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On the centre of two-parameter quantum groups

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We describe Poincaré–Birkhoff–Witt bases for the two-parameter quantum groups $U = U_{r,s}(\mathfrak{sl}_n)$ following Kharchenko and show that the positive part of U has the structure of an iterated skew polynomial ring. We define an ad-invariant bilinear form on U, which plays an important role in the construction of central elements. We introduce an analogue of the Harish-Chandra homomorphism and use it to determine the centre of U.

1. Introduction

In this paper we determine the centre of the two-parameter quantum groups $U = U_{r,s}(\mathfrak{sl}_n)$, which are the same algebras as those introduced by Takeuchi in [35, 36], but with the opposite co-product. As shown in [4, 5], these quantum groups are Drinfel'd doubles and have an R-matrix. They are related to the down–up algebras in [2,3] and to the multi-parameter quantum groups of Chin and Musson [8] and Dobrev and Parashar [10]. In the analogous quantum function algebra setting, allowing two parameters unifies the Drinfel'd–Jimbo quantum groups ($r = q$, $s = q^{-1}$) in [11] with the Dipper–Donkin quantum groups ($r = 1$, $s = q^{-1}$) in [9].

For the one-parameter quantum groups $U_q(\mathfrak{g})$ corresponding to finite-dimensional simple Lie algebras $\mathfrak{g}$, there is a sizeable literature [7, 15, 21–28, 30–32, 37–39] dealing with Poincaré–Birkhoff–Witt (PBW) bases. For the multi-parameter quantum groups associated with $\mathfrak{g}$ of classical type, Kharchenko [21] constructed PBW bases by first determining Gröbner–Shirshov bases for them. We show in this paper that Kharchenko's results, when applied to the algebra $U = U_{r,s}(\mathfrak{sl}_n)$, yield useful commutation relations, which enable us to prove that the positive part U^+ of U has the structure of an iterated skew polynomial ring. As a consequence of that result, U^+ modulo any prime ideal is a domain. The commutation relations also play an essential role in [6], where finite-dimensional restricted two-parameter quantum groups $\mathfrak{u}_{r,s}(\mathfrak{gl}_n)$ and $\mathfrak{u}_{r,s}(\mathfrak{sl}_n)$ are constructed when r and s are roots of unity. These restricted quantum groups are Drinfel'd doubles and are ribbon Hopf algebras under suitable restrictions on r and s.

Much work has been done on the centre of quantum groups for finite-dimensional simple Lie algebras [1, 12, 19, 28, 29, 34, 37], and also for (generalized) Kac–Moody (super)algebras [13, 16, 20]. The approach taken in many of these papers (and adopted here as well) is to define a bilinear form on the quantum group which is invariant under the adjoint action. This quantum version of the Killing form is often referred to in the one-parameter setting as the *Rosso form* (see [34]). The next step involves constructing an analogue ξ of the Harish-Chandra map. It is straightforward to show that the map ξ is an injective algebra homomorphism. The main difficulty lies in determining the image of ξ and in finding enough central elements to prove that the map ξ is surjective. In the two-parameter case, a new phenomenon arises: the n odd and n even cases behave differently. Additional central elements arise when n is even, which complicates the description in that case.

Our paper is organized as follows. In § 2, we briefly recall the definition and basic properties of the two-parameter quantum group $U = U_{r,s}(\mathfrak{sl}_n)$. In § 3, we describe the commutation relations which determine a Gröbner–Shirshov basis and allow a PBW basis to be constructed, and we prove that the positive part of U has an iterated skew polynomial ring structure. The next section is devoted to the construction of a bilinear form and the proof of its invariance under the adjoint action. In the final section, we define a Harish-Chandra homomorphism ξ and determine the centre of U by specifying the image of ξ and constructing central elements explicitly.

2. Two-parameter quantum groups

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Assume that Φ is a finite root system of type A_{n-1} with Π a base of simple roots. We regard Φ as a subset of a Euclidean space $\mathbb{R}^n$ with an inner product $\langle\cdot,\cdot\rangle$. We let $\epsilon_1,\dots,\epsilon_n$ denote an orthonormal basis of $\mathbb{R}^n$, and suppose that $\Pi = \{\alpha_j = \epsilon_j - \epsilon_{j+1} \mid j = 1,\dots,n-1\}$ and that $\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leqslant i \neq j \leqslant n\}$.

Fix non-zero elements r, s in the field $\mathbb{K}$. Here we assume $r \neq s$. Let $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ be the unital associative algebra over $\mathbb{K}$ generated by elements e_j, f_j $(1 \leqslant j < n)$, and $a_i^{\pm 1}$, $b_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$, which satisfy the following relations:

(R1) the $a_i^{\pm 1}$, $b_j^{\pm 1}$ all commute with one another and $a_i a_i^{-1} = b_j b_j^{-1} = 1$;

(R2) $a_i e_j = r^{\langle \epsilon_i, \alpha_j\rangle} e_j a_i$ and $a_i f_j = r^{-\langle \epsilon_i, \alpha_j\rangle} f_j a_i$;

(R3) $b_i e_j = s^{\langle \epsilon_i, \alpha_j\rangle} e_j b_i$ and $b_i f_j = s^{-\langle \epsilon_i, \alpha_j\rangle} f_j b_i$;

(R4) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(a_i b_{i+1} - a_{i+1} b_i)$;

(R5) $[e_i, e_j] = [f_i, f_j] = 0$ if $|i-j| > 1$;

(R6)
$$e_i^2 e_{i+1} - (r+s) e_i e_{i+1} e_i + rs e_{i+1} e_i^2 = 0,$$
$$e_i e_{i+1}^2 - (r+s) e_{i+1} e_i e_{i+1} + rs e_{i+1}^2 e_i = 0;$$

(R7) $f_i^2 f_{i+1} - (r^{-1}+s^{-1}) f_i f_{i+1} f_i + r^{-1}s^{-1} f_{i+1} f_i^2 = 0,$
$f_i f_{i+1}^2 - (r^{-1}+s^{-1}) f_{i+1} f_i f_{i+1} + r^{-1}s^{-1} f_{i+1}^2 f_i = 0.$

Let $U = U_{r,s}(\mathfrak{sl}_n)$ be the subalgebra of $\tilde{U} = U_{r,s}(\mathfrak{gl}_n)$ generated by the elements e_j, f_j, $\omega_j^{\pm 1}$ and $(\omega_j')^{\pm 1}$ $(1 \leqslant j < n)$, where

$$\omega_j = a_j b_{j+1} \quad \text{and} \quad \omega_j' = a_{j+1} b_j.$$

These elements satisfy (R5)–(R7) along with the following relations:

(R1′) the $\omega_i^{\pm 1}, (\omega_j')^{\pm 1}$ all commute with one another and $\omega_i \omega_i^{-1} = \omega_j'(\omega_j')^{-1} = 1$;

(R2′) $\omega_i e_j = r^{\langle \epsilon_i, \alpha_j \rangle} s^{\langle \epsilon_{i+1}, \alpha_j \rangle} e_j \omega_i$ and $\omega_i f_j = r^{-\langle \epsilon_i, \alpha_j \rangle} s^{-\langle \epsilon_{i+1}, \alpha_j \rangle} f_j \omega_i$;

(R3′) $\omega_i' e_j = r^{\langle \epsilon_{i+1}, \alpha_j \rangle} s^{\langle \epsilon_i, \alpha_j \rangle} e_j \omega_i'$ and $\omega_i' f_j = r^{-\langle \epsilon_{i+1}, \alpha_j \rangle} s^{-\langle \epsilon_i, \alpha_j \rangle} f_j \omega_i'$;

(R4′) $[e_i, f_j] = \dfrac{\delta_{i,j}}{r-s}(\omega_i - \omega_i')$.

Let U^+ and U^- be the subalgebras generated by the elements e_i and f_i, respectively, and let $\tilde{U}^0$ and U^0 be the subalgebras generated by the elements $a_i^{\pm 1}$, $b_i^{\pm 1}$, $1 \leqslant i \leqslant n$ and $\omega_i^{\pm 1}$, $(\omega_i')^{\pm 1}$, $1 \leqslant i < n$, respectively. It now follows from the defining relations that $\tilde{U}$ has a triangular decomposition: $\tilde{U} = U^- \tilde{U}^0 U^+$. Similarly, we have $U = U^- U^0 U^+$.

The algebras $\tilde{U}$ and U are Hopf algebras, where the $a_i^{\pm}$, $b_i^{\pm}$ are group-like elements, and the remaining co-products are determined by

$$\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i'.$$

This forces the co-unit and antipode maps to be

$$\begin{aligned} &\varepsilon(a_i) = \varepsilon(b_i) = 1, \quad S(a_i) = a_i^{-1}, \qquad S(b_i) = b_i^{-1}, \\ &\varepsilon(e_i) = \varepsilon(f_i) = 0, \quad S(e_i) = -\omega_i^{-1} e_i, \quad S(f_i) = -f_i(\omega_i')^{-1}. \end{aligned}$$

Let $Q = \mathbb{Z}\Phi$ denote the root lattice and set $Q^+ = \bigoplus_{i=1}^{n-1} \mathbb{Z}_{\geqslant 0}\alpha_i$. Then, for any $\zeta = \sum_{i=1}^{n-1} \zeta_i \alpha_i \in Q$, we adopt the shorthand

$$\omega_\zeta = \omega_1^{\zeta_1} \cdots \omega_{n-1}^{\zeta_{n-1}}, \qquad \omega_\zeta' = (\omega_1')^{\zeta_1} \cdots (\omega_{n-1}')^{\zeta_{n-1}}. \tag{2.1}$$

Lemma 2.1 (Benkart and Witherspoon [4, lemma 1.3]). *Suppose that*

$$\zeta = \sum_{i=1}^{n-1} \zeta_i \alpha_i \in Q.$$

Then

$$\begin{aligned} \omega_\zeta e_i &= r^{-\langle \epsilon_{i+1}, \zeta \rangle} s^{-\langle \epsilon_i, \zeta \rangle} e_i \omega_\zeta, \quad \omega_\zeta f_i = r^{\langle \epsilon_{i+1}, \zeta \rangle} s^{\langle \epsilon_i, \zeta \rangle} f_i \omega_\zeta, \\ \omega_\zeta' e_i &= r^{-\langle \epsilon_i, \zeta \rangle} s^{-\langle \epsilon_{i+1}, \zeta \rangle} e_i \omega_\zeta', \quad \omega_\zeta' f_i = r^{\langle \epsilon_i, \zeta \rangle} s^{\langle \epsilon_{i+1}, \zeta \rangle} f_i \omega_\zeta'. \end{aligned}$$

There is a grading on U with the degrees of the generators given by

$$\deg e_i = \alpha_i, \qquad \deg f_i = -\alpha_i, \qquad \deg \omega_i = \deg \omega_i' = 0.$$

Then, since the defining relations are homogeneous under this grading, the algebra U has a Q-grading:

$$U = \bigoplus_{\zeta \in Q} U_\zeta.$$

We also have

$$U^+ = \bigoplus_{\zeta \in Q^+} U^+_\zeta \quad \text{and} \quad U^- = \bigoplus_{\zeta \in Q^+} U^-_{-\zeta},$$

where $U^+_\zeta = U^+ \cap U_\zeta$ and $U^-_{-\zeta} = U^- \cap U_{-\zeta}$.

Let $\Lambda = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ be the weight lattice of $\mathfrak{gl}_n$. Corresponding to any $\lambda \in \Lambda$ is an algebra homomorphism $\varrho^\lambda : \tilde{U}^0 \to \mathbb{K}$ given by

$$\varrho^\lambda(a_i) = r^{\langle \epsilon_i, \lambda \rangle} \quad \text{and} \quad \varrho^\lambda(b_i) = s^{\langle \epsilon_i, \lambda \rangle}. \tag{2.2}$$

For any $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \Lambda$, we write

$$a_\lambda = a_1^{\lambda_1} \cdots a_n^{\lambda_n} \quad \text{and} \quad b_\lambda = b_1^{\lambda_1} \cdots b_n^{\lambda_n}. \tag{2.3}$$

Let $\Lambda_{\mathfrak{sl}} = \bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i$ be the weight lattice of $\mathfrak{sl}_n$, where ϖ_i is the fundamental weight

$$\varpi_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{n} \sum_{j=1}^n \epsilon_j,$$

and let

$$\Lambda^+_{\mathfrak{sl}} = \{\lambda \in \Lambda_{\mathfrak{sl}} \mid \langle \alpha_i, \lambda \rangle \geqslant 0 \text{ for } 1 \leqslant i < n\} = \left\{ \sum_{i=1}^{n-1} l_i \varpi_i \;\middle|\; l_i \in \mathbb{Z}_{\geqslant 0} \right\}$$

denote the set of dominant weights for $\mathfrak{sl}_n$. We fix the nth roots $r^{1/n}$ and $s^{1/n}$ of r and s, respectively, and define, for any $\lambda \in \Lambda_{\mathfrak{sl}}$, an algebra homomorphism $\varrho^\lambda : U^0 \to \mathbb{K}$ by

$$\varrho^\lambda(\omega_j) = r^{\langle \epsilon_j, \lambda \rangle} s^{\langle \epsilon_{j+1}, \lambda \rangle} \quad \text{and} \quad \varrho^\lambda(\omega_j') = r^{\langle \epsilon_{j+1}, \lambda \rangle} s^{\langle \epsilon_j, \lambda \rangle}. \tag{2.4}$$

In particular, if λ belongs to Λ, then the definition of $\varrho^\lambda(\omega_j)$ and $\varrho^\lambda(\omega_j')$ coming from (2.2) coincides with (2.4).

Associated with any algebra homomorphism $\psi : U^0 \to \mathbb{K}$ is the Verma module $M(\psi)$ with highest weight ψ and its unique irreducible quotient $L(\psi)$. When the highest weight is given by the homomorphism ϱ^λ for $\lambda \in \Lambda_{\mathfrak{sl}}$, we simply write $M(\lambda)$ and $L(\lambda)$ instead of $M(\varrho^\lambda)$ and $L(\varrho^\lambda)$.

