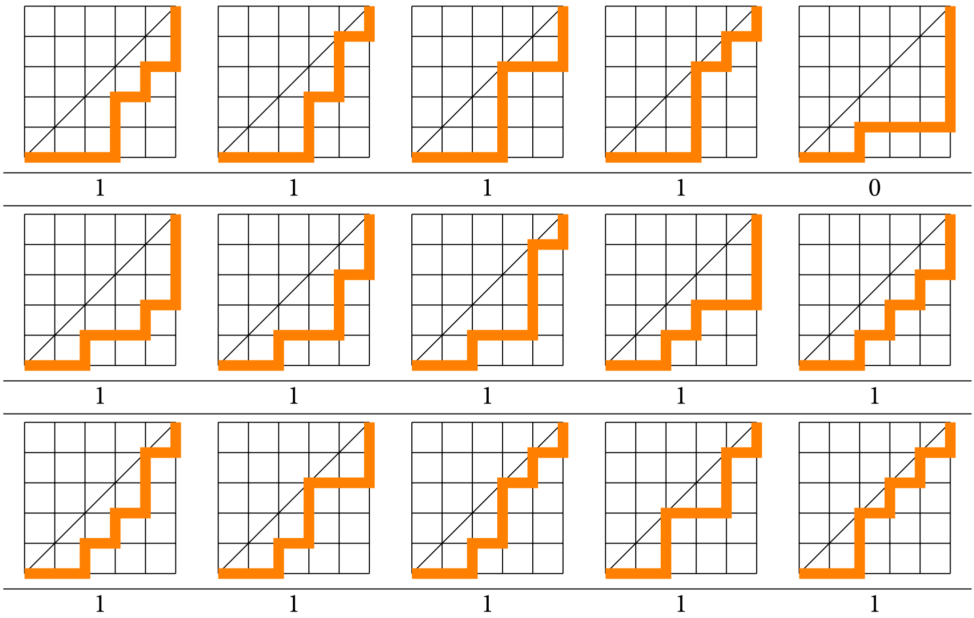
Example 3.14. Consider $\alpha = 5\alpha_1 + 5\alpha_2$ of $\mathcal{H}(3)$. Then we have the following representatives of equivalence classes of primitive Dyck paths and the corresponding contribution multiplicities.



-1	0	0	0	-1



0	1	1	1	1



Since $\alpha/2$ is not an integral weight, the correction term is zero. Thus we have $\text{mult}\,(\alpha) = 16$.

4. Sharper upper bound I

The next goal of this paper is to obtain sharper upper bounds for root multiplicities by considering cancellation among paths with opposite contribution multiplicities. In this section, we will develop a procedure to obtain such bounds, which depends on a choice of a certain family of Dyck paths. In the next section, we will explicitly make a careful choice of such a family of Dyck paths.

We begin with a lemma, which guarantees the existence of a family with desired properties.

Lemma 4.1. *For each $L_{u\times v}$ of type (-1), we can choose a primitive Dyck path $M_{u\times v}$ of size $u \times v$ which contains no framed subpaths of type (-1) or of type (0).*

Proof. Suppose that $L_{u\times v}$ is of type (-1). We may assume that $u \le v$. Since

$$\frac{A_{2n+1}-1}{A_{2n}} = \frac{aA_{2n}-A_{2n-1}}{A_{2n}} = a - \frac{A_{2n-1}}{A_{2n}} < a,$$

we have $1 \le v/u < a$. Let $E_{u\times v}$ be the Dyck path that is closest to the diagonal joining $(0,0)$ and (u,v). Then $E_{u\times v}$ is given by a concatenation of subpaths of sizes $1 \times s$ with $1 \le s \le a < A_2 = a+1$. Therefore, if $E_{u\times v}$ is primitive, we can put $M_{u\times v} = E_{u\times v}$. If $E_{u\times v}$ is not primitive, then $E_{u\times v}$ meets with the diagonal other than $(0,0)$ and (u,v). Each of these intersection points is incident with a vertical edge and a horizontal edge. Immediately after the horizontal edge, $E_{u\times v}$ travels s steps in the north for some $s = 1, 2, \ldots, a-1$. We switch the order of the vertical edge and horizontal edge, so that the resulting Dyck path, say $M_{u\times v}$, does not touch the diagonal. Also there are $s+1 < A_2$ vertical edges after the new horizontal edge. Hence $M_{u\times v}$ is primitive and does not contain any framed subpaths of type (-1) or of type (0). □

Remark 4.2. The choice of $M_{u\times v}$ made in the proof of Lemma 4.1 is not optimal for upper bounds for root multiplicities. In Section 5, we will investigate how to make an optimal choice of $M_{u\times v}$ to obtain sharp upper bounds for root multiplicities.

