Vertex operator algebras associated to type G affine Lie algebras

Jonathan D. Axtell, Kyu-Hwan Lee *

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

ARTICLE INFO

ABSTRACT

In this paper, we study representations of the vertex operator algebra $L(k, 0)$ at one-third admissible levels $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$ for the affine algebra of type $G_2^{(1)}$. We first determine singular vectors and then obtain a description of the associative algebra $A(L(k, 0))$ using the singular vectors. We then prove that there are only finitely many irreducible $A(L(k, 0))$-modules from the category $\mathcal{O}$. Applying the $A(V)$-theory, we prove that there are only finitely many irreducible weak $L(k, 0)$-modules from the category $\mathcal{O}$ and that such an $L(k, 0)$-module is completely reducible. Our result supports the conjecture made by Adamović and Milas (1995) [2].

© 2011 Elsevier Inc. All rights reserved.

Introduction

Vertex operator algebras (VOA) are mathematical counterparts of chiral algebras in conformal field theory. An important family of examples comes from representations of affine Lie algebras. More precisely, if we let $\hat{\mathfrak{g}}$ be an affine Lie algebra, the irreducible $\hat{\mathfrak{g}}$-module $L(k, 0)$ with highest weight $k\Lambda_0$, $k \in \mathbb{C}$, is a VOA, whenever $k \neq -h^\vee$, the negative of the dual Coxeter number.

The representation theory of $L(k, 0)$ varies depending on values of $k \in \mathbb{C}$. If k is a positive integer, the VOA $L(k, 0)$ has only finitely many irreducible modules which coincide with the irreducible highest weight integrable $\hat{\mathfrak{g}}$-modules of level k, and the category of $\mathbb{Z}_+$-graded weak $L(k, 0)$-modules is semisimple. If $k \notin \mathbb{Q}$ or $k < -h^\vee$, categories of $L(k, 0)$-modules are quite different from those corresponding to positive integer values. (For example, see [10,11].)

For some rational values of k, the category of weak $L(k, 0)$-modules which are in the category $\mathcal{O}$ as $\hat{\mathfrak{g}}$-modules has a structure similar to that for the category of $\mathbb{Z}_+$-graded weak modules for positive integer values. Such rational values are called *admissible levels*. This notion was defined in the

---

* Corresponding author.
E-mail addresses: axtell@math.uconn.edu (J.D. Axtell), khlee@math.uconn.edu (K.-H. Lee).

important works of Kac and Wakimoto [7,8]. Various cases have been studied with different generality by many authors. Adamović studied the case of admissible half-integer levels for type $C_l^{(1)}$ [1]. The case of all admissible levels of type $A_1^{(1)}$ was studied by Adamović and Milas [2], and by Dong, Li and Mason [3]. In his recent papers [14,15], Perše studied admissible half-integer levels for type $A_l^{(1)}$ and $B_l^{(1)}$.

In these developments, the $A(V)$-theory has played an important role. The associative algebra $A(V)$ associated to a vertex operator algebra V was introduced by Frenkel and Zhu (see [5,16]). It was shown that the irreducible modules of $A(V)$ are in one-to-one correspondence with irreducible $\mathbb{Z}_+$-graded weak modules of V. This fact gives an elegant method for the classification of representations of V, and was exploited in the works mentioned above.

In this paper, we study one-third admissible levels $-\frac{5}{3}\Lambda_0, -\frac{4}{3}\Lambda_0, -\frac{2}{3}\Lambda_0$ for type $G_2^{(1)}$ adopting the method of [1,2,13–15]. We first determine singular vectors (Proposition 2.3) and then obtain a description of the associative algebra $A(L(k,0))$ in Theorem 2.6 using the singular vectors for $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$. By constructing some polynomials in the symmetric algebra of the Cartan subalgebra, we find all the possible highest weights for irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$ (Proposition 3.6). As a result, in each case of $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$, we prove that there are only finitely many irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$. Then it follows from the one-to-one correspondence in $A(V)$-theory that there are only finitely many irreducible weak $L(k,0)$-modules from the category $\mathcal{O}$ (Theorem 3.7). In the case of irreducible $L(k,0)$-modules, our result provides a complete classification (Theorem 3.10). We also prove that such an $L(k,0)$-module is completely reducible (Theorem 3.12). Thus the VOA $L(k,0)$ is *rational in the category $\mathcal{O}$* for $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$. This result supports the conjecture made by Adamović and Milas in [2], which suggests that $L(k,0)$'s are rational in the category $\mathcal{O}$ for all admissible levels k.

Although some of our results may be generalized to higher levels k, the first difficulty is in the drastic growth of complexity in computing singular vectors, as one can see in Appendix A. It seems to be necessary to find a different approach to the problem for higher levels. The first-named author will consider singular vectors for other admissible weights in his subsequent paper.

1. Preliminaries

1.1. Vertex operator algebras

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (VOA). This means that V is a $\mathbb{Z}$-graded vector space, $V = \bigoplus_{n \in \mathbb{Z}} V_n$, Y is the *vertex operator map*, $Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]]$, $\mathbf{1} \in V_0$ is the *vacuum vector*, and $\omega \in V_2$ is the *conformal vector*, all of which satisfy the usual axioms. See [3,4,12] for more details. By an ideal in the vertex operator algebra V we mean a subspace I of V satisfying $Y(a, x)I \subseteq I[[x, x^{-1}]]$ for any $a \in V$. Given an ideal I in V such that $\mathbf{1} \notin I$, $\omega \notin I$, the quotient V/I naturally becomes a vertex operator algebra. Let (M, Y_M) be a weak module for the vertex operator algebra V. We thus have a vector space M and a map $Y_M(\cdot, x) : V \to (\text{End } M)[[x, x^{-1}]]$, which satisfy the usual set of axioms (cf. [3]). For a fixed element $a \in V$, we write $Y_M(a, x) = \sum_{m \in \mathbb{Z}} a(m)x^{-m-1}$, and for the conformal element ω we write $Y_M(\omega, x) = \sum_{m \in \mathbb{Z}} \omega(m)x^{-m-1} = \sum_{m \in \mathbb{Z}} L_m x^{-m-2}$. In particular, V is a weak module over itself with $Y = Y_V$.

A $\mathbb{Z}_+$-*graded weak V-module* is a weak V-module M together with a $\mathbb{Z}_+$-gradation $M = \bigoplus_{n=0}^{\infty} M_n$ such that

$$a(m)M_n \subseteq M_{n+r-m-1} \quad \text{for } a \in V_r \text{ and } m, n, r \in \mathbb{Z},$$

where $M_n = 0$ for $n < 0$ by definition. A weak V-module M is called a V-*module* if L_0 acts semisimply on M with a decomposition into L_0-eigenspaces $M = \bigoplus_{\alpha \in \mathbb{C}} M_\alpha$ such that for any $\alpha \in \mathbb{C}$, $\dim M_\alpha < \infty$ and $M_{\alpha+n} = 0$ for $n \in \mathbb{Z}$ sufficiently small.

We define bilinear maps $* : V \times V \to V$ and $\circ : V \times V \to V$ as follows. For any homogeneous $a \in V_n$, we write $\deg(a) = n$, and for any $b \in V$, we define

$$a * b = \mathrm{Res}_x \frac{(1+x)^{\deg a}}{x} Y(a,x)b,$$

and

$$a \circ b = \mathrm{Res}_x \frac{(1+x)^{\deg a}}{x^2} Y(a,x)b,$$

and extend both definitions by linearity to $V \times V$. Denote by $O(V)$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. For $a \in V$, denote by $[a]$ the image of a under the projection of V onto $A(V)$. The map $a \mapsto [a]$ will be called *Zhu's map*. The multiplication $*$ induces the multiplication on $A(V)$, and $A(V)$ has a structure of an associative algebra. This fact can be found in [5,16].

Proposition 1.1. *(See [5].) Let I be an ideal of the vertex operator algebra V such that $\mathbf{1} \notin I$, $\omega \notin I$. Then the associative algebra $A(V/I)$ is isomorphic to $A(V)/A(I)$, where $A(I)$ is the image of I in $A(V)$.*

Given a weak module M and homogeneous $a \in V$, we recall that we write $Y_M(a,x) = \sum_{m \in \mathbb{Z}} a(m) x^{-m-1}$. We define $o(a) = a(\deg a - 1) \in \mathrm{End}(M)$ and extend this map linearly to V.

Theorem 1.2. *(See [16].)*

(1) *Let $M = \bigoplus_{n=0}^{\infty} M_n$ be a $\mathbb{Z}_+$-graded weak V-module. Then M_0 is an $A(V)$-module defined as follows:*

$$[a] \cdot v = o(a)v$$

for any $a \in V$ and $v \in M_0$.
(2) *Let U be an $A(V)$-module. Then there exists a $\mathbb{Z}_+$-graded weak V-module M such that the $A(V)$-modules M_0 and U are isomorphic.*
(3) *The equivalence classes of the irreducible $A(V)$-modules and the equivalence classes of the irreducible $\mathbb{Z}_+$-graded weak V-modules are in bijective correspondence.*

1.2. Affine Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$, with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta_+ \subset \Delta$ the set of positive roots, θ the highest root and $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form, normalized by the condition $(\theta, \theta) = 2$. Denote by $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots of $\mathfrak{g}$, and by $\Pi^\vee = \{h_1, \ldots, h_l\}$ the set of simple coroots of $\mathfrak{g}$. The affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the vector space

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

equipped with the bracket operation

$$[a \otimes t^m, b \otimes t^n] = [a,b] \otimes t^{m+n} + m(a,b)\delta_{m+n,0}K, \quad a, b \in \mathfrak{g}, m, n \in \mathbb{Z},$$

together with the condition that K is a nonzero central element. We consider $\mathfrak{h}$ as a subalgebra of $\hat{\mathfrak{g}}$, and set

$$\hat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K \subset \hat{\mathfrak{g}}.$$

Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the corresponding triangular decomposition of $\hat{\mathfrak{g}}$.

We set $h^\vee$ to be the dual Coxeter number of $\hat{\mathfrak{g}}$. Denote by $\widehat{\Delta}$ the set of roots of $\hat{\mathfrak{g}}$, by $\widehat{\Delta}_+$ the set of positive roots of $\hat{\mathfrak{g}}$, and by $\widehat{\Pi}$ the set of simple roots of $\hat{\mathfrak{g}}$. We also denote by $\widehat{\Delta}^{\mathrm{re}}$ the set of real roots of $\hat{\mathfrak{g}}$ and let $\widehat{\Delta}_+^{\mathrm{re}} = \widehat{\Delta}^{\mathrm{re}} \cap \widehat{\Delta}_+$. The coroot corresponding to a real root $\alpha \in \widehat{\Delta}^{\mathrm{re}}$ will be denoted by $\alpha^\vee$. Let $\widehat{Q} = \bigoplus_{\alpha \in \widehat{\Pi}} \mathbb{Z}\alpha$ be the root lattice, and let $\widehat{Q}_+ = \bigoplus_{\alpha \in \widehat{\Pi}} \mathbb{Z}_+ \alpha \subset \widehat{Q}$. For any $\lambda \in \hat{\mathfrak{h}}^*$, we set

$$D(\lambda) = \{\lambda - \alpha \mid \alpha \in \widehat{Q}_+\}.$$

We say that a $\hat{\mathfrak{g}}$-module M belongs to the *category* $\mathcal{O}$ if the Cartan subalgebra $\hat{\mathfrak{h}}$ acts semisimply on M with finite-dimensional weight spaces and there exits a finite number of elements $\nu_1, \dots, \nu_k \in \hat{\mathfrak{h}}^*$ such that $\nu \in \bigcup_{i=1}^k D(\nu_i)$ for every weight ν of M. We denote by $M(\lambda)$ the Verma module for $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$, and by $L(\lambda)$ the irreducible $\hat{\mathfrak{g}}$-module with highest weight λ. Let U be a $\mathfrak{g}$-module, and let $k \in \mathbb{C}$. We set $\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\mathbb{C}[t]$ and $\hat{\mathfrak{g}}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$. Let $\hat{\mathfrak{g}}_+$ act trivially on U and K as scalar multiplication by k. Considering U as a $\mathfrak{g} \oplus \mathbb{C}K \oplus \hat{\mathfrak{g}}_+$-module, we have the induced $\hat{\mathfrak{g}}$-module

$$N(k, U) = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g} \oplus \mathbb{C}K \oplus \hat{\mathfrak{g}}_+)} U.$$

For a fixed $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the irreducible highest weight $\mathfrak{g}$-module with highest weight μ. Denote by P_+ the set of dominant integral weights of $\mathfrak{g}$, and by $\omega_1, \dots, \omega_l \in P_+$ the fundamental weights of $\mathfrak{g}$. We will write $N(k, \mu) = N(k, V(\mu))$. Denote by $J(k, \mu)$ the maximal proper submodule of $N(k, \mu)$ and $L(k, \mu) = N(k, \mu)/J(k, \mu)$. We define $\Lambda_0 \in \hat{\mathfrak{h}}^*$ by $\Lambda_0(K) = 1$ and $\Lambda_0(h) = 0$ for any $h \in \mathfrak{h}$. Then $N(k, \mu)$ is a highest weight module with highest weight $k\Lambda_0 + \mu$, and a quotient of the Verma module $M(k\Lambda_0 + \mu)$. We also obtain $L(k, \mu) \cong L(k\Lambda_0 + \mu)$.