LEMMA 2.2 (Benkart and Witherspoon [5]). *We assume that rs^{-1} is not a root of unity, and let v_λ be a highest weight vector of $M(\lambda)$ for $\lambda \in \Lambda^+_{\mathfrak{sl}}$. The irreducible module $L(\lambda)$ is then given by*

$$L(\lambda) = M(\lambda) \Big/ \left(\sum_{i=1}^{n-1} U f_i^{\langle \lambda, \alpha_i \rangle + 1} v_\lambda \right).$$

Let W be the Weyl group of the root system Φ, and let $\sigma_i \in W$ denote the reflection corresponding to α_i for each $1 \leqslant i < n$. Thus,

$$\sigma_i(\lambda) = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i \quad \text{for } \lambda \in \Lambda, \tag{2.5}$$

and σ_i also acts on $\Lambda_{\mathfrak{sl}}$, according to the same formula.

Let M be a finite-dimensional U-module on which U^0 acts semi-simply. Then

$$M = \bigoplus_{\chi} M_{\chi},$$

where each $\chi : U^0 \to \mathbb{K}$ is an algebra homomorphism, and

$$M_{\chi} = \{m \in M \mid \omega_i m = \chi(\omega_i)m \text{ and } \omega_i' m = \chi(\omega_i')m \text{ for all } i\}.$$

For brevity we write M_λ for the weight space M_{ϱ^λ} for $\lambda \in \Lambda_{\mathfrak{sl}}$.

Proposition 2.3. *Assume that rs^{-1} is not a root of unity and that $\lambda \in \Lambda_{\mathfrak{sl}}^+$. Then*

$$\dim L(\lambda)_\mu = \dim L(\lambda)_{\sigma(\mu)}$$

for all $\mu \in \Lambda_{\mathfrak{sl}}$ and $\sigma \in W$.

Proof. This is an immediate consequence of [5, proposition 2.8 and the proof of lemma 2.12]. □

3. PBW-type bases

From now on we assume that $r + s \neq 0$ (or equivalently, $r^{-1} + s^{-1} \neq 0$), and the ordering $(k, l) < (i, j)$ always means relative to the lexicographic ordering.

We define inductively

$$\mathcal{E}_{j,j} = e_j \quad \text{and} \quad \mathcal{E}_{i,j} = e_i \mathcal{E}_{i-1,j} - r^{-1} \mathcal{E}_{i-1,j} e_i, \quad i > j. \tag{3.1}$$

The defining relations for U^+ in (R6) can be reformulated as saying

$$\mathcal{E}_{i+1,i} e_i = s^{-1} e_i \mathcal{E}_{i+1,i}, \tag{3.2}$$

$$e_{i+1} \mathcal{E}_{i+1,i} = s^{-1} \mathcal{E}_{i+1,i} e_{i+1}. \tag{3.3}$$

Even though the relations in the following theorem can be deduced from [21, theorem A_n], we include a self-contained proof in the appendix for the convenience of the reader.

Theorem 3.1 (Kharchenko [21]). *Assume that $(i, j) > (k, l)$ in the lexicographic order. Then the following relations hold in the algebra U^+:*

(1) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} - \mathcal{E}_{i,l} = 0$ *if* $j = k + 1$;

(2) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0$ *if* $i > k \geqslant l > j$ *or* $j > k + 1$;

(3) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0$ *if* $i = k \geqslant j > l$ *or* $i > k \geqslant j = l$;

(4) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} = 0$ *if* $i > k \geqslant j > l$.

Let $E = \{e_1, e_2, \ldots, e_{n-1}\}$ be the set of generators of the algebra U^+. We introduce a linear ordering $\prec$ on E by saying $e_i \prec e_j$ if and only if $i < j$. We extend this ordering to the set of monomials in E so that it becomes the *degree-lexicographic ordering*; that is, for $u = u_1 u_2 \cdots u_p$ and $v = v_1 v_2 \cdots v_q$ with $u_i, v_j \in E$, we have $u \prec v$ if and only if $p < q$ or $p = q$ and $u_i \prec v_i$ for the first i such that $u_i \neq v_i$. Let $\mathcal{A}_E$ be the free associative algebra generated by E and $\mathcal{S} \subset \mathcal{A}_E$ be the set consisting of the following elements:

$$\begin{aligned} \mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} \quad & \text{if} \quad i > k \geqslant l > j \text{ or } j > k+1, \\ \mathcal{E}_{i,j}\mathcal{E}_{k,l} - s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} \quad & \text{if } i = k \geqslant j > l \text{ or } i > k \geqslant j = l, \\ \mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} \quad & \text{if } i > k \geqslant j > l. \end{aligned}$$

The elements of $\mathcal{S}$ just correspond to relations (2)–(4) of theorem 3.1. Note that we may take $\mathcal{S}$ to be the set of defining relations for the algebra U^+, since $\mathcal{S}$ contains all the (original) defining relations (R5) and (R6) of U^+, and the other relations in $\mathcal{S}$ are all consequences of (R5) and (R6).

The following theorem is a special case of in [21, theorem $\boldsymbol{A}_n$] and its consequences. Also, one can prove it using an argument similar to that in [7] or [39, 40].

THEOREM 3.2 (Kharchenko [21]). *Assume that* $r, s \in \mathbb{K}^\times$ *and* $r + s \neq 0$. *Then*

(i) *the set* $\mathcal{S}$ *is a Gröbner–Shirshov basis for the algebra* U^+ *with respect to the degree-lexicographic ordering,*

(ii) $\mathcal{B}_0 = \{\mathcal{E}_{i_1,j_1}\mathcal{E}_{i_2,j_2} \cdots \mathcal{E}_{i_p,j_p} \mid (i_1, j_1) \leqslant (i_2, j_2) \leqslant \cdots \leqslant (i_p, j_p)\}$ *(lexicographical ordering) is a linear basis of the algebra* U^+*,*

(iii) $\mathcal{B}_1 = \{e_{i_1,j_1}e_{i_2,j_2} \cdots e_{i_p,j_p} \mid (i_1, j_1) \leqslant (i_2, j_2) \leqslant \cdots \leqslant (i_p, j_p)\}$ *(lexicographical ordering) is a linear basis of the algebra* U^+*, where* $e_{i,j} = e_i e_{i-1} \cdots e_j$ *for* $i \geqslant j$.

REMARK 3.3. If we define $\mathcal{F}_{i,j}$ inductively by

$$\mathcal{F}_{j,j} = f_j \quad \text{and} \quad \mathcal{F}_{i,j} = f_i \mathcal{F}_{i-1,j} - s\mathcal{F}_{i-1,j} f_i, \quad i > j,$$

and denote by $f_{i,j}$ the monomial $f_{i,j} = f_i f_{i-1} \cdots f_j$, $i \geqslant j$, then we have linear bases for the algebra U^- as in theorem 3.2. Note that $\tilde{U}^0$ and U^0, which are group algebras, have obvious linear bases. Combining these bases using the triangular decomposition $\tilde{U} = U^- \tilde{U}^0 U^+$ and $U = U^- U^0 U^+$, we obtain PBW bases for the entire algebras $\tilde{U}$ and U, respectively.

Now we turn our attention to showing that the algebra U^+ is an iterated skew polynomial ring over $\mathbb{K}$ and that any prime ideal P of U^+ is *completely prime* (that is, U^+/P is a domain) when r and s are 'generic' (see proposition 3.6 for the precise statement). Our approach is similar to that in [33], which treats the one-parameter quantum group case. Recall that if φ is an automorphism of an algebra R, then $\vartheta \in \mathrm{End}(R)$ is a φ*-derivation* if $\vartheta(ab) = \vartheta(a)b + \varphi(a)\vartheta(b)$ for all $a, b \in R$. The skew polynomial ring $R[x; \varphi, \vartheta]$ consists of polynomials $\sum_i a_i x^i$ over R, where $xa = \varphi(a)x + \vartheta(a)$ for all $a \in R$.

For each (i,j), $1 \leqslant j \leqslant i < n$, we define an algebra automorphism $\varphi_{i,j}$ of U by

$$\varphi_{i,j}(u) = \omega_{\alpha_i+\cdots+\alpha_j} u \omega^{-1}_{\alpha_i+\cdots+\alpha_j} \quad \text{for all } u \in U.$$

Using lemma 2.1, one can check that if $(k,l) < (i,j)$, then

$$\varphi_{i,j}(\mathcal{E}_{k,l}) = \begin{cases} r^{-1}\mathcal{E}_{k,l} & \text{if } j = k+1, \\ \mathcal{E}_{k,l} & \text{if } i > k \geqslant l > j \text{ or } j > k+1, \\ s^{-1}\mathcal{E}_{k,l} & \text{if } i = k \geqslant j > l \text{ or } i > k \geqslant j = l, \\ r^{-1}s^{-1}\mathcal{E}_{k,l} & \text{if } i > k \geqslant j > l. \end{cases}$$

Hence, the automorphism $\varphi_{i,j}$ preserves the subalgebra $U^+_{i,j}$ of U^+ generated by the vectors $\mathcal{E}_{k,l}$ for $(k,l) < (i,j)$. We denote the induced automorphism of $U^+_{i,j}$ by the same symbol $\varphi_{i,j}$.

Now we define a $\varphi_{i,j}$-derivation $\vartheta_{i,j}$ on $U^+_{i,j}$ by

$$\vartheta_{i,j}(\mathcal{E}_{k,l}) = \mathcal{E}_{i,j}\mathcal{E}_{k,l} - \varphi_{i,j}(\mathcal{E}_{k,l})\mathcal{E}_{i,j} = \begin{cases} \mathcal{E}_{i,l}, & j = k+1, \\ (r^{-1} - s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l}, & i > k \geqslant j > l, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\vartheta_{i,j}$ is indeed a $\varphi_{i,j}$-derivation (cf. [33, lemma 3, p. 62]). With $\varphi_{i,j}$ and $\vartheta_{i,j}$ at hand, the next proposition follows immediately.

PROPOSITION 3.4. *The algebra U^+ is an iterated skew polynomial ring whose structure is given by*

$$U^+ = \mathbb{K}[\mathcal{E}_{1,1}][\mathcal{E}_{2,1}; \varphi_{2,1}, \vartheta_{2,1}] \cdots [\mathcal{E}_{n-1,n-1}; \varphi_{n-1,n-1}, \vartheta_{n-1,n-1}]. \tag{3.4}$$

Proof. Note that all the relations in theorem 3.1 can be condensed into a single expression:

$$\mathcal{E}_{i,j}\mathcal{E}_{k,l} = \varphi_{i,j}(\mathcal{E}_{k,l})\mathcal{E}_{i,j} + \vartheta_{i,j}(\mathcal{E}_{k,l}), \quad (i,j) > (k,l). \tag{3.5}$$

The proposition then easily follows from theorem 3.2. □

The other result of this section requires an additional lemma.

LEMMA 3.5. *The automorphism $\varphi_{i,j}$ and the $\varphi_{i,j}$-derivation $\vartheta_{i,j}$ of $U^+_{i,j}$ satisfy*

$$\varphi_{i,j}\vartheta_{i,j} = rs^{-1}\vartheta_{i,j}\varphi_{i,j}.$$

Proof. For $(k,l) < (i,j)$, the definitions imply that

$$(\varphi_{i,j}\vartheta_{i,j})(\mathcal{E}_{k,l}) = \begin{cases} s^{-1}\mathcal{E}_{i,l} & \text{if } j = k+1, \\ (r^{-1} - s^{-1})s^{-2}\mathcal{E}_{k,j}\mathcal{E}_{i,l} & \text{if } i > k \geqslant j > l, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, for $(k,l) < (i,j)$,

$$(\vartheta_{i,j}\varphi_{i,j})(\mathcal{E}_{k,l}) = \begin{cases} r^{-1}\mathcal{E}_{i,l} & \text{if } j = k+1, \\ (r^{-1} - s^{-1})r^{-1}s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,l} & \text{if } i > k \geqslant j > l, \\ 0 & \text{otherwise.} \end{cases}$$

Comparing these two calculations, we arrive at the result. □

We now obtain the following proposition.

PROPOSITION 3.6. *Assume that the subgroup of* $\mathbb{K}^\times$ *generated by* r *and* s *is torsion-free. Then all prime ideals of* U^+ *are completely prime.*

Proof. The proof follows directly from proposition 3.4, lemma 3.5 and [14, theorem 2.3]. □

4. An invariant bilinear form on U

Assume that B is the subalgebra of U generated by e_j, $\omega_j^{\pm 1}$, $1 \leqslant j < n$, and B' is the subalgebra of U generated by f_j, $(\omega_j')^{\pm 1}$, $1 \leqslant j < n$. We recall some results in [4].

PROPOSITION 4.1 (Benkart and Witherspoon [4, lemma 2.2]). *There is a Hopf pairing* $(\cdot,\cdot)$ *on* $B' \times B$ *such that, for* $x_1, x_2 \in B$, $y_1, y_2 \in B'$, *the following properties hold:*

(i) $(1, x_1) = \varepsilon(x_1)$, $(y_1, 1) = \varepsilon(y_1)$;

(ii) $(y_1, x_1x_2) = (\Delta^{\mathrm{op}}(y_1), x_1 \otimes x_2)$, $(y_1y_2, x_1) = (y_1 \otimes y_2, \Delta(x_1))$;

(iii) $(S^{-1}(y_1), x_1) = (y_1, S(x_1))$;

(iv) $(f_i, e_j) = \dfrac{\delta_{i,j}}{s-r}$;

(v) $(\omega_i', \omega_j) = ({\omega'}_i^{-1}, {\omega_j}^{-1}) = r^{\langle \epsilon_j, \alpha_i\rangle} s^{\langle \epsilon_{j+1}, \alpha_i\rangle} = r^{-\langle \epsilon_{i+1}, \alpha_j\rangle} s^{-\langle \epsilon_i, \alpha_j\rangle}$,
$({\omega_i'}^{-1}, \omega_j) = (\omega_i', {\omega_j}^{-1}) = r^{-\langle \epsilon_j, \alpha_i\rangle} s^{-\langle \epsilon_{j+1}, \alpha_i\rangle} = r^{\langle \epsilon_{i+1}, \alpha_j\rangle} s^{\langle \epsilon_i, \alpha_j\rangle}$.