For the rest of this section, we fix $M_{u\times v}$ for each $L_{u\times v}$ of type (-1) that satisfies the conditions in Lemma 4.1. A subpath of the form $M_{u\times v}$ is to be called of type $(1c)$. Let $\Theta(\alpha)$ be the set of equivalence classes of primitive Dyck paths with weight $\alpha = r\alpha_1 + s\alpha_2$. We define a function

$$\Phi : \{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = -1\} \longrightarrow \{\mathcal{D} : c(\mathcal{D}) = 1\} \tag{4.3}$$

as follows: Choose a representative Dyck path $D \in \mathcal{D}$, and we travel from $(0,0)$ to (r,s) along D. As soon as we encounter with a subpath of type (-1) or $(1c)$, we stop traveling and define $\Phi(\mathcal{D})$ as the equivalence class containing the resulting Dyck path obtained from D by replacing the subpath $L_{u\times v}$ or $M_{u\times v}$ with the corresponding $M_{u\times v}$ or $L_{u\times v}$, respectively.

Remark 4.4. Note that $\Phi(\mathcal{D})$ may not be primitive. The definition of Φ depends on the choice of $M_{u\times v}$ and on the choice of representatives D. In general, Φ is not injective.

Now we state the main theorem of this section.

Theorem 4.5. *Let α be a positive root of $\mathcal{H}(a)$. Then we have*

$$\text{mult}\,(\alpha) \le \#\{\mathcal{D} : \textit{primitive},\ \text{wt}(\mathcal{D}) = \alpha,\ c(\mathcal{D}) = 1,\ \mathcal{D} \textit{ is not an image under } \Phi\}.$$

Proof. We set

$$\begin{aligned}
N_1 &= \#\{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = 1,\ \mathcal{D} \notin \text{Im}\,\Phi\},\\
N_2 &= \#\{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = 1,\ \mathcal{D} \in \text{Im}\,\Phi\},\\
N_3 &= \#\{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = -1,\ \Phi(\mathcal{D}) \text{ is primitive}\},\\
N_4 &= \#\{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = -1,\ \Phi(\mathcal{D}) \ \text{ is non-primitive}\},\\
N_5 &= \#\{\mathcal{D} \in \Theta(\alpha/2) : c(\mathcal{D}) = -1\}.
\end{aligned}$$

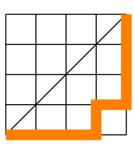
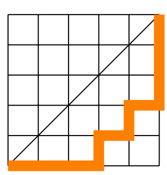
Then the identity in Theorem 3.7 can be written as

$$\text{mult}\,(\alpha) = N_1 + N_2 - N_3 - N_4 + N_5.$$

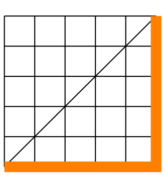
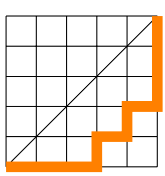
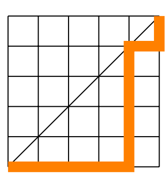
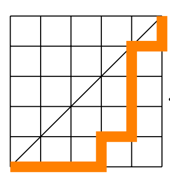
Note that we are proving $\text{mult}\,(\alpha) \le N_1$. Clearly, $N_2 - N_3 \le 0$, and we have only to show that $-N_4 + N_5 \le 0$.