1.3. Admissible weights

Let $\widehat{\Delta}^{\vee,\mathrm{re}}$ (respectively, $\widehat{\Delta}_+^{\vee,\mathrm{re}}$) be the set of real (respectively, positive real) coroots of $\hat{\mathfrak{g}}$, and $\widehat{\Pi}^\vee$ the set of simple coroots. For $\lambda \in \hat{\mathfrak{h}}^*$, we define

$$\widehat{\Delta}_\lambda^{\vee,\mathrm{re}} = \left\{\alpha^\vee \in \widehat{\Delta}^{\vee,\mathrm{re}} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\right\}, \quad \text{and} \quad \widehat{\Delta}_{\lambda,+}^{\vee,\mathrm{re}} = \widehat{\Delta}_\lambda^{\vee,\mathrm{re}} \cap \widehat{\Delta}_+^{\vee,\mathrm{re}},$$

and we set

$$\widehat{\Pi}_\lambda^\vee = \left\{\alpha^\vee \in \widehat{\Delta}_{\lambda,+}^{\vee,\mathrm{re}} \mid \alpha^\vee \text{ is not decomposable into a sum of elements from } \widehat{\Delta}_{\lambda,+}^{\vee,\mathrm{re}}\right\}.$$

Let $\widehat{W}$ denote the Weyl group of $\hat{\mathfrak{g}}$. For each $\alpha \in \widehat{\Delta}^{\mathrm{re}}$, we have a reflection $r_\alpha \in \widehat{W}$. Define $\rho \in \hat{\mathfrak{h}}^*$ in the usual way, and we recall the shifted action of an element $w \in \widehat{W}$ on $\hat{\mathfrak{h}}^*$, given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

A weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if

$$\langle \lambda + \rho, \alpha^\vee \rangle \notin -\mathbb{Z}_+ \quad \text{for all } \alpha^\vee \in \widehat{\Delta}_+^{\vee,\mathrm{re}} \quad \text{and} \quad \mathbb{Q}\widehat{\Delta}_\lambda^{\vee,\mathrm{re}} = \mathbb{Q}\widehat{\Pi}^\vee.$$

The irreducible $\hat{\mathfrak{g}}$-module $L(\lambda)$ is called *admissible* if the weight $\lambda \in \hat{\mathfrak{h}}^*$ is admissible. Given a $\hat{\mathfrak{g}}$-module M from the category $\mathcal{O}$, we call a weight vector $v \in M$ a *singular vector* if $\hat{\mathfrak{n}}_+.v = 0$.

Proposition 1.3. *(See [7].) Let λ be an admissible weight. Then*

$$L(\lambda) = M(\lambda) \Big/ \bigg(\sum_{\alpha^\vee \in \widehat{\Pi}_\lambda^\vee} \mathcal{U}(\hat{\mathfrak{g}}) v_\alpha \bigg),$$

where $v_\alpha \in M(\lambda)$ is a singular vector of weight $r_\alpha \cdot \lambda$.

Proposition 1.4. *(See [8].) Let M be a $\hat{\mathfrak{g}}$-module from the category $\mathcal{O}$. If every irreducible subquotient $L(\nu)$ of M is admissible, then M is completely reducible.*

1.4. $N(k, 0)$ and $L(k, 0)$ as VOA's

We identify the one-dimensional trivial $\mathfrak{g}$-module $V(0)$ with $\mathbb{C}$. Write $\mathbf{1} = 1 \otimes 1 \in N(k, 0)$. The $\hat{\mathfrak{g}}$-module $N(k, 0)$ is spanned by the elements of the form

$$a_1(-n_1 - 1) \cdots a_m(-n_m - 1)\mathbf{1},$$

where $a_1, \ldots, a_m \in \mathfrak{g}$ and $n_1, \ldots, n_m \in \mathbb{Z}_+$, with $a(n)$ denoting the element $a \otimes t^n$ for $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

The vector space $N(k, 0)$ admits a VOA structure, which we now describe. The vertex operator map $Y(\cdot, x) : N(k, 0) \to \mathrm{End}(N(k, 0))[[x, x^{-1}]]$ is uniquely determined by defining $Y(\mathbf{1}, x)$ to be the identity operator on $N(k, 0)$ and

$$Y\big(a(-1)\mathbf{1}, x\big) = \sum_{n \in \mathbb{Z}} a(n)x^{-n-1} \quad \text{for } a \in \mathfrak{g}.$$

In the case that $k \neq -h^\vee$, the module $N(k, 0)$ has a conformal vector

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} \big(a^i(-1)\big)^2 \mathbf{1},$$

where $\{a^i\}_{i=1,\ldots,\dim \mathfrak{g}}$ is an arbitrary orthonormal basis of $\mathfrak{g}$ with respect to the normalized Killing form $(\cdot, \cdot)$. Then it is well known that the quadruple $(N(k, 0), Y, \mathbf{1}, \omega)$ defined above is a vertex operator algebra.

Proposition 1.5. *(See [5].) The associative algebra $A(N(k, 0))$ is canonically isomorphic to $\mathcal{U}(\mathfrak{g})$. The isomorphism is given by $F : A(N(k, 0)) \to \mathcal{U}(\mathfrak{g})$,*

$$F\big(\big[a_1(-n_1 - 1) \cdots a_m(-n_m - 1)\mathbf{1}\big]\big) = (-1)^{n_1 + \cdots + n_m} a_1 \cdots a_m,$$

for $a_1, \ldots, a_m \in \mathfrak{g}$ and $n_1, \ldots, n_m \in \mathbb{Z}_+$.

Since every $\hat{\mathfrak{g}}$-submodule of $N(k, 0)$ is also an ideal in the VOA $N(k, 0)$, the module $L(k, 0)$ is a VOA for any $k \neq -h^\vee$.

Proposition 1.6. *(See [14].) Assume that the maximal $\hat{\mathfrak{g}}$-submodule of $N(k, 0)$ is generated by a singular vector v_0. Then we have*

$$A\big(L(k, 0)\big) \cong \mathcal{U}(\mathfrak{g}) / \big\langle F\big([v_0]\big)\big\rangle,$$

where $\langle F([v_0])\rangle$ is the two-sided ideal of $\mathcal{U}(\mathfrak{g})$ generated by $F([v_0])$. In particular, a $\mathfrak{g}$-module U is an $A(L(k, 0))$-module if and only if $F([v_0])U = 0$.

2. Affine Lie algebra of type $G_2^{(1)}$

2.1. Admissible weights

Let

$$\Delta = \left\{ \begin{array}{lll} \pm\frac{1}{\sqrt{3}}(\epsilon_1 - \epsilon_2), & \pm\frac{1}{\sqrt{3}}(\epsilon_1 - \epsilon_3), & \pm\frac{1}{\sqrt{3}}(\epsilon_2 - \epsilon_3), \\ \pm\frac{1}{\sqrt{3}}(2\epsilon_1 - \epsilon_2 - \epsilon_3), & \pm\frac{1}{\sqrt{3}}(2\epsilon_2 - \epsilon_1 - \epsilon_3), & \pm\frac{1}{\sqrt{3}}(2\epsilon_3 - \epsilon_1 - \epsilon_2) \end{array} \right\}$$

be the root system of type G_2. We fix the set of positive roots

$$\Delta_+ = \left\{ \begin{array}{lll} \frac{1}{\sqrt{3}}(\epsilon_1 - \epsilon_2), & \frac{1}{\sqrt{3}}(\epsilon_3 - \epsilon_1), & \frac{1}{\sqrt{3}}(\epsilon_3 - \epsilon_2), \\ \frac{1}{\sqrt{3}}(-2\epsilon_1 + \epsilon_2 + \epsilon_3), & \frac{1}{\sqrt{3}}(-2\epsilon_2 + \epsilon_1 + \epsilon_3), & \frac{1}{\sqrt{3}}(2\epsilon_3 - \epsilon_1 - \epsilon_2) \end{array} \right\}.$$

Then the simple roots are $\alpha = \frac{1}{\sqrt{3}}(\epsilon_1 - \epsilon_2)$ and $\beta = \frac{1}{\sqrt{3}}(-2\epsilon_1 + \epsilon_2 + \epsilon_3)$, and the highest root is $\theta = \frac{1}{\sqrt{3}}(2\epsilon_3 - \epsilon_1 - \epsilon_2) = 3\alpha + 2\beta$. Let $\mathfrak{g}$ be the simple Lie algebra over $\mathbb{C}$, associated with the root system of type G_2. Let $E_{10}, E_{01}, F_{10}, F_{01}, H_{10}, H_{01}$ be Chevalley generators of $\mathfrak{g}$, where E_{10} is a root vector for α, E_{01} is a root vector for β, and so on. We fix the root vectors:

$$E_{11} = [E_{10}, E_{01}],$$

$$E_{21} = \frac{1}{2}[E_{11}, E_{10}] = \frac{1}{2}\big[[E_{10}, E_{01}], E_{10}\big],$$

$$E_{31} = \frac{1}{3}[E_{21}, E_{10}] = \frac{1}{6}\big[[[E_{10}, E_{01}], E_{10}], E_{10}\big],$$

$$E_{32} = [E_{31}, E_{01}] = \frac{1}{6}\big[[[[E_{10}, E_{01}], E_{10}], E_{10}], E_{01}\big],$$

$$F_{11} = [F_{01}, F_{10}],$$

$$F_{21} = \frac{1}{2}[F_{10}, F_{11}] = \frac{1}{2}\big[F_{10}, [F_{01}, F_{10}]\big],$$

$$F_{31} = \frac{1}{3}[F_{10}, F_{21}] = \frac{1}{6}\big[F_{10}, [F_{10}, [F_{01}, F_{10}]]\big],$$

$$F_{32} = [F_{01}, F_{31}] = \frac{1}{6}\big[F_{01}, [F_{10}, [F_{10}, [F_{01}, F_{10}]]]\big]. \tag{2.1}$$

We set $H_{ij} = [E_{ij}, F_{ij}]$ for any positive root $i\alpha + j\beta \in \Delta_+$. Then one can check that H_{ij} is the coroot corresponding to $i\alpha + j\beta$, i.e. $H_{ij} = (i\alpha + j\beta)^\vee$. For a complete multiplication table, we refer the reader to Table 22.1 in [6, p. 346], where we have

$$X_1 = E_{10}, \quad X_2 = E_{01}, \quad X_3 = E_{11}, \quad X_4 = -E_{21}, \quad X_5 = -E_{31}, \quad X_6 = -E_{32},$$

$$Y_1 = F_{10}, \quad Y_2 = F_{01}, \quad Y_3 = F_{11}, \quad Y_4 = -F_{21}, \quad Y_5 = -F_{31}, \quad Y_6 = -F_{32}.$$

All admissible weights for arbitrary affine Lie algebras have been completely classified in [8]. The next proposition provides a description of the "vacuum" admissible weights for $G_2^{(1)}$ at one-third levels. This is a special case of Proposition 1.2 in [9]. We provide a proof for completeness.

Lemma 2.2. *The weight* $\lambda_{3n+i} = (n - 2 + \frac{i}{3})\Lambda_0$ *is admissible for* $n \in \mathbb{Z}_+$, $i = 1, 2$, *and we have*

$$\widehat{\Pi}^{\vee}_{\lambda_{3n+i}} = \left\{ (\delta - (2\alpha + \beta))^{\vee}, \alpha^{\vee}, \beta^{\vee} \right\},$$

where δ *is the canonical imaginary root. Furthermore,*

$$\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle = 1 \quad \text{for } \gamma = \alpha, \beta;$$

$$\langle \lambda_{3n+i} + \rho, (\delta - (2\alpha + \beta))^{\vee} \rangle = 3n + i + 1 \quad \text{for } i = 1, 2.$$

Proof. We have to show

$$\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle \notin -\mathbb{Z}_+ \quad \text{for any } \gamma \in \widehat{\Delta}^{\text{re}}_+$$

and

$$\mathbb{Q}\widehat{\Delta}^{\vee,\text{re}}_{\lambda_{3n+i}} = \mathbb{Q}\widehat{\Pi}^{\vee}.$$

Any positive real root $\gamma \in \widehat{\Delta}^{\text{re}}_+$ of $\hat{\mathfrak{g}}$ is of the form $\gamma = \bar{\gamma} + m\delta$, for $m > 0$ and $\bar{\gamma} \in \Delta$, or $m = 0$ and $\bar{\gamma} \in \Delta_+$. Denote by $\bar{\rho}$ the sum of fundamental weights of $\mathfrak{g}$. Then we can choose $\rho = h^{\vee}\Lambda_0 + \bar{\rho} = 4\Lambda_0 + \bar{\rho}$.