It is easy to prove for $\lambda \in Q$ that

$$\varrho^\lambda(\omega_\mu') = (\omega_\mu', \omega_{-\lambda}) \quad \text{and} \quad \varrho^\lambda(\omega_\mu) = (\omega_\lambda', \omega_\mu). \tag{4.1}$$

From the definition of the co-product, it is apparent that

$$\Delta(x) \in \bigoplus_{0 \leqslant \nu \leqslant \mu} U^+_{\mu-\nu}\omega_\nu \otimes U^+_\nu \quad \text{for any } x \in U^+_\mu,$$

where '$\leqslant$' is the usual partial order on $Q : \nu \leqslant \mu$ if $\mu - \nu \in Q^+$. Thus, for each i, $1 \leqslant i < n$, there are elements $p_i(x)$ and $p_i'(x)$ in $U^+_{\mu-\alpha_i}$ such that the component of $\Delta(x)$ in $U^+_{\mu-\alpha_i}\omega_i \otimes U^+_{\alpha_i}$ is equal to $p_i(x)\omega_i \otimes e_i$, and the component of $\Delta(x)$ in $U^+_{\alpha_i}\omega_{\mu-\alpha_i} \otimes U^+_{\mu-\alpha_i}$ is equal to $e_i\omega_{\mu-\alpha_i} \otimes p_i'(x)$. Therefore, for $x \in U^+_\mu$, we can write

$$\begin{aligned}\Delta(x) &= x \otimes 1 + \sum_{i=1}^{n-1} p_i(x)\omega_i \otimes e_i + \varsigma_1 \\ &= \omega_\mu \otimes x + \sum_{i=1}^{n-1} e_i\omega_{\mu-\alpha_i} \otimes p_i'(x) + \varsigma_2,\end{aligned}$$

where ς_1 and ς_2 are the sums of terms involving products of more than one e_j in the second factor and in the first factor, respectively.

LEMMA 4.2 (Benkart and Witherspoon [4, lemma 4.6]). *For all $x \in U_\zeta^+$ and all $y \in U^-$, the following hold:*

(i) $(f_i y, x) = (f_i, e_i)(y, p_i'(x)) = (s-r)^{-1}(y, p_i'(x))$;

(ii) $(y f_i, x) = (f_i, e_i)(y, p_i(x)) = (s-r)^{-1}(y, p_i(x))$;

(iii) $f_i x - x f_i = (s-r)^{-1}(p_i(x)\omega_i - \omega_i' p_i'(x))$.

COROLLARY 4.3. *If $\zeta, \zeta' \in Q^+$ with $\zeta \neq \zeta'$, then $(y, x) = 0$ for all $x \in U_\zeta^+$ and $y \in U_{-\zeta'}^-$.*

LEMMA 4.4. *Assume that rs^{-1} is not a root of unity and $\zeta \in Q^+$ is non-zero.*

(a) *If $y \in U_{-\zeta}^-$ and $[e_i, y] = 0$ for all i, then $y = 0$.*

(b) *If $x \in U_\zeta^+$ and $[f_i, x] = 0$ for all i, then $x = 0$.*

Proof. Assume that $y \in U_{-\zeta}^-$ and that $[e_i, y] = 0$ holds for all i. From the definition of $M(\lambda)$ and lemma 2.2, we can find a sufficiently large $\lambda \in \Lambda_{\mathfrak{sl}}^+$ such that the map

$$U_{-\zeta}^- \hookrightarrow L(\lambda), \qquad u \mapsto u v_\lambda,$$

is injective, where v_λ is a highest weight vector of $L(\lambda)$. Then

$$U y v_\lambda = U^- U^0 U^+ y v_\lambda = U^- y U^0 U^+ v_\lambda = U^- y v_\lambda \subsetneq L(\lambda)$$

so that $U y v_\lambda$ is a proper submodule of $L(\lambda)$, which must be 0 by the irreducibility of $L(\lambda)$. Thus, $y v_\lambda = 0$ and $y = 0$ by the injectivity of the map above. We can now apply the anti-automorphism τ of U defined by

$$\tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(\omega_i) = \omega_i \quad \text{and} \quad \tau(\omega_i') = \omega_i',$$

to obtain the second assertion. □

LEMMA 4.5. *Assume that rs^{-1} is not a root of unity. For $\zeta \in Q^+$, the spaces U_ζ^+ and $U_{-\zeta}^-$ are non-degenerately paired.*

Proof. We use induction on ζ with respect to the partial order $\leqslant$ on Q. The claim holds for $\zeta = 0$, since $U_0^- = \mathbb{K}1 = U_0^+$ and $(1, 1) = 1$. Assume now that $\zeta > 0$, and suppose that the claim holds for all ν with $0 \leqslant \nu < \zeta$. Let $x \in U_\zeta^+$ with $(y, x) = 0$ for all $y \in U_{-\zeta}^-$. In particular, we have, for all $y \in U_{-(\zeta-\alpha_i)}^-$, that

$$(f_i y, x) = 0 \quad \text{and} \quad (y f_i, x) = 0 \quad \text{for all } 1 \leqslant i < n.$$

It follows from lemma 4.2(i) and (ii) that $(y, p_i'(x)) = 0$ and $(y, p_i(x)) = 0$. By the induction hypothesis, we have $p_i'(x) = p_i(x) = 0$, and it follows from lemma 4.2(iii) that $f_i x = x f_i$ for all i. Lemma 4.4 now applies, to give $x = 0$, as desired. □

In what follows, ρ will denote the half-sum of the positive roots. Thus,

$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha = \sum_{i=1}^{n-1} \varpi_i = \tfrac{1}{2}((n-1)\epsilon_1 + (n-3)\epsilon_2 + \cdots + ((n-1) - 2(n-1))\epsilon_n). \quad (4.2)$$

It is evident from the triangular decomposition that there is a vector-space isomorphism

$$\bigoplus_{\mu,\nu\in Q^+} (U^-_{-\nu}{\omega'_\nu}^{-1}) \otimes U^0 \otimes U^+_\mu \xrightarrow{\sim} U.$$

This guarantees that the bilinear form which we introduce next is well defined.

Definition 4.6. Set

$$\langle (y{\omega'_\nu}^{-1})\omega'_\eta\omega_\phi x \mid (y_1{\omega'_{\nu_1}}^{-1})\omega'_{\eta_1}\omega_{\phi_1}x_1\rangle = (y,x_1)(y_1,x)(\omega'_\eta,\omega_{\phi_1})(\omega'_{\eta_1},\omega_\phi)(rs^{-1})^{\langle\rho,\nu\rangle}$$

for all $x \in U^+_\mu$, $x_1 \in U^+_{\mu_1}$, $y \in U^-_{-\nu}$, $y_1 \in U^-_{-\nu_1}$, $\mu,\mu_1,\nu,\nu_1 \in Q^+$, and all $\eta,\eta_1,\phi,\phi_1 \in Q$. Extend this linearly to a bilinear form $\langle\cdot,\cdot\rangle : U \times U \to \mathbb{K}$ on all of U.

Note that

$$\begin{aligned}\langle (y{\omega'_\nu}^{-1})\omega'_\eta\omega_\phi x \mid (y_1{\omega'_{\nu_1}}^{-1})\omega'_{\eta_1}\omega_{\phi_1}x_1\rangle \\ = \langle y{\omega'_\nu}^{-1} \mid x_1\rangle \cdot \langle \omega'_\eta\omega_\phi \mid \omega'_{\eta_1}\omega_{\phi_1}\rangle \cdot \langle x \mid y_1{\omega'_{\nu_1}}^{-1}\rangle.\end{aligned} \tag{4.3}$$

So the form respects the decomposition

$$\bigoplus_{\mu,\nu\in Q^+} (U^-_{-\nu}{\omega'_\nu}^{-1}) \otimes U^0 \otimes U^+_\mu \xrightarrow{\sim} U.$$

The following lemma is an immediate consequence of the above definition and corollary 4.3.

Lemma 4.7. *Assume that* $\mu,\mu_1,\nu,\nu_1 \in Q^+$. *Then*

$$\langle U^-_{-\nu}U^0U^+_\mu \mid U^-_{-\nu_1}U^0U^+_{\mu_1}\rangle = 0$$

unless $\mu = \nu_1$ *and* $\nu = \mu_1$.

Since U is a Hopf algebra, it acts on itself via the adjoint representation,

$$\mathrm{ad}(u)v = \sum_{(u)} u_{(1)}vS(u_{(2)}),$$

where $u, v \in U$ and $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$.

Proposition 4.8. *The bilinear form* $\langle\cdot\mid\cdot\rangle$ *is* ad*-invariant, i.e.*

$$\langle \mathrm{ad}(u)v \mid v_1\rangle = \langle v \mid \mathrm{ad}(S(u))v_1\rangle$$

for all $u, v, v_1 \in U$.

Proof. It suffices to assume u is one of the generators ω_i, ω'_i, e_i, f_i. Also, without loss of generality, we may suppose that

$$v = (y{\omega'_\nu}^{-1})\omega'_\eta\omega_\phi x \quad \text{and} \quad v_1 = (y_1{\omega'_{\nu_1}}^{-1})\omega'_{\eta_1}\omega_{\phi_1}x_1,$$

where $x \in U^+_\mu$, $y \in U^-_{-\nu}$, $x_1 \in U^+_{\mu_1}$, $y_1 \in U^-_{-\nu_1}$ and $\mu,\nu,\mu_1,\nu_1 \in Q^+$.

CASE 1 ($u = \omega_i$). From the definition, $\mathrm{ad}(\omega_i)v = \omega_i v \omega_i^{-1} = r^{\langle \epsilon_i, \mu-\nu\rangle} s^{\langle \epsilon_{i+1}, \mu-\nu\rangle} v$ so that

$$\langle \mathrm{ad}(\omega_i)v \mid v_1\rangle = r^{\langle \epsilon_i, \mu-\nu\rangle} s^{\langle \epsilon_{i+1}, \mu-\nu\rangle} \langle v \mid v_1\rangle.$$

On the other hand, we have

$$\mathrm{ad}(S(\omega_i))v_1 = \omega_i^{-1} v_1 \omega_i = r^{\langle \epsilon_i, \nu_1-\mu_1\rangle} s^{\langle \epsilon_{i+1}, \nu_1-\mu_1\rangle} v_1,$$

which implies that

$$\langle v \mid \mathrm{ad}(S(\omega_i))v_1\rangle = r^{\langle \epsilon_i, \nu_1-\mu_1\rangle} s^{\langle \epsilon_{i+1}, \nu_1-\mu_1\rangle} \langle v \mid v_1\rangle.$$

If $\langle v \mid v_1\rangle \neq 0$, then we must have $\nu = \mu_1$ and $\nu_1 = \mu$ by lemma 4.7. Thus, $\mu - \nu = \nu_1 - \mu_1$ and $\langle \mathrm{ad}(\omega_i)v \mid v_1\rangle = \langle v \mid \mathrm{ad}(S(\omega_i))v_1\rangle$.

CASE 2 ($u = \omega_i'$). We have only to replace ω_i by ω_i' and interchange ϵ_i and ϵ_{i+1} in the argument of case 1.

CASE 3 ($u = e_i$). This case is similar to case 4, below, so we omit the calculation.

CASE 4 ($u = f_i$). Using lemmas 2.1 and 4.2(iii), we get

$$\begin{aligned}
\mathrm{ad}(f_i)v &= vS(f_i) + f_i v S(\omega_i') = -v f_i (\omega_i')^{-1} + f_i v (\omega_i')^{-1} \\
&= -y(\omega_\nu')^{-1}\omega_\eta'\omega_\phi x f_i (\omega_i')^{-1} + f_i y (\omega_\nu')^{-1}\omega_\eta'\omega_\phi x (\omega_i')^{-1} \\
&= -y(\omega_\nu')^{-1}\omega_\eta'\omega_\phi f_i x (\omega_i')^{-1} + (s-r)^{-1} y (\omega_\nu')^{-1}\omega_\eta'\omega_\phi p_i(x)\omega_i(\omega_i')^{-1} \\
&\quad - (s-r)^{-1} y (\omega_\nu')^{-1}\omega_\eta'\omega_\phi \omega_i' p_i'(x)(\omega_i')^{-1} + f_i y (\omega_\nu')^{-1}\omega_\eta'\omega_\phi x (\omega_i')^{-1} \\
&= -r^{\langle \epsilon_i, \eta-\nu\rangle} r^{\langle \epsilon_{i+1}, \phi+\mu\rangle} s^{\langle \epsilon_i, \phi+\mu\rangle} s^{\langle \epsilon_{i+1}, \eta-\nu\rangle} y f_i (\omega_{\nu+\alpha_i}')^{-1}\omega_\eta'\omega_\phi x \\
&\quad + r^{\langle \epsilon_{i+1}, \mu\rangle} s^{\langle \epsilon_i, \mu\rangle} f_i y (\omega_{\nu+\alpha_i}')^{-1}\omega_\eta'\omega_\phi x \\
&\quad + (s-r)^{-1} r^{-\langle \alpha_i, \mu-\alpha_i\rangle} s^{\langle \alpha_i, \mu-\alpha_i\rangle} y (\omega_\nu')^{-1}\omega_{\eta-\alpha_i}'\omega_{\phi+\alpha_i} p_i(x) \\
&\quad - (s-r)^{-1} r^{\langle \epsilon_{i+1}, \mu-\alpha_i\rangle} s^{\langle \epsilon_i, \mu-\alpha_i\rangle} y (\omega_\nu')^{-1}\omega_\eta'\omega_\phi p_i'(x).
\end{aligned}$$

Now

$$\mathrm{ad}(S(f_i))v_1 = \mathrm{ad}(-f_i(\omega_i')^{-1})v_1 = -r^{-\langle \epsilon_{i+1}, \mu_1-\nu_1\rangle} s^{-\langle \epsilon_i, \mu_1-\nu_1\rangle}\,\mathrm{ad}(f_i)v_1.$$

We apply the previous calculation of $\mathrm{ad}(f_i)v$ with v replaced by v_1 to see that

$$\begin{aligned}
\mathrm{ad}(S(f_i))v_1 &= r^{\langle \epsilon_i, \eta_1-\nu_1\rangle} r^{\langle \epsilon_{i+1}, \phi_1+\nu_1\rangle} s^{\langle \epsilon_i, \phi_1+\nu_1\rangle} s^{\langle \epsilon_{i+1}, \eta_1-\nu_1\rangle} y_1 f_i (\omega_{\nu_1+\alpha_i}')^{-1}\omega_{\eta_1}'\omega_{\phi_1} x_1 \\
&\quad - r^{\langle \epsilon_{i+1}, \nu_1\rangle} s^{\langle \epsilon_i, \nu_1\rangle} f_i y_1 (\omega_{\nu_1+\alpha_i}')^{-1}\omega_{\eta_1}'\omega_{\phi_1} x_1 \\
&\quad - (s-r)^{-1} r^{-\langle \epsilon_i, \mu_1-\alpha_i\rangle} r^{\langle \epsilon_{i+1}, \nu_1-\alpha_i\rangle} s^{\langle \epsilon_i, \nu_1-\alpha_i\rangle} s^{-\langle \epsilon_{i+1}, \mu_1-\alpha_i\rangle} \\
&\qquad \times y_1 (\omega_{\nu_1}')^{-1}\omega_{\eta_1-\alpha_i}'\omega_{\phi_1+\alpha_i} p_i(x_1) \\
&\quad + (s-r)^{-1} r^{\langle \epsilon_{i+1}, \nu_1-\alpha_i\rangle} s^{\langle \epsilon_i, \nu_1-\alpha_i\rangle} y_1 (\omega_{\nu_1}')^{-1}\omega_{\eta_1}'\omega_{\phi_1} p_i'(x_1).
\end{aligned}$$

It follows from lemma 4.7 that $\langle \mathrm{ad}(f_i)v \mid v_1\rangle$ and $\langle v \mid \mathrm{ad}(S(f_i))v_1\rangle$ can be non-zero when either (a) $\nu + \alpha_i = \mu_1$ and $\nu_1 = \mu$, or (b) $\nu = \mu_1$ and $\nu_1 = \mu - \alpha_i$.