Suppose that $\mathcal{D}_0$ is primitive with $\text{wt}(\mathcal{D}_0) = \alpha/2$ and $c(\mathcal{D}_0) = -1$. By Lemma 3.5, D_0 has a framed subpath $L_{u\times v}$ of type (-1). Thus $D_0 \neq \Phi(D_0)$. Set $D := \pi(D_0\,\Phi(D_0))$. Then D is primitive, and we have $\text{wt}(D) = \alpha$ and $c(D) = -1$. Moreover $\Phi(D)$ is non-primitive by construction and the correspondence $D_0 \mapsto D$ induces an injective map from $\{\mathcal{D} \in \Theta(\alpha/2) : c(\mathcal{D}) = -1\}$ to $\{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = -1,\ \Phi(\mathcal{D}) \text{ is non-primitive}\}$. Thus we have $N_5 \le N_4$. □

Example 4.6. Consider again $\alpha = 5\alpha_1 + 5\alpha_2 \in \mathcal{H}(3)$. We choose $M_{4\times 4}$ and $M_{5\times 5}$ to be

 and , respectively.

Then the map Φ gives

 $\longrightarrow$  and  $\longrightarrow$ . (4.7)

In the first correspondence, the whole path is $L_{5\times 5}$ and it is simply replaced by $M_{5\times 5}$; in the second, the subpath $L_{4\times 4}$ is replaced by $M_{4\times 4}$. There are 18 primitive equivalence classes of paths with contribution multiplicity 1 as one can see from Example 3.14. Since two of them are in the image of Φ as shown in (4.7), we actually obtain an equality $16 = \text{mult}(\alpha) = \#\{\mathcal{D} \in \Theta(\alpha) : c(\mathcal{D}) = 1, \mathcal{D} \notin \text{Im}\,\Phi\}$.

5. Sharper upper bound II

The upper bound in Theorem 4.5 depends on the choice of Dyck paths $M_{u\times v}$ and the resulting function Φ. In this section, we will make an optimal choice of $M_{u\times v}$ so that Φ may become close to an injection and consequently produce sharp upper bounds for root multiplicities.

Recall that we have defined the sequences $\{A_n\}, \{B_n\}$ by

$$\begin{aligned} &A_0 = 0, \quad A_1 = 1, \quad A_{n+2} = aA_{n+1} - A_n + 1 \quad \text{for } n \geq 0,\\ &B_0 = 0, \quad B_1 = 1, \quad B_{n+2} = aB_{n+1} - B_n \quad \text{for } n \geq 0. \end{aligned}$$

Lemma 5.1. *We have, for $i = 1, 2, \ldots,$*

$$A_{i+1}A_{i-1} = A_i^2 - A_i.$$

Proof. We use induction. If $i = 1$, then the assertion is clearly true. Assume that we have $A_iA_{i-2} = A_{i-1}^2 - A_{i-1}$. Then we obtain

$$\begin{aligned} A_{i+1}A_{i-1} &= (aA_i - A_{i-1} + 1)A_{i-1} = aA_iA_{i-1} - A_{i-1}^2 + A_{i-1}\\ &= aA_iA_{i-1} - A_iA_{i-2} = A_i(aA_{i-1} - A_{i-2}) = A_i(A_i - 1) = A_i^2 - A_i. \end{aligned}$$

□

For any positive integers u, v, denote by $E_{u\times v}$ the Dyck path of size $u \times v$ that is closest to the diagonal joining $(0, 0)$ and (u, v). For any integer $n \geq 2$, we define

$$M_{A_{2n}\times(A_{2n+1}-i)} = \begin{cases} L_{A_{2n-1}\times(A_{2n}-2)}E_{(A_{2n}-2A_{2n-1})\times(A_{2n+1}-2A_{2n}+2)}L_{A_{2n-1}\times(A_{2n}-1)}, & \text{for } i = 1;\\ L_{A_{2n-1}\times(A_{2n}-3)}E_{(A_{2n}-2A_{2n-1})\times(A_{2n+1}-2A_{2n}+4-i)}L_{A_{2n-1}\times(A_{2n}-1)}, & \text{for } 2 \leq i \leq a+2;\\ L_{(A_{2n-1}+1)\times j_1}E_{(A_{2n}-2A_{2n-1}-1)\times(A_{2n+1}-A_{2n}+1-j_1-i)}L_{A_{2n-1}\times(A_{2n}-1)}, & \text{for } a+3 \leq i \leq \dfrac{(a+3)A_{2n}}{A_{2n-1}};\\ L_{(A_{2n-1}+1)\times j_1}E_{(A_{2n}-2A_{2n-1}-1)\times(A_{2n+1}-j_1-j_2-i)}L_{A_{2n-1}\times j_2}, & \text{for } \dfrac{(a+3)A_{2n}}{A_{2n-1}} < i \leq A_{2n+1}-A_{2n}, \end{cases}$$