We have

$$\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle = \left\langle \left(n + 2 + \frac{i}{3} \right)\Lambda_0 + \bar{\rho}, (\bar{\gamma} + m\delta)^{\vee} \right\rangle$$

$$= \frac{2}{(\bar{\gamma}, \bar{\gamma})} \left(m\left(n + 2 + \frac{i}{3} \right) + (\bar{\rho}, \bar{\gamma}) \right).$$

If $m = 0$, then it is trivial that $\langle \lambda_{3n+i}, \gamma^{\vee} \rangle \notin -\mathbb{Z}_+$. Suppose that $m \geqslant 1$. If $(\bar{\gamma}, \bar{\gamma}) = 2$ and $m \not\equiv 0$ (mod 3), then $\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle \notin -\mathbb{Z}_+$. If $(\bar{\gamma}, \bar{\gamma}) = 2$, and $m \equiv 0$ (mod 3), then $m \geqslant 3$, and since $(\bar{\rho}, \bar{\gamma}) \geqslant -3$ for any $\bar{\gamma} \in \Delta$, we have

$$\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle = m\left(n + 2 + \frac{i}{3} \right) + (\bar{\rho}, \bar{\gamma}) \geqslant 3\left(n + 2 + \frac{1}{3} \right) - 3 = 3n + 4 \geqslant 4,$$

which implies $\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle \notin -\mathbb{Z}_+$. If $(\bar{\gamma}, \bar{\gamma}) = \frac{2}{3}$, then $(\bar{\rho}, \bar{\gamma}) \geqslant -\frac{5}{3}$. We have

$$\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle = 3\left(m\left(n + 2 + \frac{i}{3} \right) + (\bar{\rho}, \bar{\gamma}) \right) \geqslant 3\left(n + \frac{7}{3} + (\bar{\rho}, \bar{\gamma}) \right) \geqslant 3\left(n + \frac{7}{3} - \frac{5}{3} \right)$$

$$= 3n + 2 \geqslant 2,$$

which implies $\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle \notin -\mathbb{Z}_+$. Thus, $\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle \notin -\mathbb{Z}_+$ for any $\gamma \in \widehat{\Delta}^{\text{re}}_+$.

One can easily see that

$$\widehat{\Delta}^{\vee,\text{re}}_{\lambda_{3n+i},+} = \left\{ m\delta + \bar{\gamma} \mid m > 0, \ m \equiv 0 \ (\text{mod } 3), \ (\bar{\gamma}, \bar{\gamma}) = 2 \right\}$$

$$\cup \left\{ m\delta + \bar{\gamma} \mid m > 0, \ (\bar{\gamma}, \bar{\gamma}) = 2/3 \right\} \cup \Delta_+.$$

Then we obtain

$$\widehat{\Pi}^{\vee}_{\lambda_{3n+i}} = \left\{ \left(\delta - (2\alpha + \beta)\right)^{\vee}, \alpha^{\vee}, \beta^{\vee} \right\},$$

and we see that $\mathbb{Q}\widehat{\Delta}^{\vee,\mathrm{re}}_{\lambda_{3n+i}} = \mathbb{Q}\widehat{\Pi}^{\vee}_{\lambda_{3n+i}} = \mathbb{Q}\widehat{\Pi}^{\vee}$. Through direct calculations, we get

$$\langle \lambda_{3n+i} + \rho, \gamma^{\vee} \rangle = 1 \quad \text{for } \gamma = \alpha, \beta, \quad \text{and}$$

$$\langle \lambda_{3n+i} + \rho, \left(\delta - (2\alpha + \beta)\right)^{\vee} \rangle = 3n + i + 1. \qquad \square$$

2.2. Singular vectors

In what follows, let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $G_2^{(1)}$ and $\mathcal{U}(\hat{\mathfrak{g}})$ its universal enveloping algebra.

We write $X^i(-m) = X(-m)^i$ for elements in $\mathcal{U}(\hat{\mathfrak{g}})$. We set

$$a = E_{21}(-1),$$

$$b = E_{31}(-1)E_{11}(-1) - E_{32}(-1)E_{10}(-1),$$

$$c = E_{31}^2(-1)E_{01}(-1) - E_{32}(-1)E_{31}(-1)H_{01}(-1) - E_{32}^2(-1)F_{01}(-1),$$

$$w = E_{31}(-1)E_{32}(-2) - E_{32}(-1)E_{31}(-2),$$

and define

$$u = \frac{1}{3}a^2 - b, \quad \text{and} \quad v = \frac{2}{9}a^3 - ab - 3c.$$

The following proposition determines singular vectors for the first three admissible weights, i.e. $-\frac{5}{3}\Lambda_0, -\frac{4}{3}\Lambda_0, -\frac{2}{3}\Lambda_0$, respectively.

Proposition 2.3. *The vector $v_k \in N(k, 0)$ is a singular vector for the given value of k:*

$$v_k = \begin{cases} u.\mathbf{1} & \text{for } k = -\frac{5}{3}, \\ (v + w).\mathbf{1} & \text{for } k = -\frac{4}{3}, \\ u(v - w).\mathbf{1} & \text{for } k = -\frac{2}{3}. \end{cases}$$

The proof will be given in Appendix A. As one can see in the proof, the computational difficulty increases as the level k goes up. A different approach will be used in a subsequent work of the first-named author on higher levels.

2.3. Description of Zhu's algebra

Proposition 2.4. *The maximal $\hat{\mathfrak{g}}$-submodule $J(k, 0)$ of $N(k, 0)$ is generated by the vector v_k for $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$, respectively, where v_k's are given in Proposition 2.3.*

Proof. Let $\lambda_{3n+i} = (-2 + n + \frac{i}{3})\Lambda_0 = k\Lambda_0$ as before. It follows from Proposition 1.3 and Lemma 2.2 that the maximal submodule of the Verma module $M(\lambda_{3n+i})$ is generated by three singular vectors with weights

$$r_{\delta-(2\alpha+\beta)} \cdot \lambda_{3n+i}, \qquad r_\alpha \cdot \lambda_{3n+i}, \quad r_\beta \cdot \lambda_{3n+i}, \quad \text{respectively.}$$

We consider the three cases

$$n = 0,\ i = 1,\ k = -5/3; \qquad n = 0,\ i = 2,\ k = -4/3; \qquad n = 1,\ i = 1,\ k = -2/3.$$

In each case, there is a singular vector $u_k \in M(\lambda_{3n+i})$ of weight $r_{\delta-(2\alpha+\beta)}.\lambda_{3n+i}$, whose image under the projection of $M(\lambda_{3n+i})$ onto $N(k, 0)$ is the singular vector v_k given in Proposition 2.3.

The other singular vectors have weights

$$r_\alpha \cdot \lambda_{3n+i} = \lambda_{3n+i} - \langle \lambda_{3n+i} + \rho, \alpha^\vee \rangle \alpha = \lambda_{3n+i} - \alpha, \quad \text{and}$$

$$r_\beta \cdot \lambda_{3n+i} = \lambda_{3n+i} - \langle \lambda_{3n+i} + \rho, \beta^\vee \rangle \beta = \lambda_{3n+i} - \beta,$$

so the images of these vectors under the projection of $M(\lambda_{3n+i})$ onto $N(k, 0)$ are 0 from the definition. Therefore the maximal submodule of $N(k, 0)$ is generated by the singular vector v_k, i.e. $J(k, 0) = \mathcal{U}(\hat{\mathfrak{g}})v_k$. □

Now we consider the image of a singular vector v_k under Zhu's map

$$[\cdot]: N(k, 0) \to A\big(N(k, 0)\big) \cong \mathcal{U}(\mathfrak{g}),$$

which is defined in Section 1. We recall that the vertex algebra $N(k, 0)$ is (linearly) isomorphic to the associative algebra $\mathcal{U}(\hat{\mathfrak{g}}_-)$. We thus have an induced map from $\mathcal{U}(\hat{\mathfrak{g}}_-)$ to $\mathcal{U}(\mathfrak{g})$ and a commutative diagram of linear maps:

$$
\begin{array}{ccc}
\mathcal{U}(\hat{\mathfrak{g}}_-) & \simeq & N(k, 0) \\
\downarrow & & \downarrow \\
\mathcal{U}(\mathfrak{g}) & \simeq & A\big(N(k, 0)\big).
\end{array}
$$

We will identify $N(k, 0)$ with $\mathcal{U}(\hat{\mathfrak{g}}_-)$ and $A(N(k, 0))$ with $\mathcal{U}(\mathfrak{g})$. We have:

$$[a] = E_{21},$$

$$[b] = E_{31}E_{11} - E_{32}E_{10},$$

$$[c] = E_{31}^2 E_{01} - E_{32}E_{31}H_{01} - E_{32}^2 F_{01}.$$

We also have:

$$[u] = \frac{1}{3}[a]^2 - [b],$$

$$[v] = \frac{2}{9}[a]^3 - [a][b] - 3[c],$$

$$[w] = 0,$$

$$\big[u(v - w)\big] = [u][v] = \frac{2}{27}[a]^5 - \frac{5}{9}[a]^3[b] - [a]^2[c] + [a][b]^2 + 3[b][c]. \tag{2.5}$$

The following theorem is now a consequence of Propositions 1.6 and 2.4.

Theorem 2.6. *The associative algebra $A(L(k,0))$ is isomorphic to $\mathcal{U}(\mathfrak{g})/I_k$, where I_k is the two-sided ideal of $\mathcal{U}(\mathfrak{g})$ generated by the vector $[v_k]$, where*

$$
[v_k] = \begin{cases} [u] & \text{for } k = -\frac{5}{3}, \\ [v] & \text{for } k = -\frac{4}{3}, \\ [uv] & \text{for } k = -\frac{2}{3}. \end{cases}
$$

3. Irreducible modules

In this section we adopt the method from [1,2,13–15] in oder to classify irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations.

3.1. Modules for associative algebra $A(L(k,0))$

Denote by $_L$ the adjoint action of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{U}(\mathfrak{g})$ defined by $X_L f = [X, f]$ for $X \in \mathfrak{g}$ and $f \in \mathcal{U}(\mathfrak{g})$. We also write $(ad\,X)f = X_L f = [X, f]$. Then $ad\,X$ is a derivation on $\mathcal{U}(\mathfrak{g})$. Let $R(k)$ be a $\mathcal{U}(\mathfrak{g})$-submodule of $\mathcal{U}(\mathfrak{g})$ generated by the vector $[v_k]$, where $[v_k]$ is given in Theorem 2.6. It is straightforward to see that $R(k)$ is an irreducible finite-dimensional $\mathcal{U}(\mathfrak{g})$-module isomorphic to $V((3k + 7)(2\alpha + \beta))$. Let $R(k)_0$ be the zero-weight subspace of $R(k)$.

Proposition 3.1. *(See [1,2].) Let $V(\mu)$ be an irreducible highest weight $\mathcal{U}(\mathfrak{g})$-module with highest weight vector v_μ for $\mu \in \mathfrak{h}^*$. Then the following statements are equivalent*:

(1) $V(\mu)$ *is an $A(L(k,0))$-module,*
(2) $R(k) \cdot V(\mu) = 0$,
(3) $R(k)_0 \cdot v_\mu = 0$.

Let $r \in R(k)_0$. Then there exists a unique polynomial $p_r \in S(\mathfrak{h})$, where $S(\mathfrak{h})$ is the symmetric algebra of $\mathfrak{h}$, such that

$$
r \cdot v_\mu = p_r(\mu)v_\mu.
$$

Set $\mathcal{P}(k)_0 = \{p_r \mid r \in \mathcal{R}(k)_0\}$. Then we have:

Corollary 3.2. *There is a bijective correspondence between*

(1) *the set of irreducible $A(L(k,0))$-modules $V(\mu)$ from the category $\mathcal{O}$, and*
(2) *the set of weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathcal{P}(k)_0$.*

3.2. Polynomials in $\mathcal{P}(k)_0$

We now determine some polynomials in the set $\mathcal{P}(k)_0$ for the cases $k = -\frac{5}{3}$, $k = -\frac{4}{3}$, $k = -\frac{2}{3}$, respectively. We will use some computational lemmas which we collect and prove in Appendix B.

Lemma 3.3 *(Case: $k = -\frac{5}{3}$). We let*

(1) $q(H) = H_{21}(H_{21} + 2)$,
(2) $p_1(H) = H_{10}(H_{10} - 1)$, *and*
(3) $p_2(H) = \frac{1}{3}H_{11}(H_{11} - 1) + 3H_{01}$.

Then $q(H), p_1(H), p_2(H) \in \mathcal{P}(-\frac{5}{3})_0$.

Proof. (1) We show that $(E_{21}^2 F_{21}^4)_L[u] \equiv Cq(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$ for some $C \neq 0$. Using Lemma B.12 and Lemma B.13, we have

$$\left(E_{21}^2 F_{21}^4\right)_L[u] = \left(E_{21}^2 F_{21}^4\right)_L\left(\frac{1}{3}[a]^2 - [b]\right)$$

$$\equiv 4!2!\left(\frac{1}{3}H_{21}(H_{21} - 1) + H_{21}\right)$$

$$\equiv 4!2!\frac{1}{3}H_{21}(H_{21} + 2) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+},$$

which is what we wanted to show.