(a) By lemma 4.2(i), (ii), we have

$$\begin{aligned}\langle \mathrm{ad}(f_i)v \mid v_1\rangle &= -r^{\langle \epsilon_i,\eta-\nu\rangle} r^{\langle \epsilon_{i+1},\phi+\mu\rangle} s^{\langle \epsilon_i,\phi+\mu\rangle} s^{\langle \epsilon_{i+1},\eta-\nu\rangle} \\ &\qquad \times (yf_i,x_1)(y_1,x)(\omega'_\eta,\omega_{\phi_1})(\omega'_{\eta_1},\omega_\phi)(rs^{-1})^{\langle \rho,\nu+\alpha_i\rangle} \\ &\quad + r^{\langle \epsilon_{i+1},\mu\rangle} s^{\langle \epsilon_i,\mu\rangle}(f_iy,x_1)(y_1,x)(\omega'_\eta,\omega_{\phi_1})(\omega'_{\eta_1},\omega_\phi)(rs^{-1})^{\langle \rho,\nu+\alpha_i\rangle} \\ &= A\times (y_1,x)(\omega'_\eta,\omega_{\phi_1})(\omega'_{\eta_1},\omega_\phi)(rs^{-1})^{\langle \rho,\nu\rangle},\end{aligned}$$

where

$$\begin{aligned}A = -(s-r)^{-1} r^{\langle \epsilon_i,\eta-\nu\rangle} r^{\langle \epsilon_{i+1},\phi+\mu\rangle} s^{\langle \epsilon_i,\phi+\mu\rangle} s^{\langle \epsilon_{i+1},\eta-\nu\rangle} rs^{-1}(y,p_i(x_1)) \\ + (s-r)^{-1} r^{\langle \epsilon_{i+1},\mu\rangle} s^{\langle \epsilon_i,\mu\rangle} rs^{-1}(y,p'_i(x_1)).\end{aligned}$$

Similarly,

$$\langle v \mid \mathrm{ad}(S(f_i))v_1\rangle = B\times (y_1,x)(\omega'_\eta,\omega_{\phi_1})(\omega'_{\eta_1},\omega_\phi)(rs^{-1})^{\langle \rho,\nu\rangle},$$

where

$$\begin{aligned}B = -(s-r)^{-1} r^{-\langle \epsilon_i,\mu_1-\alpha_i\rangle} r^{\langle \epsilon_{i+1},\nu_1-\alpha_i\rangle} s^{\langle \epsilon_i,\nu_1-\alpha_i\rangle} s^{-\langle \epsilon_{i+1},\mu_1-\alpha_i\rangle} \\ \times (\omega'_\eta,\omega_i)((\omega'_i)^{-1},\omega_\phi)(y,p_i(x_1)) \\ + (s-r)^{-1} r^{\langle \epsilon_{i+1},\nu_1-\alpha_i\rangle} s^{\langle \epsilon_i,\nu_1-\alpha_i\rangle}(y,p'_i(x_1)).\end{aligned}$$

Comparing both sides, we conclude that $\langle \mathrm{ad}(f_i)v \mid v_1\rangle = \langle v \mid \mathrm{ad}(S(f_i))v_1\rangle$.

(b) An argument analogous to that for (a) can be used in this case. □

Remark 4.9. It was shown in [4] that U is isomorphic to the Drinfel'd double $D(B,(B')^{\mathrm{coop}})$, where B is the Hopf subalgebra of U generated by the elements $\omega_j^{\pm1}$, e_j, $1\leqslant j<n$, and $(B')^{\mathrm{coop}}$ is the subalgebra of U generated by the elements $(\omega'_j)^{\pm1}$, f_j, $1\leqslant j<n$, but with the opposite co-product. This realization of U allows us to define the *Rosso form* R on U according to [18, p. 77]:

$$R\langle a\otimes b \mid a'\otimes b'\rangle = (b',S(a))(S^{-1}(b),a') \quad \text{for } a,a'\in B \text{ and } b,b'\in (B')^{\mathrm{coop}}.$$

The Rosso form is also an ad-invariant form on U, but it does not admit the decomposition in (4.3). Rather, it has the following factorization (we suppress the tensor symbols in the notation):

$$\begin{aligned}&R\langle x\omega_\phi\omega'_\eta({\omega'_\nu}^{-1}y) \mid x_1\omega_{\phi_1}\omega'_{\eta_1}({\omega'_{\nu_1}}^{-1}y_1)\rangle \\ &\qquad = R\langle x \mid {\omega'_{\nu_1}}^{-1}y_1\rangle\cdot R\langle \omega_\phi\omega'_\eta \mid \omega_{\phi_1}\omega'_{\eta_1}\rangle\cdot R\langle {\omega'_\nu}^{-1}y \mid x_1\rangle. \qquad (4.4)\end{aligned}$$

That is to say, the form R respects the decomposition

$$\bigoplus_{\mu,\nu\in Q^+} U^+_\mu\otimes U^0\otimes({\omega'_\nu}^{-1}U^-_{-\nu}) \xrightarrow{\sim} U.$$

For $(\eta,\phi)\in Q\times Q$, we define a group homomorphism $\chi_{\eta,\phi}: Q\times Q\to\mathbb{K}^\times$ by

$$\chi_{\eta,\phi}(\eta_1,\phi_1) = (\omega'_\eta,\omega_{\phi_1})(\omega'_{\eta_1},\omega_\phi), \quad (\eta_1,\phi_1)\in Q\times Q. \qquad (4.5)$$

Lemma 4.10. *Assume that $r^k s^l = 1$ if and only if $k = l = 0$. If $\chi_{\eta,\phi} = \chi_{\eta',\phi'}$, then $(\eta, \phi) = (\eta', \phi')$.*

Proof. If $\chi_{\eta,\phi} = \chi_{\eta',\phi'}$, then

$$\chi_{\eta,\phi}(0, \alpha_j) = r^{\langle \epsilon_j, \eta \rangle} s^{\langle \epsilon_{j+1}, \eta \rangle} = \chi_{\eta',\phi'}(0, \alpha_j) = r^{\langle \epsilon_j, \eta' \rangle} s^{\langle \epsilon_{j+1}, \eta' \rangle}.$$

Since $r^{\langle \epsilon_j, \eta \rangle - \langle \epsilon_j, \eta' \rangle} s^{\langle \epsilon_{j+1}, \eta \rangle - \langle \epsilon_{j+1}, \eta' \rangle} = 1$, it must be that $\langle \epsilon_j, \eta \rangle = \langle \epsilon_j, \eta' \rangle$ for all $1 \leqslant j \leqslant n$. From this it is easy to see that $\eta = \eta'$. Similar considerations with $\chi_{\eta,\phi}(\alpha_i, 0) = \chi_{\eta',\phi'}(\alpha_i, 0)$ show that $\phi = \phi'$. □

Proposition 4.11. *Assume that $r^k s^l = 1$ if and only if $k = l = 0$. Then the bilinear form $\langle \cdot \mid \cdot \rangle$ is non-degenerate on U.*

Proof. It is sufficient to argue that if $u \in U_{-\nu}^- U^0 U_\mu^+$ and $\langle u \mid v \rangle = 0$ for all $v \in U_{-\mu}^- U^0 U_\nu^+$, then $u = 0$. Choose, for each $\mu \in Q^+$, a basis $u_1^\mu, u_2^\mu, \dots, u_{d_\mu}^\mu$, $d_\mu = \dim U_\mu^+$, of U_μ^+. Owing to lemma 4.5, we can take a dual basis $v_1^\mu, v_2^\mu, \dots, v_{d_\mu}^\mu$ of $U_{-\mu}^-$, i.e. $(v_i^\mu, u_j^\mu) = \delta_{i,j}$. Then the set

$$\{(v_i^\nu {\omega'_\nu}^{-1}) \omega'_\eta \omega_\phi u_j^\mu \mid 1 \leqslant i \leqslant d_\nu,\ 1 \leqslant j \leqslant d_\mu \text{ and } \eta, \phi \in Q\}$$

is a basis of $U_{-\nu}^- U^0 U_\mu^+$. From the definition of the bilinear form, we obtain

$$\begin{aligned}\langle (v_i^\nu {\omega'_\nu}^{-1}) \omega'_\eta \omega_\phi u_j^\mu \mid (v_k^\mu {\omega'_\mu}^{-1}) \omega_{\eta_1} \omega_{\phi_1} u_l^\nu \rangle \\ &= (v_i^\nu, u_l^\nu)(v_k^\mu, u_j^\mu)(\omega'_\eta, \omega_{\phi_1})(\omega'_{\eta_1}, \omega_\phi)(rs^{-1})^{\langle \rho, \nu \rangle} \\ &= \delta_{i,l} \delta_{j,k} (\omega'_\eta, \omega_{\phi_1})(\omega'_{\eta_1}, \omega_\phi)(rs^{-1})^{\langle \rho, \nu \rangle}.\end{aligned}$$

Now write $u = \sum_{i,j,\eta,\phi} \theta_{i,j,\eta,\phi} (v_i^\nu {\omega'_\nu}^{-1}) \omega'_\eta \omega_\phi u_j^\mu$, and take $v = (v_k^\mu {\omega'_\mu}^{-1}) \omega'_{\eta_1} \omega_{\phi_1} u_l^\nu$ with $1 \leqslant k \leqslant d_\mu$ and $1 \leqslant l \leqslant d_\nu$ and $\eta_1, \phi_1 \in Q$. From the assumption $\langle u \mid v \rangle = 0$ we have

$$\sum_{\eta,\phi} \theta_{l,k,\eta,\phi} (\omega'_\eta, \omega_{\phi_1})(\omega'_{\eta_1}, \omega_\phi)(rs^{-1})^{\langle \rho, \nu \rangle} = 0 \tag{4.6}$$

for all $1 \leqslant k \leqslant d_\mu$ and $1 \leqslant l \leqslant d_\nu$ and for all $\eta_1, \phi_1 \in Q$. Equation (4.6) can be written as

$$\sum_{\eta,\phi} \theta_{l,k,\eta,\phi} (rs^{-1})^{\langle \rho, \nu \rangle} \chi_{\eta,\phi} = 0$$

for each k and l (where $1 \leqslant k \leqslant d_\mu$ and $1 \leqslant l \leqslant d_\nu$). It follows from lemma 4.10 and the linear independence of distinct characters (Dedekind's theorem; see, for example, [17, p. 280]) that $\theta_{l,k,\eta,\phi} = 0$ for all $\eta, \phi \in Q$ and for all l and k. Hence, we have $u = 0$ as desired. □

5. The centre of $U = U_{r,s}(\mathfrak{sl}_n)$

Throughout this section we make the following assumption:

$$r^k s^l = 1 \quad \text{if and only if } k = l = 0. \tag{5.1}$$

Under this hypothesis, we see that, for $\zeta \in Q$,

$$U_\zeta = \{z \in U \mid \omega_i z \omega_i^{-1} = r^{\langle \epsilon_i, \zeta \rangle} s^{\langle \epsilon_{i+1}, \zeta \rangle} z \text{ and } \omega'_i z (\omega'_i)^{-1} = r^{\langle \epsilon_{i+1}, \zeta \rangle} s^{\langle \epsilon_i, \zeta \rangle} z\}. \tag{5.2}$$

We denote the centre of U by $\mathfrak{Z}$. Since any central element of U must commute with ω_i and ω_i' for all i, it follows from (5.2) that $\mathfrak{Z} \subset U_0$. We define an algebra automorphism $\gamma^{-\rho} : U^0 \to U^0$ by

$$\gamma^{-\rho}(a_i) = r^{-\langle \rho, \epsilon_i \rangle} a_i \quad \text{and} \quad \gamma^{-\rho}(b_i) = s^{-\langle \rho, \epsilon_i \rangle} b_i. \tag{5.3}$$

Thus,

$$\gamma^{-\rho}(\omega_i'\omega_i^{-1}) = (rs^{-1})^{\langle \rho, \alpha_i \rangle} \omega_i'\omega_i^{-1}. \tag{5.4}$$

Definition 5.1. The *Harish-Chandra homomorphism* $\xi : \mathfrak{Z} \to U^0$ is the restriction to $\mathfrak{Z}$ of the map

$$\gamma^{-\rho} \circ \pi : U_0 \xrightarrow{\pi} U^0 \xrightarrow{\gamma^{-\rho}} U^0,$$

where $\pi : U_0 \to U^0$ is the canonical projection.