where j_1 is the integer satisfying

$$\frac{j_1}{A_{2n-1}+1} \leq \frac{A_{2n+1}-i}{A_{2n}} < \frac{j_1+1}{A_{2n-1}+1}$$

and j_2 is the integer satisfying

$$\frac{j_2-1}{A_{2n-1}+1} < \frac{A_{2n+1}-i}{A_{2n}} \leq \frac{j_2}{A_{2n-1}+1}.$$

For $1 \leq i \leq A_{2n+1} - A_{2n} - \frac{A_{2n}}{A_{2n-1}+1}$, we define $M_{(A_{2n+1}-i)\times A_{2n}}$ to be the transpose of $M_{A_{2n}\times(A_{2n+1}-i)}$. For $A_{2n+1} - A_{2n} - \frac{A_{2n}}{A_{2n-1}+1} < i < A_{2n+1} - A_{2n}$, define $M_{(A_{2n+1}-i)\times A_{2n}}$ by

$$L_{j_2\times A_{2n-1}}E_{(A_{2n+1}-j_1-j_2-i)\times(A_{2n}-2A_{2n-1}-2)}L_{j_1\times(A_{2n-1}+2)},$$

where j_2 is the integer satisfying

$$\frac{A_{2n-1}}{j_2} \leq \frac{A_{2n}}{A_{2n+1}-i} < \frac{A_{2n-1}}{j_2-1}$$

and j_1 is the integer satisfying

$$\frac{A_{2n-1}+2}{j_1+1} < \frac{A_{2n}}{A_{2n+1}-i} \leq \frac{A_{2n-1}+2}{j_1}.$$

Lemma 5.2. *The paths $M_{A_{2n}\times(A_{2n+1}-i)}$ and $M_{(A_{2n+1}-i)\times A_{2n}}$ are Dyck paths for each $n \geq 2$ and $1 \leq i \leq A_{2n+1} - A_{2n}$.*

Proof. Since the other case is similar, we only consider $M_{A_{2n}\times(A_{2n+1}-i)}$. First we need to show that $\frac{A_{2n+1}-i}{A_{2n}} \leq \frac{A_{2n}-1}{A_{2n-1}}$ which is equivalent to

$$A_{2n+1}A_{2n-1} - iA_{2n-1} \leq A_{2n}^2 - A_{2n}.$$

By Lemma 5.1, this becomes $iA_{2n-1} \geq 0$, which is obvious.

Next, we consider $\frac{A_{2n+1}-1}{A_{2n}} \geq \frac{A_{2n}-2}{A_{2n-1}}$ which is equivalent to

$$A_{2n+1}A_{2n-1} - A_{2n-1} \geq A_{2n}^2 - 2A_{2n}.$$

Again by Lemma 5.1, this becomes $A_{2n-1} \leq A_{2n}$, which is clearly true. The remaining cases can be checked in a similar way. □

For $n \geq 2$ and $k \in \{1, 2, \ldots, A_{2n+1} - A_{2n} - 1\}$, we define

$$M_{(A_{2n}+k)\times(A_{2n+1}-i)} = \begin{cases} L_{(A_{2n-1}+1)\times j_1} E_{(A_{2n}-2A_{2n-1}+k-1)\times(A_{2n+1}-A_{2n}+1-j_1-i)} L_{A_{2n-1}\times(A_{2n}-1)}, & \text{for } 1 \leq i \leq p; \\ L_{(A_{2n-1}+1)\times j_1} E_{(A_{2n}-2A_{2n-1}+k-1)\times(A_{2n+1}-j_1-j_2-i)} L_{A_{2n-1}\times j_2}, & \text{for } p < i \leq A_{2n+1} - A_{2n} - k, \end{cases}$$