(2) We will show that $(E_{10}^2 F_{31}^2)_L[u] \equiv Cp_1(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$ for some $C \neq 0$. We again use Lemma B.12 and Lemma B.13 to obtain:

$$\left(E_{10}^2 F_{31}^2\right)_L\left(\frac{1}{3}[a]^3 - [b]\right) \equiv (2!)^2\frac{1}{3}H_{10}(H_{10} - 1) \equiv \frac{4}{3}p_1(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}.$$

(3) In this case we show that $(E_{11}^2 F_{32}^2)_L[u] \equiv Cp_2(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$ for some $C \neq 0$. Similarly to the first two cases we compute:

$$\left(E_{11}^2 F_{32}^2\right)_L\left(\frac{1}{3}[a]^2 - [b]\right) \equiv (2!)^2\left(\frac{1}{3}H_{11}(H_{11} - 1) + 3H_{01}\right)$$

$$\equiv Cp_2(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}. \qquad \square$$

We now give polynomials for the next case.

Lemma 3.4 (Case: $k = -\frac{4}{3}$). Let

(1) $q(H) = \frac{2}{9}H_{21}(H_{21} - 1)(H_{21} - 2) + H_{21}(H_{21} - 2) + 3H_{01}(H_{01} + 2)$,
(2) $p_1(H) = H_{10}(H_{10} - 1)(H_{10} - 2)$,
(3) $p_2(H) = \frac{2}{9}H_{11}(H_{11} - 1)(H_{11} - 2) + 6H_{01}H_{32}$.

Then $p_1(H), p_2(H), q(H) \in \mathcal{P}(-\frac{4}{3})_0$.

Proof. (1) We show that $(E_{21}^3 F_{21}^6)_L[v] \equiv Cq(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$ for some constant $C \neq 0$. By Lemma B.13, we have:

$$\left(E_{21}^3 F_{21}^6\right)_L[v] = \left(E_{21}^3 F_{21}^6\right)_L\left(\frac{2}{9}[a]^3 - [a][b] - 3[c]\right)$$

$$\equiv -3!6!\frac{2}{9}H_{21}(H_{21} - 1)(H_{21} - 2) - \frac{3!}{2!}\frac{6!}{4!}(H_{21} - 2)\left(E_{21}^2 F_{21}^4\right)_L[b]$$

$$- 3\left(E_{21}^3 F_{21}^6\right)_L[c] \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}.$$

By Lemma B.12, we thus have:

$$\left(E_{21}^3 F_{21}^6\right)_L[v] \equiv -3!6!\frac{2}{9}H_{21}(H_{21} - 1)(H_{21} - 2) + 3!6!(H_{21} - 2)H_{21} + 3!6!H_{01}(H_{01} + 2)$$

$$\equiv Cq(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}.$$

(2) We will show that $(E_{10}^3 F_{31}^3)_L[v] \equiv C p_1(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$ for some constant $C \neq 0$. Using Lemma B.13, we obtain:

$$(E_{10}^3 F_{31}^3)_L \left(\frac{2}{9}[a]^3 - [a][b] - 3[c] \right)$$

$$\equiv \frac{2}{9}(3!)^2 H_{10}(H_{10} - 1)(H_{10} - 2) + \frac{3!}{2!}\frac{3!}{2!}(H_{10} - 2)(E_{10}^2 F_{31}^2)_L[b]$$

$$- 3(E_{10}^3 F_{31}^3)_L[c] \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}.$$

By Lemma B.12, we thus have

$$(E_{10}^3 F_{31}^3)_L \left(\frac{2}{9}[a]^3 - [a][b] - 3[c] \right) \equiv \frac{2}{9}(3!)^2 H_{10}(H_{10} - 1)(H_{10} - 2)$$

$$\equiv C p_1(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}.$$

(3) Finally, we show that $(E_{11}^3 F_{32}^3)_L[v] \equiv C p_2(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$ for some constant $C \neq 0$. Since $H_{11} + H_{31} = 2H_{32}$, we have

$$(E_{11}^3 F_{32}^3)_L v' \equiv \frac{2}{9}(3!)^2 H_{11}(H_{11} - 1)(H_{11} - 2) - \frac{3!}{2!}\frac{3!}{2!}(H_{11} - 2)(E_{11}^2 F_{32}^2)_L[b] - 3(E_{11}^3 F_{32}^3)_L[c]$$

$$\equiv (3!)^2 \left(\frac{2}{9}H_{11}(H_{11} - 1)(H_{11} - 2) + 3(H_{11} - 2)H_{01} + 3H_{01}(H_{31} + 2) \right)$$

$$\equiv (3!)^2 \left(\frac{2}{9}H_{11}(H_{11} - 1)(H_{11} - 2) + 6H_{01}H_{32} \right)$$

$$\equiv C p_2(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}. \qquad \square$$

The last case is presented below.

Lemma 3.5 *(Case: $k = -\frac{2}{3}$). We let*

$$q(H) = \frac{2}{27}H_{21}(H_{21} - 1)(H_{21} - 2)(H_{21} - 3)(H_{21} - 4) + \frac{5}{9}H_{21}(H_{21} - 2)(H_{21} - 3)(H_{21} - 4)$$

$$+ (H_{21} - 3)(H_{21} - 4)H_{01}(H_{01} + 2) + 2H_{21}(H_{21} - 4)(H_{11} - 1)$$

$$+ 2(H_{21} - 4)H_{10}(H_{10} - 1) - 6(H_{21} - 4)H_{01}(H_{01} + 1) + 6(H_{21} - 3)H_{01}(H_{01} + 2),$$

$$p_1(H) = H_{10}(H_{10} - 1)(H_{10} - 2)(H_{10} - 3)(H_{10} - 4),$$

$$p_2(H) = \frac{2}{27}H_{11}(H_{11} - 1)(H_{11} - 2)(H_{11} - 3)(H_{11} - 4) + \frac{5}{3}(H_{11} - 2)(H_{11} - 3)(H_{11} - 4)H_{01}$$

$$+ (H_{11} - 3)(H_{11} - 4)H_{01}(H_{11} - 3) + 18(H_{11} - 4)H_{01}(H_{01} - 1)$$

$$- 2(H_{11} - 3)(H_{11} - 4)H_{01} + 18H_{01}(H_{01} - 1)(H_{31} + 2).$$

Then $p_1(H), p_2(H), q(H) \in \mathcal{P}(-\frac{2}{3})_0$.

Proof. First recall from (2.5) that

$$\big[u(v-w)\big] = [u][v] = \frac{2}{27}[a]^5 - \frac{5}{9}[a]^3[b] - [a]^2[c] + [a][b]^2 + 3[b][c].$$

We will show that $(E_{21}^5 F_{21}^{10})_L([u][v]) \equiv -5!10!q(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}$.

Using Lemmas B.1, B.11, we have:

$$
\begin{aligned}
(F_{21}^{10})_L([u][v]) &= (F_{21}^{10})_L\left(\frac{2}{27}[a]^5 - \frac{5}{9}[a]^3[b] - [a]^2[c] + [a][b]^2 + 3[b][c]\right) \\
&= \frac{2}{27}\frac{10!}{(2!)^5}(-2)^5 F_{21}^5 - \frac{5}{9}\frac{10!}{(2!)^3 4!}(-2)^3 F_{21}^3 (F_{21}^4)_L[b] \\
&\quad - \frac{10!}{(2!)^2 6!}(-2)^2 F_{21}^2 (F_{21}^6)_L[c] \\
&\quad + \frac{10!}{2!8!}(-2)F_{21}(F_{21}^8)_L[b]^2 + 3(F_{21}^{10})[b][c] \\
&= -\frac{2}{27}10!F_{21}^5 + \frac{5}{9}\frac{10!}{4!}F_{21}^3 (F_{21}^4)_L[b] - \frac{10!}{6!}F_{21}^2 (F_{21}^6)_L[c] \\
&\quad - \frac{10!}{8!}F_{21}(F_{21}^8)_L[b]^2 + 3(F_{21}^{10})[b][c].
\end{aligned}
$$

Now using Lemma B.3, we obtain:

$$
\begin{aligned}
\frac{1}{10!}(E_{21}^5 F_{21}^{10})_L([u][v]) &= -\frac{2}{27}5!H_{21}(H_{21}-1)(H_{21}-2)(H_{21}-3)(H_{21}-4) \\
&\quad + \frac{5}{9}\frac{5!}{2!}(H_{21}-2)(H_{21}-3)(H_{21}-4)\frac{1}{4!}(E_{21}^2 F_{21}^4)_L[b] \\
&\quad - \frac{5!}{3!}(H_{21}-3)(H_{21}-4)\frac{1}{6!}(E_{21}^3 F_{21}^6)_L[c] \\
&\quad - \frac{5!}{4!}(H_{21}-4)\frac{1}{8!}(E_{21}^4 F_{21}^8)_L[b]^2 + 3\frac{1}{10!}(E_{21}^5 F_{21}^{10})_L([b][c]).
\end{aligned}
$$

Combining this with Lemmas B.12, B.13, B.14, we obtain:

$$
\begin{aligned}
\frac{1}{10!}(E_{21}^5 F_{21}^{10})_L([u][v]) &\equiv -\frac{2}{27}5!H_{21}(H_{21}-1)(H_{21}-2)(H_{21}-3)(H_{21}-4) \\
&\quad + \frac{5}{9}\frac{5!}{2!}(H_{21}-2)(H_{21}-3)(H_{21}-4)(-2)H_{21} \\
&\quad - \frac{5!}{3!}(H_{21}-3)(H_{21}-4)3!H_{01}(H_{01}+2) \\
&\quad - \frac{5!}{4!}(H_{21}-4)4!\big(2H_{21}H_{11} + 2H_{10}(H_{10}-1) - 6H_{01}(H_{01}+1)\big) \\
&\quad + 3\,5!(-2)H_{01}(H_{01}+2)(H_{21}-3) \\
&\equiv -5!q(H) \pmod{\mathcal{U}(\mathfrak{g})\mathfrak{n}_+}.
\end{aligned}
$$

The proofs for $p_1(H)$ and $p_2(H)$ are similar, and we omit the details. □

3.3. Finiteness of the number of irreducible modules

We are now able to obtain the following result for the associative algebra $A(L(k,0))$. For convenience, if $\mu \in \mathfrak{h}^*$, we write $\mu_{ij} = \mu(H_{ij})$. We will identify $\mu \in \mathfrak{h}^*$ with the pair (μ_{10}, μ_{01}).

Proposition 3.6. *There are finitely many irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$ for each of $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$. Moreover, the possible highest weights $\mu = (\mu_{10}, \mu_{01})$ for irreducible $A(L(k,0))$-modules are as follows*:

(1) *if $k = -\frac{5}{3}$, then $\mu = (0,0)$, $(0, -\frac{2}{3})$ or $(1, -\frac{4}{3})$;*
(2) *if $k = -\frac{4}{3}$, then $\mu = (0,0)$, $(0, -\frac{2}{3})$, $(0, -\frac{1}{3})$, $(1,0)$, $(1, -\frac{4}{3})$ or $(2, -\frac{5}{3})$;*
(3) *if $k = -\frac{2}{3}$, then $\mu = (0,0)$, $(0, -\frac{2}{3})$, $(0, -\frac{1}{3})$, $(0, \frac{1}{3})$, $(0,1)$, $(1,0)$, $(1, -\frac{4}{3})$, $(1, -\frac{2}{3})$, $(2,0)$, $(2, -\frac{5}{3})$, $(2, -\frac{4}{3})$ or $(4, -\frac{7}{3})$.*

Proof. (1) It follows from Corollary 3.2 that highest weights $\mu \in \mathfrak{h}^*$ of irreducible $A(L(-\frac{5}{3}, 0))$-modules satisfy $p(\mu) = 0$ for all $p \in \mathcal{P}_0(-\frac{5}{3})$. Lemma 3.3 implies that $p_1(\mu) = p_2(\mu) = q(\mu) = 0$ for such weights μ. Let $\mu \in \mathfrak{h}^*$. The equation $p_1(\mu) = 0$ is

$$\mu_{10}(\mu_{10} - 1) = 0,$$

which implies $\mu_{10} = 0$ or 1.

First suppose $\mu_{10} = 0$. Then from $q(\mu) = 0$ we must have $\mu_{01} = 0$ or $-\frac{2}{3}$. Similarly, from $p_2(\mu) = 0$, we also get $\mu_{01} = 0$ or $-\frac{2}{3}$. So the weight μ must be of the form $\mu = (\mu_{10}, \mu_{01}) = (0, 0)$ or $(0, -\frac{2}{3})$ in this case. Now suppose $\mu_{10} = 1$. The equation $q(\mu) = 0$ gives $\mu_{01} = -\frac{2}{3}$ or $-\frac{4}{3}$, and the equation $p_2(\mu) = 0$ gives $\mu_{01} = 0$ or $-\frac{4}{3}$. So the only possibility is $\mu = (\mu_{10}, \mu_{01}) = (1, -\frac{4}{3})$. Altogether, this gives only three possible weights μ such that $p_1(\mu) = p_2(\mu) = q(\mu) = 0$:

$$\mu = (\mu_{10}, \mu_{01}) = (0, 0), \left(0, -\frac{2}{3}\right) \text{ or } \left(1, -\frac{4}{3}\right).$$

(2) Similarly to the part (1), we use the polynomials of Lemma 3.4. Using a computer algebra system, we calculate the common zeros of the polynomials $q(H), p_1(H), p_2(H)$ to obtain the following list of possible highest weights:

$$\mu = (\mu_{10}, \mu_{01}) = (0, 0), \left(0, -\frac{2}{3}\right), \left(0, -\frac{1}{3}\right), (1, 0), \left(1, -\frac{4}{3}\right) \text{ or } \left(2, -\frac{5}{3}\right).$$

(3) For this part, we use Lemma 3.5. Using a computer algebra system, we again compute the common zeros of the polynomials $q(H), p_1(H), p_2(H)$ to obtain the following list of possible highest weights:

$$\mu = (\mu_{10}, \mu_{01}) = (0, 0), \left(0, -\frac{2}{3}\right), \left(0, -\frac{1}{3}\right), \left(0, \frac{1}{3}\right), (0, 1), (1, 0), \left(1, -\frac{4}{3}\right), \left(1, -\frac{2}{3}\right),$$
$$(2, 0), \left(2, -\frac{5}{3}\right), \left(2, -\frac{4}{3}\right) \text{ or } \left(4, -\frac{7}{3}\right). \quad \square$$

Now we apply the $A(V)$-theory (Theorem 1.2), and obtain our main result in the following theorem.