Proposition 5.2. *ξ is an injective algebra homomorphism.*

Proof. Note that $U_0 = U^0 \oplus K$, where $K = \bigoplus_{\nu > 0} U_{-\nu}^- U^0 U_\nu^+$ is the two-sided ideal in U_0 which is the kernel of π, and hence of ξ. Thus, ξ is an algebra homomorphism. Assume that $z \in \mathfrak{Z}$ and $\xi(z) = 0$. Writing $z = \sum_{\nu \in Q^+} z_\nu$ with $z_\nu \in U_{-\nu}^- U^0 U_\nu^+$, we have $z_0 = 0$. Fix any $\nu \in Q^+ \setminus \{0\}$ minimal with the property that $z_\nu \neq 0$. Also choose bases $\{y_k\}$ and $\{x_l\}$ for $U_{-\nu}^-$ and U_ν^+, respectively. We may write $z_\nu = \sum_{k,l} y_k t_{k,l} x_l$ for some $t_{k,l} \in U^0$. Then

$$\begin{aligned} 0 &= e_i z - z e_i \\ &= \sum_{\gamma \neq \nu} (e_i z_\gamma - z_\gamma e_i) + \sum_{k,l} (e_i y_k - y_k e_i) t_{k,l} x_l + \sum_{k,l} y_k (e_i t_{k,l} x_l - t_{k,l} x_l e_i). \end{aligned}$$

Note that $e_i y_k - y_k e_i \in U_{-(\nu-\alpha_i)}^- U^0$. Recalling the minimality of ν, we see that only the second term belongs to $U_{-(\nu-\alpha_i)}^- U^0 U_\nu^+$. Therefore, we have

$$\sum_{k,l} (e_i y_k - y_k e_i) t_{k,l} x_l = 0.$$

By the triangular decomposition of U and the fact that $\{x_l\}$ is a basis of U_ν^+, we get $\sum_k e_i y_k t_{k,l} = \sum_k y_k e_i t_{k,l}$ for each l and for all $1 \leqslant i < n$.

Now we fix l and consider the irreducible module $L(\lambda)$ for $\lambda \in \Lambda_{\mathfrak{sl}}^+$. Let v_λ be the highest weight vector of $L(\lambda)$, and set $m = \sum_k y_k t_{k,l} v_\lambda$. Then, for each i,

$$e_i m = \sum_k e_i y_k t_{k,l} v_\lambda = \sum_k y_k e_i t_{k,l} v_\lambda = 0.$$

Hence, m generates a proper submodule of $L(\lambda)$. The irreducibility of $L(\lambda)$ forces $m = 0$. Choosing an appropriate $\lambda \in \Lambda_{\mathfrak{sl}}^+$ with lemma 2.2 in mind, we have

$$\sum_k y_k t_{k,l} = 0.$$

Since $\{y_k\}$ is a basis for $U_{-\nu}^-$, it must be that $t_{k,l} = 0$ for each k. But l can be arbitrary, so we get $z_\nu = 0$, which is a contradiction. □

Proposition 5.3. *If n is even, set*

$$\mathfrak{z} = \omega_1'\omega_3'\cdots\omega_{n-1}'\omega_1\omega_3\cdots\omega_{n-1} = a_1\cdots a_n b_1\cdots b_n. \tag{5.5}$$

Then $\mathfrak{z}$ is central and $\xi(\mathfrak{z}) = \mathfrak{z}$.

Proof. We have

$$e_i\mathfrak{z} = r^{-\langle\epsilon_1+\epsilon_2+\cdots+\epsilon_n,\alpha_i\rangle} s^{-\langle\epsilon_1+\epsilon_2+\cdots+\epsilon_n,\alpha_i\rangle}\mathfrak{z}e_i = \mathfrak{z}e_i \quad \text{for all } 1 \leqslant i < n.$$

Similarly, $f_i\mathfrak{z} = \mathfrak{z}f_i$ for all $1 \leqslant i < n$, so that $\mathfrak{z}$ is central. Finally, observe that

$$\xi(\mathfrak{z}) = r^{-\langle\rho,\epsilon_1+\epsilon_2+\cdots+\epsilon_n\rangle} s^{-\langle\rho,\epsilon_1+\epsilon_2+\cdots+\epsilon_n\rangle}\mathfrak{z} = \mathfrak{z}.$$

□

By introducing appropriate factors into the definition of the homomorphism ϱ^λ in (2.2), we are able to obtain a duality between U^0 and its characters. Thus, for any $\lambda, \mu \in \Lambda_{\mathfrak{sl}}$, we let $\varrho^{\lambda,\mu} : U^0 \to \mathbb{K}$ be the algebra homomorphism defined by

$$\left.\begin{aligned} \varrho^{\lambda,\mu}(\omega_j) &= r^{\langle\epsilon_j,\lambda\rangle} s^{\langle\epsilon_{j+1},\lambda\rangle}(rs^{-1})^{\langle\alpha_j,\mu\rangle}, \\ \varrho^{\lambda,\mu}(\omega_j') &= r^{\langle\epsilon_{j+1},\lambda\rangle} s^{\langle\epsilon_j,\lambda\rangle}(rs^{-1})^{\langle\alpha_j,\mu\rangle}. \end{aligned}\right\} \tag{5.6}$$

In particular, $\varrho^{\lambda,0}$ is just the homomorphism ϱ^λ on U^0.

Lemma 5.4. *Assume that $u = \omega_\eta'\omega_\phi$ with $\eta, \phi \in Q$. If $\varrho^{\lambda,\mu}(u) = 1$ for all $\lambda, \mu \in \Lambda_{\mathfrak{sl}}$, then $u = 1$.*

Proof. We write $\eta = \sum_i \eta_i\alpha_i$ and $\phi = \sum_i \phi_i\alpha_i$. Then $\varrho^{\varpi_i,0}(u) = \varrho^{\varpi_i,0}(\omega_\eta'\omega_\phi) = r^{A_i}s^{B_i} = 1$ for each $1 \leqslant i < n$, where

$$\begin{aligned} A_i &= \langle\epsilon_2,\varpi_i\rangle\eta_1 + \cdots + \langle\epsilon_n,\varpi_i\rangle\eta_{n-1} + \langle\epsilon_1,\varpi_i\rangle\phi_1 + \cdots + \langle\epsilon_{n-1},\varpi_i\rangle\phi_{n-1}, \\ B_i &= \langle\epsilon_1,\varpi_i\rangle\eta_1 + \cdots + \langle\epsilon_{n-1},\varpi_i\rangle\eta_{n-1} + \langle\epsilon_2,\varpi_i\rangle\phi_1 + \cdots + \langle\epsilon_n,\varpi_i\rangle\phi_{n-1}. \end{aligned}$$

It follows from assumption (5.1) that $A_i = B_i = 0$. It is now straightforward to see from the definitions that, for $1 \leqslant i < n$,

$$A_i = \sum_{j=1}^{i-1}\eta_j - \frac{i}{n}\sum_{j=1}^{n-1}\eta_j + \sum_{j=1}^{i}\phi_j - \frac{i}{n}\sum_{j=1}^{n-1}\phi_j = 0,$$

$$B_i = \sum_{j=1}^{i}\eta_j - \frac{i}{n}\sum_{j=1}^{n-1}\eta_j + \sum_{j=1}^{i-1}\phi_j - \frac{i}{n}\sum_{j=1}^{n-1}\phi_j = 0.$$

After elementary manipulations we have $\eta_i = \phi_i$ for all $1 \leqslant i < n$ and $\eta_2 = \eta_4 = \cdots = 0$ and

$$\eta_1 = \eta_3 = \cdots = \frac{2}{n}\sum_{j=1}^{n-1}\eta_j = \frac{2}{n}l\eta_1,$$

where $l = \frac{1}{2}n$ if n is even and $l = \frac{1}{2}(n-1)$ if n is odd. Therefore, $u = 1$ when n is odd, and $u = \mathfrak{z}^{\eta_1}$, $\eta_1 \in \mathbb{Z}$, when n is even. Now, when n is even,

$$1 = \varrho^{0,\varpi_1}(u) = (\varrho^{0,\varpi_1}(\mathfrak{z}))^{\eta_1} = (rs^{-1})^{2\eta_1}.$$

Thus, $\eta_1 = 0$, and $u = 1$ as desired. □

Corollary 5.5. *Assume that* $u \in U^0$. *If* $\varrho^{\lambda,\mu}(u) = 0$ *for all* $(\lambda, \mu) \in \Lambda_{\mathfrak{sl}} \times \Lambda_{\mathfrak{sl}}$, *then* $u = 0$.

Proof. Corresponding to each $(\eta, \phi) \in Q \times Q$ is the character on the group $\Lambda_{\mathfrak{sl}} \times \Lambda_{\mathfrak{sl}}$ defined by

$$(\lambda, \mu) \mapsto \varrho^{\lambda,\mu}(\omega'_\eta \omega_\phi).$$

It follows from lemma 5.4 that different (η, ϕ) give rise to different characters.

Suppose now that $u = \sum \theta_{\eta,\phi} \omega'_\eta \omega_\phi$, where $\theta_{\eta,\phi} \in \mathbb{K}$. By assumption,

$$\sum \theta_{\eta,\phi} \varrho^{\lambda,\mu}(\omega'_\eta \omega_\phi) = 0$$

for all $(\lambda, \mu) \in \Lambda_{\mathfrak{sl}} \times \Lambda_{\mathfrak{sl}}$. By the linear independence of different characters, $\theta_{\eta,\phi} = 0$ for all $(\eta, \phi) \in Q \times Q$, and so $u = 0$. □

Set

$$U^0_\flat = \bigoplus_{\eta \in Q} \mathbb{K} \omega'_\eta \omega_{-\eta}, \tag{5.7}$$

$$U^0_\natural = \begin{cases} U^0_\flat & \text{if } n \text{ is odd,} \\ \bigoplus \mathbb{K} \omega'_\eta \omega_\phi, & \text{if } n \text{ is even,} \end{cases} \tag{5.8}$$

where, in the even case, the sum is over the pairs $(\eta, \phi) \in Q \times Q$ which satisfy the following condition: if $\eta = \sum_{i=1}^{n-1} \eta_i \alpha_i$ and $\phi = \sum_{i=1}^{n-1} \phi_i \alpha_i$, then

$$\left.\begin{aligned} &\eta_1 + \phi_1 = \eta_3 + \phi_3 = \cdots = \eta_{n-1} + \phi_{n-1}, \\ &\eta_2 + \phi_2 = \eta_4 + \phi_4 = \cdots = \eta_{n-2} + \phi_{n-2} = 0. \end{aligned}\right\} \tag{5.9}$$

Clearly, $U^0_\flat \subsetneq U^0_\natural$ when n is even, as $\mathfrak{z} \in U^0_\natural \setminus U^0_\flat$.

There is an action of the Weyl group W on U^0 defined by

$$\sigma(a_\lambda b_\mu) = a_{\sigma(\lambda)} b_{\sigma(\mu)} \tag{5.10}$$

for all $\lambda, \mu \in \Lambda$ and $\sigma \in W$. We want to know the effect of this action on a product $\omega'_\eta \omega_\phi$, where $\eta = \sum_{i=1}^{n-1} \eta_i \alpha_i$ and $\phi = \sum_{i=1}^{n-1} \phi_i \alpha_i$. For this, write $\omega'_\eta \omega_\phi = a_\mu b_\nu$, where $\mu = \sum_{i=1}^{n} \mu_i \epsilon_i$, $\nu = \sum_{i=1}^{n} \nu_i \epsilon_i$, and

$$\mu_i = \eta_{i-1} + \phi_i, \qquad \nu_i = \eta_i + \phi_{i-1} \tag{5.11}$$

for all $1 \leqslant i \leqslant n$ (where $\eta_0 = \eta_n = \phi_0 = \phi_n = 0$). Then, for the simple reflection σ_k, we have

$$\begin{aligned} \sigma_k(\omega'_\eta \omega_\phi) &= \sigma_k(a_\mu b_\nu) \\ &= a_\mu b_\nu a_{\alpha_k}^{-\langle \mu, \alpha_k \rangle} b_{\alpha_k}^{-\langle \nu, \alpha_k \rangle} \\ &= \omega'_\eta \omega_\phi (a_k a_{k+1}^{-1})^{-\langle \mu, \alpha_k \rangle} (b_k b_{k+1}^{-1})^{-\langle \nu, \alpha_k \rangle} \\ &= \omega'_\eta \omega_\phi (a_k b_{k+1})^{-\langle \mu, \alpha_k \rangle} (a_{k+1} b_k)^{\langle \mu, \alpha_k \rangle} (b_k^{-1} b_{k+1})^{\langle \mu+\nu, \alpha_k \rangle} \\ &= \omega'_\eta \omega_\phi (\omega'_k \omega_k^{-1})^{\mu_k - \mu_{k+1}} (b_k^{-1} b_{k+1})^{\mu_k + \nu_k - \mu_{k+1} - \nu_{k+1}} \\ &= \omega'_\eta \omega_\phi (\omega'_k \omega_k^{-1})^{\eta_{k-1} - \eta_k + \phi_k - \phi_{k+1}} (b_k^{-1} b_{k+1})^{\eta_{k-1} + \phi_{k-1} - \eta_{k+1} - \phi_{k+1}}. \end{aligned} \tag{5.12}$$

From this it is apparent that the subalgebras $U^0_\flat$ and $U^0_\natural$ of U^0 are closed under the W-action. Moreover, the W-action on $U^0_\flat$ amounts to

$$\sigma(\omega'_\eta\omega_{-\eta}) = \omega'_{\sigma(\eta)}\omega_{-\sigma(\eta)} \quad \text{for all } \sigma \in W \text{ and } \eta \in Q.$$

PROPOSITION 5.6. *We have*

$$\varrho^{\sigma(\lambda),\mu}(u) = \varrho^{\lambda,\mu}(\sigma^{-1}(u)) \tag{5.13}$$

for all $u \in U^0_\natural$, $\sigma \in W$ *and* $\lambda, \mu \in \Lambda_{\mathfrak{sl}}$.