where j_1 is the integer satisfying

$$\frac{j_1}{A_{2n-1}+1} \leq \frac{A_{2n+1}-i}{A_{2n}+k} < \frac{j_1+1}{A_{2n-1}+1},$$

j_2 is the integer satisfying

$$\frac{j_2-1}{A_{2n-1}+1} < \frac{A_{2n+1}-i}{A_{2n}+k} \leq \frac{j_2}{A_{2n-1}+1},$$

and p is the integer satisfying

$$\frac{(A_{2n}-3)-1}{A_{2n-1}+1} < \frac{A_{2n+1}-p}{A_{2n}+k} \leq \frac{(A_{2n}-3)}{A_{2n-1}+1}.$$

For $1 \leq i \leq A_{2n+1} - A_{2n} - k - \frac{A_{2n}+k}{A_{2n-1}+1}$, we define $M_{(A_{2n+1}-i)\times(A_{2n}+k)}$ to be the transpose of $M_{(A_{2n}+k)\times(A_{2n+1}-i)}$. For $A_{2n+1} - A_{2n} - k - \frac{A_{2n}+k}{A_{2n-1}+1} < i < A_{2n+1} - A_{2n} - k$, define $M_{(A_{2n+1}-i)\times(A_{2n}+k)}$ by

$$L_{j_2\times A_{2n-1}} E_{(A_{2n+1}-j_1-j_2-i)\times(A_{2n}-2A_{2n-1}+k-2)} L_{j_1\times(A_{2n-1}+2)},$$

where j_2 is the integer satisfying

$$\frac{A_{2n-1}}{j_2} \leq \frac{A_{2n}+k}{A_{2n+1}-i} < \frac{A_{2n-1}}{j_2-1}$$

and j_1 is the integer satisfying

$$\frac{A_{2n-1}+2}{j_1+1} < \frac{A_{2n}+k}{A_{2n+1}-i} \le \frac{A_{2n-1}+2}{j_1}.$$

Lemma 5.3. *The paths $M_{(A_{2n}+k)\times(A_{2n+1}-i)}$ and $M_{(A_{2n+1}-i)\times(A_{2n}+k)}$ are Dyck paths for each $n \ge 2$, $1 \le k \le A_{2n+1} - A_{2n} - 1$ and $1 \le i \le A_{2n+1} - A_{2n} - k$.*

Proof. The proof is even simpler than the proof of Lemma 5.2, so we will omit it. □

We recall the definition of mutation of Dyck paths, which is developed by Lee and Schiffler [17], Rupel [21], and Lee et al. [16].

Definition 5.4. Consider the bijective function $\phi : \{1^a2, 1^{a-1}2, \dots, 12, 2\} \longrightarrow \{1, 12, \dots, 12^{a-1}, 12^a\}$ defined by

$$\phi(1^a2) = 1, \quad \phi(1^{a-1}2) = 12, \quad \dots \quad \phi(12) = 12^{a-1}, \quad \phi(2) = 12^a.$$

Suppose that a finite sequence S is obtained by concatenating (copies of) $1^a2, 1^{a-1}2, \cdots, 12, 2$. Let $\phi(S)$ be the sequence obtained from S by replacing each subsequence 1^a2 (resp. $1^{a-1}2, \cdots, 2$) with $\phi(1^a2)$ (resp. $\phi(1^{a-1}2), \cdots, \phi(2)$). We call $\phi(S)$ the *mutation* of S.

Lemma 5.5. *Let u and v be positive integers with $0 \le av - u \le v \le u$. Then $\phi(E_{u\times v}) = E_{v\times(av-u)}$.*

Proof. See [16, p. 68]. □

Now, for each (u, v) with $A_2 \le \min(u, v), \max(u, v) < A_3$, we choose a primitive Dyck path $M_{u\times v}$ of size $u \times v$ such that $M_{u\times v} \ne E_{u\times v}$ except for $(u, v) = (A_2, A_3 - 1)$ and $(A_3 - 1, A_2)$ and which contains no framed subpaths of type (-1) or of type (0). A subpath of the form $M_{u\times v}$ with $A_{2n} \le \min(u, v), \max(u, v) < A_{2n+1}$ $(n \ge 1)$ is said to be of type $(1s)$. More specifically, a subpath of the form $M_{u\times v}$ with $A_2 \le \min(u, v), \max(u, v) < A_3$ is said to be of type $(1s1)$. If $n \ge 2$, then a subpath of the form $M_{u\times v}$ with $A_{2n} \le \min(u, v), \max(u, v) < A_{2n+1}$ is said to be of type $(1s2)$. Here we do not further specify the paths $M_{u\times v}$ of type $(1s1)$ since the conditions given above are enough to obtain the main result (Proposition 5.6), whereas the paths $M_{u\times v}$ of type $(1s2)$ have been deliberately chosen in this section.