Theorem 3.7. *There are finitely many irreducible weak modules from the category $\mathcal{O}$ for each of the following simple vertex operator algebras: $L(-\frac{5}{3}, 0)$, $L(-\frac{4}{3}, 0)$, $L(-\frac{2}{3}, 0)$.*

Remark 3.8. This theorem provides further evidence for the conjecture of Adamović and Milas in [2], mentioned in the introduction. Furthermore, if $L(\lambda)$ is an irreducible module of the VOA $L(k, 0)$, for $k = -\frac{5}{3}$, $-\frac{4}{3}$, or $-\frac{2}{3}$, then we recall from Section 1.2 that we must have $L(\lambda) \cong L(k\Lambda_0, \mu)$ for the values of $\mu \in \mathfrak{h}^*$ given in Proposition 3.6.

In the case of irreducible $L(k, 0)$-modules, we obtain a complete classification. We state this result in the following proposition and theorem.

Proposition 3.9. *The complete list of irreducible finite-dimensional $A(L(k, 0))$-modules $V(\mu)$ for each k is as follows:*

(1) *if $k = -\frac{5}{3}$, then $V(\mu) = V(0)$,*
(2) *if $k = -\frac{4}{3}$, then $V(\mu) = V(0)$ or $V(\omega_1)$,*
(3) *if $k = -\frac{2}{3}$, then $V(\mu) = V(0)$, $V(\omega_1)$, $V(\omega_2)$, or $V(2\omega_1)$,*

where ω_1, ω_2 are the fundamental weights of $\mathfrak{g}$.

Proof. Among the list of weights in Proposition 3.6, we need only to consider dominant integral weights, i.e. those weights $\mu = (m_1, m_2)$ with $m_1, m_2 \in \mathbb{Z}_+$. Notice that the weights of the singular vectors $[v_k]$ are $2\omega_1$, $3\omega_1$ and $5\omega_1$, respectively. Considering the set of weights of $V(\mu)$ listed above, we see that each singular vector $[v_k]$ annihilates the corresponding modules $V(\mu)$. Now the proposition follows from Proposition 1.6. □

We again apply the $A(V)$-theory (Theorem 1.2), and obtain the following theorem.

Theorem 3.10. *The complete list of irreducible $L(k, 0)$-modules $L(k, \mu)$ for each k is as follows:*

(1) *if $k = -\frac{5}{3}$, then $L(k, \mu) = L(k, 0)$,*
(2) *if $k = -\frac{4}{3}$, then $L(k, \mu) = L(k, 0)$ or $L(k, \omega_1)$,*
(3) *if $k = -\frac{2}{3}$, then $L(k, \mu) = L(k, 0)$, $L(k, \omega_1)$, $L(k, \omega_2)$, or $L(k, 2\omega_1)$.*

3.4. Semisimplicity of weak modules from the category $\mathcal{O}$

In this subsection we show that the category of weak $L(k, 0)$-modules from the category $\mathcal{O}$ is semisimple.

Lemma 3.11. *Assume that $\lambda = k\Lambda_0 + \mu$ for $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$, where $\mu \in \mathfrak{h}^*$ is one of the values given in Proposition* 3.6 *for each k. Then the weights λ are admissible.*

Proof. The proof is essentially the same as Lemma 2.2. Let us write $\widehat{\Pi}_0^\vee = \{(\delta - (2\alpha + \beta))^\vee, \alpha^\vee, \beta^\vee\}$, $\widehat{\Pi}_1^\vee = \{(\delta - (3\alpha + \beta))^\vee, \alpha^\vee, (\alpha + \beta)^\vee\}$, and $\widehat{\Pi}_2^\vee = \{(\delta - \theta)^\vee, \alpha^\vee, (\alpha + \beta)^\vee\}$. Since the proof for the other cases are similar, we consider only the case $k = -\frac{5}{3}$. From Lemma 2.2, we already know that $\lambda = -\frac{5}{3}\Lambda_0 + \mu$ is admissible for $\mu = (0, 0)$, with $\widehat{\Pi}_\lambda^\vee = \widehat{\Pi}_0^\vee$.

If $\mu = (0, -\frac{2}{3})$, we have to show that

$$\left\langle -\frac{5}{3}\Lambda_0 + \mu + \rho, \gamma^\vee \right\rangle \notin -\mathbb{Z}_+ \quad \text{for any } \gamma \in \widehat{\Delta}_+^{\mathrm{re}} \quad \text{and} \quad \mathbb{Q}\widehat{\Delta}_\lambda^{\vee,\mathrm{re}} = \mathbb{Q}\widehat{\Pi}^\vee.$$

Recall that $\rho = 4\Lambda_0 + \bar\rho$; also $\gamma \in \widehat\Delta_+^{\mathrm{re}}$ must have the form $\gamma = \bar\gamma + m\delta$, for $m > 0$ and $\bar\gamma \in \Delta$, or $m = 0$ and $\bar\gamma \in \Delta_+$. We then have:

$$\left\langle -\frac{5}{3}\Lambda_0 + \mu + \rho, \gamma^\vee \right\rangle = \left\langle \frac{7}{3}\Lambda_0 + \mu + \bar\rho, (\bar\gamma + m\delta)^\vee \right\rangle$$

$$= \frac{2}{(\bar\gamma, \bar\gamma)} \frac{7}{3} m + \langle \mu, \bar\gamma^\vee \rangle + \langle \bar\rho, \bar\gamma^\vee \rangle.$$

We may then check that $\langle -\frac{5}{3}\Lambda_0 + \mu + \rho, \gamma^\vee \rangle \geqslant \frac{1}{3}$, so that $\langle -\frac{5}{3}\Lambda_0 + \mu + \rho, \gamma^\vee \rangle \notin -\mathbb{Z}_+$. One may also verify that $\widehat\Pi_\lambda^\vee = \widehat\Pi_1^\vee$ so that $\mathbb{Q}\widehat\Delta_\lambda^{\vee,\mathrm{re}} = \mathbb{Q}\widehat\Pi^\vee$.

Similarly, one can show that $\lambda = -\frac{5}{3}\Lambda_0 + \mu$ is admissible for $\mu = (1, -\frac{4}{3})$ and that $\widehat\Pi_\lambda^\vee = \widehat\Pi_2^\vee$. $\quad\square$

Theorem 3.12. *Let M be a weak $L(k, 0)$-module from the category $\mathcal{O}$, for $k = -\frac{5}{3}, -\frac{4}{3}$, or $-\frac{2}{3}$. Then M is completely reducible.*

Proof. Let $L(\lambda)$ be an irreducible subquotient of M. Then $L(\lambda)$ is an $L(k, 0)$-module, and we see from Remark 3.8 that λ must be a weight of the form $k\Lambda_0 + \mu$, where μ is given in Proposition 3.6 for $k = -\frac{5}{3}, -\frac{4}{3}, -\frac{2}{3}$, respectively. From Lemma 3.11 it follows that such a λ is admissible. Now Proposition 1.4 implies that M is completely reducible. $\quad\square$

Acknowledgments

Appendix A. Proof of Proposition 2.3

In this appendix, we prove Proposition 2.3. We first give a few lemmas.

Lemma A.1.

(1) *We have*

$$\big[a, E_{10}(0)\big] = 3E_{31}(-1), \qquad \big[b, E_{10}(0)\big] = 2E_{31}(-1)E_{21}(-1),$$

$$\big[c, E_{10}(0)\big] = E_{32}(-1)E_{31}(-1)E_{10}(-1) - E_{31}^2(-1)E_{11}(-1),$$

$$\big[u, E_{10}(0)\big] = 0, \qquad \big[v, E_{10}(0)\big] = 0, \qquad \big[w, E_{10}(0)\big] = 0.$$

(2) *Each of the elements $a, b, c, u, v, w \in \mathcal{U}(\hat{\mathfrak{g}})$ commutes with $E_{01}(0)$.*

Proof. (1) Using the multiplication table in (2.1), it is easy to see $[a, E_{10}(0)] = 3E_{31}(-1)$. Next, we have

$$\big[b, E_{10}(0)\big] = \big[E_{31}(-1)E_{11}(-1) - E_{32}(-1)E_{10}(-1), E_{10}(0)\big]$$

$$= E_{31}(-1)\big[E_{11}(-1), E_{10}(0)\big] + \big[E_{31}(-1), E_{10}(0)\big]E_{11}(-1)$$

$$- E_{32}(-1)\big[E_{10}(-1), E_{10}(0)\big] - \big[E_{32}(-1), E_{10}(0)\big]E_{10}(-1)$$

$$= 2E_{31}(-1)E_{21}(-1).$$

Starting with the definition

$$\left[c, E_{10}(0)\right] = \left[E_{31}^2(-1)E_{01}(-1) - E_{32}(-1)E_{31}(-1)H_{01}(-1) - E_{32}^2(-1)F_{01}(-1), E_{10}(0)\right],$$

we consider each term separately and obtain

$$\begin{aligned}
&\left[E_{31}^2(-1)E_{01}(-1), E_{10}(0)\right] \\
&\quad = E_{31}^2(-1)\left[E_{01}(-1), E_{10}(0)\right] + E_{31}(-1)\left[E_{31}(-1), E_{10}(0)\right]E_{01}(-1) \\
&\qquad + \left[E_{31}(-1), E_{10}(0)\right]E_{31}(-1)E_{01}(-1) \\
&\quad = -E_{31}^2(-1)E_{11}(-1), \\
&\left[E_{32}(-1)E_{31}(-1)H_{01}(-1), E_{10}(0)\right] \\
&\quad = E_{32}(-1)E_{31}(-1)\left[H_{01}(-1), E_{10}(0)\right] + E_{32}(-1)\left[E_{31}(-1), E_{10}(0)\right]H_{01}(-1) \\
&\qquad + \left[E_{32}(-1), E_{10}(0)\right]E_{31}(-1)H_{01}(-1) \\
&\quad = -E_{32}(-1)E_{31}(-1)E_{10}(-1),
\end{aligned}$$

and

$$\begin{aligned}
&\left[E_{32}^2(-1)F_{01}(-1), E_{10}(0)\right] \\
&\quad = E_{32}^2(-1)\left[F_{01}(-1), E_{10}(0)\right] + E_{32}(-1)\left[E_{32}(-1), E_{10}(0)\right]F_{01}(-1) \\
&\qquad + \left[E_{32}(-1), E_{10}(0)\right]E_{32}(-1)F_{01}(-1) \\
&\quad = 0.
\end{aligned}$$

Therefore, we obtain

$$\left[c, E_{10}(0)\right] = E_{32}(-1)E_{31}(-1)E_{10}(-1) - E_{31}^2(-1)E_{11}(-1).$$

Next, we get

$$\begin{aligned}
\left[u, E_{10}(0)\right] &= \frac{1}{3}\left[a^2, E_{10}(0)\right] - \left[b, E_{10}(0)\right] \\
&= \frac{1}{3}a\left[a, E_{10}(0)\right] + \frac{1}{3}\left[a, E_{10}(0)\right]a - \left[b, E_{10}(0)\right] \\
&= E_{21}(-1)E_{31}(-1) + E_{31}(-1)E_{21}(-1) - 2E_{31}(-1)E_{21}(-1) \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\left[v, E_{10}(0)\right] &= \frac{2}{9}\left[a^3, E_{10}(0)\right] - \left[ab, E_{10}(0)\right] - 3\left[c, E_{10}(0)\right] \\
&= 2E_{21}^2(-1)E_{31}(-1) - a\left[b, E_{10}(0)\right] - \left[a, E_{10}(0)\right]b - 3\left[c, E_{10}(0)\right] \\
&= 2E_{21}^2(-1)E_{31}(-1) - 2E_{21}(-1)E_{31}(-1)E_{21}(-1)
\end{aligned}$$

$$- 3E_{31}(-1)\big\{E_{31}(-1)E_{11}(-1) - E_{32}(-1)E_{10}(-1)\big\}$$
$$- 3\big\{E_{32}(-1)E_{31}(-1)E_{10}(-1) - E_{31}^2(-1)E_{11}(-1)\big\} = 0.$$

Finally, it is easy to see $[w, E_{10}(0)] = 0$.