Proof. First, we show that $\varrho^{\sigma(\lambda),0}(u) = \varrho^{\lambda,0}(\sigma^{-1}(u))$. Since

$$\varrho^{\sigma_i(\varpi_j),0}(a_k) = r^{\langle\epsilon_k,\sigma_i(\varpi_j)\rangle} = r^{\langle\sigma_i(\epsilon_k),\varpi_j\rangle} = \varrho^{\varpi_j,0}(\sigma_i(a_k))$$

and

$$\varrho^{\sigma_i(\varpi_j),0}(b_k) = s^{\langle\epsilon_k,\sigma_i(\varpi_j)\rangle} = s^{\langle\sigma_i(\epsilon_k),\varpi_j\rangle} = \varrho^{\varpi_j,0}(\sigma_i(b_k))$$

for $1 \leqslant i, j < n$ and $1 \leqslant k \leqslant n$, we see that (5.13) holds in this case. Next we argue that $\varrho^{0,\mu}(u) = \varrho^{0,\mu}(\sigma^{-1}(u))$. It is sufficient to suppose that $u = \omega'_\eta\omega_\phi$ and $\sigma = \sigma_k$ for some k. Then (5.12) shows that

$$\sigma_k(\omega'_\eta\omega_\phi) = \omega'_\eta\omega_\phi(\omega'_k\omega_k^{-1})^{\eta_{k-1}-\eta_k+\phi_k-\phi_{k+1}}.$$

Now, using the definition of $\varrho^{0,\mu}$, we have $\varrho^{0,\mu}(\sigma_k(\omega'_\eta\omega_\phi)) = \varrho^{0,\mu}(\omega'_\eta\omega_\phi)$. Finally, since $\varrho^{\lambda,\mu}(u) = \varrho^{\lambda,0}(u)\varrho^{0,\mu}(u)$, the assertion follows. □

We define

$$(U^0_\natural)^W = \{u \in U^0_\natural \mid \sigma(u) = u,\ \forall\sigma \in W\} \quad \text{and} \quad (U^0_\flat)^W = U^0_\flat \cap (U^0_\natural)^W. \tag{5.14}$$

LEMMA 5.7. *Assume that* $u \in U^0$ *and* $\varrho^{\lambda,\mu}(u) = \varrho^{\sigma(\lambda),\mu}(u)$ *for all* $\lambda, \mu \in \Lambda_{\mathfrak{sl}}$ *and* $\sigma \in W$. *Then* $u \in (U^0_\natural)^W$.

Proof. Suppose that $u = \sum_{(\eta,\phi)}\theta_{\eta,\phi}\omega'_\eta\omega_\phi \in U^0$ satisfies $\varrho^{\lambda,\mu}(u) = \varrho^{\sigma(\lambda),\mu}(u)$ for all $\lambda, \mu \in \Lambda_{\mathfrak{sl}}$ and $\sigma \in W$. Then

$$\sum_{(\eta,\phi)}\theta_{\eta,\phi}\varrho^{\lambda,\mu}(\omega'_\eta\omega_\phi) = \sum_{(\zeta,\psi)}\theta_{\zeta,\psi}\varrho^{\sigma_i(\lambda),\mu}(\omega'_\zeta\omega_\psi)$$

for all $\lambda, \mu \in \Lambda_{\mathfrak{sl}}$. If $\kappa_{\eta,\phi}$ and $\kappa^i_{\zeta,\psi}$ are the characters on $\Lambda_{\mathfrak{sl}} \times \Lambda_{\mathfrak{sl}}$ defined by

$$\kappa_{\eta,\phi}(\lambda,\mu) = \varrho^{\lambda,\mu}(\omega'_\eta\omega_\phi) \quad \text{and} \quad \kappa^i_{\zeta,\psi}(\lambda,\mu) = \varrho^{\sigma_i(\lambda),\mu}(\omega'_\zeta\omega_\psi),$$

then

$$\sum_{(\eta,\phi)}\theta_{\eta,\phi}\kappa_{\eta,\phi} = \sum_{(\zeta,\psi)}\theta_{\zeta,\psi}\kappa^i_{\zeta,\psi}. \tag{5.15}$$

Each side of (5.15) is a linear combination of different characters by lemma 5.4. Now, if $\theta_{\eta,\phi} \neq 0$, then $\kappa_{\eta,\phi} = \kappa^i_{\zeta,\psi}$ for some (ζ,ψ). Moreover, for each $1 \leqslant j < n$,

$$\begin{aligned}\kappa_{\eta,\phi}(0,\varpi_j) &= \varrho^{0,\varpi_j}(\omega'_\eta\omega_\phi) = (rs^{-1})^{\langle\eta+\phi,\varpi_j\rangle}\\ &= \kappa^i_{\zeta,\psi}(0,\varpi_j) = \varrho^{0,\varpi_j}(\omega'_\zeta\omega_\psi) = (rs^{-1})^{\langle\zeta+\psi,\varpi_j\rangle}.\end{aligned}$$

Thus, $\langle \eta+\phi, \varpi_j\rangle = \langle \zeta+\psi, \varpi_j\rangle$ for all j, and so

$$\eta+\phi = \zeta+\psi. \tag{5.16}$$

If $\eta=\sum_j \eta_j\alpha_j$, $\phi=\sum_j \phi_j\alpha_j$, $\zeta=\sum_j \zeta_j\alpha_j$ and $\psi=\sum_j \psi_j\alpha_j$, then the equation $\kappa_{\eta,\phi}(\varpi_i,0)=\kappa^i_{\zeta,\psi}(\varpi_i,0)$ along with (5.16) yields

$$\eta_{i-1}+\phi_{i-1}+\phi_i = \zeta_i+\psi_{i-1}+\psi_{i+1} \quad \text{and} \quad \eta_{i-1}+\eta_i+\phi_{i-1} = \zeta_{i-1}+\zeta_{i+1}+\psi_i$$

(with the convention that $\eta_0=\eta_n=\phi_0=\phi_n=\zeta_0=\zeta_n=\psi_0=\psi_n=0$). Thus,

$$\eta_{i-1}+\phi_{i-1} = \eta_{i+1}+\phi_{i+1}, \quad 1\leqslant i<n. \tag{5.17}$$

This implies that if $\theta_{\eta,\phi}\neq 0$, then $\omega'_\eta\omega_\phi\in U^0_\natural$. As a result, $u\in U^0_\natural$.

By proposition 5.6, $\varrho^{\lambda,\mu}(u)=\varrho^{\sigma(\lambda),\mu}(u)=\varrho^{\lambda,\mu}(\sigma^{-1}(u))$ for all $\lambda,\mu\in\Lambda_{\mathfrak{sl}}$ and $\sigma\in W$. But then $u=\sigma^{-1}(u)$ by corollary 5.5, so $u\in(U^0_\natural)^W$, as claimed. □

PROPOSITION 5.8. *The image of the centre $\mathfrak{Z}$ of U under the Harish-Chandra homomorphism satisfies*

$$\xi(\mathfrak{Z})\subseteq (U^0_\natural)^W.$$

Proof. Assume that $z\in\mathfrak{Z}$. Choose $\mu,\lambda\in\Lambda_{\mathfrak{sl}}$ and assume that $\langle\lambda,\alpha_i\rangle\geqslant 0$ for some (fixed) value i. Let $v_{\lambda,\mu}\in M(\varrho^{\lambda,\mu})$ be the highest weight vector. Then

$$zv_{\lambda,\mu}=\pi(z)v_{\lambda,\mu}=\varrho^{\lambda,\mu}(\pi(z))v_{\lambda,\mu}=\varrho^{\lambda+\rho,\mu}(\xi(z))v_{\lambda,\mu}$$

for all $z\in\mathfrak{Z}$. Thus, z acts as the scalar $\varrho^{\lambda+\rho,\mu}(\xi(z))$ on $M(\varrho^{\lambda,\mu})$.

Using [5, lemma 2.3], it is easy to see that

$$e_i f_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu}=\left([\langle\lambda,\alpha_i\rangle+1]f_i^{\langle\lambda,\alpha_i\rangle}\frac{r^{-\langle\lambda,\alpha_i\rangle}\omega_i-s^{-\langle\lambda,\alpha_i\rangle}\omega'_i}{r-s}\right)v_{\lambda,\mu}=0,$$

where, for $k\geqslant 1$,

$$[k]=\frac{r^k-s^k}{r-s}. \tag{5.18}$$

Thus, $e_j f_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu}=0$ for all $1\leqslant j<n$. Note that

$$\begin{aligned} zf_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu} &= \pi(z)f_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu} \\ &= \varrho^{\sigma_i(\lambda+\rho)-\rho,\mu}(\pi(z))f_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu} \\ &= \varrho^{\sigma_i(\lambda+\rho),\mu}(\xi(z))f_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu}. \end{aligned}$$

On the other hand, since z acts as the scalar $\varrho^{\lambda+\rho,\mu}(\xi(z))$ on $M(\varrho^{\lambda,\mu})$,

$$zf_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu}=\varrho^{\lambda+\rho,\mu}(\xi(z))f_i^{\langle\lambda,\alpha_i\rangle+1}v_{\lambda,\mu}.$$

Therefore,

$$\varrho^{\lambda+\rho,\mu}(\xi(z))=\varrho^{\sigma_i(\lambda+\rho),\mu}(\xi(z)). \tag{5.19}$$

Now we claim that (5.19) holds for an arbitrary choice of $\lambda\in\Lambda_{\mathfrak{sl}}$. Indeed, if $\langle\lambda,\alpha_i\rangle=-1$, then $\lambda+\rho=\sigma_i(\lambda+\rho)$, and so (5.19) holds trivially. For λ such that $\langle\lambda,\alpha_i\rangle<-1$, we let $\lambda'=\sigma_i(\lambda+\rho)-\rho$. Then $\langle\lambda',\alpha_i\rangle\geqslant 0$ and we may apply (5.19) to λ'. Substituting $\lambda' = \sigma_i(\lambda+\rho)-\rho$ into the result, we see that (5.19) holds for this case also.

Since i can be arbitrary, and W is generated by the reflections σ_i, we deduce that

$$\varrho^{\lambda,\mu}(\xi(z)) = \varrho^{\sigma(\lambda),\mu}(\xi(z)) \tag{5.20}$$

for all $\lambda,\mu \in \Lambda_{\mathfrak{sl}}$ and for all $\sigma \in W$. The assertion of the proposition then follows immediately from lemma 5.7. □

LEMMA 5.9. *$z \in \mathfrak{Z}$ if and only if $\mathrm{ad}(x)z = (\imath\circ\varepsilon)(x)z$ for all $x \in U$, where $\varepsilon : U \to \mathbb{K}$ is the co-unit and $\imath : \mathbb{K} \to U$ is the unit of U.*

Proof. Let $z \in \mathfrak{Z}$. Then, for all $x \in U$,

$$\mathrm{ad}(x)z = \sum_{(x)} x_{(1)} z S(x_{(2)}) = z \sum_{(x)} x_{(1)} S(x_{(2)}) = (\imath\circ\varepsilon)(x)z.$$

Conversely, assume that $\mathrm{ad}(x)z = (\imath\circ\varepsilon)(x)z$ for all $x \in U$. Then

$$\omega_i z \omega_i^{-1} = \mathrm{ad}(\omega_i)z = (\imath\circ\varepsilon)(\omega_i)z = z.$$

Similarly, $\omega_i' z (\omega_i')^{-1} = z$. Furthermore,

$$0 = (\imath\circ\varepsilon)(e_i)z = \mathrm{ad}(e_i)z = e_i z + \omega_i z(-\omega_i^{-1})e_i = e_i z - z e_i$$

and

$$0 = (\imath\circ\varepsilon)(f_i)z = \mathrm{ad}(f_i)z = z(-f_i(\omega_i')^{-1}) + f_i z(\omega_i')^{-1} = (-zf_i + f_i z)(\omega_i')^{-1}.$$

Hence, $z \in \mathfrak{Z}$. □

LEMMA 5.10. *Assume that $\Psi : U_{-\mu}^- \times U_\nu^+ \to \mathbb{K}$ is a bilinear map, and let $(\eta,\phi) \in Q\times Q$. There then exists $u \in U_{-\nu}^- U^0 U_\mu^+$ such that*

$$\langle u \mid (y{\omega_\mu'}^{-1})\omega_{\eta_1}'\omega_{\phi_1} x\rangle = (\omega_{\eta_1}',\omega_\phi)(\omega_\eta',\omega_{\phi_1})\Psi(y,x) \tag{5.21}$$

for all $x \in U_\nu^+$, $y \in U_{-\mu}^-$ and $(\eta_1,\phi_1) \in Q\times Q$.

Proof. As in the proof of proposition 4.11, for each $\mu \in Q^+$ we choose an arbitrary basis $u_1^\mu, u_2^\mu, \dots, u_{d_\mu}^\mu$ ($d_\mu = \dim U_\mu^+$) of U_μ^+ and a dual basis $v_1^\mu, v_2^\mu, \dots, v_{d_\mu}^\mu$ of $U_{-\mu}^-$ such that $(v_i^\mu, u_j^\mu) = \delta_{i,j}$. If we set

$$u = \sum_{i,j} \Psi(v_j^\mu, u_i^\nu) v_i^\nu (\omega_\nu')^{-1}\omega_\eta'\omega_\phi u_j^\mu (rs^{-1})^{-\langle\rho,\nu\rangle},$$

then it is straightforward to verify that u satisfies equation (5.21). □

We define a U-module structure on the dual space U^* by $(x\cdot f)(v) = f(\mathrm{ad}(S(x))v)$ for $f \in U^*$ and $x \in U$. Also we define a map $\beta : U \to U^*$ by setting

$$\beta(u)(v) = \langle u \mid v\rangle \quad \text{for } u,v \in U. \tag{5.22}$$

Then β is an injective U-module homomorphism by propositions 4.8 and 4.11, where the U-module structure on U is given by the adjoint action.

DEFINITION 5.11. Assume that M is a finite-dimensional U-module. For each $m \in M$ and $f \in M^*$, we define $c_{f,m} \in U^*$ by $c_{f,m}(v) = f(v \cdot m)$, $v \in U$.