The following proposition shows that our choice of $M_{u\times v}$ made above for type $(1s2)$ is optimal in the sense that the resulting map Φ in (4.3) is as close to an injection as possible. Before stating the proposition, we define one more terminology: For a Dyck path D, a pair of two subpaths of D is said to be *disjoint* if the two subpaths share no edges.

Proposition 5.6. *Suppose that $M_{u\times v} \ne E_{u\times v}$ for $A_2 \le \min(u, v), \max(u, v) < A_3$ except for $(u, v) = (A_2, A_3 - 1)$ and $(A_3 - 1, A_2)$. Consider a Dyck path D. Then each subpath of type $(1s2)$ of D is disjoint from all other subpaths of type $(1s)$.*

Proof. First we show that any pair of two distinct subpaths of type $(1s2)$ is disjoint. Suppose that two distinct subpaths of type $(1s2)$ are not disjoint. Let one of the two subpaths be $L_{u_1\times v_1}E_{u_2\times v_2}L_{u_3\times v_3}$ and the other $L_{u_1'\times v_1'}E_{u_2'\times v_2'}L_{u_3'\times v_3'}$. Without loss of generality, assume that the first subpath starts before the second one does. Since $u_1, v_1, u_3, v_3, u_1', v_1', u_3', v_3' > 2$, the only possibility is that $u_3 = u_1'$ and $v_3 = v_1'$. We will check that this never happens. By symmetry we further assume that $u_3 \le v_3$.

Then, by definition of j_2, we get

$$\begin{aligned} j_2 - 1 &< \frac{A_{2n-1}+1}{A_{2n}}\left(A_{2n+1} - \frac{(a+3)A_{2n}}{A_{2n-1}}\right) \\ &= \frac{A_{2n-1}A_{2n+1}}{A_{2n}} + \frac{A_{2n+1}}{A_{2n}} - (a+3) - \frac{a+3}{A_{2n-1}} \\ &\overset{\text{Lemma 5.1}}{=} A_{2n} - 1 + \frac{A_{2n+1}}{A_{2n}} - (a+3) - \frac{a+3}{A_{2n-1}} \\ &< A_{2n} - 1 + a - (a+3) = A_{2n} - 4, \end{aligned}$$

which implies that $j_2 \leq A_{2n} - 4$. So if $u_3 = u_1' = A_{2n-1}$ then $v_3 \neq v_1'$. For other cases, it is easy to see that if $u_3 = u_1'$ then $v_3 \neq v_1'$.

Next we show that each subpath of type $(1s2)$ is disjoint from any subpath of type $(1s1)$. Let W be a subpath of type $(1s2)$ and V a subpath of type $(1s1)$. Note that W is of the form $L_{u_1\times v_1}E_{u_2\times v_2}L_{u_3\times v_3}$ with $\min(u_1, v_1, u_3, v_3) \geq A_3$, and that V is of the form $M_{u\times v}$ with $A_2 \leq \min(u, v), \max(u, v) < A_3$. So if W and V are not disjoint, then $M_{u\times v}$ should be a subpath of $E_{u_2\times v_2}$. This happens only when $M_{u\times v} = E_{u\times v}$, in other words, $M_{u\times v}$ is closest to the diagonal. Hence $(u, v) = (A_2, A_3-1)$ or (A_3-1, A_2). Without loss of generality, let $(u, v) = (A_3 - 1, A_2)$. However $(\phi \circ \phi)(E_{u\times v})$ is not defined, because $aA_2 - (A_3 - 1) = A_1 = 1$ and $aA_1 - A_2 < 0$. On the other hand, it is straightforward to check that $(\phi \circ \phi)(E_{u_2\times v_2})$ is well defined, which implies that $E_{u\times v}$ cannot be a subpath of $E_{u_2\times v_2}$. □

Funding

This work was supported by NRF Grant # 2014-021261 and by NRF Grant # 2010-0010753. This work was partially supported by a grant from the Simons Foundation (#318706). This work was partially supported by NSA grant H98230-14-1-0323.