(2) The equalities $[a, E_{01}(0)] = 0$, $[b, E_{01}(0)] = 0$, $[c, E_{01}(0)] = 0$ can be proved as in the part (1), and we omit the details. Then it immediately follows that $[u, E_{01}(0)] = 0$ and $[v, E_{01}(0)] = 0$. Since $w = \frac{1}{3}[a, b]$, we also obtain $[w, E_{01}(0)] = 0$. $\square$

Lemma A.2. *We have*

$$\big[a, F_{32}(1)\big] = -F_{11}(0),$$

$$\big[b, F_{32}(1)\big] = E_{31}(-1)F_{21}(0) - E_{11}(-1)F_{01}(0) - E_{10}(-1)H_{32}(0) + (K+1)E_{10}(-1),$$

$$\big[c, F_{32}(1)\big] = E_{32}(-1)E_{31}(-1)F_{32}(0) + E_{32}(-1)H_{01}(-1)F_{01}(0) - 2E_{32}(-1)F_{01}(-1)H_{32}(0)$$
$$+ (2K+2)E_{32}(-1)F_{01}(-1) + E_{31}^2(-1)F_{31}(0) - 2E_{31}(-1)E_{01}(-1)F_{01}(0)$$
$$- E_{31}(-1)H_{01}(-1)H_{32}(0) + (K+1)E_{31}(-1)H_{01}(-1),$$

$$\big[u, F_{32}(1)\big] = -\left(K + \frac{5}{3}\right)E_{10}(-1) - E_{31}(-1)F_{21}(0) - \frac{2}{3}E_{21}(-1)F_{11}(0)$$
$$+ E_{11}(-1)F_{01}(0) + E_{10}(-1)H_{32}(0),$$

$$\big[v, F_{32}(1)\big] = -E_{32}(-1)E_{10}(-1)F_{11}(0) - 3E_{32}(-1)F_{01}(-1)$$
$$+ \frac{4}{3}E_{31}(-2) + E_{31}(-1)E_{11}(-1)F_{11}(0) - E_{31}(-1)H_{11}(-1)$$
$$- \frac{2}{3}a^2F_{11}(0) - \frac{1}{3}aE_{10}(-1) - a\big[b, F_{32}(1)\big] - 3\big[c, F_{32}(1)\big],$$

$$\big[w, F_{32}(1)\big] = -E_{32}(-2)F_{01}(0) + E_{32}(-1)F_{01}(-1)$$
$$- E_{31}(-2)H_{32}(0) + E_{31}(-1)H_{32}(-1) + KE_{31}(-2).$$

Proof. We only prove the equalities for $[b, F_{32}(1)]$ and $[u, F_{32}(1)]$. The other equalities can be proved similarly. We obtain

$$\big[b, F_{32}(1)\big] = \big[E_{31}(-1)E_{11}(-1) - E_{32}(-1)E_{10}(-1), F_{32}(1)\big]$$
$$= E_{31}(-1)\big[E_{11}(-1), F_{32}(1)\big] + \big[E_{31}(-1), F_{32}(1)\big]E_{11}(-1)$$
$$- E_{32}(-1)\big[E_{10}(-1), F_{32}(1)\big] - \big[E_{32}(-1), F_{32}(1)\big]E_{10}(-1)$$
$$= E_{31}(-1)F_{21}(0) - F_{01}(0)E_{11}(-1) - \big\{H_{32}(0) - K\big\}E_{10}(-1)$$
$$= E_{31}(-1)F_{21}(0) - E_{11}(-1)F_{01}(0) - E_{10}(-1)H_{32}(0) + (K+1)E_{10}(-1),$$

and

$$\big[u, F_{32}(1)\big] = \frac{1}{3}a\big[a, F_{32}(1)\big] + \frac{1}{3}\big[a, F_{32}(1)\big]a - \big[b, F_{32}(1)\big]$$
$$= -\frac{2}{3}E_{21}(-1)F_{11}(0) - \frac{2}{3}E_{10}(-1)$$

$$-E_{31}(-1)F_{21}(0) + E_{11}(-1)F_{01}(0) + E_{10}(-1)H_{32}(0) - (K+1)E_{10}(-1)$$

$$= -\left(K + \frac{5}{3}\right)E_{10}(-1) - E_{31}(-1)F_{21}(0) - \frac{2}{3}E_{21}(-1)F_{11}(0)$$

$$+ E_{11}(-1)F_{01}(0) + E_{10}(-1)H_{32}(0). \qquad \Box$$

We need one more lemma.

Lemma A.3. *We have the following commutator relations*:

$$\left[H_{32}(0), v - w\right] = 3(v - w), \qquad \left[F_{01}(0), v - w\right] = 0,$$

$$\left[F_{11}(0), v - w\right] = \left(\frac{1}{3}a^2 - 2b\right)E_{10}(-1) + aE_{31}(-1)H_{10}(-1)$$

$$- 5aE_{31}(-2) + 5E_{31}(-1)E_{21}(-2)$$

$$+ 3E_{31}^2(-1)F_{10}(-1) + 3E_{32}(-1)E_{31}(-1)F_{11}(-1)$$

$$- 3aE_{32}(-1)F_{01}(-1),$$

$$\left[F_{21}(0), v - w\right] = \left(-\frac{2}{3}a^2 + b\right)H_{21}(-1) + \frac{2}{3}aE_{21}(-2)$$

$$- 2aE_{31}(-1)F_{10}(-1) - 2aE_{32}(-1)F_{11}(-1)$$

$$+ 3E_{31}(-1)E_{11}(-1)H_{01}(-1) + 3E_{32}(-1)E_{10}(-1)H_{01}(-1)$$

$$- 6E_{31}(-1)E_{10}(-1)E_{01}(-1) + 6E_{32}(-1)E_{11}(-1)F_{01}(-1)$$

$$+ 4E_{11}(-1)E_{31}(-2) - 4E_{10}(-1)E_{32}(-2).$$

Proof. Since the proofs of the other equalities are similar, we only provide a proof for $F_{11}(0)$. We first have

$$\left[F_{11}(0), v - w\right] = \left[F_{11}(0), \frac{2}{9}a^3 - ab - 3c - w\right].$$

Considering each term separately, we get

$$\left[F_{11}(0), a^3\right] = 6a^2 E_{10}(-1) - 18aE_{31}(-2),$$

$$\left[F_{11}(0), ab\right] = \left[F_{11}(0), a\right]b + a\left[F_{11}(0), b\right]$$

$$= -2E_{10}(-1)b - a\left\{-E_{31}(-1)H_{11}(-1) + aE_{10}(-1) - 3E_{32}(-1)F_{01}(-1)\right\},$$

$$\left[F_{11}(0), c\right] = -E_{31}^2(-1)F_{10}(-1) + aE_{31}(-1)H_{01}(-1)$$

$$- E_{32}(-1)E_{31}(-1)F_{11}(-1) + 2aE_{32}(-1)F_{01}(-1),$$

$$\left[F_{11}(0), w\right] = aE_{31}(-2) - E_{31}(-1)E_{21}(-2).$$

Using two more relations

$$\left[E_{10}(-1), b\right] = -2E_{31}(-1)E_{21}(-2) \quad \text{and} \quad H_{11} = H_{10} + 3H_{01},$$

one can now obtain the result for $[F_{11}(0), v - w]$. $\quad \Box$

We now prove Proposition 2.3. For convenience, we state the proposition again:

Proposition A.4. *The vector $v_k \in N(k, 0)$ is a singular vector for the given value of k:*

$$v_k = \begin{cases} u.\mathbf{1} & \text{for } k = -\frac{5}{3}, \\ (v + w).\mathbf{1} & \text{for } k = -\frac{4}{3}, \\ u(v - w).\mathbf{1} & \text{for } k = -\frac{2}{3}. \end{cases}$$

Proof. To show that each vector v_k is a singular vector, it suffices to check that $E_{10}(0).v_k = 0$, $E_{01}(0).v_k = 0$, and $F_{32}(1).v_k = 0$ for each k. Assume that $k = -\frac{5}{3}$. By Lemma A.1, we obtain

$$E_{10}(0).v_k = E_{10}(0)u.\mathbf{1} = -\big[u, E_{10}(0)\big].\mathbf{1} = 0,$$

and similarly we get $E_{01}(0).v_k = 0$. Now we consider $F_{32}(1)$ and obtain by Lemma A.2

$$F_{32}(1).v_k = -\big[u, F_{32}(1)\big].\mathbf{1} = 0.$$

Assume that $k = -\frac{4}{3}$. It follows from Lemma A.1 that $E_{10}(0).v_k = 0$ and $E_{01}(0).v_k = 0$. We also obtain from Lemma A.2

$$
\begin{aligned}
F_{32}(1).v_k &= -\big[v + w, F_{32}(1)\big] \\
&= 3E_{32}(-1)F_{01}(-1) - \frac{4}{3}E_{31}(-2) + E_{31}(-1)H_{11}(-1) + \frac{1}{3}aE_{10}(-1) + (k+1)aE_{10}(-1) \\
&\quad + 3(2k+2)E_{32}(-1)F_{01}(-1) + 3(k+1)E_{31}(-1)H_{01}(-1) - E_{32}(-1)F_{01}(-1) \\
&\quad - E_{31}(-1)H_{32}(-1) - kE_{31}(-2) \\
&= 3E_{32}(-1)F_{01}(-1) - 2E_{32}(-1)F_{01}(-1) - E_{32}(-1)F_{01}(-1) \\
&\quad - \frac{4}{3}E_{31}(-2) + \frac{4}{3}E_{31}(-2) + \frac{1}{3}aE_{10}(-1) - \frac{1}{3}aE_{10}(-1) \\
&\quad + E_{31}(-1)H_{11}(-1) - E_{31}(-1)H_{01}(-1) - E_{31}(-1)H_{32}(-1) \\
&= 0,
\end{aligned}
$$

where we drop .$\mathbf{1}$ from the notation and use the equalities

$$H_{11} = H_{10} + 3H_{01} \quad \text{and} \quad H_{32} = H_{10} + 2H_{01}.$$

Assume that $k = -\frac{2}{3}$. We will continue to drop .$\mathbf{1}$ from the notation. It again follows from Lemma A.1 that $E_{10}(0).v_k = 0$ and $E_{01}(0).v_k = 0$. We now consider $F_{32}(1)$ and have

$$F_{32}(1).v_k = \big[F_{32}(1), u(v - w)\big] = \big[F_{32}(1), u\big](v - w) + u\big[F_{32}(1), v - w\big].$$

We first compute $[F_{32}(1), u](v - w)$. We use Lemma A.2 and obtain:

$$\big[F_{32}(1), u\big](v - w)$$

$$= \left(k + \frac{5}{3}\right) E_{10}(-1)(v - w) + E_{31}(-1) F_{21}(0)(v - w)$$

$$+ \frac{2}{3} E_{21}(-1) F_{11}(0)(v - w) - E_{11}(-1) F_{01}(0)(v - w) - E_{10}(-1) H_{32}(0)(v - w)$$

$$= \left(k + \frac{5}{3}\right) E_{10}(-1)(v - w) + E_{31}(-1)\big[F_{21}(0), v - w\big]$$

$$+ \frac{2}{3} E_{21}(-1)\big[F_{11}(0), v - w\big] - E_{11}(-1)\big[F_{01}(0), v - w\big] - E_{10}(-1)\big[H_{32}(0), v - w\big].$$

Now using Lemma A.3 and the fact that $H_{21} = 2H_{10} + 3H_{01}$ along with the relation $[a, b] = 3w$, we obtain the following:

$$\big[F_{32}(1), u\big](v - w)$$

$$= \frac{2}{9}\left(k - \frac{1}{3}\right) a^3 E_{10}(-1) - k \cdot ba E_{10}(-1) - (3k + 2) E_{10}(-1)c$$

$$- \left(4k + \frac{8}{3}\right) w E_{10}(-1) - 6u E_{31}(-1) H_{01}(-1) - 6u E_{32}(-1) F_{01}(-1)$$

$$- 2u E_{31}(-1) H_{10}(-1) + \left(2k + \frac{4}{3}\right) E_{31}(-1) a E_{21}(-2) + 3k \cdot b E_{31}(-2)$$

$$- \left(2k + \frac{2}{3}\right) a^2 E_{31}(-2)$$

$$= -\frac{2}{9} a^3 E_{10}(-1) + \frac{2}{3} ba E_{10}(-1) - 6u E_{31}(-1) H_{01}(-1)$$

$$- 6u E_{32}(-1) F_{01}(-1) - 2u E_{31}(-1) H_{10}(-1) + 2u E_{31}(-2)$$

$$= -\frac{2}{3} ua E_{10}(-1) - 6u E_{31}(-1) H_{10}(-1) - 6u E_{32}(-1) F_{01}(-1)$$

$$- 2u E_{31}(-1) H_{10}(-1) + 2u E_{31}(-2),$$

where the second equality is obtained by substituting $k = -\frac{2}{3}$.