PROPOSITION 5.12. *Assume that M is a finite-dimensional U-module such that*

$$M = \bigoplus_{\lambda \in \mathrm{wt}(M)} M_\lambda \quad \textit{and} \quad \mathrm{wt}(M) \subset Q.$$

For each $f \in M^*$ and $m \in M$, there exists a unique $u \in U$ such that

$$c_{f,m}(v) = \langle u \mid v \rangle \quad \textit{for all } v \in U.$$

Proof. The uniqueness follows immediately from proposition 4.11. Since $c_{f,m}$ depends linearly on m, we may assume that $m \in M_\lambda$ for some $\lambda \in Q$. For

$$v = (y {\omega'_\mu}^{-1}) \omega'_{\eta_1} \omega_{\phi_1} x, \quad x \in U^+_\nu, \quad y \in U^-_{-\mu}, \quad (\eta_1, \phi_1) \in Q \times Q,$$

we have

$$\begin{aligned} c_{f,m}(v) &= c_{f,m}((y {\omega'_\mu}^{-1}) \omega'_{\eta_1} \omega_{\phi_1} x) \\ &= f((y {\omega'_\mu}^{-1}) \omega'_{\eta_1} \omega_{\phi_1} x m) \\ &= \varrho^{\nu+\lambda}(\omega'_{\eta_1} \omega_{\phi_1}) f((y {\omega'_\mu}^{-1}) x m). \end{aligned}$$

Note that $(y, x) \mapsto f((y {\omega'_\mu}^{-1}) x m)$ is bilinear, and (4.1) gives us

$$(\omega'_{\eta_1}, \omega_{-\nu-\lambda}) = \varrho^{\nu+\lambda}(\omega'_{\eta_1}) \quad \text{and} \quad (\omega'_{\nu+\lambda}, \omega_{\phi_1}) = \varrho^{\nu+\lambda}(\omega_{\phi_1}).$$

Thus,

$$c_{f,m}(v) = (\omega'_{\eta_1}, \omega_{-\nu-\lambda})(\omega'_{\nu+\lambda}, \omega_{\phi_1}) f(y(\omega'_\mu)^{-1} x m),$$

and lemma 5.10 enables us to find $u_{\nu\mu} \in U^-_{-\nu} U^0 U^+_\mu$ such that $c_{f,m}(v) = \langle u_{\nu\mu} \mid v \rangle$ for all $v \in U^-_{-\mu} U^0 U^+_\nu$.

Now, for an arbitrary $v \in U$, we write $v = \sum_{(\mu,\nu)} v_{\mu\nu}$ with $v_{\mu\nu} \in U^-_{-\mu} U^0 U^+_\nu$. Since M is finite-dimensional, there is a finite set $\mathcal{F}$ of pairs $(\mu, \nu) \in Q \times Q$ such that

$$c_{f,m}(v) = c_{f,m}\Bigg(\sum_{(\mu,\nu) \in \mathcal{F}} v_{\mu\nu} \Bigg) \quad \text{for all } v \in U.$$

Setting $u = \sum_{(\mu,\nu) \in \mathcal{F}} u_{\nu\mu}$ and using lemma 4.7, we have

$$\begin{aligned} c_{f,m}(v) = c_{f,m}\Bigg(\sum_{(\mu,\nu) \in \mathcal{F}} v_{\mu\nu} \Bigg) &= \sum_{(\mu,\nu) \in \mathcal{F}} c_{f,m}(v_{\mu\nu}) \\ = \sum_{(\mu,\nu) \in \mathcal{F}} \langle u_{\nu\mu} \mid v_{\mu\nu} \rangle &= \sum_{(\mu,\nu) \in \mathcal{F}} \langle u_{\nu\mu} \mid v \rangle = \langle u \mid v \rangle. \end{aligned}$$

This completes the proof. □

The category $\mathcal{O}$ of representations of U is naturally defined. We refer the reader to [4, § 4] for the precise definition. All highest weight modules with weights in $\Lambda_{\mathfrak{sl}}$, such as the Verma modules $M(\lambda)$ and the irreducible modules $L(\lambda)$ for $\lambda \in \Lambda_{\mathfrak{sl}}$, belong to category $\mathcal{O}$.

Assume that M is any U-module in category $\mathcal{O}$, and define a linear map $\Theta : M \to M$ by

$$\Theta(m) = (rs^{-1})^{-\langle \rho, \lambda \rangle} m \tag{5.23}$$

for all $m \in M_\lambda$, $\lambda \in \Lambda_{\mathfrak{sl}}$. We claim that

$$\Theta u = S^2(u)\Theta \quad \text{for all } u \in U. \tag{5.24}$$

Indeed, we have only to check this holds when u is one of the generators e_i, f_i, ω_i or ω_i', and for them the verification of (5.24) is straightforward.

For $\lambda \in \Lambda_{\mathfrak{sl}}^+$, we define $f_\lambda \in U^*$ as given by the following trace map:

$$f_\lambda(u) = \mathrm{tr}_{L(\lambda)}(u\Theta), \quad u \in U.$$

Lemma 5.13. *Assume that $\lambda \in \Lambda_{\mathfrak{sl}}^+ \cap Q$. Then $f_\lambda \in \mathrm{Im}(\beta)$, where β is defined in equation (5.22).*

Proof. Let $k = \dim L(\lambda)$, and fix a basis $\{m_i\}$ for $L(\lambda)$ and its dual basis $\{f_i\}$ for $L(\lambda)^*$. We now have

$$f_\lambda(v) = \mathrm{tr}_{L(\lambda)}(v\Theta) = \sum_{i=1}^{k} c_{f_i, \Theta m_i}(v).$$

By proposition 5.12, we can find $u_i \in U$ such that $c_{f_i,\Theta m_i}(v) = \langle u_i \mid v \rangle$ for each i, $1 \leqslant i \leqslant k$. Set $u = \sum_{i=1}^{k} u_i$ such that

$$\beta(u)(v) = \sum_{i=1}^{k} \langle u_i \mid v \rangle = \sum_{i=1}^{k} c_{f_i,\Theta m_i}(v) = f_\lambda(v).$$

Thus, $f_\lambda \in \mathrm{Im}(\beta)$. □

Proposition 5.14. *The element $z_\lambda := \beta^{-1}(f_\lambda)$ is contained in the centre $\mathfrak{Z}$ for each $\lambda \in \Lambda_{\mathfrak{sl}}^+ \cap Q$.*

Proof. Using (5.24), we have, for all $x \in U$,

$$\begin{aligned}
(S^{-1}(x) f_\lambda)(u) &= f_\lambda(\mathrm{ad}(x)u) \\
&= \mathrm{tr}_{L(\lambda)}\Big(\sum_{(x)} x_{(1)} u S(x_{(2)})\Theta\Big) \\
&= \mathrm{tr}_{L(\lambda)}\Big(u \sum_{(x)} S(x_{(2)})\Theta x_{(1)}\Big) \\
&= \mathrm{tr}_{L(\lambda)}\Big(u \sum_{(x)} S(x_{(2)}) S^2(x_{(1)})\Theta\Big) \\
&= \mathrm{tr}_{L(\lambda)}\Big(u S\Big(\sum_{(x)} S(x_{(1)}) x_{(2)}\Big)\Theta\Big) \\
&= (\imath \circ \varepsilon)(x)\, \mathrm{tr}_{L(\lambda)}(u\Theta) = (\imath \circ \varepsilon)(x) f_\lambda(u).
\end{aligned}$$

Substituting x for $S^{-1}(x)$ in the above, we deduce from $\varepsilon \circ S = \varepsilon$ the relation

$$x f_\lambda = (\imath \circ \epsilon)(x) f_\lambda.$$

We can write

$$x f_\lambda = x\beta(\beta^{-1}(f_\lambda)) = \beta(\mathrm{ad}(S(x))\beta^{-1}(f_\lambda))$$

and

$$(\imath \circ \varepsilon)(x) f_\lambda = (\imath \circ \varepsilon)(x)\beta(\beta^{-1}(f_\lambda)) = \beta((\imath \circ \varepsilon)(x)\beta^{-1}(f_\lambda)).$$

Since β is injective, $\mathrm{ad}(S(x))\beta^{-1}(f_\lambda) = (\imath \circ \varepsilon)(x)\beta^{-1}(f_\lambda)$. Since $\varepsilon \circ S^{-1} = \varepsilon$, substituting x for $S(x)$, we obtain

$$\mathrm{ad}(x)\beta^{-1}(f_\lambda) = (\imath \circ \varepsilon)(x)\beta^{-1}(f_\lambda) \quad \text{for all } x \in U.$$

Therefore, we may conclude from lemma 5.9 that $\beta^{-1}(f_\lambda) \in \mathfrak{Z}$. □

This brings us to our main result on the centre of U.

THEOREM 5.15. *Assume that r and s satisfy condition (5.1).*

(i) *If n is odd, then the map $\xi : \mathfrak{Z} \to (U^0_\natural)^W = (U^0_\flat)^W$ is an isomorphism.*

(ii) *If n is even, the centre $\mathfrak{Z}$ is isomorphic under ξ to a subalgebra of $(U^0_\natural)^W$ containing $\mathbb{K}[\mathfrak{z}, \mathfrak{z}^{-1}] \otimes (U^0_\flat)^W$, i.e. $\mathbb{K}[\mathfrak{z}, \mathfrak{z}^{-1}] \otimes (U^0_\flat)^W \subseteq \xi(\mathfrak{Z}) \subseteq (U^0_\natural)^W$, where the element $\mathfrak{z} \in \mathfrak{Z}$ is defined in (5.5).*

Proof. We set $z_\lambda = \beta^{-1}(f_\lambda)$ for $\lambda \in \Lambda^+_{\mathfrak{sl}} \cap Q$ and write

$$z_\lambda = \sum_{\nu \geqslant 0} z_{\lambda,\nu} \quad \text{and} \quad z_{\lambda,0} = \sum_{(\eta,\phi)\in Q\times Q} \theta_{\eta,\phi}\omega'_\eta\omega_\phi,$$

where $z_{\lambda,\nu} \in U^-_{-\nu}U^0U^+_\nu$ and $\theta_{\eta,\phi} \in \mathbb{K}$. Then, for $(\eta_1, \phi_1) \in Q \times Q$,

$$\langle z_\lambda \mid \omega'_{\eta_1}\omega_{\phi_1}\rangle = \langle z_{\lambda,0} \mid \omega'_{\eta_1}\omega_{\phi_1}\rangle = \sum_{(\eta,\phi)} \theta_{\eta,\phi}(\omega'_{\eta_1}, \omega_\phi)(\omega'_\eta, \omega_{\phi_1}).$$

On the other hand,

$$\begin{aligned}\langle z_\lambda \mid \omega'_{\eta_1}\omega_{\phi_1}\rangle &= \beta(z_\lambda)(\omega'_{\eta_1}\omega_{\phi_1}) = f_\lambda(\omega'_{\eta_1}\omega_{\phi_1}) = \mathrm{tr}_{L(\lambda)}(\omega'_{\eta_1}\omega_{\phi_1}\Theta)\\ &= \sum_{\mu\leqslant\lambda} \dim(L(\lambda)_\mu)(rs^{-1})^{-\langle\rho,\mu\rangle}\varrho^\mu(\omega'_{\eta_1}\omega_{\phi_1})\\ &= \sum_{\mu\leqslant\lambda} \dim(L(\lambda)_\mu)(rs^{-1})^{-\langle\rho,\mu\rangle}(\omega'_{\eta_1}, \omega_{-\mu})(\omega'_\mu, \omega_{\phi_1}).\end{aligned}$$

Now we may write

$$\sum_{(\eta,\phi)} \theta_{\eta,\phi}\chi_{\eta,\phi} = \sum_{\mu\leqslant\nu} \dim(L(\lambda)_\mu)(rs^{-1})^{-\langle\rho,\mu\rangle}\chi_{\mu,-\mu},$$

where the characters $\chi_{\eta,\phi}$ are defined in (4.5). By assumption (5.1), lemma 4.10 and the linear independence of distinct characters, we obtain

$$\theta_{\eta,\phi} = \begin{cases} \dim(L(\lambda)_\eta)(rs^{-1})^{-\langle\rho,\eta\rangle} & \text{if } \eta+\phi=0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$z_{\lambda,0} = \sum_{\mu\leqslant\lambda} \dim(L(\lambda)_\mu)(rs^{-1})^{-\langle\rho,\mu\rangle}\omega'_\mu\omega_{-\mu},$$

and, by (5.4),

$$\xi(z_\lambda) = \varrho^{-\rho}(z_{\lambda,0}) = \sum_{\mu\leqslant\lambda} \dim(L(\lambda)_\mu)\omega'_\mu\omega_{-\mu}. \tag{5.25}$$

Note that $\mathfrak{z} = \xi(\mathfrak{z}) \in (U^0_\natural)^W$ when n is even. By propositions 5.2 and 5.8, it is sufficient to show that $(U^0_\flat)^W \subseteq \xi(\mathfrak{Z})$. For $\lambda \in \Lambda^+_{\mathfrak{sl}} \cap Q$, we define

$$\operatorname{av}(\lambda) = \frac{1}{|W|}\sum_{\sigma\in W}\sigma(\omega'_\lambda\omega_{-\lambda}) = \frac{1}{|W|}\sum_{\sigma\in W}\omega'_{\sigma(\lambda)}\omega_{-\sigma(\lambda)}. \tag{5.26}$$

Remembering that, for each $\eta \in Q$, there exists $\sigma \in W$ such that $\sigma(\eta) \in \Lambda^+_{\mathfrak{sl}} \cap Q$, we see that the set $\{\operatorname{av}(\lambda) \mid \lambda \in \Lambda^+_{\mathfrak{sl}} \cap Q\}$ forms a basis of $(U^0_\flat)^W$. Thus, we have only to show that $\operatorname{av}(\lambda) \in \operatorname{Im}(\xi)$ for all $\lambda \in \Lambda^+_{\mathfrak{sl}} \cap Q$. We use induction on λ. If $\lambda = 0$, $\operatorname{av}(0) = 1 = \xi(1)$. Assume that $\lambda > 0$. Since $\dim L(\lambda)_\mu = \dim L(\lambda)_{\sigma(\mu)}$ for all $\sigma \in W$ (proposition 2.3) and $\dim L(\lambda)_\lambda = 1$, we can rewrite (5.25) to obtain

$$\xi(z_\lambda) = |W|\operatorname{av}(\lambda) + |W|\sum \dim(L(\lambda)_\mu)\operatorname{av}(\mu),$$

where the sum is over μ such that $\mu < \lambda$ and $\mu \in \Lambda^+_{\mathfrak{sl}} \cap Q$. By the induction hypothesis, we get $\operatorname{av}(\lambda) \in \operatorname{Im}(\xi)$. This completes the proof. □