References

[1] Berman, S., Moody, R. V. (1979). Lie algebra multiplicities. *Proc. Am. Math. Soc.* 76(2):223–228.
[2] Borcherds, R. E. (1990). The monster Lie algebra. *Adv. Math.* 83:30–47.
[3] Carbone, L., Freyn, W., Lee, K.-H. (2014). *Dimensions of Imaginary Root Spaces of Hyperbolic Kac–Moody Algebras*. Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, Contemporary Mathematics, Vol. 623. Providence, RI: American Mathematical Society, pp. 23–40.
[4] Feingold, A. J. (1980). A hyperbolic GCM Lie algebra and the Fibonacci numbers. *Proc. Am. Math. Soc.* 80(3): 379–385.
[5] Feingold, A. J., Frenkel, I. B. (1983). A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2. *Math. Ann.* 263(1):87–144.
[6] Frenkel, I. B. (1985). *Representations of Kac–Moody Algebras and Dual Resonance Models*. Applications of group theory in physics and mathematical physics (Chicago, 1982). Lectures in Application Mathematics, Vol. 21. Providence, RI: American Mathematical Society, pp. 325–353.
[7] Garland, H., Lepowsky, J. (1976). Lie algebra homology and the Macdonald–Kac formulas. *Invent. Math.* 34(1):37–76.
[8] Kac, V. G. (1990). *Infinite-Dimensional Lie Algebras*. 3rd ed. Cambridge: Cambridge University Press.
[9] Kac, V. G., Moody, R. V., Wakimoto, M. (1988). *On E_{10}*, Differential Geometrical Methods in Theoretical Physics (Como, 1987), NATO Advanced Science Institute Series C Mathematics Physics Science, Vol. 250. Dordrecht: Kluwer Academic Publisher, pp. 109–128.
[10] Kang, S.-J. (1993). Kac–Moody lie algebras, spectral sequences, and the witt formula. *Trans. Am. Math. Soc.* 339(2):463–493.
[11] Kang, S.-J. (1993). Root multiplicities of the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$. *J. Algebra* 160(2):492–523.
[12] Kang, S.-J. (1994). On the hyperbolic Kac–Moody lie algebra $HA_1^{(1)}$. *Trans. Am. Math. Soc.* 341(2):623–638.
[13] Kang, S.-J. (1994). Root multiplicities of Kac–Moody algebras. *Duke Math. J.* 74(3):635–666.
[14] Kang, S.-J., Melville, D. J. (1995). Rank 2 symmetric hyperbolic Kac–Moody algebras. *Nagoya Math. J.* 140:41–75.
[15] Kim, H. H., Lee, K.-H. (2013). Root multiplicities of hyperbolic Kac–Moody algebras and Fourier coefficients of modular forms. *Ramanujan J.* 32:329–352.
[16] Lee, K., Li, L., Zelevinsky, A. (2014). Greedy elements in rank 2 cluster algebras. *Selecta Math. (N.S.)* 20(1):57–82.

[17] Lee, K., Schiffler, R. (2013). A combinatorial formula for rank 2 cluster variables. *J. Algebra Comb.* 37:67–85.
[18] Lepowsky, J., Moody, R. V. (1979). Hyperbolic Lie algebras and quasiregular cusps on Hilbert modular surfaces. *Math. Ann.* 245(1):63–88.
[19] Niemann, P. (2002). Some generalized Kac–Moody algebras with known root multiplicities. *Mem. Am. Math. Soc.* 157(746).
[20] Peterson, D. H. (1983). Freudenthal-type formulas for root and weight multiplicities (preprint (unpublished)).
[21] Rupel, D. (2012). Proof of the Kontsevich non-commutative cluster positivity conjecture. *C. R. Math. Acad. Sci. Paris* 350:929–932.