Now we finally compute $u[F_{32}(1), v - w]$. From Lemma A.2 and $H_{11} = H_{10} + 3H_{01}$, we obtain:

$$u\big[F_{32}(1), v - w\big] = (6k + 10) u E_{32}(-1) F_{01}(-1) + \left(k + \frac{4}{3}\right) ua E_{10}(-1)$$

$$+ 2u E_{31}(-1) H_{10}(-1) + (3k + 8) u E_{31}(-1) H_{01}(-1) + \left(k - \frac{4}{3}\right) u E_{31}(-2)$$

$$= 6u E_{32}(-1) F_{01}(-1) + \frac{2}{3} ua E_{10}(-1)$$

$$+ 2u E_{31}(-1) H_{10}(-1) + 6u E_{31}(-1) H_{01}(-1) - 2u E_{31}(-2),$$

where we again substitute $k = -\frac{2}{3}$. Now it is clear that

$$F_{32}(1).v_k = \big[F_{32}(1), u\big](v - w) + u\big[F_{32}(1), v - w\big] = 0. \qquad \square$$

Appendix B. Lemmas for construction of polynomials

The following results will be useful.

Lemma B.1. *(See [14].) Let $X \in \mathfrak{g}$ and let $Y_1, \ldots, Y_m \in \mathcal{U}(\mathfrak{g})$. Then*

$$\left(X^n\right)_L (Y_1 \ldots Y_m) = \sum_{\substack{(k_1, \ldots, k_m) \in (\mathbb{Z}_+)^m \\ \sum k_i = n}} \binom{n}{k_1 \ldots k_m} \left(X^{k_1}\right)_L Y_1 \ldots \left(X^{k_m}\right)_L Y_m,$$

where $\binom{n}{k_1 \ldots k_n} = \frac{n!}{k_1! \cdots k_m!}$.

Proof. This can be seen most readily by considering an exponential generating function. Given a derivation D of $\mathcal{U}(\mathfrak{g})$, we may form the generating function

$$\exp(Dt) = 1 + Dt + \frac{D^2}{2!}t^2 + \cdots \in \left(\mathrm{End}\,\mathcal{U}(\mathfrak{g})\right)[[t]].$$

Applying this to a $Y \in \mathcal{U}(\mathfrak{g})$, we obtain an element $\exp(Dt)Y \in \mathcal{U}(\mathfrak{g})[[t]]$. The lemma is a direct consequence of the fact that $\exp(Dt)$ satisfies the identity

$$\exp(Dt)(Y_1 \cdots Y_n) = \exp(Dt)Y_1 \cdots \exp(Dt)Y_n. \tag{B.2}$$

(See [12].) To obtain the lemma, replace D with the adjoint action $X_L (= ad\, X)$ in Eq. (B.2), and equate the coefficient of t^n on both sides. Finally, multiplying both coefficients by $n!$, we obtain the identity in the lemma. $\square$

Lemma B.3.

(1) $(E_{ij}^m)_L(F_{ij}^m) \in m! H_{ij}(H_{ij} - 1) \cdots (H_{ij} - m + 1) + \mathcal{U}(\mathfrak{g}) E_{ij}$, for all $i\alpha + j\beta \in \Delta_+$.
(2) *Suppose $X \in \mathcal{U}(\mathfrak{g})_0$, the zero-weight subspace of $\mathcal{U}(\mathfrak{g})$. Then $X \in \mathfrak{n}_- \mathcal{U}(\mathfrak{g})$ if and only if $X \in \mathcal{U}(\mathfrak{g}) \mathfrak{n}_+$.*
(3) *For $Y \in \mathcal{U}(\mathfrak{g})$ and $n > r > 0$, we have*

$$\left(E_{ij}^n\right)_L \left(F_{ij}^r Y\right) \in F_{ij} \mathcal{U}(\mathfrak{g}) + \frac{n!}{(n-r)!}(H_{ij} - n + r) \cdots (H_{ij} - n + 1)\left(E_{ij}^{n-r}\right)_L Y + \mathcal{U}(\mathfrak{g}) E_{ij}.$$

Proof. Part (1) follows from direct computation and part (2) follows by considering a PBW basis given in triangular form for $\mathcal{U}(\mathfrak{g})_0$. For part (3), we consider an exponential generating function. For simplicity, let us write E, H, F, in place of E_{ij}, H_{ij}, F_{ij}. We then have:

$$\exp\left((adE)t\right)F^r Y = \left(\exp\left((adE)t\right)F\right)^r \exp\left((adE)t\right)Y$$

$$= \left(F + Ht - Et^2\right)^r \exp\left((adE)t\right)Y. \tag{B.4}$$

One can check

$$\left(F + Ht - Et^2\right)^r \in F\mathcal{U}(\mathfrak{g})[[t]] + \sum_{i=0}^{r} \binom{r}{i}(-1)^i (H - i)(H - i - 1) \cdots (H - r + 1) E^i t^{r+i}. \tag{B.5}$$

For convenience, we introduce the notation $(x)_{(i)} = x(x - 1) \cdots (x - i + 1)$ for $i > 0$, and $(x)_{(0)} = 1$. Then we have

$$(x)_{(i)} = (-1)^i(-x+i-1)_{(i)}, \tag{B.6}$$

$$(x+y)_{(m)} = \sum_{i=0}^{m} \binom{m}{i}(x)_{(i)}(y)_{(m-i)}. \tag{B.7}$$

We obtain the following identity using (B.6) and (B.7):

$$
\begin{aligned}
(x-n+r)_{(r)} &= (-1)^r\big(n-r-(x-r+1)\big)_{(r)} \\
&= (-1)^r \sum_{i=0}^{r} \binom{r}{i}(n-r)_{(i)}\big(-(x-r+1)\big)_{(r-i)} \\
&= (-1)^r \sum_{i=0}^{r} \binom{r}{i}(n-r)_{(i)}(-1)^{r-i}(x-i)_{(r-i)} \\
&= \sum_{i=0}^{r} \binom{r}{i}(-1)^i(n-r)_{(i)}(x-i)_{(r-i)}.
\end{aligned} \tag{B.8}
$$

Using this notation we combine Eqs. (B.4) and (B.5) to write:

$$\exp\big((adE)t\big)F^rY \in F\mathcal{U}(\mathfrak{g})[[t]] + \sum_{i=0}^{r} \binom{r}{i}(-1)^i(H-i)_{(r-i)}E^i t^{r+i}\exp\big((adE)t\big)Y.$$

Taking the coefficient of t^n on both sides gives:

$$
\begin{aligned}
\frac{1}{n!}(adE)^n\big(F^rY\big) &\in F\mathcal{U}(\mathfrak{g}) + \sum_{i=0}^{r} \binom{r}{i}(-1)^i(H-i)_{(r-i)}E^i \frac{1}{(n-r-i)!}(adE)^{n-r-i}Y \\
&\subseteq F\mathcal{U}(\mathfrak{g}) + \sum_{i=0}^{r} \binom{r}{i}(-1)^i(H-i)_{(r-i)}\frac{1}{(n-r-i)!}(adE)^{n-r}Y + \mathcal{U}(\mathfrak{g})E.
\end{aligned} \tag{B.9}
$$

With the substitution $x = H$, we obtain from (B.8)

$$(H-n+r)_{(r)} = \sum_{i=0}^{r} \binom{r}{i}(-1)^i \frac{(n-r)!}{(n-r-i)!}(H-i)_{(r-i)}. \tag{B.10}$$

After multiplying (B.9) by $n!$, we use the identity (B.10) to obtain

$$(adE)^n\big(F^rY\big) \in F\mathcal{U}(\mathfrak{g}) + \frac{n!}{(n-r)!}(H-n+r)_{(r)}(adE)^{n-r}Y + \mathcal{U}(\mathfrak{g})E.$$

This proves part (3).   □

The following lemmas will be needed for the construction of certain polynomials.

Lemma B.11. *The following identities hold in* $\mathcal{U}(\mathfrak{g})$. *First we have*:

$$\left(F_{21}^2\right)_L[a] = -2F_{21},$$

$$\left(F_{21}^4\right)_L[b] = 4!(F_{31}F_{11} - F_{32}F_{10}),$$

$$\left(F_{21}^6\right)_L[c] = -6!\left(F_{31}^2 F_{01} - F_{32}F_{31}H_{01} - F_{32}^2 E_{01}\right), \quad and$$

$$\left(F_{21}^3\right)_L[a] = \left(F_{21}^5\right)_L[b] = \left(F_{21}^7\right)_L[c] = 0.$$

Next we have:

$$(F_{31})_L[a] = F_{10},$$

$$\left(F_{31}^2\right)_L[b] = -2!(F_{31}E_{11} - F_{21}E_{01}),$$

$$\left(F_{31}^3\right)_L[c] = 3!\left(F_{31}(H_{32}+1)E_{01} + F_{32}E_{01}^2 - F_{31}^2 E_{32}\right), \quad and$$

$$\left(F_{31}^2\right)_L[a] = \left(F_{31}^3\right)_L[b] = \left(F_{31}^4\right)_L[c] = 0.$$

Finally we have:

$$(F_{32})_L[a] = F_{11},$$

$$\left(F_{32}^2\right)_L[b] = 2!(F_{32}E_{10} - F_{21}F_{01}),$$

$$\left(F_{32}^3\right)_L[c] = -3!\left(F_{32}F_{01}(H_{31}+2) + F_{31}F_{01}^2 - F_{32}^2 E_{31}\right), \quad and$$

$$\left(F_{32}^2\right)_L[a] = \left(F_{32}^3\right)_L[b] = \left(F_{32}^4\right)_L[c] = 0.$$

Proof. Using Lemma B.1, we have:

$$\left(F_{21}^4\right)_L[b] = \binom{4}{3\ 1}\left(F_{21}^3\right)_L E_{31}(F_{21L}E_{11}) - \binom{4}{3\ 1}\left(F_{21}^3\right)_L E_{32}(F_{21L}E_{10})$$

$$+ \binom{4}{2\ 2}\left(F_{21}^2\right)_L E_{31}\left(F_{21}^2\right)_L E_{11} - \binom{4}{2\ 2}\left(F_{21}^2\right)_L E_{32}\left(F_{21}^2\right)_L E_{10}$$

$$= \binom{4}{3\ 1}(6F_{32})(2F_{10}) - \binom{4}{3\ 1}(-6F_{31})(-2F_{11})$$

$$+ \binom{4}{2\ 2}(-2F_{11})(-6F_{31}) - \binom{4}{2\ 2}(2F_{10})(6F_{32})$$

$$= 4!F_{31}F_{11} - 4!F_{32}F_{10}.$$

We also have:

$$\left(F_{21}^5\right)_L[b] = \binom{5}{3\ 2}\left(F_{21}^3\right)_L E_{31}\left(F_{21}^2\right)_L E_{11} - \binom{5}{3\ 2}\left(F_{21}^3\right)_L E_{32}\left(F_{21}^2\right)_L E_{10}$$

$$= \binom{5}{3\ 2}(6F_{32})(-6F_{31}) - \binom{5}{3\ 2}(-6F_{31})(6F_{32})$$

$$= 0.$$

The other cases are proved similarly. □

Lemma B.12. *The following identities hold in* $\mathcal{U}(\mathfrak{g})$. *First we have*:

$$\frac{1}{2}\left(E_{21}F_{21}^2\right)_L[a] = -H_{21},$$

$$\frac{1}{4!}\left(E_{21}^2 F_{21}^4\right)_L[b] \equiv -2H_{21} \quad \left(\mathrm{mod}\,\mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right),$$

$$\frac{1}{6!}\left(E_{21}^3 F_{21}^6\right)_L[c] \equiv 3!H_{01}(H_{01}+2) \quad \left(\mathrm{mod}\,\mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right).$$

Next:

$$(E_{10}F_{31})_L[a] = H_{10}, \qquad \frac{1}{2}\left(E_{10}^2 F_{31}^2\right)_L[b] \equiv \frac{1}{3!}\left(E_{10}^3 F_{31}^3\right)_L[c] \equiv 0 \quad \left(\mathrm{mod}\,\mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right).$$

Finally:

$$(E_{11}F_{32})_L[a] = H_{11},$$

$$\frac{1}{2}\left(E_{11}^2 F_{32}^2\right)_L[b] \equiv -6H_{01} \quad \left(\mathrm{mod}\,\mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right),$$

$$\frac{1}{3!}\left(E_{11}^3 F_{32}^3\right)_L[c] \equiv -6H_{01}(H_{31}+2) \quad \left(\mathrm{mod}\,\mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right).$$

Proof. Using Lemmas B.1, B.11, we have:

$$\frac{1}{4!}\left(E_{21}^2 F_{21}^4\right)_L[b] = \left(E_{21}^2\right)_L(F_{31}F_{11} - F_{32}F_{10})$$

$$= \binom{2}{2\,0}\left(\left(E_{21}^2\right)_L F_{31}\right)F_{11} - \binom{2}{2\,0}\left(\left(F_{21}^2\right)_L F_{32}\right)F_{10} + \binom{2}{1\,1}(F_{21L}F_{31})(F_{21L}F_{11})$$

$$- \binom{2}{1\,1}(F_{21L}F_{32})(F_{21L}F_{10}) + \binom{2}{0\,2}F_{31}\left(F_{21}^2\right)_L F_{11} - \binom{2}{0\,2}F_{32}\left(F_{21}^2\right)_L F_{10}$$

$$= \binom{2}{2\,0}(-2E_{11})F_{11} - \binom{2}{2\,0}(2E_{10})F_{10} + \binom{2}{1\,1}(-F_{10})(-2E_{10})$$

$$- \binom{2}{1\,1}(-F_{11})(2E_{11}) + \binom{2}{0\,2}F_{31}(-6E_{31}) - \binom{2}{0\,2}F_{32}(6E_{32})$$

$$= -2\binom{2}{2\,0}(H_{11} + F_{11}E_{11}) - 2\binom{2}{2\,0}(H_{10} + F_{10}E_{10}) + 2\binom{2}{1\,1}F_{10}E_{10}$$

$$+ 2\binom{2}{1\,1}F_{11}E_{11} - 6\binom{2}{0\,2}F_{31}6E_{31} - 6\binom{2}{0\,2}F_{32}E_{32}$$

$$\equiv -2H_{11} - 2H_{10} \equiv -2H_{21} \quad \left(\mathrm{mod}\,\mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right).$$

The other cases follow in the same way. $\square$

Lemma B.13. *Suppose that $n, r, s, t \in \mathbb{Z}_+$ and $n = r + 2s + 3t$. Then the following hold in $\mathcal{U}(\mathfrak{g})$:*

$$\left(E_{21}^n F_{21}^{2n}\right)_L \left([a]^r [b]^s [c]^t\right)$$

$$\equiv (-1)^r \frac{n!}{(n-r)!} \frac{(2n)!}{(2n-2r)!} (H_{21} - n + 1) \cdots (H_{21} - n + r) \left(E_{21}^{n-r} F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right),$$

$$\left(E_{10}^n F_{31}^n\right)_L \left([a]^r [b]^s [c]^t\right)$$

$$\equiv \left(\frac{n!}{(n-r)!}\right)^2 (H_{10} - n + 1) \cdots (H_{10} - n + r) \left(E_{10}^{n-r} F_{31}^{n-r}\right)_L \left([b]^s [c]^t\right),$$

$$\left(E_{11}^n F_{32}^n\right)_L \left([a]^r [b]^s [c]^t\right)$$

$$\equiv \left(\frac{n!}{(n-r)!}\right)^2 (H_{11} - n + 1) \cdots (H_{11} - n + r) \left(E_{11}^{n-r} F_{32}^{n-r}\right)_L \left([b]^s [c]^t\right),$$

where all the congruences are modulo $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$.