EXAMPLE 5.16. The centre $\mathfrak{Z}$ of $U = U_{r,s}(\mathfrak{sl}_2)$ has a basis of monomials $\mathfrak{z}^i\mathcal{C}^j$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}_{\geqslant 0}$, where $\mathfrak{z} = \omega'\omega$ (we omit the subscript since there is only one of them), and $\mathcal{C}$ is the Casimir element,

$$\mathcal{C} = ef + \frac{s\omega + r\omega'}{(r-s)^2} = fe + \frac{r\omega + s\omega'}{(r-s)^2}.$$

Now

$$\xi(\mathfrak{z}) = \mathfrak{z} \quad \text{and} \quad \xi(\mathcal{C}) = \frac{(rs)^{1/2}}{(r-s)^2}(\omega + \omega').$$

Thus, the monomials $\mathfrak{z}^i\mathfrak{c}^j$, $i \in \mathbb{Z}$, $j \in \mathbb{Z}_{\geqslant 0}$, where $\mathfrak{c} = \omega + \omega'$, give a basis for $\xi(\mathfrak{Z})$. The subalgebra $(U^0_\flat)^W$ consists of polynomials in $\mathfrak{a} := \omega'\omega^{-1} + (\omega')^{-1}\omega = 2\operatorname{av}(\alpha)$. Observe that $\mathfrak{a} + 2 = \mathfrak{z}^{-1}\mathfrak{c}^2 \in \xi(\mathfrak{Z})$, but we cannot express $\mathfrak{c}$ as an element of $\mathbb{K}[\mathfrak{z},\mathfrak{z}^{-1}] \otimes (U^0_\flat)^W$. Since $\sigma((\omega')^\ell\omega^m) = (\omega')^m\omega^\ell$, we see that $(U^0)^W$ has as a basis the sums $(\omega')^\ell\omega^m + (\omega')^m\omega^\ell$ for all $\ell, m \in \mathbb{Z}$, and hence $\mathbb{K}[\mathfrak{z},\mathfrak{z}^{-1}] \otimes (U^0_\flat)^W \subsetneq \xi(\mathfrak{Z}) = (U^0_\natural)^W = (U^0)^W$, as no conditions are imposed by (5.9).

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Appendix A.

LEMMA A.1. *The relations*

(i) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0$, *for* $i \geqslant j > k+1 \geqslant l+1$,

(ii) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} - \mathcal{E}_{i,l} = 0$, *for* $i \geqslant j = k+1 \geqslant l+1$,

(iii) $\mathcal{E}_{i,j}e_j - s^{-1}e_j\mathcal{E}_{i,j} = 0$, *for* $i > j$,

hold in U^+.

Proof. The equations in (i) are obvious.

For (ii), we fix j and l with $j > l$ and use induction on i. If $i = j$, this is just the definition of $\mathcal{E}_{i,l}$ from (3.1). Assume that $i > j$. We then have

$$\begin{aligned}\mathcal{E}_{i,j}\mathcal{E}_{j-1,l} &= e_i\mathcal{E}_{i-1,j}\mathcal{E}_{j-1,l} - r^{-1}\mathcal{E}_{i-1,j}e_i\mathcal{E}_{j-1,l} \\ &= r^{-1}e_i\mathcal{E}_{j-1,l}\mathcal{E}_{i-1,j} + e_i\mathcal{E}_{i-1,l} - r^{-2}\mathcal{E}_{j-1,l}\mathcal{E}_{i-1,j}e_i - r^{-1}\mathcal{E}_{i-1,l}e_i \\ &= r^{-1}\mathcal{E}_{j-1,l}\mathcal{E}_{i,j} + \mathcal{E}_{i,l}\end{aligned}$$

by part (i) and the induction hypothesis.

To establish (iii), we fix j and use induction on i. When $i = j+1$, the relation is simply (3.2) with j instead of i. Assume that $i > j+1$. We then have

$$\begin{aligned}\mathcal{E}_{i,j}e_j &= e_i\mathcal{E}_{i-1,j}e_j - r^{-1}\mathcal{E}_{i-1,j}e_je_i \\ &= s^{-1}e_je_i\mathcal{E}_{i-1,j} - r^{-1}s^{-1}e_j\mathcal{E}_{i-1,j}e_i \\ &= s^{-1}e_j\mathcal{E}_{i,j}\end{aligned}$$

by (i) and induction. □

LEMMA A.2. *In* U^+,

(i) $\mathcal{E}_{i,j}\mathcal{E}_{j,l} - r^{-1}s^{-1}\mathcal{E}_{j,l}\mathcal{E}_{i,j} + (r^{-1} - s^{-1})e_j\mathcal{E}_{i,l} = 0$, *for* $i > j > l$,

(ii) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - \mathcal{E}_{k,l}\mathcal{E}_{i,j} = 0$, *for* $i > k \geqslant l > j$.

Proof. The following expression can be easily verified by induction on l:

$$\mathcal{E}_{i,j}\mathcal{E}_{j,l} - r^{-1}s^{-1}\mathcal{E}_{j,l}\mathcal{E}_{i,j} + r^{-1}\mathcal{E}_{i,l}e_j - s^{-1}e_j\mathcal{E}_{i,l} = 0, \quad i > j > l. \tag{A 1}$$

We claim that

$$\mathcal{E}_{j+1,j-1}e_j - e_j\mathcal{E}_{j+1,j-1} = 0. \tag{A 2}$$

Indeed, we have $e_j\mathcal{E}_{j,j-1} = s^{-1}\mathcal{E}_{j,j-1}e_j$ as in (3.3), and using this we get

$$\begin{aligned}
&\mathcal{E}_{j+1,j}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}\mathcal{E}_{j+1,j} \\
&\quad = e_{j+1}e_j\mathcal{E}_{j,j-1} - r^{-1}e_je_{j+1}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}e_{j+1}e_j + r^{-2}s^{-1}\mathcal{E}_{j,j-1}e_je_{j+1} \\
&\quad = s^{-1}e_{j+1}\mathcal{E}_{j,j-1}e_j - r^{-1}e_je_{j+1}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}e_{j+1}e_j + r^{-2}e_j\mathcal{E}_{j,j-1}e_{j+1} \\
&\quad = s^{-1}\mathcal{E}_{j+1,j-1}e_j - r^{-1}e_j\mathcal{E}_{j+1,j-1}.
\end{aligned}$$

On the other hand, we also have, from (A 1),

$$\mathcal{E}_{j+1,j}\mathcal{E}_{j,j-1} - r^{-1}s^{-1}\mathcal{E}_{j,j-1}\mathcal{E}_{j+1,j} = s^{-1}e_j\mathcal{E}_{j+1,j-1} - r^{-1}\mathcal{E}_{j+1,j-1}e_j,$$

such that

$$(r^{-1}+s^{-1})\mathcal{E}_{j+1,j-1}e_j - (r^{-1}+s^{-1})e_j\mathcal{E}_{j+1,j-1} = 0.$$

Since we have assumed that $r^{-1}+s^{-1} \neq 0$, this implies (A 2).

Now to demonstrate that

$$\mathcal{E}_{i,j}e_k - e_k\mathcal{E}_{i,j} = 0, \quad i > k > j, \tag{A 3}$$

we fix k, and assume first that $j = k-1$. The argument proceeds by induction on i. If $i = k+1$, then the expression in (A 3) becomes (A 2) (with k instead of j there). When $i > k+1$,

$$\begin{aligned}
\mathcal{E}_{i,k-1}e_k &= e_i\mathcal{E}_{i-1,k-1}e_k - r^{-1}\mathcal{E}_{i-1,k-1}e_ke_i \\
&= e_ke_i\mathcal{E}_{i-1,k-1} - r^{-1}e_k\mathcal{E}_{i-1,k-1}e_i = e_k\mathcal{E}_{i,k-1}.
\end{aligned}$$

For the case $j < k-1$, we have by induction on j,

$$\begin{aligned}
\mathcal{E}_{i,j}e_k &= \mathcal{E}_{i,j+1}e_je_k - r^{-1}e_j\mathcal{E}_{i,j+1}e_k \\
&= e_k\mathcal{E}_{i,j+1}e_j - r^{-1}e_ke_j\mathcal{E}_{i,j+1} \\
&= e_k\mathcal{E}_{i,j},
\end{aligned}$$

so that (A 3) is verified.

As a consequence, the relations in part (i) follow from (A 1) and (A 3), while those in (ii) can be derived easily from (A 3) by fixing i, j and k and using induction on l. □

LEMMA A.3. *The relations*

(i) $\mathcal{E}_{i,j}\mathcal{E}_{k,j} - s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j} = 0$, *for* $i > k > j$,

(ii) $\mathcal{E}_{i,j}\mathcal{E}_{k,l} - r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (r^{-1}-s^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l} = 0$ *for* $i > k > j > l$,

hold in U^+.

Proof. Part (i) follows from lemmas A.1(iii) and A.2(ii). For (ii), we apply induction on l. When $l = j-1$, part (i), and lemmas A.1(ii) and A.2(ii) imply that

$$\begin{aligned}
&\mathcal{E}_{i,j}\mathcal{E}_{k,j-1} \\
&\quad = \mathcal{E}_{i,j}\mathcal{E}_{k,j}e_{j-1} - r^{-1}\mathcal{E}_{i,j}e_{j-1}\mathcal{E}_{k,j} \\
&\quad = s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j}e_{j-1} - r^{-1}\mathcal{E}_{i,j}e_{j-1}\mathcal{E}_{k,j} \\
&\quad = r^{-1}s^{-1}\mathcal{E}_{k,j}e_{j-1}\mathcal{E}_{i,j} + s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j-1} - r^{-2}e_{j-1}\mathcal{E}_{i,j}\mathcal{E}_{k,j} - r^{-1}\mathcal{E}_{i,j-1}\mathcal{E}_{k,j} \\
&\quad = r^{-1}s^{-1}\mathcal{E}_{k,j}e_{j-1}\mathcal{E}_{i,j} + s^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j-1} - r^{-2}s^{-1}e_{j-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j} - r^{-1}\mathcal{E}_{k,j}\mathcal{E}_{i,j-1} \\
&\quad = r^{-1}s^{-1}\mathcal{E}_{k,j-1}\mathcal{E}_{i,j} + (s^{-1} - r^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,j-1}.
\end{aligned}$$

Now assume that $l < j-1$. Then $\mathcal{E}_{i,j}e_l = e_l\mathcal{E}_{i,j}$ and $\mathcal{E}_{k,j}e_l = e_l\mathcal{E}_{k,j}$ by lemma A.1(i) and so, by lemma A.1(ii), we obtain

$$\begin{aligned}
\mathcal{E}_{i,j}\mathcal{E}_{k,l} &= \mathcal{E}_{i,j}\mathcal{E}_{k,l+1}e_l - r^{-1}\mathcal{E}_{i,j}e_l\mathcal{E}_{k,l+1} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,l+1}e_l\mathcal{E}_{i,j} + (s^{-1} - r^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l+1}e_l \\
&\qquad - r^{-2}s^{-1}e_l\mathcal{E}_{k,l+1}\mathcal{E}_{i,j} - r^{-1}(s^{-1} - r^{-1})e_l\mathcal{E}_{k,j}\mathcal{E}_{i,l+1} \\
&= r^{-1}s^{-1}\mathcal{E}_{k,l}\mathcal{E}_{i,j} + (s^{-1} - r^{-1})\mathcal{E}_{k,j}\mathcal{E}_{i,l}
\end{aligned}$$

by the induction assumption. □

LEMMA A.4. *In U^+,*

$$\mathcal{E}_{i,j}\mathcal{E}_{i,l} - s^{-1}\mathcal{E}_{i,l}\mathcal{E}_{i,j} = 0, \quad i \geqslant j > l. \tag{A 4}$$

Proof. First consider the case $i = j$. If $l = i-1$, the above relation is merely the defining relation in (3.3). Assume that $l < i-1$. By induction on l, we have

$$\begin{aligned}
e_i\mathcal{E}_{i,l} &= e_i\mathcal{E}_{i,l+1}e_l - r^{-1}e_ie_l\mathcal{E}_{i,l+1} \\
&= s^{-1}\mathcal{E}_{i,l+1}e_le_i - r^{-1}s^{-1}e_l\mathcal{E}_{i,l+1}e_i \\
&= s^{-1}\mathcal{E}_{i,l}e_i.
\end{aligned}$$

When $i > j$, by induction on j and lemma A.2(ii), we get

$$\begin{aligned}
\mathcal{E}_{i,j}\mathcal{E}_{i,l} &= \mathcal{E}_{i,j+1}e_j\mathcal{E}_{i,l} - r^{-1}e_j\mathcal{E}_{i,j+1}\mathcal{E}_{i,l} \\
&= \mathcal{E}_{i,j+1}\mathcal{E}_{i,l}e_j - r^{-1}s^{-1}e_j\mathcal{E}_{i,l}\mathcal{E}_{i,j+1}, \\
&= s^{-1}\mathcal{E}_{i,l}\mathcal{E}_{i,j+1}e_j - r^{-1}s^{-1}\mathcal{E}_{i,l}e_j\mathcal{E}_{i,j+1} \\
&= s^{-1}\mathcal{E}_{i,l}\mathcal{E}_{i,j}.
\end{aligned}$$

□

The proof of theorem 3.1 is now complete because we have

(1) $\Longleftrightarrow$ lemma A.1(ii);

(2) $\Longleftrightarrow$ lemma A.1(i) and lemma A.2(ii);

(3) $\Longleftrightarrow$ lemma A.1(iii), lemma A.3(i), and lemma A.4;

(4) $\Longleftrightarrow$ lemma A.2(i) and lemma A.3(ii).

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