Proof. We prove only the first case. Using Lemma B.1 we have:

$$\left(F_{21}^{2n}\right)_L \left([a]^r [b]^s [c]^t\right) = \frac{(2n)!}{(2r)!(4s+6t)!} \left(F_{21}^{2r}\right)_L \left([a]^r\right) \left(F_{21}^{4s+6t}\right)_L \left([b]^s [c]^t\right)$$

$$= \frac{(2n)!}{(2r)!(2n-2r)!} \frac{(2r)!}{(2!)^r} \left(\left(F_{21}^2\right)_L [a]\right)^r \left(F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right)$$

$$= \frac{(2n)!}{2^r (2n-2r)!} \left(\left(F_{21}^2\right)_L [a]\right)^r \left(F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right)$$

$$= \frac{(2n)!}{2^r (2n-2r)!} (-2F_{21})^r \left(F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right)$$

$$= (-1)^r \frac{(2n)!}{(2n-2r)!} (F_{21})^r \left(F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right),$$

since $(F_{21}^3)_L[a] = (F_{21}^5)_L[b] = (F_{21}^7)_L[c] = 0$ and $(F_{21}^2)_L[a] = -2F_{21}$ by Lemma B.11.

Then we use Lemma B.3(3) with $Y = (F_{21}^{2(n-r)})_L([b]^s[c]^t)$ to obtain:

$$\left(E_{21}^n F_{21}^{2n}\right)_L \left([a]^r [b]^s [c]^t\right)$$

$$= (-1)^r \frac{(2n)!}{(2n-2r)!} \left(E_{21}^n\right)_L \left(F_{21}^r \left(F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right)\right)$$

$$\in (-1)^r \frac{n!}{(n-r)!} \frac{(2n)!}{(2n-2r)!} (H_{21} - n + r) \cdots (H_{21} - n + 1) \left(E_{21}^{n-r} F_{21}^{2(n-r)}\right)_L \left([b]^s [c]^t\right)$$

$$+ F_{21}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g}) E_{21}.$$

Now it follows from Lemma B.3 (2) that we have

$$\left(E_{21}^n F_{21}^{2n}\right)_L\left([a]^r[b]^s[c]^t\right)$$

$$\equiv (-1)^r \frac{n!}{(n-r)!}\frac{(2n)!}{(2n-2r)!}(H_{21}-n+r)\cdots(H_{21}-n+1)\left(E_{21}^{n-r}F_{21}^{2(n-r)}\right)_L\left([b]^s[c]^t\right)$$

$$\left(\bmod \mathcal{U}(\mathfrak{g})\mathfrak{n}_+\right). \quad \square$$

We give one more lemma.

Lemma B.14. *The following hold*:

$$\left(E_{21}^4 F_{21}^8\right)_L\left([b]^2\right) \equiv 4!8!\left(2H_{21}H_{11}+2H_{10}(H_{10}-1)-6H_{01}(H_{01}+1)\right),$$

$$\left(E_{21}^5 F_{21}^{10}\right)_L\left([b][c]\right) \equiv -5!10!2H_{01}(H_{01}+2)(H_{21}-3),$$

$$\left(E_{10}^4 F_{31}^4\right)_L\left([b]^2\right) \equiv \left(E_{10}^5 F_{31}^5\right)_L\left([b][c]\right) \equiv 0,$$

$$\left(E_{11}^4 F_{32}^4\right)_L\left([b]^2\right) \equiv (4!)^2 2\left(9H_{01}(H_{01}-1)-H_{01}(H_{11}-3)\right),$$

$$\left(E_{11}^5 F_{32}^5\right)_L\left([b][c]\right) \equiv (5!)^2 6H_{01}(H_{01}-1)(H_{31}+2),$$

where all the congruences are modulo $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$.

Proof. We prove the first part only. From Lemma B.11, we have:

$$\left(F_{21}^8\right)_L\left([b]^2\right) = \binom{8}{4\,4}\left(\left(F_{21}^4\right)_L[b]\right)^2$$

$$= 8!\left(F_{31}F_{11}-F_{32}F_{10}\right)^2$$

$$= 8!\left(F_{31}^2 F_{11}^2 - 2F_{32}F_{31}F_{11}F_{10}-2F_{32}F_{31}F_{21}+F_{32}^2 F_{10}^2\right).$$

We thus obtain:

$$\frac{1}{8!}\left(E_{21}^4 F_{21}^8\right)_L\left([b]^2\right) = \left(E_{21}^4\right)_L\left(F_{31}^2 F_{11}^2 - 2F_{32}F_{31}F_{11}F_{10}-2F_{32}F_{31}F_{21}+F_{32}^2 F_{10}^2\right).$$

This equals the following element modulo $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$:

$$\binom{4}{3100}(-6E_{32})(-F_{10})F_{11}^2 - 2\binom{4}{3100}(6E_{31})(-F_{10})F_{11}F_{10}$$

$$+ \binom{4}{3010}(-6E_{32})F_{31}(-2E_{10})F_{11} - 2\binom{4}{3010}(6E_{31})F_{31}(-2E_{10})F_{10}$$

$$+ \binom{4}{2200}(-2E_{11})(-2E_{11})F_{11}^2 - 2\binom{4}{2200}(2E_{10})(-2E_{11})F_{11}F_{10}$$

$$+ \binom{4}{2110}(-2E_{11})(-F_{10})(-2E_{10})F_{11} - 2\binom{4}{2110}(2E_{10})(-F_{10})(-2E_{10})F_{10}$$

$$+ \binom{4}{2020}(-2E_{11})F_{31}(-6E_{31})F_{11} - 2\binom{4}{2020}(2E_{10})F_{31}(-6E_{31})F_{10}$$

$$- 2\binom{4}{3100}(6E_{31})(-F_{10})F_{21} + \binom{4}{3100}(6E_{31})(-F_{11})F_{10}^2$$

$$- 2\binom{4}{3010}(6E_{31})F_{31}H_{21} + \binom{4}{3010}(6E_{31})F_{32}(-2E_{11})F_{10}$$

$$- 2\binom{4}{2200}(2E_{10})(-2E_{11})F_{21} + \binom{4}{2200}(2E_{10})(2E_{10})F_{10}^2$$

$$- 2\binom{4}{2110}(2E_{10})(-F_{10})H_{21} + \binom{4}{2110}(2E_{10})(-F_{11})(-2E_{11})F_{10}$$

$$+ \binom{4}{2020}(2E_{10})F_{32}(-6E_{32})F_{10},$$

where we have omitted the term $\binom{4}{1300}(-F_{10})(-6E_{32})F_{11}^2 \in F_{10}\mathcal{U}(\mathfrak{g})$, which belongs to $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$ by Lemma B.3(2), as well as similar terms which also belong to $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$.

We now see that $\frac{1}{4!8!}(E_{21}^4 F_{21}^8)_L([b]^2)$ is equal to the following element modulo $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$:

$$E_{32}F_{10}F_{10}^2 + E_{31}F_{10}F_{11}F_{10} + 2E_{31}F_{10}F_{21} - E_{31}F_{11}F_{10}^2$$

$$+ 2E_{32}F_{31}E_{10}F_{11} + 4E_{31}F_{31}E_{10}F_{10} - 2E_{31}F_{31}H_{21} - 2E_{31}F_{32}E_{11}F_{10}$$

$$+ E_{11}^2F_{11}^2 + 2E_{10}E_{11}F_{11}F_{10} + 2E_{10}E_{11}F_{21} + E_{10}^2F_{10}^2$$

$$- 2E_{11}F_{10}E_{10}F_{11} - 4E_{10}F_{10}E_{10}F_{10} + 2E_{10}F_{10}H_{21} + 2E_{10}F_{11}E_{11}F_{10}$$

$$+ 3E_{11}F_{31}E_{31}F_{11} + 6E_{10}F_{31}E_{31}F_{10} - 3E_{10}F_{32}E_{32}F_{10}.$$

Again modulo $\mathcal{U}(\mathfrak{g})\mathfrak{n}_+$, this is equal to

$$6H_{01} - 4H_{10} + 2H_{21} + 2H_{10}(H_{10} - 1) + 2H_{11}(H_{11} - 1) + 4H_{31}H_{10}$$

$$- 2H_{21}H_{31} - 18H_{01} + 2H_{10}(H_{11} + 1) - 4H_{10} - 4H_{10}^2 + 2H_{10}H_{21}.$$

After simplifying terms in this expression, we finally obtain:

$$\frac{1}{4!8!}\left(E_{21}^4 F_{21}^8\right)_L\left([b]^2\right) \equiv 2H_{21}H_{11} - 6H_{01}(H_{01} + 1) + 2H_{10}(H_{10} - 1) \quad (\text{mod } \mathcal{U}(\mathfrak{g})\mathfrak{n}_+). \qquad \square$$

References

[1] D. Adamović, Some rational vertex algebras, Glas. Mat. Ser. III 29 (1) (1994) 25–40.

[2] D. Adamović, A. Milas, Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$, Math. Res. Lett. 2 (1995) 563–575.

[3] C. Dong, H. Li, G. Mason, Vertex operator algebras associated to admissible representations of $\widehat{sl_2}$, Comm. Math. Phys. 184 (1997) 65–93.

[4] I. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Pure Appl. Math., vol. 134, Academic Press, Boston, 1998.

[5] I.B. Frenkel, Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1) (1992) 123–168.

[6] W. Fulton, J. Harris, Representation Theory. A First Course, Grad. Texts in Math. (Readings in Mathematics), vol. 129, Springer-Verlag, New York, 1991.

[7] V.G. Kac, M. Wakimoto, Modular invariant representations of infinite-dimensional Lie algebras and superalgebras, Proc. Natl. Acad. Sci. USA 85 (14) (1988) 4956–4960.

[8] V.G. Kac, M. Wakimoto, Classification of modular invariant representations of affine algebras, in: Infinite-Dimensional Lie Algebras and Groups, Luminy-Marseille, 1988, in: Adv. Ser. Math. Phys., vol. 7, World Sci. Publ., Teaneck, NJ, 1989, pp. 138–177.

[9] V.G. Kac, M. Wakimoto, On rationality of W-algebras, Transform. Groups 13 (3–4) (2008) 671–713.

[10] D. Kazhdan, G. Lusztig, Tensor structures arising from affine Lie algebras. I, II, J. Amer. Math. Soc. 6 (4) (1993) 905–947, 949–1011.

[11] D. Kazhdan, G. Lusztig, Tensor structures arising from affine Lie algebras. III, IV, J. Amer. Math. Soc. 7 (2) (1994) 335–381, 383–453.

[12] J. Lepowsky, H. Li, Introduction to Vertex Operator Algebras and Their Representations, Progr. Math., vol. 227, Birkhäuser, Boston, Inc., Boston, MA, 2004.

[13] A. Meurman, M. Primc, Annihilating fields of standard modules of $sl(2, \mathbb{C})$ and combinatorial identities, Mem. Amer. Math. Soc. 137 (1999).

[14] O. Perše, Vertex operator algebras associated to type B affine Lie algebras on admissible half-integer levels, J. Algebra 307 (1) (2007) 215–248.

[15] O. Perše, Vertex operator algebras associated to certain admissible modules for affine Lie algebras of type A, Glas. Mat. Ser. III 43 (63) (1) (2008) 41–57.

[16] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1) (1996) 237–302.